$\mathbb{R}^{\mathbf{1},\boldsymbol{n}} = \mathbb{R} \times \mathbb{R}^n$: $(n+1)$-dimensional **Minkowski spacetime**

$$\boldsymbol{p} = (t, \boldsymbol{x}) \in \mathbb{R}^{1,n}$$

$$t: \text{ time coordinate}$$

$$\boldsymbol{x} = (x_1, \ldots, x_n) : \text{space coordinate}$$

Minkowski norm:

$$\begin{aligned} |(t, \boldsymbol{x})|^2 &= t^2 - |\boldsymbol{x}|^2 \\ &= t^2 - (x_1^2 + \cdots + x_n^2). \end{aligned}$$

$$\begin{aligned} (t, \boldsymbol{x}) \text{ is } \textbf{timelike} &: |(t, \boldsymbol{x})|^2 > 0 \\ (t, \boldsymbol{x}) \text{ is } \textbf{lightlike} &: |(t, \boldsymbol{x})| = 0 \\ (t, \boldsymbol{x}) \text{ is } \textbf{spacelike} &: |(t, \boldsymbol{x})|^2 < 0. \end{aligned}$$

$(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ are **timelike separated** if $(s, \boldsymbol{x}) - (t, \boldsymbol{y})$ is timelike (similarly **spacelike separated**).

Timelike separated means the events are **causally related**; a signal (traveling slower than $c = 1$) can reach one event from the other.

$$\boldsymbol{F'} : \text{ second reference frame,}$$

moving with constant velocity $\boldsymbol{v} \in \mathbb{R}^n$ with respect to the first F. An observer in the second frame measures coordinates $(t', \boldsymbol{x'})$, synchronized so that $t = t' = 0$ when the two frames coincide. Write

$$\boldsymbol{v} = (\tanh \rho)\boldsymbol{u},$$

where $\tanh \rho = |\boldsymbol{v}|$ (with $c = 1$) and $\boldsymbol{u}$ is a unit vector. The **Lorentz transformation** (for t'):

$$t' = (\cosh \rho)t - (\sinh \rho)\boldsymbol{x} \cdot \boldsymbol{u}.$$

Two timelike separated events occur in the **same order** for any observers. (Otherwise causality would be violated.) Two spacelike separated events can always occur in either order in suitable reference frames.

Main problem. Given k events in $\mathbb{R}^{1,n}$, in what different orders can they occur for different observers? How many such orders are there? (suggested by ??, NSA)

E.g., given three events $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}$ in $\mathbb{R}^{1,1}$, there do not exist three observers who see them in the orders

$$\boldsymbol{p_1} < \boldsymbol{p_2} < \boldsymbol{p_3}$$

(i.e., $\boldsymbol{p_1}$ before $\boldsymbol{p_2}$ before $\boldsymbol{p_3}$ in time)

$$\boldsymbol{p_2} < \boldsymbol{p_3} < \boldsymbol{p_1}$$

$$\boldsymbol{p_3} < \boldsymbol{p_1} < \boldsymbol{p_2}.$$

First assume all the events $\boldsymbol{p_i}$ are spacelike separated (easy to construct). Let $(t_i, \boldsymbol{x_i})$ be the coordinates of $\boldsymbol{p_i}$ with respect to a fixed observer. An observer moving at velocity $\boldsymbol{v} = (\tanh\rho)\boldsymbol{u}$ sees $\boldsymbol{p_i}$ occur at time

$$t_i' = (\cosh\rho)t_i - (\sinh\rho)\boldsymbol{x_i}\cdot\boldsymbol{u}.$$

Hence $\boldsymbol{p_i}$ occurs simultaneously to $\boldsymbol{p_j}$ for this observer if $t_i' = t_j'$, i.e.,

$$\begin{aligned}(\cosh\rho)t_i &- (\sinh\rho)\boldsymbol{x_i}\cdot\boldsymbol{u}\\ &= (\cosh\rho)t_j - (\sinh\rho)\boldsymbol{x_j}\cdot\boldsymbol{u}.\end{aligned}$$

Equivalently,

$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j})\cdot\boldsymbol{v}.$$

The set of such velocities $\boldsymbol{v}$ forms a **hyperplane** in $\mathbb{R}^n$. The different sides of the hyperplanes determine whether $\boldsymbol{p_i}$ occurs before or after $\boldsymbol{p_j}$.

