Order Polynomials

March 31, 2021

Slides available at:
www-math.mit.edu/~rstan/transparencies/ordpoly.pdf

Basic notation

$$\begin{aligned} \mathbb{N} &= \{0, 1, 2, \dots\} \\ \mathbb{P} &= \{1, 2, 3, \dots\} \\ [n] &= \{1, 2, \dots, n\}, \text{ for } n \in \mathbb{N} \end{aligned}$$

In particular, $[0] = \emptyset$.

Background on Eulerian polynomials

$\boldsymbol{w} = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$

descent of w: an index $1 \le i \le n-1$ such that $a_i > a_{i+1}$

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$w = 692478513 \in \mathfrak{S}_9$: $\mathrm{des}(w) = 3$

Eulerian polynomials

Definition. Let $n \geq 1$. Define the **Eulerian polynomial** $A_n(x)$ by

$$\mathbf{A_n(x)} = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}.$$

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Example. $n = 3$

w	$\mathrm{des}(w)$
123	0
213	1
312	1
132	1
231	1
321	2

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$$\Rightarrow A_3(x) = 1 + 4x + x^2$$

Slight alternative definition

Note. Some people define

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{1+\mathrm{des}(w)}.$$

Eulerian numbers

$$\begin{aligned}
A_1(x) &= 1 \\
A_2(x) &= 1 + x \\
A_3(x) &= 1 + 4x + x^2 \\
A_4(x) &= 1 + 11x + 11x^2 + x^3 \\
A_5(x) &= 1 + 26x + 66x^2 + 26x^3 + x^4
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Define $A_n(x) = \sum_{m=0}^{n-1} \mathbf{A(n,m)} x^m$. Call $A(n,m)$ an **Eulerian number** (the number of $w \in \mathfrak{S}_n$ with m descents).

Symmetry of Eulerian polynomials

Proposition. $x^{n-1}A_n(1/x) = A_n(x)$

Equivalently, $A(n, m) = A(n, n-1-m)$.

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Proof. $\mathrm{des}(a_1 a_2 \cdots a_n) = n - 1 - \mathrm{des}(a_n, \ldots, a_2, a_1)$

Note also

$$\mathrm{des}(a_1 a_2 \cdots a_n) = n - 1 - \mathrm{des}(n+1-a_1, n+1-a_2, \ldots, n+1-a_n).$$

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$$\begin{aligned}\sum_{k\geq 0} (k+1)^2 x^k &= \frac{d}{dx}\frac{x}{(1-x)^2} \\ &= \frac{1+x}{(1-x)^3}.\end{aligned}$$

More generating functions

Similarly,

$$\sum_{k\geq 0}(k+1)^3x^k = \frac{1+4x+x^2}{(1-x)^4}$$

$$\sum_{k\geq 0}(k+1)^4x^k = \frac{1+11x+11x^2+x^3}{(1-x)^5},$$

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etc.

Numerators are the Eulerian polynomials!.

Generating function for $(k+1)^n$

Theorem (**Carlitz-Riordan**, 1953, though "essentially" known earlier). *For all* $n \geq 1$, *we have*

$$\sum_{k\geq 0}(k+1)^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

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Naive proof. Induction on n. True for $n=1$. Assume for n, i.e.,

$$\sum_{k\geq 0}(k+1)^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

Apply $\frac{d}{dx}x$. Get (after some computation)

$$\sum_{k\geq 0}(k+1)^{n+1} x^k = \frac{(1+nx)A_n(x)+(x-x^2)A_n'(x)}{(1-x)^{n+2}}.$$

Proof (cont.)

$$\sum_{k\geq 0}(k+1)^{n+1}x^k = \frac{(1+nx)A_n(x)+(x-x^2)A_n'(x)}{(1-x)^{n+2}}.$$

Multiply by $(1-x)^{n+2}$ and take coefficient of x^m. On the right-hand side we get

$$\begin{aligned}A(n,m) + nA(n,m-1) + mA(n,m) - (m-1)A(n,m-1)\\ = (m+1)A(n,m) + (n-m+1)A(n,m-1).\end{aligned}$$

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To show: this expression equals $A(n+1,m)$.

