Two Analogues of Pascal's Triangle

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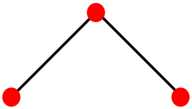
We draw diagrams upside-down from the usual convention, so $\hat{0}$ is at the top.

Construction of P_{23}



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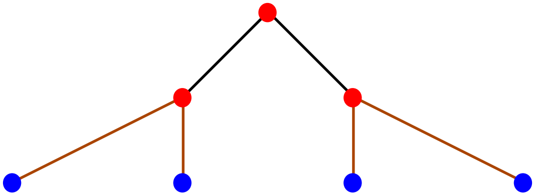

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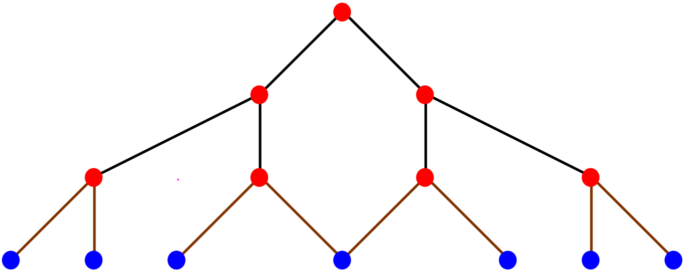

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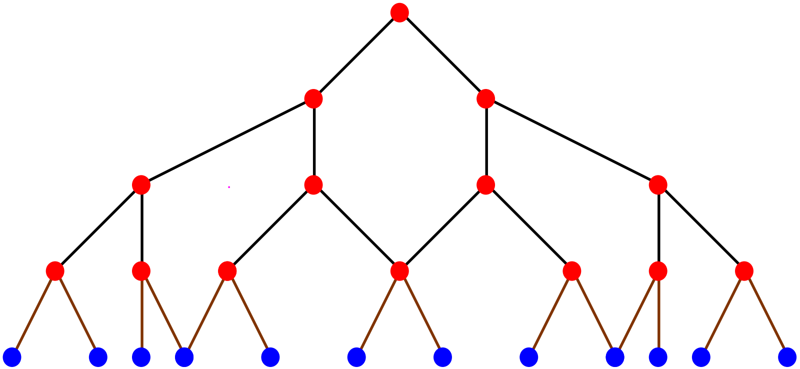

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$$p_{ib}(n) = ip_{ib}(n-1) - (i-1)p_{ib}(n-b).$$

Initial conditions: $p_{ib}(n) = i^n$, $0 \leq n \leq b-1$

$$\Rightarrow \sum_{n\geq 0} p_{ib}(n)x^n = \frac{1}{1-ix+(i-1)x^b}.$$

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For $t \in P_{ib}$, let $\boldsymbol{e(t)}$ be the number of saturated chains from $\hat{0}$ to t.

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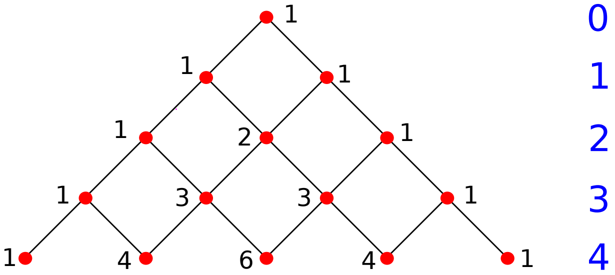
Example. P_{22}



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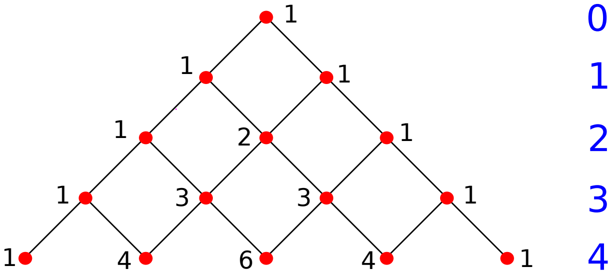

Pascal's triangle

Pascal's triangle

rows 0–4:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

kth entry in row n, beginning with $k = 0$: $\binom{n}{k}$

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 & & & 1 & & 1 & & & \\
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 & 1 & & 3 & & 3 & & 1 & \\
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1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

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$$\sum_k \binom{n}{k} x^k = (1+x)^n$$

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$$\sum_k \binom{n}{k}^3 = ??$$

Even worse! Generating function is not algebraic.

