Plethysm and Kronecker Products

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Intended audience

Talk aimed at those with a general knowledge of symmetric functions but no specialized knowledge of plethysm and Kronecker product.

Introduction

- **plethysm** and **Kronecker product**: the two most important operations in the theory of symmetric functions that are not understood combinatorially
- Plethysm due to **D. E. Littlewood**
- **Internal product** of symmetric functions: the symmetric function operation corresponding to Kronecker product, due to **J. H. Redfield** and **D. E. Littlewood**
- We will give a survey of their history and basic properties.

Dudley Ernest Littlewood

- 7 September 1903 – 6 October 1979
- tutor at Trinity College: **J. E. Littlewood** (no relation)
- 1948–1970: chair of mathematics at University College of North Wales, Bangor

Plethysm

- introduced by **D. E. Littlewood** in 1936
- name suggested by **M. L. Clark** in 1944
- after Greek **plethysmos** ($\pi\lambda\eta\theta\upsilon\sigma\mu\acute{o}\varsigma$) for "multiplication"

Polynomial representations

V, W: finite-dimensional vector spaces/$\mathbb{C}$

polynomial representation

$$\varphi : \mathrm{GL}(V) \to \mathrm{GL}(W) (\text{example}) :$$

$$\varphi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

Definition of plethysm

$\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{X}$: vector spaces/$\mathbb{C}$ of dimensions m, n, p

$\boldsymbol{\varphi}: \mathrm{GL}(V) \to \mathrm{GL}(W)$: polynomial representation with character $\boldsymbol{f} \in \Lambda_n$, so $\mathrm{tr}\,\varphi(A) = f(x_1, \ldots, x_m)$ if A has eigenvalues $x_1, \ldots, x_m$

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$\Rightarrow \psi\varphi\colon \mathrm{GL}(V) \to \mathrm{GL}(X)$ is a polynomial representation. Let $\boldsymbol{g}[\boldsymbol{f}]$ (or $\boldsymbol{g} \circ \boldsymbol{f}$) denote its character, the **plethysm** of f and g.

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$\Rightarrow$ if $f = \sum_{u \in I} u$ ($\boldsymbol{I}$ = set of monomials) then

$$g[f] = g(u\colon u \in I).$$

Extension of defintions

Can extend definition of $g[f]$ to **any** symmetric functions f, g using

$$\begin{aligned} f[p_n] \;=\; p_n[f] &= f(x_1^n, x_2^n, \ldots) \\ (af+bg)[h] &= af[h]+bg[h],\ \ a,b\in\mathbb{Q} \\ (fg)[h] &= f[h]\cdot g[h], \end{aligned}$$

where $\boldsymbol{p_n} = x_1^n + x_2^n + \cdots$.

Examples

Note. Can let $m, n \to \infty$ and define $g[f]$ in **infinitely** many variables $x_1, x_2, \ldots$ (**stabilization**).

$h_2 = \sum_{i \leq j} x_i x_j$, so $f[h_2] = f(x_1^2, x_1 x_2, x_1 x_3, \ldots)$.

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By RSK, $\displaystyle\prod_{i \leq j} (1 - x_i x_j)^{-1} = \sum_\lambda s_{2\lambda}$. Since $\displaystyle\prod_i (1 - x_i)^{-1} = 1 + h_1 + h_2 + \cdots$, we get

$$h_n[h_2] = \sum_{\lambda \vdash n} s_{2\lambda},$$

i.e., the character of $S^n(S^2V)$.

Schur positivity

$\varphi: \mathrm{GL}(V) \to \mathrm{GL}(W)$: polynomial representation with character $f \in \Lambda_n$

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No combinatorial proof known, even for $f = h_m$, $g = h_n$.