Theorem. *The number of different orders in which the events* $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ *can occur is the number* $\boldsymbol{r(\mathcal{A})}$ *of regions of the arrangement*

$$\boldsymbol{\mathcal{A}} = \mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$$

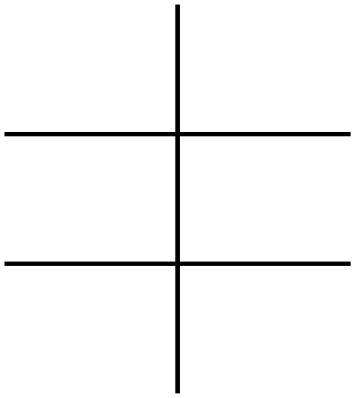
with hyperplanes

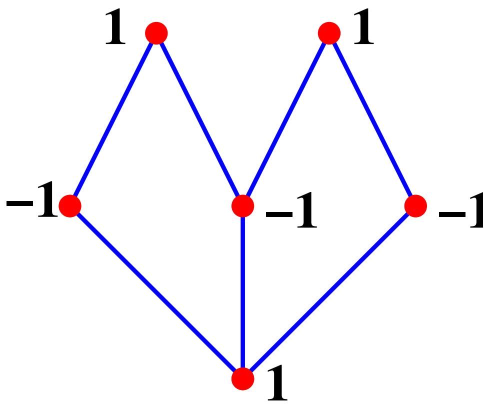
$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j}) \cdot \boldsymbol{v}, \quad 1 \leq i < j \leq k.$$

In general hopeless.

Assume therefore that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are **generic** (can be defined precisely).

Main tool: **intersection poset** $\boldsymbol{L_{\mathcal{A}}}$, set of all **nonempty** intersections of hyperplanes in $\mathcal{A}$ (including $\hat{\mathbf{0}} = \mathbb{R}^n$), ordered by reverse inclusion.





Möbius function $\boldsymbol{\mu} : L_{\mathcal{A}} \to \mathbb{Z}$

$$\begin{aligned} \mu(\hat{0}) &= 1 \\ \forall y > \hat{0},\ \sum_{x \leq y} \mu(x) &= 0. \end{aligned}$$

Characteristic polynomial:

$$\boldsymbol{\chi_{\mathcal{A}}(t)} = \sum_{x \in L_{\mathcal{A}}} \mu(x) t^{\dim(x)}.$$

Above example: $\chi_{\mathcal{A}}(t) = t^2 - 3t + 2$.

Theorem (Zaslavsky). Let $\dim(\mathcal{A}) = \boldsymbol{n}$. Then

$$\begin{aligned} r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) \\ &= \sum_{x \in L_{\mathcal{A}}} |\mu(x)|. \end{aligned}$$

Example. **Braid arrangement** $\boldsymbol{\mathcal{B}_k}$: $z_i = z_j$, $1 \leq i < j \leq k$.

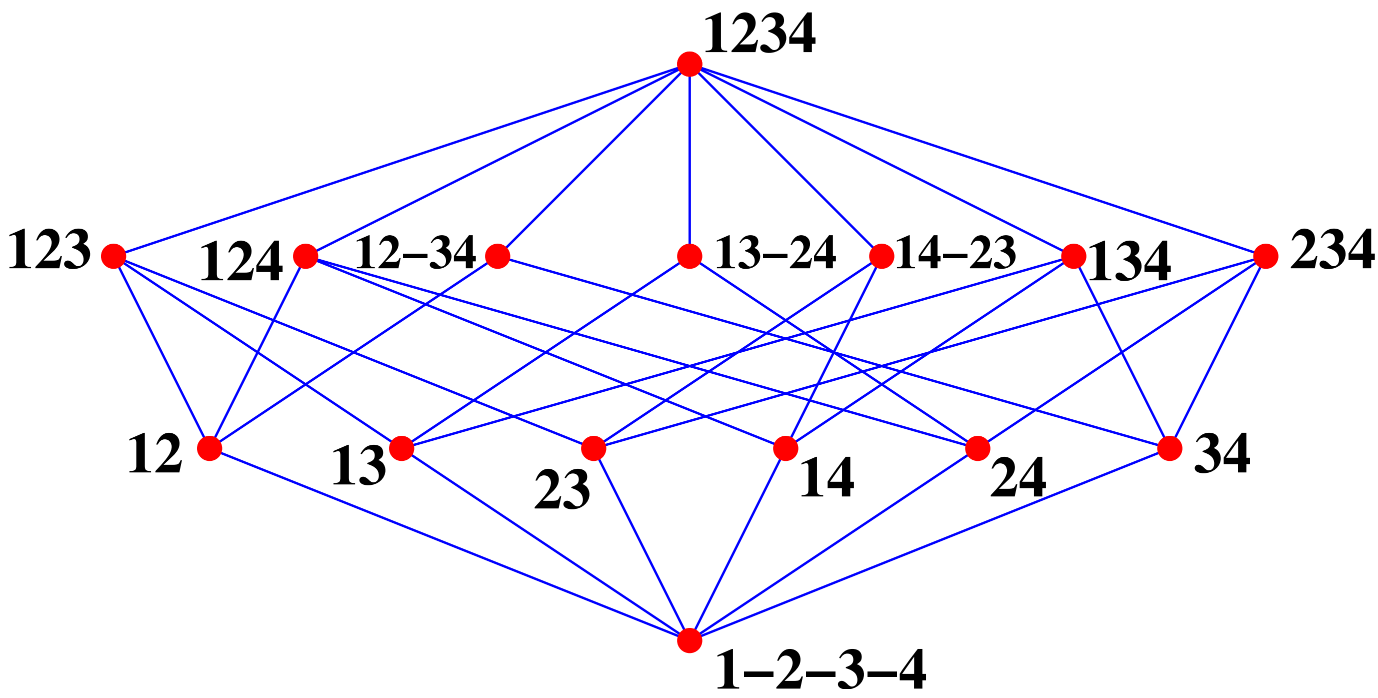
Intersection poset: $L_{\mathcal{B}_k} = \Pi_k$, the lattice of partitions of $\{1, 2, \ldots, k\}$ ordered by refinement.