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

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 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & {\color{blue}1} & & 1 & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

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Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

				1				
		1		1		1		
	1		2		2		1	
1								1

$$\vdots$$

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				1				
		1		1		1		
	1	1	2	1	2	1	1	
1								1
				⋮				

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & 1 & & & 1 & & & 1 & & & & \\
 & & 1 & & 1 & & 2 & 1 & 2 & & 1 & & 1 & & \\
1 & 2 & & 3 & & 3 & & & 3 & & 3 & 2 & & 1 & \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

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1

1 1 1

1 1 2 1 2 1 1

1 1 2 1 3 2 3 1 3 2 3 1 2 1 1

⋮

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\begin{array}{ccccccccccccccc}
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 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
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Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$

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- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k = 0$) in row n. Write

$$P_n(x) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $P_{n+1}(x) = (1 + x + x^2) P_n(x^2)$, since $x\, P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2) P_n(x^2)$ to summing two consecutive entries.

Stern analogue of binomial theorem

Corollary. $P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$
u_2(n) := \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots
$$

Sums of squares

							1							
			1				1				1			
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1

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$$\sum_{n \geq 0} u_2(n) x^n = \frac{1-2x}{1-5x+2x^2}$$

Proof

$$u_2(n+1) = \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots$$

$$= 3u_2(n) + 2 \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .$$

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Thus define $\boldsymbol{u_{1,1}(n)} := \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$
\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle \right) \left\langle {n \atop k} \right\rangle \\
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Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

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$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

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Much nicer than $\sum_k \binom{n}{k}^3$

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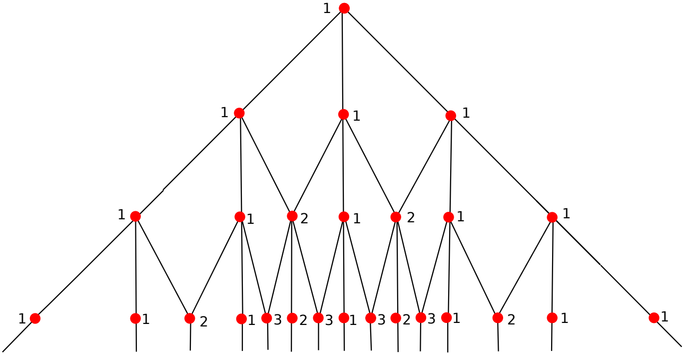
Much more can be said!

The Stern poset



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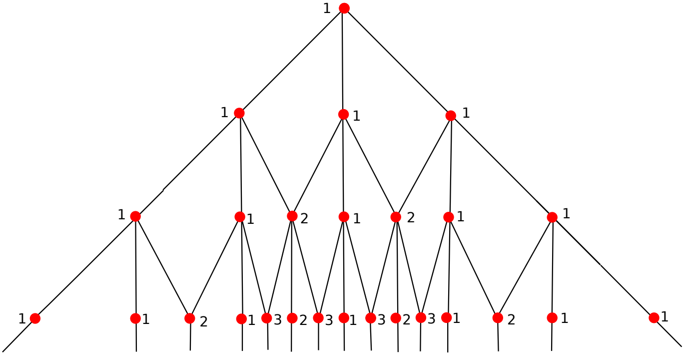
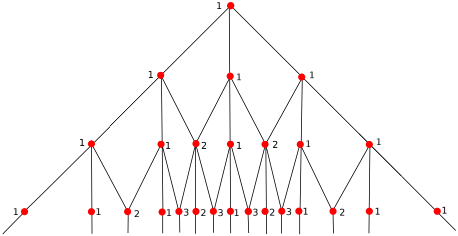


P_{32}

“Binomial theorem” for the Stern poset



Label t by $e(t)$. Then the kth label (beginning with $k = 0$) at rank n is $\left\langle {n \atop k} \right\rangle$:

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2\cdot 2^i}\right).$$

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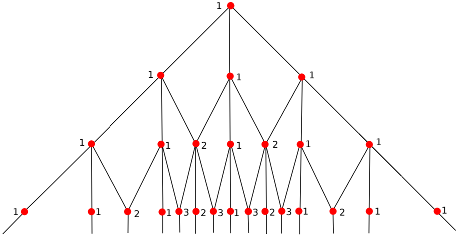

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Similar product formulas for all P_{ib}.