Schur-Weyl duality for plethysm

$\boldsymbol{N(\mathfrak{S}_k^m)}$: normalizer of $\mathfrak{S}_k^m$ in $\mathfrak{S}_{km}$, the **wreath product** $\mathfrak{S}_k \wr \mathfrak{S}_m$, or order $k!^m \cdot m!$

ch(ψ): the **Frobenius characteristic** of the class function ψ of $\mathfrak{S}_n$, i.e.,

$$\mathrm{ch}(\psi) = \sum_{\lambda \vdash n} \langle \psi, \chi^\lambda \rangle s_\lambda.$$

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Theorem (**Specht**). Special case:

$$\mathrm{ch}\left(1_{N(\mathfrak{S}_k^m)}^{\mathfrak{S}_{km}}\right) = h_m[h_k]$$

$\blacksquare$

Main open problem

Find a combinatorial interpretation of $\langle s_\lambda[s_\mu], s_\nu \rangle$, especially the case $\langle h_m[h_n], s_\nu \rangle$.

E.g., $h_2[h_n] = \sum_{k=0}^{\lfloor n/2 \rfloor} s_{2(n-k),2k}$.

$h_3[h_n]$ known, but quickly gets more complicated.

Plethystic inverses

Note $p_1 = s_1 = \sum x_i$ and $g[s_1] = s_1[g] = g$. We say that f and g are **plethystic inverses**, denoted $f = \boldsymbol{g}^{[-1]}$, if

$$f[g] = g[f] = s_1.$$

Note. $f[g] = s_1 \Leftrightarrow g[f] = s_1$.

Lyndon symmetric function L_n

C_n: cyclic subgroup of $\mathfrak{S}_n$ generated by $(1, 2, \ldots, n)$

ζ: character of C_n defined by $\zeta(1, 2, \ldots, n) = e^{2\pi i/n}$

Lyndon symmetric function:

$$\begin{aligned} L_n &= \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d} \\ &= \mathrm{ch}\,\mathrm{ind}_{C_n}^{\mathfrak{S}_n} e^{2\pi i/n} \end{aligned}$$

Cadogan's theorem

$f = e_1 - e_2 + e_3 - e_4 + \cdots$

$g = L_1 + L_2 + L_3 + \cdots$

Theorem (**Cadogan**, 1971). $g = f^{[-1]}$

Lyndon basis

Extend L_n to a basis $\{L_\lambda\}$ for the ring Λ of symmetric functions:

Let $\boldsymbol{m, k} \geq 1$, and $\boldsymbol{\langle k^m \rangle} = (k, k, \ldots, k)$ (m times). Define

$$\begin{aligned} \boldsymbol{L_{\langle k^m \rangle}} &= h_m[L_k] \\ \boldsymbol{L_{\langle 1^{m_1}, 2^{m_2}, \ldots \rangle}} &= L_{\langle 1^{m_1} \rangle} L_{\langle 2^{m_2} \rangle} \cdots . \end{aligned}$$

Equivalently, for fixed m,

$$\sum_{k \geq 0} L_{\langle k^m \rangle} t^k = \exp \sum_{n \geq 1} \frac{1}{n} L_n(p_i \to p_{mi}) t^i.$$

Cycle type

Fix $n \geq 1$. Let $S \subseteq [n-1]$.

F_S : Gessel fundamental quasisymmetric function

Example. $n = 6, S = \{1, 3, 4\}$:

$$F_S = \sum_{1 \leq i_1 < i_2 \leq i_3 < i_4 < i_5 \leq i_6} x_{i_1} \cdots x_{i_6}.$$

Theorem (**Gessel-Reutenauer**, 1993). We have

$$\sum_{\substack{w \in \mathfrak{S}_n \\ \text{type}(w) = \lambda}} F_{D(w)} = L_\lambda.$$

An example

Example. $\lambda = (2,2)$:

w	$D(w)$
2143	1,3
3412	2
4321	1,2,3

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$$L_{(2,2)} = s_{(2,2)} + s_{(1,1,1,1)} = (F_{1,3} + F_2) + F_{1,2,3}$$

Free Lie algebras

$\boldsymbol{A}$: the alphabet $x_1, \ldots, x_n$

$\mathbb{C}\langle \boldsymbol{A} \rangle$: free associative algebra over $\mathbb{C}$ generated by A

$\boldsymbol{\mathcal{L}}[\boldsymbol{A}]$: smallest subalgebra of $\mathbb{C}\langle A \rangle$ containing $x_1, \ldots, x_n$ and closed under the **Lie bracket** $[u, v] = uv - vu$ (**free Lie algebra**)

