Some aspects of (r, k)-parking functions

Richard P. Stanley
University of Miami and M.I.T.

and

Yinghui Wang (王颖慧)
M.I.T.

Basic definition

(r, k)-parking function: a sequence $(a_1, \ldots, a_n)$ of positive integers whose decreasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies

$$b_i \leq k + (i-1)r.$$

$\mathbf{PF}_n^{(r,k)}$: set of (r, k)-parking functions of length n

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Example. $(8,4,8,2)$ is **not** a $(2,3)$-parking function, since $(2,4,8,8) \not\leq (3,5,7,9)$ (termwise).

Parking scenario

Cars $C_1, \ldots, C_{rn}$ need to park in spaces $1, 2, \ldots, rn + k - 1$.

preference vector $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$, $1 \leq a_i \leq rn + k - 1$, where cars $C_{r(i-1)+1}, \ldots, C_{ri}$ all prefer a_i.

Cars go one at a time to their preferred space and then park in first available space.

Easy: all cars can park if and only if α is an (r, k)-parking function.

Number of (r, k)-parking functions

Theorem (**Steck** 1969, essentially).

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Proof. Completely analogous to Pollak's proof for $r = k = 1$.

Parking function symmetric function

$\mathfrak{S}_n$ acts on $\mathrm{PF}_n^{(r,k)}$ by permuting coordinates. Let $\boldsymbol{F_n^{(r,k)}}$ denote the Frobenius characteristic of this action.

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Equivalently,

$$F_n^{(r,k)} = \sum_{\beta} h_{m_1(\beta)} h_{m_2(\beta)} \cdots ,$$

where $\boldsymbol{\beta}$ runs over all **weakly increasing** (r, k)-parking functions, and $m_i(\beta)$ is the number of i's in β.

An example

Let $r = 1, k = 2, n = 3$. The weakly increasing $(1, 2)$-parking functions (a, b, c) of length three, i.e, $(a, b, c) \leq (2, 3, 4)$:

$$111 \quad 112 \quad 113 \quad 114 \quad 122 \quad 123 \quad 124$$

$$133 \quad 134 \quad 222 \quad 223 \quad 224 \quad 233 \quad 234$$

Hence

$$F_3^{(2,1)} = 2h_3 + 8h_2h_1 + 4h_1^3.$$

Basis expansions

$F_n^{(r,k)}$ has "nice" expansions in terms of the six classical bases m, p, h, e, s, f.

E.g.,

$$F_n^{(r,k)} = \frac{k}{rn+k} \sum_{\lambda \vdash n} \binom{rn+k}{d_1(\lambda), \ldots, d_n(\lambda), rn+k-\ell(\lambda)} h_\lambda$$

$$= k \sum_{\lambda \vdash n} z_\lambda^{-1} (rn+k)^{\ell(\lambda)-1} p_\lambda,$$

where $\boldsymbol{d_i(\lambda)}$ is the number of parts of λ equal to i.

A generating function

$$\mathcal{P}^{(r,k)}(t) := \sum_{n \geq 0} F_n^{(r,k)} t^n$$

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Proof: simple factorization argument.

Negative exponents

What about $\left(\mathcal{P}^{r,1}(t)\right)^k$ for $k < 0$?

Simplest case: $r = 1$ and $k = -1$.

Motivation

Let

$$\begin{aligned} \boldsymbol{A(t)} &= \sum_{n\geq 0} a_n t^n \\ \boldsymbol{B(t)} &= \sum_{n\geq 0} b_n t^n \\ &= \frac{1}{1-A(t)} = \sum_{k\geq 0} A(t)^k. \end{aligned}$$

Thus a_n counts **“prime”** objects and b_n all objects.

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Suggests: $1 - \frac{1}{\mathcal{P}^{(1,1)}(t)}$ might be connected with **"prime"** parking functions.

Prime parking functions

Definition (**I. Gessel**). A parking function is **prime** if it remains a parking function when we delete a 1 from it.

Note. A sequence $b_1 \leq b_2 \leq \cdots \leq b_n$ is an increasing parking function if and only if $1 \leq b_1 \leq \cdots \leq b_n$ is an increasing prime parking function.

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$$\Rightarrow \boldsymbol{\mathcal{PF}_4^{(1,1)}} = h_4 + 2h_3h_1 + h_2^2 + h_2h_1^2$$

Factorization of increasing PF's

1	2	3	4	5	6	7	8	9	10	11
1	1	3	3	4	4	7	8	8	9	10

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$$\to (1,1),\ (1,1,2,2),\ (1),\ (1,1,2,3)$$

Theorem. $\left(\mathcal{P}^{(1,1)}(t)\right)^{-1} = 1 - \sum_{n\geq 1} \mathrm{PPF}_n t^n$

A more complicated example

Coefficient of t^5 in $-\mathcal{P}^{(1,1)}(t)^{-2}$ is

$$2h_3h_1^2 + 2h_2^2h_1 + 4h_3h_2 + 4h_4h_1 + 2h_5.$$

Frobenius characteristic of the action of $\mathfrak{S}_5$ on all sequences $(a_1, \ldots, a_5) \in \mathbb{P}^{(1,1)}$ whose increasing rearrangement $b_1 \leq \cdots \leq b_5$ satisfies either of the conditions

1. $b_1 = b_2 = 1$, $b_3 \leq 2$, $b_4 \leq 3$, $b_5 \leq 3$, or
2. $b_1 = b_2 = b_3 = 2$, $b_4 \leq 3$, $b_5 \leq 4$.

Parking function basis

Write $\boldsymbol{F_n} = F_n^{(1,1)}$ (simplest case), with $F_0 = 1$.
For $\lambda = (\lambda_1, \lambda_2, \dots)$ write

$$\boldsymbol{F_\lambda} = F_{\lambda_1} F_{\lambda_2} \cdots .$$

Easy. $\{F_\lambda : \lambda \vdash n\}$ is a $\mathbb{Z}$-basis for Λ_n (homogeneous symmetric functions of degree n with integer coefficients).

Some problems

- Expand F_λ in the classical bases m, h, e, p, s, f, and vice versa.
- Formula or combinatorial interpretation of $\langle F_\lambda, F_\mu \rangle$.

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Very little is known.

Scalar products

Theorem.

$$\langle F_n, F_\lambda \rangle = \frac{1}{n+1} \prod_{i \geq 1} \frac{1}{\lambda_i + 1} \binom{(n+1)(\lambda_i+1) + \lambda_i - 1}{\lambda_i}$$

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In general $\langle F_\lambda, F_\mu \rangle$ has large prime factors. Is there a combinatorial interpretation, even for $\frac{1}{n+1}\binom{n(n+3)}{n}$?

Three expansions

d_i: number of parts of λ equal to i

$$e_n = \frac{1}{n+1}\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}\binom{n+\ell(\lambda)}{d_1, d_2, \ldots, rn}F_\lambda$$

$$p_n = \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)+1}\binom{n+\ell(\lambda)-1}{d_1, d_2, \ldots, rn-1}F_\lambda$$

$$h_n = \frac{1}{n-1}\sum_{\lambda\vdash n}(-1)^{\ell(\lambda)+1}\binom{n+\ell(\lambda)-2}{d_1, d_2, \ldots, rn-2}F_\lambda$$

The last slide

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