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$$\begin{aligned} A(n,m)+nA(n,m-1)+mA(n,m)-(m-1)A(n,m-1) \\ = (m+1)A(n,m)+(n-m+1)A(n,m-1). \end{aligned}$$

To show: this expression equals $A(n+1,m)$.

How to get a permutation $\mathfrak{S}_{n+1}$ with m descents by inserting $n+1$ into a permutation $w = a_1a_2\cdots a_n \in \mathfrak{S}_n$?

Proof (cont.)

$$(m+1)A(n,m) + (n-m+1)A(n,m-1).$$

- If $a_i > a_{i+1}$, then inserting $n+1$ between a_i and a_{i+1} leaves the number of descents the same, as does inserting $n+1$ after a_n. To get m descents, we have $\mathrm{des}(w) = m$. This gives $(m+1)A(n,m)$ choices.

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Thus $(m+1)A(n,m) + (n-m+1)A(n,m-1) = A(n+1,m)$.
The proof follows by induction. □

A better proof.

Definition. Let $\boldsymbol{w} = a_1 \cdots a_n \in \mathfrak{S}_n$. Define a function $\boldsymbol{f} \colon [n] \to \mathbb{N} = \{0, 1, \dots\}$ to be **w-compatible** if the following two conditions hold:

(a) $f(a_1) \le f(a_2) \le \cdots \le f(a_n)$ (i.e., f is **weakly increasing along w**)

(b) $f(a_i) < f(a_{i+1})$ if $a_i > a_{i+1}$ (i.e., f is **strictly increasing along descents**)

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Fundamental theorem on descents (**P. A. MacMahon**). *Every function $f : [n] \to \mathbb{N}$ is compatible with a unique $w \in \mathfrak{S}_n$.*

Proof of fundamental theorem

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Proof by example.

i	1	2	3	4	5	6	7	8	9
$f(i)$	4	1	7	4	8	3	1	8	4

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In order for $f(a_i) < f(a_{i+1})$ if $a_i < a_{i+1}$, we must arrange the sets on which f is constant in increasing order. Thus

$$w = 2,7,6,1,4,9,3,5,8. \quad \square$$

Number of w-compatible $f : [n] \to [m]$

Let $w \in \mathfrak{S}_n$ and $m \geq 0$.

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Recall $\left(\!\binom{a}{b}\!\right)$ denotes the number of b element multisets whose elements belong to some a-element set. We have (Combinatorics 101) $\left(\!\binom{a}{b}\!\right) = \binom{a+b-1}{b}$.

Theorem. *We have*

$$\#\mathcal{A}_m(w) = \binom{m+n-1-\operatorname{des}(w)}{n} = \left(\!\binom{m-\operatorname{des}(w)}{n}\!\right),$$

a polynomial in m of degree n. Moreover,

$$\sum_{k\geq 0} \#\mathcal{A}_{k+1}(w)x^k = \frac{x^{\operatorname{des}(w)}}{(1-x)^{n+1}}.$$

Proof by example

Let $w = 2751634$. Then $\#\mathcal{A}_m(w) =$

$$\#\{1 \le \underbrace{f(2) \le f(7)} < \underbrace{f(5)}_{-1} < \underbrace{f(1) \le f(6)}_{-2} < \underbrace{f(3) \le f(4)}_{-3=-\mathrm{des}(w)} \le m\}$$

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Let $g(2) = f(2), g(7) = f(7), g(5) = f(5) - 1, g(1) = f(1) - 2$, etc. (**compression**). Thus $\#\mathcal{A}_m(w) =$

$$\#\{1 \le g(2) \le g(7) \le g(5) \le g(1) \le g(6) \le g(3) \le g(4) \le m - 3\}$$
$$= \left(\!\!\binom{m-3}{7}\!\!\right).$$

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$$= \left(\!\!\binom{m-3}{7}\!\!\right).$$

In general, $\#\mathcal{A}_m(w) = \left(\!\!\binom{m - \mathrm{des}(w)}{n}\!\!\right)$.