A partition such as 134–26–5 corresponds to the intersection

$$z_1 = z_3 = z_4, \quad z_2 = z_6.$$

$$\begin{aligned} \chi_{\mathcal{B}_k}(t) &= t(t-1)\cdots(t-k+1) \\ &= \sum_{i=1}^{k} (-1)^{k-i} c(k,i) t^i. \end{aligned}$$

where $\boldsymbol{c(k,i)}$ is a **signless Stirling number of the first kind** (number of permutations of $\{1, 2, \ldots, k\}$ with i cycles).


1234
123
124
12–34
13–24
14–23
134
234
12
13
23
14
24
34
1–2–3–4

Recall: want $r(\mathcal{A})$, where $\mathcal{A}$ is given by:

$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j}) \cdot \boldsymbol{v}, \ \ 1 \leq i < j \leq k.$$

By genericity, no $n+1$ of these hyperplanes intersect. Any partition of $\{1, 2, \ldots, k\}$ with at least $k-n$ blocks defines an intersection. E.g., 134–26–5 defines

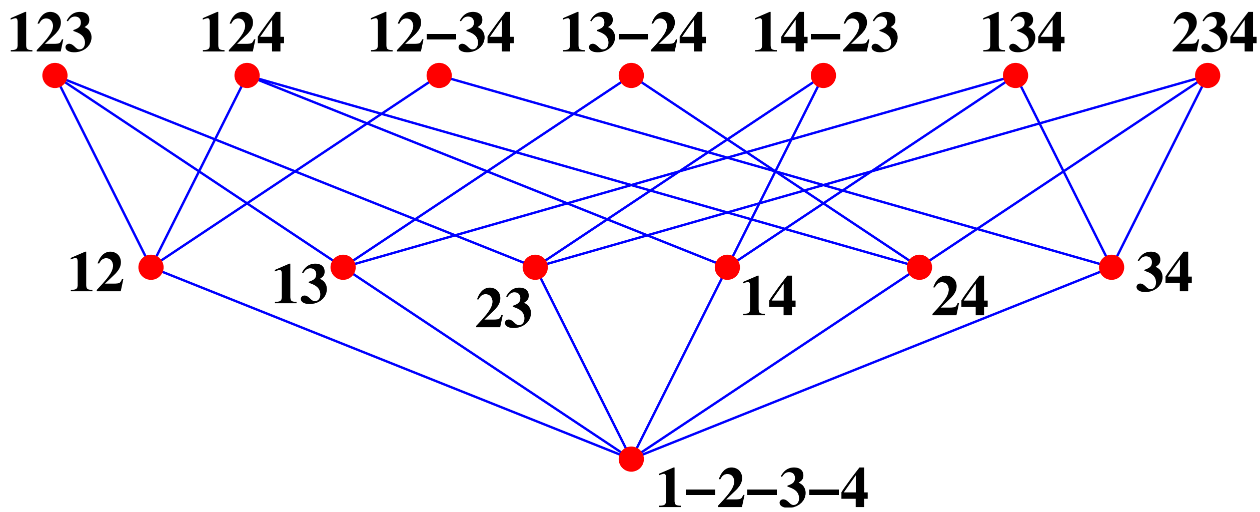
$$(\boldsymbol{x_1} - \boldsymbol{x_3}) \cdot \boldsymbol{v} = t_1 - t_3, \ (\boldsymbol{x_3} - \boldsymbol{x_4}) \cdot \boldsymbol{v} = t_3 - t_4$$

$$(\Rightarrow (\boldsymbol{x_1} - \boldsymbol{x_4}) \cdot \boldsymbol{v} = t_1 - t_4)$$

$$(\boldsymbol{x_2} - \boldsymbol{x_6}) \cdot \boldsymbol{v} = t_2 - t_6.$$

Hence $L(\mathcal{A})$ consists of all partitions of $\{1, 2, \ldots, k\}$ with **at most** $k-n$ blocks (rank n **truncation** of Π_k).