$\mathbf{Lie}_n$

$\mathbf{Lie}_n$: multilinear subspace of $\mathbb{C}\langle A\rangle$ (degree one in each x_i)

basis: $[x_1,[x_{w(2)},[x_{w(3)},[\cdots]\cdots]]$, $w\in\mathfrak{S}_{[2,n]}$

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Note. Can be extended to L_λ (decomposition of $\mathbb{C}\langle A\rangle$)

Partition lattices

$\boldsymbol{\Pi_n}$: poset (lattice) of partitions of $\{1,\ldots,n\}$, ordered by refinement

$\boldsymbol{\widetilde{\Pi}_n}$: $\Pi_n - \{\hat{0},\hat{1}\}$

$\boldsymbol{\Delta(\Pi_n)}$: set of chains of $\widetilde{\Pi_n}$ (a simplicial complex)

$\boldsymbol{\widetilde{H}_i}(\Pi_n)$: ith reduced homology group of $\Delta(\Pi_n)$, say over $\mathbb{C}$

Homology and $\mathfrak{S}_n$-action

Theorem. (a) $\widetilde{H}_i(\Pi_n) = 0$ unless $i = n - 3$, and $\dim \widetilde{H}_{n-3}(\Pi_n) = (n-1)!$.

(b) Action of $\mathfrak{S}_n$ on $\widetilde{H}_{n-3}(\Pi_n)$ has Frobenius characteristic ωL_n.

Lower truncations of Π_n

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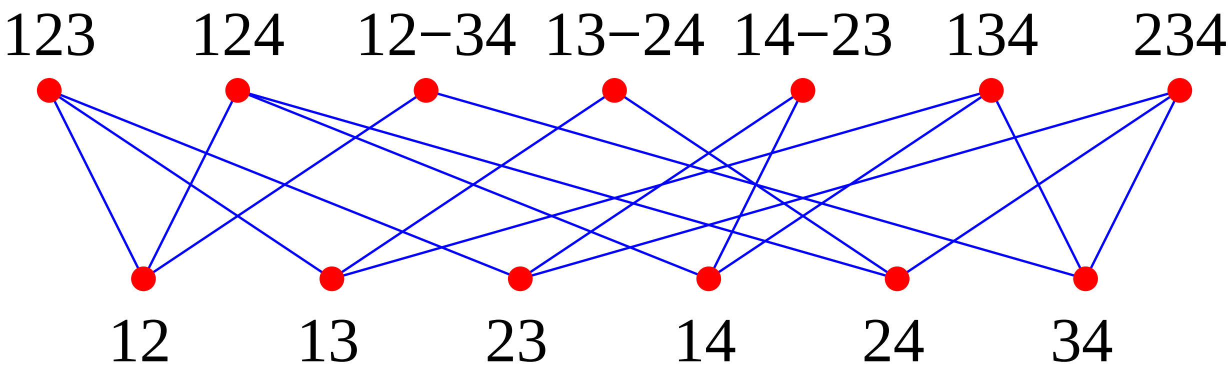
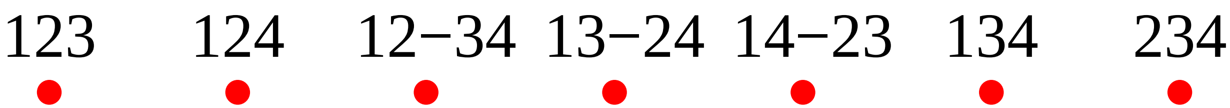


$$\widetilde{\Pi}_4 = \widetilde{\Pi}_4(2)$$



$$\widetilde{\Pi}_4(1)$$

$\mathfrak{S}_n$-action on lower truncations

Theorem (**Sundaram**, 1994) *The Frobenius characteristic of the action of $\mathfrak{S}_n$ on the top homology of $\widetilde{\Pi}_n(r)$ is the degree n term in the plethysm*

$$(\omega(L_{r+1} - L_r + \cdots + (-1)^r L_1))[h_1 + \cdots + h_n].$$


PART
2

Tensor product of characters

χ, ψ: characters (or any class functions) of $\mathfrak{S}_n$

$\chi \otimes \psi$ (or $\chi\psi$): **tensor** (or **Kronecker**) **product** of χ and ψ, i.e.,

$$(\chi \otimes \psi)(w) = \chi(w)\psi(w).$$

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$f\colon \mathfrak{S}_n \to \mathrm{GL}(V)$: representation with character χ