To prove: $\sum_{k\geq 0} \#\mathcal{A}_{k+1}(w)x^k = \frac{x^{\mathrm{des}(w)}}{(1-x)^{n+1}}$

Recall $\left(\!\binom{a}{b}\!\right) = \binom{a+b-1}{b}$. Then

$$\begin{aligned}
\sum_{k\geq 0} \#\mathcal{A}_{k+1}(w)x^k &= \sum_{k\geq 0} \left(\!\binom{k+1-\mathrm{des}(w)}{n}\!\right) x^k \\
&= \sum_{k} \binom{k+n-\mathrm{des}(w)}{n} x^k \\
&= \sum_{j} \binom{j+n}{j} x^{j+\mathrm{des}(w)} \quad (k = j+\mathrm{des}(w)) \\
&= x^{\mathrm{des}(w)} \sum_{j} \binom{-(n+1)}{j} (-1)^j x^j \\
&= \frac{x^{\mathrm{des}(w)}}{(1-x)^{n+1}}. \quad \square
\end{aligned}$$

$\sum_{k\geq 0}(k+1)^n x^k$ demystified

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$[m]^{[n]}$: set of all $f : [n] \to [m]$

Since every such f is compatible with a unique $w \in \mathfrak{S}_n$, we have

$$[k+1]^{[n]} = \dot{\bigcup_{w\in\mathfrak{S}_n}} \mathcal{A}_{k+1}(w).$$

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Take cardinality of both sides, multiply by x^k, and sum on $k\geq 0$:

$$\begin{aligned}\sum_{k\geq 0}(k+1)^n x^n &= \sum_{w\in\mathfrak{S}_n}\sum_{k\geq 0}\#\mathcal{A}_{k+1}(w)x^k\\ &= \frac{\sum_{w\in\mathfrak{S}_n}x^{\mathrm{des}(w)}}{(1-x)^{n+1}}.\end{aligned}$$

Real zeros

Theorem (**Frobenius**). *Every zero (or root) of* $A_n(x)$ *is real, simple and negative.*

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Proof. Note that every real zero is negative since $A_n(x)$ has positive coefficients and constant term 1.

Induction on n. True for $n = 1$. Assume for n. Recall

$$\sum_{k\geq 0}(k+1)^{n+1}x^k = \frac{(1+nx)A_n(x)+(x-x^2)A_n'(x)}{(1-x)^{n+2}}.$$

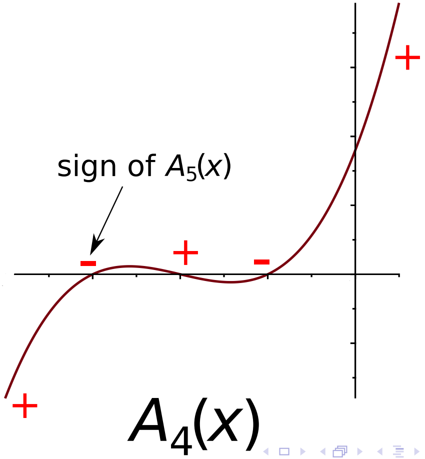
Hence

$$A_{n+1}(x) = (1+nx)A_n(x)+(x-x^2)A_n'(x).$$

NOTE. $x - x^2 < 0$ for $x < 0$.

Interlacing zeros

$$A_{n+1}(x) = (1 + nx)A_n(x) + (x - x^2)A_n'(x).$$



Newton's theorem

Theorem (**I. Newton**). *Let*

$$P(x) = \sum_{j=0}^{n} \binom{n}{j} a_j x^j \in \mathbb{R}[x].$$

If every zero of $P(x)$ is real, then $a_j^2 \geq a_{j-1}a_{j+1}$.