Rank two truncation of Π_4:



Recall: $\chi_{\mathcal{B}_k}(t) = \sum_{i=1}^{k} (-1)^{k-i} c(k,i) t^i$.
Hence:

Theorem. *For k generic spacelike separated events in $\mathbb{R}^{1,n}$, the number of orders in which these events can be seen by observers in different reference frames is*

$$r(\mathcal{A}) = c(k,k) + c(k,k-1) + \cdots + c(k,k-n).$$

(Maximum possible for **any** k events.)

What if events are not all spacelike separated?

Recall: if $(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ are timelike separated, then they appear in the same order in all reference frames.

In that case, solutions $\boldsymbol{v}$ to

$$s - t = (\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{v}$$

satisfy $|v| > 1$.

Let $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ be **any** k events in $\mathbb{R}^{1,n}$.

Separation graph: $\boldsymbol{G = G(p_1, \ldots, p_k)}$:

$V(G) = \{1, \ldots, k\}$
$E(G) = \{ij : \boldsymbol{p_i}, \boldsymbol{p_j} \text{ spacelike separated}\}$.

Theorem. *The number of different orders in which the events* $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ *can occur is the number* $\boldsymbol{r(\mathcal{A}_G)}$ *of regions of the arrangement*

$$\boldsymbol{\mathcal{A}_G} = \mathcal{A}_G(p_1, \ldots, p_k)$$

with hyperplanes

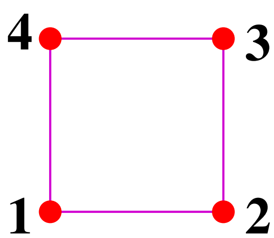
$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j}) \cdot \boldsymbol{v}, \quad ij \in E(G).$$

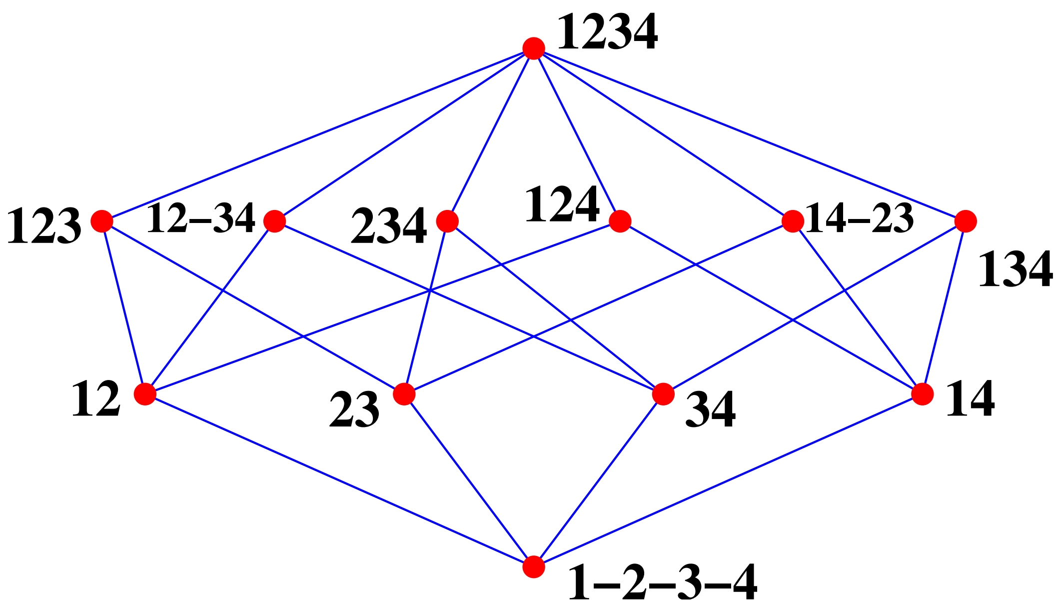
Let G be given, and $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ otherwise generic.

Graphical arrangement:

$$\mathcal{B}_G: \quad z_i = z_j,\ ij \in E(G).$$

Intersection poset: $L_{\mathcal{B}_G}$ = the lattice of partitions of $\{1, 2, \ldots, k\}$ whose blocks induce connected subgraphs of G, ordered by refinement.





$$\chi_{\mathcal{B}_G}(t) = \chi_G(t),$$

the **chromatic polynomial** of G.