$g\colon \mathfrak{S}_n \to \mathrm{GL}(W)$: representation with character ψ

$\Rightarrow \chi \otimes \psi$ is the character of the representation $f \otimes g\colon \mathfrak{S}_n \to \mathrm{GL}(V \otimes W)$ given by

$$(f \otimes g)(w) = f(w) \otimes g(w).$$

Kronecker coefficients

Let $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \vdash n$.

$$\begin{aligned} \boldsymbol{g_{\lambda\mu\nu}} &= \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle \\ &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi^\lambda(w) \chi^\mu(w) \chi^\nu(w) \end{aligned}$$

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Consequences:

- $g_{\lambda\mu\nu} \in \mathbb{N} = \{0, 1, \ldots\}$
- $g_{\lambda\mu\nu}$ is symmetric in λ, μ, ν.

Internal product

Recall for $\lambda, \mu, \nu \vdash n$,

$$g_{\lambda\mu\nu} = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle.$$

Define the *internal product* $\boldsymbol{s_\lambda * s_\mu}$ by

$$\langle s_\lambda * s_\mu, s_\nu \rangle = g_{\lambda\mu\nu}.$$

Extend to any symmetric functions by bilinearity.

Tidbits

(a) $s_\lambda * s_n = s_\lambda, \quad s_\lambda * s_{\langle 1^n \rangle} = \omega s_\lambda$

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(b) **Conjecture** (**Saxl**, 2012). Let $\boldsymbol{\delta_n} = (n-1, n-2, \ldots, 1)$ and $\lambda \vdash \binom{n}{2}$. Then $\langle s_{\delta_n} * s_{\delta_n}, s_\lambda \rangle > 0$.

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(d) $\sum_{\lambda,\mu,\nu\vdash n} g^2_{\lambda\mu\nu} = \sum_{\mu\vdash n} z_\mu$. Hence

$$\max_{\lambda,\mu,\nu\vdash n} \log g_{\lambda\mu\nu} \sim \frac{n}{2}\log n.$$

What λ, μ, ν achieve the maximum?

Generating function

Theorem (**Schur**).

$$\prod_{i,j,k}(1 - x_i y_j z_k)^{-1} = \sum_{\lambda,\mu,\nu} g_{\lambda\mu\nu} s_\lambda(x) s_\mu(y) s_\nu(z).$$

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Equivalent formulation:

Write $\boldsymbol{xy}$ for the alphabet $\{x_iy_j\}_{i,j\geq 1}$. Thus $f(xy) = f[s_1(x)s_1(y)]$. Then

$$\langle f, g*h\rangle = \langle f(xy), g(x)h(y)\rangle.$$

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What if we replace s_1 by s_n, for instance?

Vanishing

Vanishing of $g_{\lambda\mu\nu}$ not well-understood. Sample result:

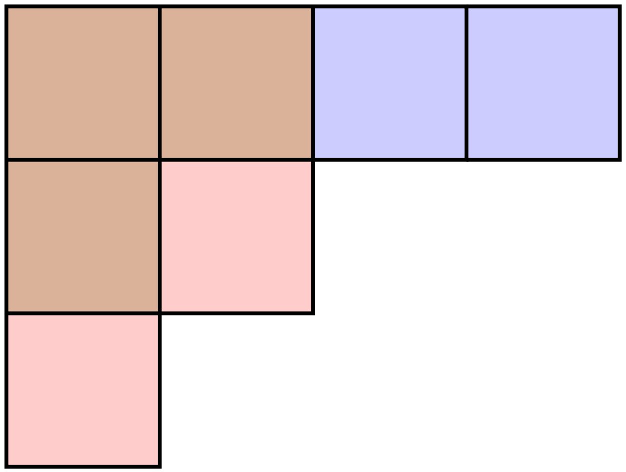
Theorem (**Dvir**, 1993). Fix $\mu, \nu \vdash n$. Then

$$\max\{\ell(\lambda) \,:\, g_{\lambda\mu\nu} \neq 0\} = |\mu \cap \nu'|$$

(intersection of diagrams).