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Note. Write $P(x) = \sum b_j x^j$, so $b_j = \binom{n}{j} a_j$. Then $a_j^2 \geq a_{j-1}a_{j+1}$ becomes

$$b_j^2 \geq b_{j-1}b_{j+1}\left(1+\frac{1}{j}\right)\left(1+\frac{1}{n-j}\right),$$

which is stronger than $b_j^2 \geq b_{j-1}b_{j+1}$.

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which is stronger than $b_j^2 \geq b_{j-1}b_{j+1}$.

Corollary. *If each $a_j > 0$ then the sequence $a_0, a_1, \ldots, a_n$ (or $b_0, b_1, \ldots, b_n$) is unimodal.*

Proof of Newton's theorem

Let $\boldsymbol{D} = \frac{d}{dx}$. By Rolle's theorem, $\boldsymbol{Q(x)} = D^{j-1}P(x)$ has only real zeros, and thus also $\boldsymbol{R(x)} = x^{n-j+1}Q(1/x)$. Again by Rolle's theorem, $D^{n-j-1}R(x)$ has only real zeros. Easy to compute:

$$D^{n-j-1}R(x) = \frac{n!}{2}\left(a_{j-1}x^2 + 2a_jx + a_{j+1}\right).$$

This quadratic polynomial has real zeros if and only if $a_j^2 \geq a_{j-1}a_{j+1}$. $\square$

Application to Eulerian polynomials

Recall: $A_n(x) = \sum_{m=0}^{n-1} \underbrace{A(n,m)}_{\text{Eulerian number}} x^m.$

Since $A_n(x)$ has only real zeros (and has positive coefficients), we get:

Corollary. *The sequence $A(n,0), A(n,1), \ldots, A(n,n-1)$ is log-concave, and hence unimodal.*

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Note. Combinatorial proof due to **Bóna** and **Ehrenborg**, 2000.

The order polynomial redux

P: p-element poset

For $n \geq 1$, define the **order polynomial** $\boldsymbol{\Omega_P(n)}$ of P by

$$\Omega_P(n) = \#\left\{f\colon P \to \{1,\ldots,n\} \;\middle|\; s \leq_P t \Rightarrow f(s) \leq_{\mathbb{Z}} f(t)\right\}.$$

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Reciprocity for order polynomials. $\overline{\Omega}_P(n) = (-1)^p \Omega_P(-n)$.

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Goal: a nice formula for $\sum_{n\geq 0} \Omega_P(n) x^n = x + \cdots$.

Reminders

Definition. Let $\boldsymbol{w} = a_1 \cdots a_n \in \mathfrak{S}_n$. Define a function $\boldsymbol{f} : [n] \to \mathbb{N} = \{0, 1, \dots\}$ to be **w-compatible** if the following two conditions hold:

(a) $f(a_1) \geq f(a_2) \geq \cdots \geq f(a_n)$ (i.e., f is **weakly decreasing along w**)

(b) $f(a_i) > f(a_{i+1})$ if $a_i > a_{i+1}$ (i.e., f is **strictly decreasing along descents**)

Fundamental theorem on descents. *Every function $f : [n] \to \mathbb{N}$ is compatible with a unique $w \in \mathfrak{S}_n$.*

Fundamental theorem on P-partitions

P: a **naturally labelled** poset on the set $[p]$, i.e., if $i <_P j$ then $i <_{\mathbb{Z}} j$. Equivalently, the permutation $12\cdots p$ is a linear extension of P.

***P*-partition**: an order-preserving map $f\colon P \to \mathbb{N}$, i.e., $i \leq_P j \Rightarrow f(i) \leq_{\mathbb{Z}} f(j)$.

$\mathcal{L}(P)$: set of linear extensions of P, regarded as permutations $a_1 a_2 \cdots a_p \in \mathfrak{S}_p$ of the elements of P

Theorem. *A function $f\colon P \to [n]$ is order-preserving if and only if it is compatible with some $w \in \mathcal{L}(P)$.*

Proof of fundamental theorem

Theorem. *A function* $f : P \to [n]$ *is order-preserving if and only if it is compatible with some* $w \in \mathcal{L}(P)$.