For $q \in \mathbb{P}$,

$$\chi_G(q) = \#\{\kappa : V \to \{1, \ldots, q\} \mid ij \in E \Rightarrow \kappa(i) \neq \kappa(j)\}.$$

Exactly as for $\mathcal{B}_n = \mathcal{B}_{K_n}$, we get:

Theorem. *Let $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k} \in \mathbb{R}^{1,n}$ be generic events with separation graph G. Let $\chi_G(t) = t^k - a_1 t^{k-1} + \cdots + a_{k-1}t$. The number of different orders in which these events can be seen by observers in different reference frames is*

$$r(\mathcal{B}_G) = 1 + a_1 + a_2 + \cdots + a_n.$$

Note added 4/4/05. The above theorem is incorrect. It is being revised.

What separation graphs G can occur?

Suppose $(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ are timelike separated, as well as $(t, \boldsymbol{y})$ and $(u, \boldsymbol{z})$, with $s < t < u$. Then

$$
\begin{aligned}
t - s &> |\boldsymbol{y} - \boldsymbol{x}| \\
u - t &> |\boldsymbol{z} - \boldsymbol{y}| \\
\Rightarrow u - s &> |\boldsymbol{y} - \boldsymbol{x}| + |\boldsymbol{z} - \boldsymbol{y}| \\
&\geq |\boldsymbol{z} - \boldsymbol{x}|,
\end{aligned}
$$

so $\boldsymbol{x}$ and $\boldsymbol{z}$ are also timelike separated.

Thus the (edge) complement $\bar{\boldsymbol{G}}$ of a separation graph is a **comparability graph**, i.e., the graph of (distinct) comparable elements of a poset P, and separation graphs G are **incomparability graphs**.

Call P a **timelike poset**.

Many known characterizations of comparability graphs. E.g.:

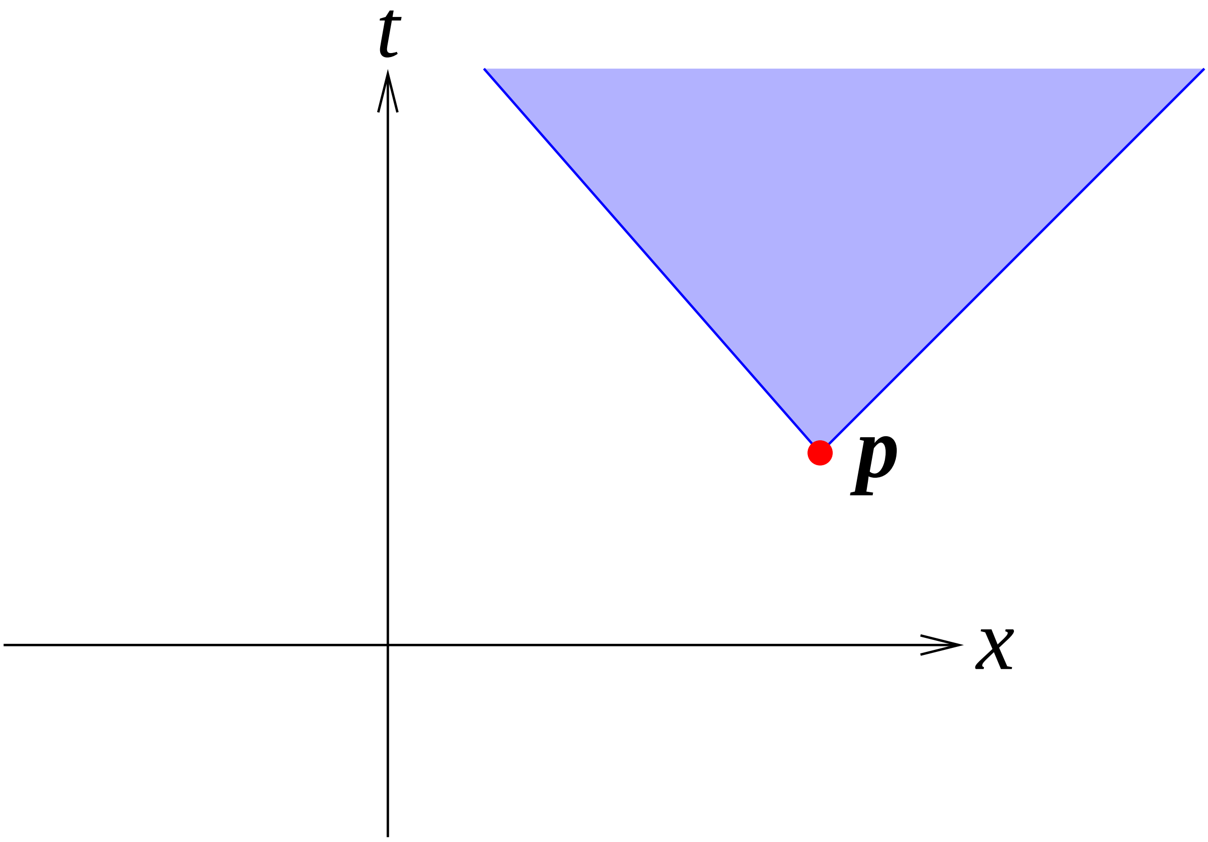
Theorem (Ghoulà-Houri, Gilmore and Hoffman). *$G = (V, E)$ is a comparability graph if and only if there is no sequence $(v_1, v_2, \ldots, v_{2n+1})$ of (not necessarily distinct) $v_i \in V$, with $n \geq 2$, such that $v_i v_{i+1} \in E$ and $v_i v_{i+2} \notin E$ (cyclically).*