Example of Dvir's theorem

$s_{41} * s_{32} = s_{41} + s_{32} + s_{\mathbf{311}} + s_{\mathbf{221}}$. Intersection of $(4,1)$ and $(3,2)' = (2,2,1)$:



Combinatorial interpretation

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Example. Let $\lambda \vdash n$. Then $\langle s_{j,1^{n-j}} * s_{k,1^{n-k}}, s_\lambda \rangle$ is the number of $(u,v,w) \in \mathfrak{S}_n^3$ such that $uvw = 1$, $D(u) = \{j\}$, $D(v) = \{k\}$, and if w is inserted into λ from right to left and from bottom to top, then a standard Young tableau results.

Conjugation action

$\mathfrak{S}_n$ acts on itself by conjugation, i.e., $w \cdot u = w^{-1}uw$. The Frobenius characteristic of this action is

$$K_n := \sum_{\lambda \vdash n} (s_\lambda * s_\lambda) = \sum_{\mu \vdash n} p_\mu.$$

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Combinatorial interpretation of $\langle K_n, s_\nu \rangle$ not known. All known proofs that K_n is Schur-positive use representation theory.

Stability

Example. For $n \geq 8$,

$$s_{n-2,2} * s_{n-2,2} = s_n + s_{n-3,1,1,1} + 2s_{n-2,2} + s_{n-1,1} + s_{n-2,1,1} + 2s_{n-3,2,1} + s_{n-4,2,2} + s_{n-3,3} + s_{n-4,3,1} + s_{n-4,4}.$$

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$\boldsymbol{\lambda[n]} := (n - |\lambda|, \lambda_1, \lambda_2, \ldots)$

Theorem (**Murnaghan**, 1937). *For any partitions α, β, γ, the Kronecker coefficient $g_{\alpha[n],\beta[n],\gamma[n]}$ stabilizes.*

Vast generalization proved by **Steven Sam** and **Andrew Snowden**, 2016.

Reduced Kronecker coefficient

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Hence $\bar{g}_{2,2,\emptyset} = 1$, $\bar{g}_{2,2,111} = 1$, $\bar{g}_{2,2,2} = 2$, etc.

$VP_{ws} = VNP$?

Algebraic complexity

Flagship problem: $\boldsymbol{VP}_{ws} \neq \boldsymbol{VNP}$.

Determinantal complexity of $\boldsymbol{f} \in \mathbb{C}[x_1, \ldots, x_n]$: smallest $n \in \mathbb{N}$ such that f is the determinant of an $n \times n$ matrix whose entries are affine linear forms in the x_i.

Theorem (**Valiant** 1979, **Toda** 1992). TFAE:

- *Determinantal complexity of an $n \times n$ permanant is superpolynomial in n.*
- $\boldsymbol{VP}_{ws} \neq \boldsymbol{VNP}$

Mulmuley and Sohoni 2001

Ω_n: closure of the orbit of $\mathrm{GL}_{n^2} \cdot \det_n$ in $\mathrm{Sym}^n \mathbb{C}^{n^2}$.

padded permanent: $x_{11}^{n-m} \mathrm{per}_m \in \mathrm{Sym}^n \mathbb{C}^{n^2}$.

Conjecture. For all $c > 0$ and infinitely many m, there exists a partition λ (i.e., an irreducible polynomial representation of GL_{n^2}) occurring in the coordinate ring $\mathbb{C}[Z_{m^c,m}]$ but not in $\mathbb{C}[\Omega_{m^c}]$.

Bürgisser, Ikenmeyer, and Panova

Theorem (**BIP** 2016) *The conjecture of Mulmuley and Sohoni is false.*

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Proof involves Kronecker product coefficients $g_{\lambda\mu\nu}$ in an essential way.

The last slide 

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