A Fibonacci product

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$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

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$v_r(n)$: sum of rth powers of coefficients of $I_n(x)$

The Fibonacci triangle $\mathcal{F}$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad \bullet \quad 1 \quad 1 \\
1 \quad 1 \quad \bullet \quad 1 \quad 2 \quad 1 \quad \bullet \quad 1 \quad 1 \\
1 \quad 1 \quad \bullet \quad 1 \quad 2 \quad 1 \quad \bullet \quad 2 \quad 2 \quad \bullet \quad 1 \quad 2 \quad 1 \quad \bullet \quad 1 \quad 1 \\
1 \quad 1 \quad \bullet \quad 1 \quad 2 \quad 1 \quad \bullet \quad 2 \quad 2 \quad \bullet \quad 1 \quad 3 \quad 2 \quad \bullet \quad 2 \quad 3 \quad 1 \quad \bullet \quad 2 \quad 2 \quad \bullet \quad 1 \quad 2 \quad 1 \quad \bullet \quad 1 \quad 1
\end{array}
$$

The Fibonacci triangle $\mathcal{F}$

														1														
											1						1											
					1						1			•			1						1					
			1		1			•			1			2			1			•			1		1			
	1		1	•	1			2			1	•	2		2	•	1			2			1	•	1		1	
1	1	•	1	2	1	•	2		2	•	1	3	2	•	2	3	1	•	2		2	•	1	2	1	•	1	1

- Copy each entry of row $n \geq 1$ to the next row.
- Add two entries if separated by at bullet (and form group of 3)
- Copy once more the middle entry of a group of 3 (group of 2)
- Adjoin 1 at beginning and end of each row after row 0.

“Binomial theorem” for $\mathcal{F}$

$\begin{bmatrix} n \\ k \end{bmatrix}$: kth entry (beginning with $k = 0$) in row n (beginning with $n = 0$) in $\mathcal{F}$

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Theorem. $$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = I_n(x) := \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

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Proof omitted.

$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}^2$

Can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

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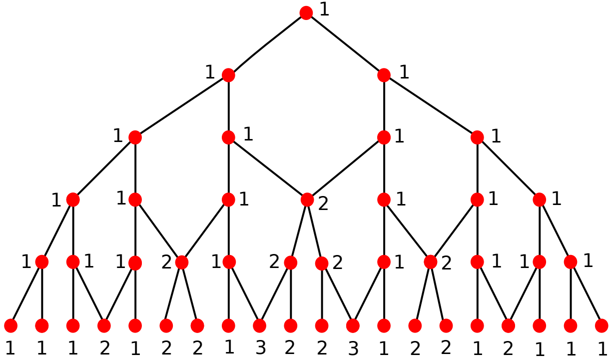
Quite a bit more complicated (automated by **D. Zeilberger**).

Theorem. $\displaystyle\sum_{n \geq 0} v_2(n) x^n = \frac{1 - 2x^2}{1 - 2x - 2x^2 + 2x^3}$, *and similarly for higher powers.*

A diagram (poset) associated with $\mathfrak{F}$



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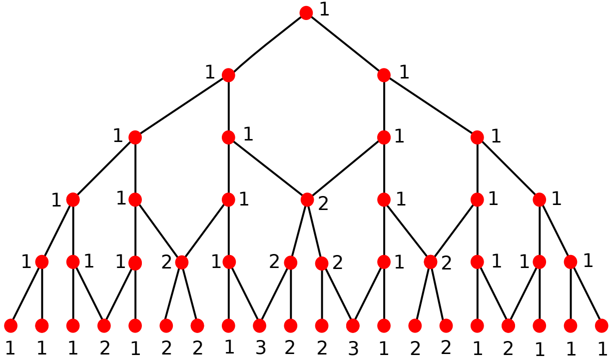
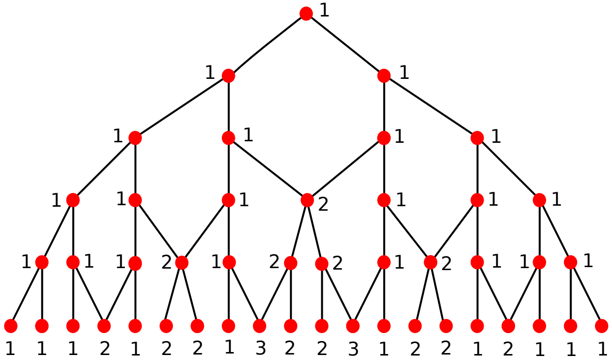


P_{23}

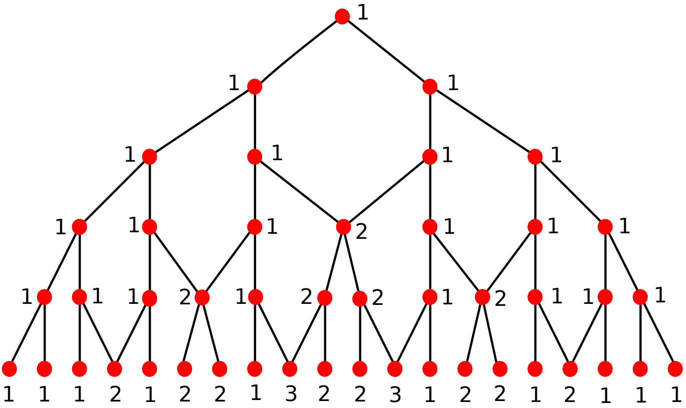
Further property



Label t by $e(t)$. Then the kth label (beginning with $k = 0$) at rank n is $\begin{bmatrix} n \\ k \end{bmatrix}$:

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right).$$

Strings of size two and three



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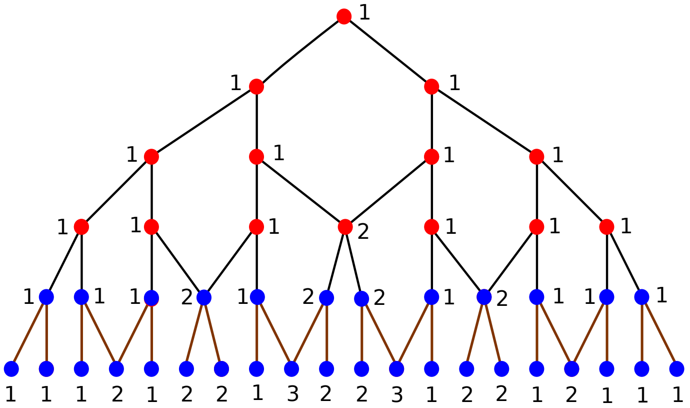

Coefficients of $I_n(x)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

Coefficient of x^m: number of ways to write m as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

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Example. Coefficient of x^8 in $(1+x)(1+x^2)(1+x^3)(1+x^5)(1+x^8)$ is 3:

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Can we see these sums from $\mathfrak{F}$? Each path from the top to a point $t \in \mathfrak{F}$ should correspond to a sum.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

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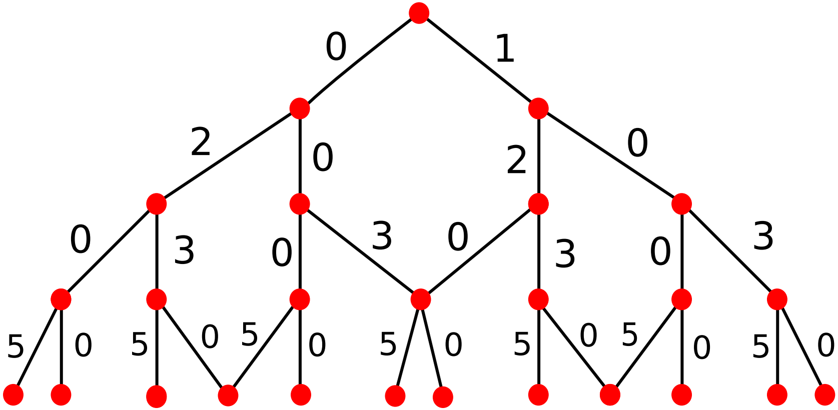
Diagram of the edge labeling



Connection with sums of Fibonacci numbers

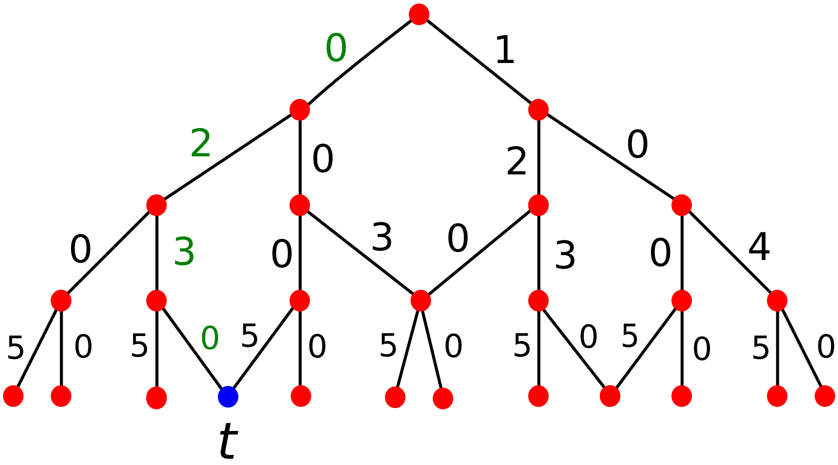
Let $t \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\sigma(t)$.

