A Survey of Unimodality and Log-Concavity

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4 October 2021

Basic definitions

Definition. (1) A sequence $a_0, \ldots, a_n$ of real numbers is **unimodal** if $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$ for some j.

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Example. $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ (strongly log-concave)

I. REAL ZEROS

Newton's theorem

Theorem (**I. Newton**). *Let*

$$\gamma_1, \dots, \gamma_n \in \mathbb{R}$$

and

$$P(x) = \prod (x + \gamma_i) = \sum a_i \binom{n}{i} x^i = \sum b_i x^i.$$

Then $a_0, a_1, \dots, a_n$ is log-concave. Same as $b_0, \dots, b_n$ strongly log-concave.

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Proof. $P^{(n-i-1)}(x)$ has real zeros

$$\Rightarrow Q(x) := x^{i+1} P^{(n-i-1)}(1/x) \text{ has real zeros}$$

$$\Rightarrow Q^{(i-1)}(x) \text{ has real zeros.}$$

But $Q^{(i-1)}(x) = \frac{n!}{2}\left(a_{i-1} + 2a_i x + a_{i+1} x^2\right)$

$$\Rightarrow a_i^2 \geq a_{i-1} a_{i+1}. \quad \square$$

Basic linear algebra

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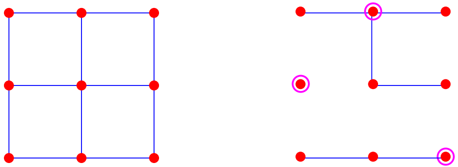
Example. $\boldsymbol{G}$: finite graph with vertex set $\boldsymbol{V}$ and $\boldsymbol{\mu_{uv}}$ edges between vertices u and v

$\boldsymbol{L}$: **Laplacian matrix** of G. Rows and columns indexed by V, with

$$L_{uv} = \begin{cases} \deg(v), & \text{if } u = v \\ -\mu_{uv}, & \text{if } u \neq v. \end{cases}$$

The Matrix-Tree theorem

Matrix-Tree Theorem (slightly expanded). $\det(I + xL) = \sum a_i x^i$, *where* $\mathbf{a_i}$ *is the number of rooted spanning forests of* G *with* i *edges. Thus* $\sum a_i x^i$ *has only real zeros, so* $a_0, a_1, \ldots, a_{\#V}$ *is strongly log-concave.*



What about unrooted spanning forests?

b_i: number of (unrooted) spanning forests of G with i edges.

More generally, let X be a finite subset of a vector space of dimension n, and let b_i be the number of i-element linearly independent subsets of X.

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Theorem (**Lenz**, 2013, based on **Huh**, 2012) $b_0, b_1, \ldots, b_n$ *is log-concave (with no external zeros).*

Proof of Huh based on **Hodge-Riemann relations** for the cohomology of certain varieties. Later generalized by **Adiprasito**, **Huh**, and **Katz** to any finite matroid (an abstract generalization of a finite subset of a vector space).

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What about **strongly** log-concave? To be discussed.

Total positivity

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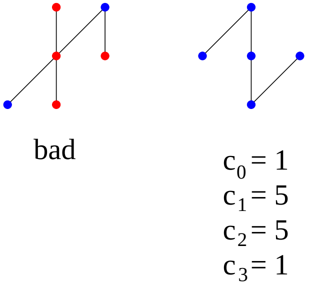
Theorem. *Let A be an $n \times n$ totally nonnegative matrix. Then all eigenvalues of A are real and nonnegative. Hence the characteristic polynomial* $\det(xI - A)$ *has only real zeros.*

An application

Let P be a finite poset (partially ordered set) with no induced $3+1$ or $2+2$, i.e., there do not exist elements $s < t < u, v$ with no other relations among them, nor elements $s < t, u < v$ with no other relationas among them. Let c_i be the number of i-element chains of P.



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Proof

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Proof. Let A be the matrix with rows and columns indexed by P, with

$$A_{st} = \begin{cases} 0, & \text{if } s \le t \\ 1, & \text{otherwise.} \end{cases}$$

Then A is totally nonnegative, and $\det(I + xA) = \sum c_i x^i$. $\square$

Two further remarks

- Can be shown that the $(2+2)$-avoiding hypothesis is unnecessary (using symmetric functions).