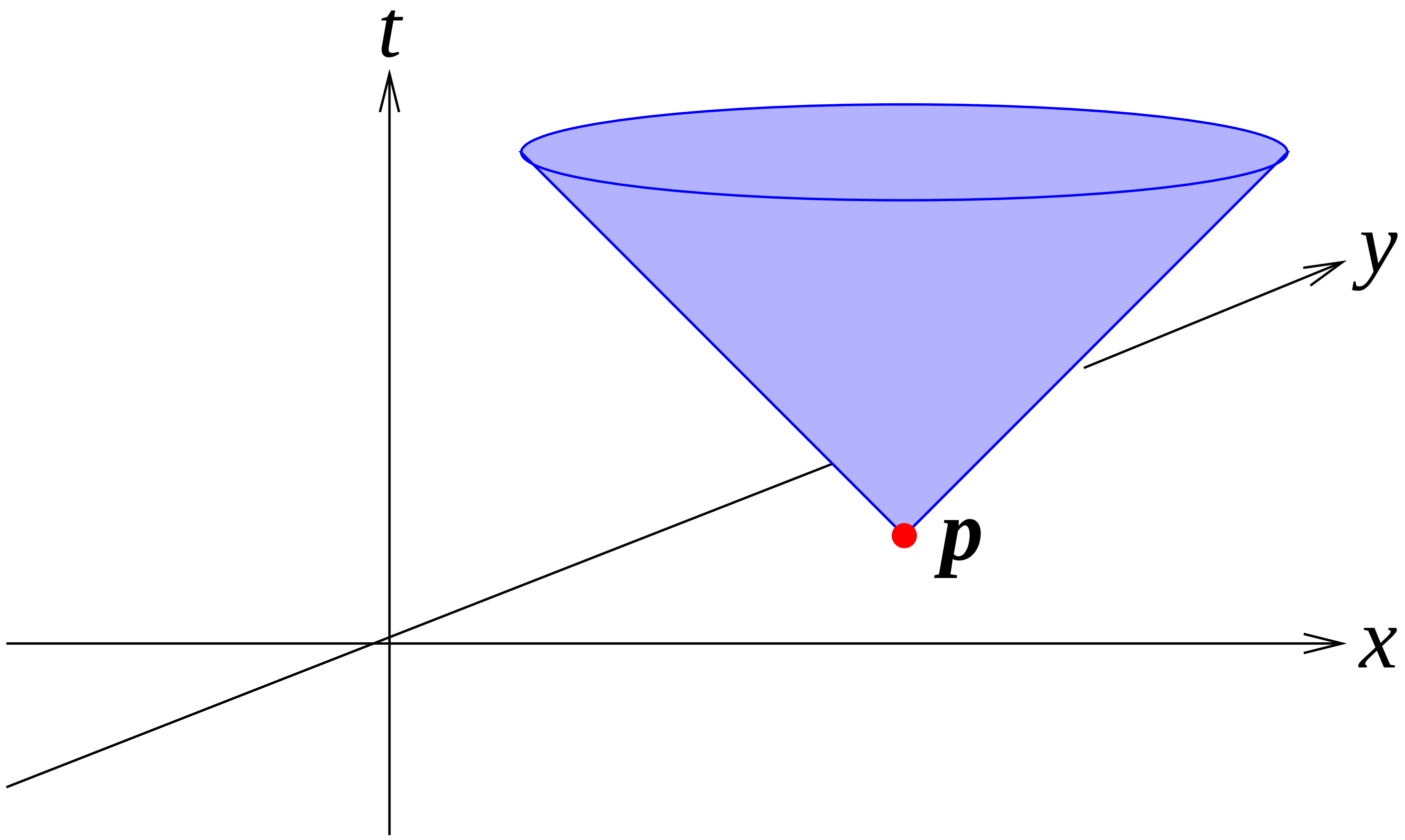
Another viewpoint.