Proof. ("If" part) Clear. In fact, if $w = a_1 a_2 \cdots a_p \in \mathcal{L}(P)$ and $f(a_1) \le f(a_2) \le \cdots \le f(a_p)$ (no condition on strict inequalities), then f is order-preserving.

"Only if" part of proof

To show: if f is compatible with some $w \notin \mathcal{L}(P)$, then f is not order-preserving.

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Let $w = a_1 a_2 \cdots a_p$. Since $w \notin \mathcal{L}(P)$, there exists $i < j$ such that $a_i >_P a_j$. Thus also $a_i >_{\mathbb{Z}} a_j$. Hence there exists $i \leq k < j$ such that $a_k >_{\mathbb{Z}} a_{k+1}$, so $f(a_k) < f(a_{k+1})$ (by compatibility).

Now $f(a_i) \leq f(a_{i+1}) \leq \cdots \leq f(a_j)$ (by compatibility), so $f(a_i) < f(a_j)$. Hence f is not order preserving. $\square$

Corollaries to fundamental theorem

$\mathcal{A}_m(P) := \{P\text{-partitions } f\colon P \to [m]\}$, $\#\mathcal{A}_m(P) = \Omega_P(m)$

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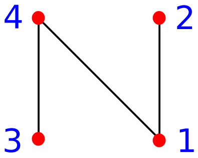
Corollary 2. $\sum_{m \geq 0} \Omega_P(m) x^m = \frac{\sum_{w \in \mathcal{L}(P)} x^{1+\mathrm{des}(w)}}{(1-x)^{p+1}}$

Proof. Follows from Corollary 1 and

$$\sum_{k \geq 0} \#\mathcal{A}_{k+1}(w) x^k = \frac{x^{\mathrm{des}(w)}}{(1-x)^{p+1}}.$$

An example



$w \in \mathcal{L}(P)$	$\mathrm{des}(w)$
1 2 3 4	0
1 3 2 4	1
1 3 4 2	1
3 1 2 4	1
3 1 4 2	2

$$\sum_{m\geq 0} \Omega_P(m) x^m = \frac{x + 3x^2 + x^3}{(1-x)^5}$$

Eulerian polynomials redux

Note. If P is a P-element antichain, then we get

$$\sum_{m\geq 0} m^p x^m = \frac{\sum_{w\in\mathfrak{S}_p} x^{1+\mathrm{des}(w)}}{(1-x)^{p+1}}.$$

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Equivalent to previous result:

$$\sum_{m\geq 0} (m+1)^p x^m = \frac{\sum_{w\in\mathfrak{S}_p} x^{\mathrm{des}(w)}}{(1-x)^{p+1}}.$$

Symmetry and real-rootedness

Recall: Eulerian polynomials $A_n(x)$ are symmetric (i.e., $x^{n-1}A_n(1/x) = A_n(x)$) and have only real roots (or zeros). What about $\boldsymbol{A_P(x)} := \sum_{w \in \mathcal{L}(P)} x^{\mathrm{des}(w)}$?

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Easy consequence of reciprocity:

Theorem. $x^k A_P(1/x) = A_P(x)$ *if and only if every maximal chain of* P *has* $p-k$ *elements. In other words,* P *is graded of rank* $p-k-1$.

Unimodality

Let $A_P(x) = \sum_{m=0}^{p-1} A(P,m)x^m$, so

$$A(P,m) = \#\{w \in \mathcal{L}(P) : \mathrm{des}(w) = m\},$$

a ***P*-Eulerian number**. $A_P(x)$ is **unimodal** if

$$A(P,0) \le A(P,1) \le \cdots \le A(P,j) \ge A(P,j+1) \ge \cdots \ge A(P,p-1)$$

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Is $A_P(x)$ always unimodal? Open.

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Note. Let $\boldsymbol{f(n)}$ be the number of nonisomorphic n-element posets. Then $f(17)$ is not known. Moreover, $f(n) = 2^{\frac{1}{4}n^2+o(1)}$.

Good special cases

Rodica Simion (1984) showed that $A_P(x)$ has only real zeros if P is a disjoint union of chains.

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