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• Multivariate generalizations of real-rooted polynomials: **stable polynomials** (**P. Brandén**) and **Lorentzian polynomials** (**P. Brandén** and **J. Huh**). Sample application:

Theorem. If I_k is the number of k-element independent sets of a matroid, then the sequence $I_0, I_1, \ldots$ is strongly log-concave. Conjectured by **Mason** in 1972. Also proved in a similar way by **Anari-Liu-Gharan-Vinzant**. (We mentioned earlier the proof by Lenz of log-concavity.)

II. ANALYTIC METHODS

Partitions

Let $p(n,k)$ be the number of partitions of n into k parts. E.g., $p(7,3) = 4$:

$$5+1+1, \quad 4+2+1, \quad 3+3+1, \quad 3+2+2.$$

$$\sum_{n\geq 0} p(n,k)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$$

$$\Rightarrow p(n,k) = \frac{1}{2\pi i}\oint \frac{s^{k-n-1}\,ds}{(1-s)(1-s^2)\cdots(1-s^k)}.$$

Theorem of Szekeres

Theorem (**G. Szekeres**, 1954) *For $n > N_0$, the sequence*

$$p(n,1), p(n,2), \dots, p(n,n)$$

is unimodal, with maximum at

$$k = c\sqrt{n}L + c^2\left(\frac{3}{2} + \frac{3}{2}L - \frac{1}{4}L^2\right) - \frac{1}{2}$$

$$+O\left(\frac{\log^4 n}{\sqrt{n}}\right)$$

$$c = \sqrt{6}/\pi, \qquad L = \log c\sqrt{n}.$$

III. ALEXANDROV-FENCHEL INEQUALITIES

Let ***K***, ***L*** be convex bodies (nonempty compact convex sets) in $\mathbb{R}^n$, and let $x, y \geq 0$. Define the **Minkowski sum**

$$xK + yL = \{x\alpha + y\beta : \alpha \in K,\ \beta \in L\}.$$

Then there exist $V_i(K, L) \geq 0$, the **(Minkowski) mixed volumes** of K and L, satisfying

$$\mathrm{Vol}(xK + yL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) x^{n-i} y^i.$$

Note $V_0 = \mathrm{Vol}(K),\ V_n = \mathrm{Vol}(L)$.

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Theorem (**Alexandrov-Fenchel**, 1936–38) $V_i^2 \geq V_{i-1} V_{i+1}$

Corollary. *Let P be an n-element poset. Fix $x \in P$. Let* $\mathbf{N_i}$ *denote the number of order-preserving bijections (linear extensions)*

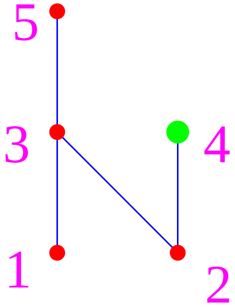
$$f : P \to \{1, 2, \ldots, n\}$$

such that $f(x) = i$. Then

$$N_i^2 \geq N_{i-1} N_{i+1}.$$

Proof. Find $K, L \subset \mathbb{R}^{n-1}$ such that $V_i(K, L) = N_{i+1}$. $\square$

An example



12345	4
12354	5
12435	3
21345	4
21354	5
21435	3
24135	4

[2]

$$(N_1, \ldots, N_5) = (0, 1, 2, 2, 2)$$

Generalizations

There are algebraic and algebraic-geometric generalizations of the Alexandrov-Fenchel inequalities with many applications.

IV. REPRESENTATIONS OF SL$(2,\mathbb{C})$ AND $\mathfrak{sl}(2,\mathbb{C})$

Representations of $\mathrm{SL}(2, \mathbb{C})$

Let

$$G = \mathrm{SL}(2, \mathbb{C}) = \{2 \times 2 \text{ complex matrices with determinant } 1\}.$$

Let $A \in G$, with eigenvalues θ, θ^{-1}. For all $n \geq 0$, there is a unique irreducible (polynomial) representation

$$\varphi_n : G \to \mathrm{GL}(V_{n+1})$$

of dimension $n + 1$, and $\varphi_n(A)$ has eigenvalues

$$\theta^{-n}, \theta^{-n+2}, \theta^{-n+4}, \ldots, \theta^{n}.$$

Every (continuous) representation is a direct sum of irreducibles.