Given $\boldsymbol{p} = (s, \boldsymbol{x}) \in \mathbb{R}^{1,n}$, define the (open) **future light cone** $\boldsymbol{C(p)}$ by

$$\begin{aligned} C(\boldsymbol{p}) &= \{\boldsymbol{q} = (t, \boldsymbol{y}) \in \mathbb{R}^{1,n} : \\ &\qquad t > s;\ \boldsymbol{p}, \boldsymbol{q} \text{ timelike sep.}\} \\ &= \{\boldsymbol{q} \in \mathbb{R}^{1,n} : \\ &\qquad t - s > |\boldsymbol{y} - \boldsymbol{x}|\}, \end{aligned}$$

a half-cone with apex $\boldsymbol{p}$, slope 45°, and opening in the t-direction.


t
p
x



Note. If $\boldsymbol{q} \in C(\boldsymbol{p})$ then $C(\boldsymbol{q}) \subset C(\boldsymbol{p})$. Thus

$$\boldsymbol{q} \in C(\boldsymbol{p}),\ \boldsymbol{r} \in C(\boldsymbol{q}) \Rightarrow \boldsymbol{r} \in C(\boldsymbol{p}),$$

so again $\bar{G}$ is a comparability graph. In fact, $\bar{G}$ is a **half-cone graph**, i.e., isomorphic to a set of half-cones, with an edge from C to C' if $C \subset C'$ or $C' \subset C$.

By intersecting the half-cones with a hyperplane $t = t_0$ for $t_0 \gg 0$, we see that half-cone orders are **sphere orders**.

What more can be said about timelike posets (or sphere orders) P? A poset P has **dimension** d if

$$P \subset \mathbb{R}^d, \quad P \not\subset \mathbb{R}^{d-1}.$$

Large literature on sphere orders and poset dimension.

Easy. The timelike posets for $n = 1$ (i.e., events in $\mathbb{R}^{1,1}$) are just the posets of dimension 2.

Theorem (Felsner, Trotter, Fishburn). Not every finite poset is a sphere order.

Suppose $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are generic space-like separated events in $\mathbb{R}^{1,n}$.

Recall: number of different orders in which the events can be seen in different reference frames is

$$r(\mathcal{A}) = c(k,k)+c(k,k-1)+\cdots+c(k,k-n).$$

What sets of orders (permutations of $1, \ldots, k$) are possible?

How many sets of permutations of the events are there?

Assume $\boldsymbol{n = 1}$, so $\boldsymbol{v} = v \in \mathbb{R}$.

As v increases from $-\infty$ to ∞, the order of the events $\boldsymbol{p_i}$ and $\boldsymbol{p_j}$ will change as v passes through

$$\boldsymbol{v_{ij}} := \frac{t_i - t_j}{x_i - x_j}.$$

Thus get a sequence

$$\boldsymbol{\Lambda} = \left(\pi_0, \pi_1, \ldots, \pi_{\binom{k}{2}}\right)$$

of permutations of $1, 2, \ldots, k$. Assume (without loss of generality) that for $v = 0$ the permutation is $1, 2, \ldots, k$, i.e.,

$$t_1 < t_2 < \cdots < t_k.$$

Varying the $\boldsymbol{p_i}$'s, the sequence Λ will change when

$$\frac{t_i - t_j}{x_i - x_j} = \frac{t_r - t_s}{x_r - x_s}.$$

Theorem. *The number of different* Λ *is the number of "effective regions" of the arrangement*

$$\frac{t_i - t_j}{x_i - x_j} = \frac{t_r - t_s}{x_r - x_s},$$

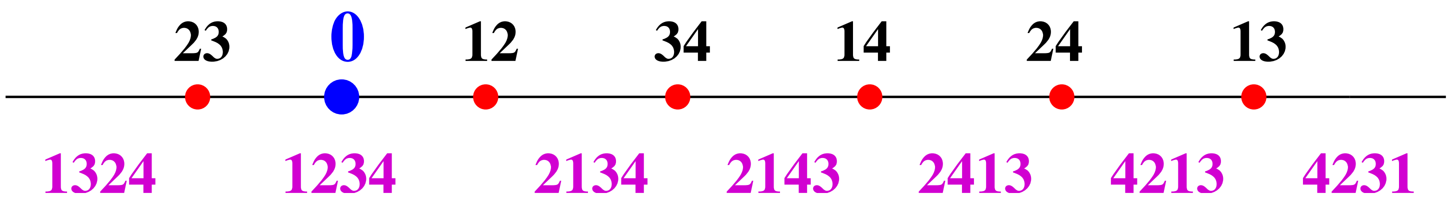
$1 \leq i < j \leq k$, $1 \leq r < s \leq k$, *of quadric hypersurfaces in* $\mathbb{R}^n \times \mathbb{R}^n$.