Connection with sums of Fibonacci numbers

Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

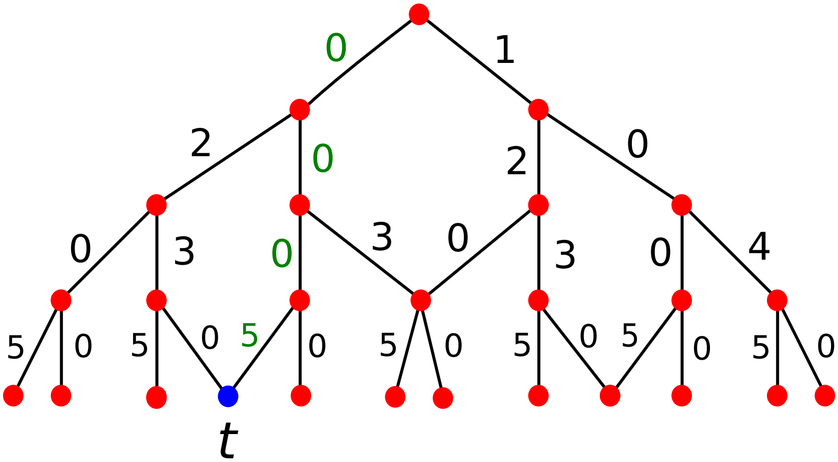
If $\text{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $\{F_2, F_3, \ldots, F_{n+1}\}$.

An example



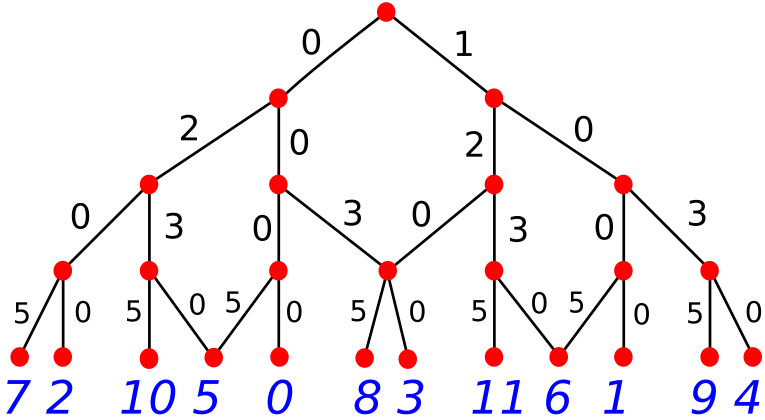
$$2 + 3 = F_3 + F_4$$

An example



$5 = F_5$

An ordering of $\mathbb{N}$



In the limit as rank $\to \infty$, get an interesting linear ordering of $\mathbb{N}$.

Second proof: factorization in a free monoid

$$
\begin{aligned}
I_n(x) &:= \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) \\
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$$\mathbf{v_2(n)} := \sum_k \begin{bmatrix} n \\ k \end{bmatrix}^2 = \# \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

A concatenation product

$$\mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

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Easy to check: $\alpha\beta \in \mathcal{M}_{n+m}$

The monoid $\mathcal{M}$

$$\boldsymbol{\mathcal{M}} := \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots ,$$

a **monoid** (semigroup with identity) under concatenation. The identity element is $\emptyset \in \mathcal{M}_0$.

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Definition. A subset $\mathcal{G} \subset \mathcal{M}$ **freely generates** $\mathcal{M}$ if every $\alpha \in \mathcal{M}$ can be written uniquely as a product of elements of $\mathcal{G}$. (We then call $\mathcal{M}$ a **free** monoid.)

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Suppose $\mathcal{G}$ freely generates $\mathcal{M}$, and let $G(x) = \sum_{n \geq 1} \#(\mathcal{M}_n \cap \mathcal{G}) x^n$. Then

$$\begin{aligned} \sum_n v_2(n) x^n &= \sum_n \#\mathcal{M}_n \cdot x^n \\ &= 1 + G(x) + G(x)^2 + \cdots \\ &= \frac{1}{1 - G(x)}. \end{aligned}$$

Free generators of $\mathcal{M}$

Theorem. *$\mathcal{M}$ is freely generated by the following elements:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

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Example. $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$: $1+2+3+5=3+8$

$G(x)$

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Let k be the number of columns of $*$'s. Length is $2k+3$. Thus

$$\begin{aligned} G(x) &= 2x + 2\sum_{k\geq 0} 2^k x^{2k+3} \\ &= 2x + \frac{2x^3}{1-2x^2}. \end{aligned}$$

Completion of proof

$$\sum_n v_2(n)x^n = \frac{1}{1-G(x)}$$
$$= \frac{1}{1-\left(2x+\frac{2x^3}{1-2x^2}\right)}$$
$$= \frac{1-2x^2}{1-2x-2x^2+2x^3} \quad \square$$

Further vistas?

What more can be said about P_{ij}?

The final slide

The final slide