Unimodal weight multiplicities

If $\varphi : G \to \mathrm{GL}(V)$ is any (finite-dimensional) representation, then

$$\operatorname{tr}\varphi(A) = \sum_{i\in\mathbb{Z}} a_i\theta^i, \quad a_i = a_{-i}$$

$$= a_0 + +a_1(\theta + \theta^{-1}) + \sum_{i\geq 2}(a_i - a_{i-2})\left(\theta^{-i} + \theta^{-i+2} + \cdots + \theta^i\right)$$

$$\Rightarrow a_i \geq a_{i-2}$$

$$\Rightarrow \{a_{2i}\}, \{a_{2i+1}\} \text{ are } \textbf{unimodal}$$

(and symmetric)

(Completely analogous construction for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.)

q-binomial coefficient

For $k, n \geq 0$ define

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \frac{(1-q^{n+k})(1-q^{n+k-1})\cdots(1-q^{n+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)},$$

a polynomial in q with nonnegative integer coefficients.

kth symmetric power

Example. $S^k(\varphi_n)$, eigenvalues

$$(\theta^{-n})^{t_0}\left(\theta^{-n+2}\right)^{t_1}\cdots(\theta^n)^{t_n},$$

$$t_0+t_1+\cdots+t_n=k, \quad t_i\geq 0$$

$$\Rightarrow \operatorname{tr}\varphi(A) =$$

$$\sum_{t_0+\cdots+t_n=k} \theta^{t_0(-n)+t_1(-n+2)+\cdots+t_n n}$$

$$= \quad \theta^{-nk}\begin{bmatrix} n+k \\ k \end{bmatrix}_{\theta^2}$$

$$= \quad \theta^{-nk}\sum_{i\geq 0} P_i(n,k)\theta^{2i},$$

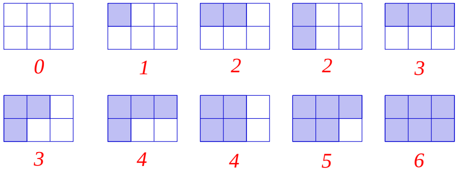
where $P_i(n,k)$ is the number of partitions of i with $\leq k$ parts, largest part $\leq n$.

Sylvester's theorem

$$\Rightarrow P_0(n,k), \ldots, P_{nk}(n,k)$$

is **unimodal** (**Sylvester**, 1878).

Combinatorial proof by **K. O'Hara**, 1990.



$$\sum_i P_i(3,2)q^i = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{(1-q^5)(1-q^4)}{(1-q^2)(1-q)}$$

Principal $\mathfrak{sl}(2,\mathbb{C})$

Example. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Then there exists a **principal $\mathfrak{sl}(2,\mathbb{C})$** $\subset \mathfrak{g}$. A representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ restricts to

$$\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V).$$

Example. $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$, φ = spin representation:

$$\Rightarrow (1+q)(1+q^2)\cdots(1+q^n)$$

has unimodal coefficients (**Dynkin** 1950, **Hughes** 1977). (No combinatorial proof known.)

Higher dimensional partitions

Recall: $\boldsymbol{P_i(n,k)}$: number of partitions of i with $\leq k$ parts, largest part $\leq n$, i.e, number of 1-dimensional integer arrays (sequences) $a_1, a_2, \ldots, a_k$ such that

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Generalize to $P_i(n_1, n_2, \ldots, n_{d+1})$: number of d-dimensional arrays $\left(a_{j_1, j_2, \ldots, a_{j_d}}\right)_{1 \leq j_r \leq n_r}$ of nonnegative integers, weakly decreasing in each coordinate, maximum entry $\leq n_{d+1}$, sum of entries $= i$.

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The case $d = 2$: **plane partitions** (**MacMahon**)

Example: $n_1 = n_2 = n_3 = 2$

$$\begin{array}{c|c|ccc|ccc|c|c} 00 & 10 & 11 & 10 & 20 & 11 & 21 & 20 & \dots & 22 \\ 00 & 00 & 00 & 10 & 00 & 10 & 00 & 10 & & 22 \end{array}$$

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$$(P_0, \dots, P_8) = (1, 1, 3, 3, 4, 3, 3, 1, 1)$$

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Proof follows from principal $\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{sl}(N, \mathbb{C})$, $N = 1 + \max n_j$, and choosing a certain irrep of $\mathfrak{sl}(N, \mathbb{C})$.