Example. $(\boldsymbol{p_1}, \ldots, \boldsymbol{p_4}) = (01, 13, 42, 77)$. Then $v_{12} = (1-0)/(3-1) = 1/2$, etc.

$$v_{23} < 0 < v_{12} < v_{34} < v_{14} < v_{24} < v_{13},$$

so

$$\Lambda = (1324, 1234, 2134, 2143, 2413, 4213, 4231).$$



In general,

$$\begin{aligned} \Lambda &= \sigma \cdot \left(\pi_0, \pi_1, \ldots, \pi_{\binom{k}{2}} \right) \\ &= \left(\sigma\pi_0, \sigma\pi_1, \ldots, \sigma\pi_{\binom{k}{2}} \right), \end{aligned}$$

where $\sigma \in \mathfrak{S}_k$, some $\pi_i = \sigma^{-1}$, and $(\pi_0, \pi_1, \ldots, \pi_{\binom{k}{2}})$ is a maximal chain in the **weak Bruhat order** of $\mathfrak{S}_k$.

Number of such $\sigma\cdot\left(\pi_0, \pi_1, \ldots, \pi_{\binom{k}{2}}\right)$ is

$$\left(1+\binom{k}{2}\right) f^{(k-1,k-2,\ldots,1)},$$

where $f^{(k-1,k-2,\ldots,1)}$ is the number of standard Young tableaux of shape $(k-1, k-2, \ldots, 1)$:

$$f^{(k-1,k-2,\ldots,1)}$$

$$= \frac{\binom{k}{2}!}{1^{k-1}\, 3^{k-2}\, 5^{k-3} \cdots (2k-3)^1}.$$

(upper bound on number of sequences $\Lambda = \Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$)

Is the converse true, i.e., does every such sequence $\sigma \cdot (\pi_0, \ldots, \pi_{\binom{k}{2}})$ arise as some Λ?

Theorem. No for $k \geq 5$.

But **yes** when $k = 3$. The different Λ are

$$(123, 132, 312, 321),\ (213, 123, 132, 312)$$
$$(231, 213, 123, 132),\ (321, 231, 213, 123)$$
$$(321, 312, 132, 123),\ (312, 132, 123, 213)$$
$$(132, 123, 213, 231),\ (123, 213, 231, 321).$$

In particular, none contain all of $123, 231, 312$.

Also **yes** for $k = 4$: $7 \cdot 16$ different Λ.

Equivalent to theory of **allowable sequences** (Goodman-Pollack).

A CLASSICAL ANALOGUE

$\boldsymbol{p_1}, \ldots, \boldsymbol{p_k} \in \mathbb{R}^n$ (Euclidean space)

At time $t = 0$, each point $\boldsymbol{p_i}$ emits a flash of light. In how many orders can these events be **observed** from different points $\boldsymbol{x}$?

Only consider $\boldsymbol{x}$ for which no two events are observed simultaneously.

$\boldsymbol{p}$ and $\boldsymbol{q}$ are observed simultaneously at points $\boldsymbol{x}$ on the perpendicular bisector of $\boldsymbol{p}$ and $\boldsymbol{q}$, with equation

$$(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{x} = \frac{1}{2}\left(|\boldsymbol{p}|^2 - |\boldsymbol{q}|^2\right).$$

Compare $(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{v} = s - t$.

Hence:

Theorem. *The number of different orders in which* $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ *can be observed is the number* $r(\mathcal{C})$ *of regions of the arrangement* $\mathcal{C}$ *with hyperplanes*

$$(\boldsymbol{p_i} - \boldsymbol{p_j}) \cdot \boldsymbol{x} = \frac{1}{2}\left(|\boldsymbol{p_i}|^2 - |\boldsymbol{p_j}|^2\right).$$

Assume $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are **generic**. Then $r(\mathcal{C})$ is computed exactly as before:

Theorem.

$$r(\mathcal{C}) = c(k,k)+c(k,k-1)+\cdots+c(k,k-n).$$

Good-Tideman (1977): voting theory

Zaslavsky (2002): arrangements

Kamiya-Orlik-Takemura-Terao (2004): ranking patterns