A conjecture

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Open for $d = 3$. Also open for $n_1 = n_2 = \cdots = n_{d+1} = 2$. In these cases, no nice way to compute P_i or $\sum P_i$.

For $n_1 = n_2 = \cdots = n_{d+1} = 2$, $\sum P_i$ is the order of the **free distributive lattice** on $d + 1$ generators (Dedekind's problem).

Projective varieties

Let **X** be an irreducible n-dimensional complex projective variety with finite quotient singularities (e.g., smooth).

$$\boldsymbol{\beta_i} = \dim_{\mathbb{C}} H^i(X;\mathbb{C})$$

$\mathfrak{sl}(2,\mathbb{C})$ acts on $H^*(X;\mathbb{C})$, and $H^i(X;\mathbb{C})$ is a weight space with weight $i - N$

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Follows from **hard Lefschetz theorem**.

Two examples

Example. $X = G_k(\mathbb{C}^{n+k})$ (**Grassmannian**). Then

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Example. (**Hessenberg varieties.**) Fix $1 \leq p \leq n-1$. For $w = w_1 \cdots w_n \in \mathfrak{S}_n$, let

$$\boldsymbol{d_p(w)} = \#\{(i,j) : w_i > w_j,\ 1 \leq j-i \leq p\}.$$

$$\begin{aligned} d_1(w) &= \#\textbf{descents} \text{ of } w \\ d_{p-1}(w) &= \#\textbf{inversions} \text{ of } w. \end{aligned}$$

Let $\boldsymbol{A_p(n,k)} = \#\{w \in \mathfrak{S}_n : d_p(w) = k\}$.

de Mari-Shayman theorem

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Theorem (**de Mari-Shayman**, 1987). The sequence

$$A_p(n,0), A_p(n,1), \ldots, A_p(n, p(2n-p-1)/2)$$

is **unimodal**.

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Proof. Construct a "generalized Hessenberg variety" X_{np} satisfying $\beta_{2k}(X_{np}) = A_p(n,k)$. □

Polytope definitions

(convex) polytope: the convex hull $\mathcal{P}$ of a finite set $S \subset \mathbb{R}^n$

$\dim \mathcal{P}$: dimension of affine span of $\mathcal{P}$ (so $\mathcal{P}$ is homeomorphic to a d-dimensional ball)

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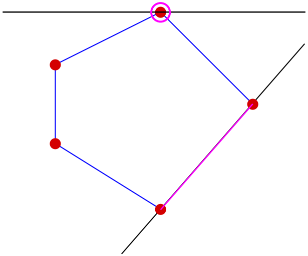
face of $\mathcal{P}$: the intersection of $\mathcal{P}$ with a supporting hyperplane H (so $\mathcal{P}$ lies on one side of H)



Simplicial polytopes and f-vectors

i-dimensional simplex: convex hull of $i+1$ affinely indepedent points in $\mathbb{R}^n$

simplicial polytope: every proper face is a simplex

E.g, the tetrahedron, octahedron, and icosahedron are simplicial, but not the cube or dodecahedron

Let $\mathcal{P}$ be a simplicial polytope, with f_i i-dimensional faces (with $f_{-1} = 0$). E.g., for the octahedron,

$$f_0 = 6, \quad f_1 = 12, \quad f_2 = 8.$$

The h-vector

$\mathcal{P}$: a simplicial polytope of dimension d

Define the ***h*-vector** $\boldsymbol{h(\mathcal{P})} = (h_0, h_1, \dots, h_d)$ of $\mathcal{P}$ by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

E.g., for the octahedron $\mathcal{O}$,,

$$(x-1)^3 + 6(x-1)^2 + 12(x-1) + 8 = x^3 + 3x^2 + 3x + 1,$$

so $h(\mathcal{O}) = (1, 3, 3, 1)$.

Conditions on h_i

Dehn-Sommerville equations (1905,1927): $h_i = h_{d-i}$

GLBC (**McMullen-Walkup**, 1971):

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor},$$

so the sequence $h_0, h_1, \ldots, h_d$ is unimodal.

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(Generalized Lower Bound Conjecture)

Even stronger condition (the **g-conjecture for simplicial polytopes**) conjectured by **McMullen** in 1971. Gave a conjectured **complete characterization** of f-vectors of simplicial polytopes.

Toric varieties

Note. Every simplicial polytope in $\mathbb{R}^n$ can be slightly perturbed to have rational vertices without affecting the combinatorial type.

Let $\boldsymbol{X(\mathcal{P})}$ be the **toric variety** corresponding to a rational simplicial polytope $\mathcal{P}$. Then $\mathcal{P}$ is an irreducible complex projective variety with finite quotient singularities. Let

$$\boldsymbol{H(P)} = H^0 \oplus H^2 \oplus H^4 \oplus \cdots \oplus H^{2d}$$

be its cohomology ring (over $\mathbb{C}$), so $\boldsymbol{\beta^{2i}} := \dim_{\mathbb{C}} H^{2i} < \infty$.

Fact. $\beta_{2i} = h_i$

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Also, $H(P)$ is generated as a $\mathbb{C}$-algebra by H^2. This and hard Lefschetz imply the g-conjecture for simplicial polytopes.

Triangulated spheres

A **triangulated sphere** is an abstract simplicial complex Δ whose geometric realization is a $(d-1)$-sphere.

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If Δ triangulates a $(d-1)$-sphere, then $(h_0, h_1, \ldots, h_d)$ is defined as before, and $h_i = h_{d-i}$.

g-conjecture for spheres

Theorem (**K. Adiprasito**, 2018). *The g-conjecture for spheres is true. In particular, if Δ triangulates a $(d-1)$-sphere then $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$ (and $h_i = h_{d-i}$).*

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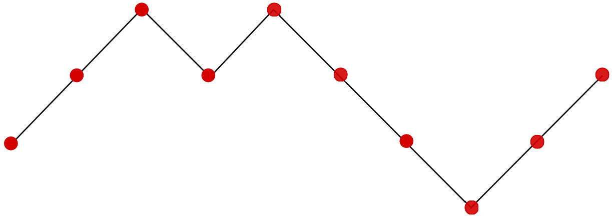
Idea of proof. There is a ring $\boldsymbol{H(\Delta)}$ (the face ring modulo a linear system of parameters) which for a certain l.s.o.p is isomorphic to $H(\mathcal{P})$ when Δ is the boundary complex of a rational simplicial polytope. Then prove a hard Lefschetz theorem for $H(\Delta)$.

V. SOME OPEN PROBLEMS

Fences

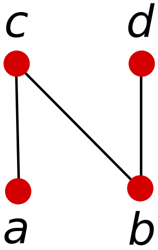
P: a p-element **fence**, i.e., a poset such as



order ideal: $I \subseteq P$ such that $t \in I, s \leq t \Rightarrow s \in I$

c_i: number of i-element order ideals of P

Conjecture of Morier-Genoud and Ovsienko



$$\varnothing, a, b, ab, bc, abc, abd, abcd$$

$$(c_0, \ldots, c_4) = (1, 2, 2, 2, 1)$$

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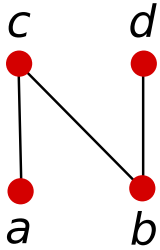

$$\varnothing, a, b, ab, bc, abc, abd, abcd$$

$$(c_0, \ldots, c_4) = (1, 2, 2, 2, 1)$$

Conjecture. For any p-element fence, the sequence $c_0, c_1, \ldots, c_p$ is unimodal.

Knots

K: a knot in $\mathbb{R}^3$

$\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$: the **Alexander polynomial** of K (a famous knot invariant).

Fact. A polynomial $\Gamma(t) \in \mathbb{Z}[t, t^{-1}]$ is the Alexander polynomial of some knot if and only if $\Gamma(1) = 1$ and $\Gamma(1/t) = \Gamma(t)$.

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Conjecture (**A. Stoimenow**, 2014) If K is alternating, then $\Delta_K(t)$ has log-concave coefficients. (Unimodality for $\Delta_K(-t)$ conjectured by **R. H. Fox** in 1962)

Genus distribution of graphs

G: finite connected graph

$g_i(G)$: number of combinatorially distinct cellular embeddings (i.e., every face is homeomorphic to an open disk) of G in an orientable surface of genus i

Fact. The sequence $g_0(G), g_1(G), g_2(G), \ldots$ (the **genus distribution** of G) has finitely many positive terms and no internal zeros.

Conjecture (**Gross-Robbins-Tucker**, 1989) The genus distribution of G is log-concave. (Known that $\sum g_i(G)t^i$ need not have only real zeros.)

The last slide

The last slide