Smith Normal Form and Combinatorics

Richard P. Stanley

June 8, 2017

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist ***P***, ***Q*** $\in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \ldots, d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

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Note. (1) Can extend to $m \times n$.

$$(2)\ \mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of det.

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- April 1883: shared *Grand prix des sciences mathématiques* with Minkowski

Row and column operations

Can put a matrix into SNF by the following operations.

- Add a multiple of a row to another row.
- Add a multiple of a column to another column.
- Multiply a row or column by a **unit** in R.

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Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Existence of SNF

PIR: principal ideal ring, e.g., $\mathbb{Z}$, $K[x]$, $\mathbb{Z}/m\mathbb{Z}$.

Theorem (**Smith**, for $\mathbb{Z}$). If R is a PIR then A has a unique SNF up to units.

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Theorem (**Smith**, for $\mathbb{Z}$). If R is a PIR then A has a unique SNF up to units.

Otherwise A "typically" does not have a SNF but may have one in special cases.

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

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$R^n/(v_1, \dots, v_n)$: **(Kasteleyn) cokernel** of A

An explicit formula for SNF

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Theorem. *$e_1 e_2 \cdots e_i$ is the gcd of all $i \times i$ minors of A.*

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Special case: e_1 is the gcd of all entries of A.

Laplacian matrices

$L(G)$: Laplacian matrix of the graph G

rows and columns indexed by vertices of G

$$L(G)_{uv} = \begin{cases} -\#(\text{edges } uv), & u \neq v \\ \deg(u), & u = v. \end{cases}$$

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reduced Laplacian matrix $\boldsymbol{L_0(G)}$: for some vertex v, remove from $\boldsymbol{L}(G)$ the row and column indexed by v

Matrix-tree theorem

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Applications to sandpile models, chip firing, etc.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

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What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Laplacian matrix of K_n

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Theorem. $\boldsymbol{L_0}(K_n) \xrightarrow{\text{SNF}} \operatorname{diag}(1, n, n, \ldots, n)$, *a refinement of Cayley's theorem that* $\kappa(K_n) = n^{n-2}$.

Proof that $L_0(K_n) \xrightarrow{\text{SNF}} \mathrm{diag}(1, n, n, \ldots, n)$

Trick: 2×2 submatrices (up to row and column permutations):

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \quad \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1 e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i | e_{i+1}$, we get the SNF $\mathrm{diag}(1, n, n, \ldots, n)$.

Chip firing

Abelian sandpile: a finite collection σ of indistinguishable chips distributed among the vertices V of a (finite) connected graph. Equivalently,

$$\sigma\colon V \to \{0, 1, 2, \dots\}.$$

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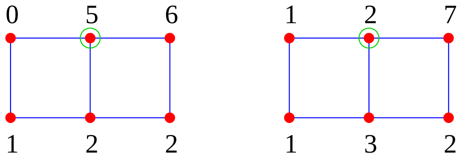
toppling of a vertex v: if $\sigma(v) \geq \deg(v)$, then send a chip to each neighboring vertex.



The sandpile group

Choose a vertex to be a **sink**, and ignore chips falling into the sink.

stable configuration: no vertex can topple

Theorem (easy). *After finitely many topples a stable configuration will be reached, which is independent of the order of topples.*

The monoid of stable configurations

Define a commutative monoid $\boldsymbol{M}$ on the stable configurations by vertex-wise addition followed by stabilization.

ideal of M: subset $J \subseteq M$ satisfying $\sigma J \subseteq J$ for all $\sigma \in M$

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Exercise. The (unique) minimal ideal of a finite commutative monoid is a group.

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sandpile group of G: the minimal ideal $\boldsymbol{K(G)}$ of the monoid M

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Theorem. *Let*

$$L_0(G) \xrightarrow{\text{SNF}} \text{diag}(e_1, \ldots, e_{n-1}).$$

Then

$$K(G) \cong \mathbb{Z}/e_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/e_{n-1}\mathbb{Z}.$$

The n-cube

C_n: graph of the n-cube

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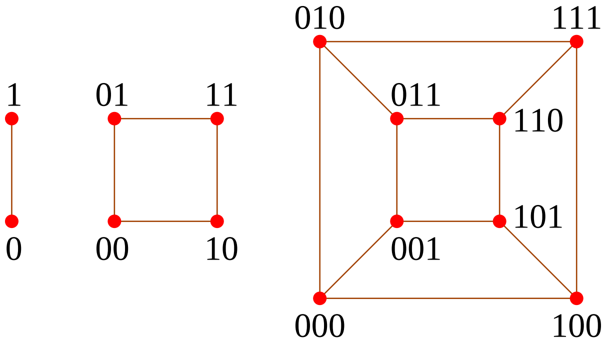
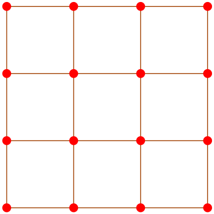


The 4-cube



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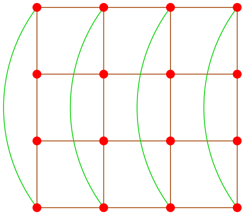

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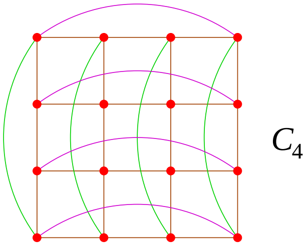

An open problem

$$\kappa(C_n) = 2^{2^n - n - 1} \prod_{i=1}^{n} i^{\binom{n}{i}}$$

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2-Sylow subgroup of $K(C_n)$ is **unknown**.

SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.

Is the question interesting?

$\mathrm{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and $\mathrm{SNF}(M) = (e_1, \ldots, e_n)$, then $e_1 = d$.

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Theorem. $\lim_{k\to\infty} p_k(n,d) = \dfrac{1}{d^{n^2}\zeta(n^2)}$

Specifying some e_i

with **Yinghui Wang**

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Two general results.

- Let $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{P}$, $\alpha_i | \alpha_{i+1}$.

  $\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_i = \alpha_i$ for $1 \le \alpha_i \le n-1$.

$$\boldsymbol{\mu(n)} = \lim_{k \to \infty} \mu_k(n).$$

  Then $\mu(n)$ exists, and $0 < \mu(n) < 1$.

Second result

- Let $\alpha_n \in \mathbb{P}$.

  $\boldsymbol{\nu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_n = \alpha_n$.

  Then

$$\lim_{k\to\infty} \nu_k(n) = 0.$$

Sample result

$\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\boldsymbol{\mu(n)} = \lim_{k\to\infty} \mu_k(n).$$

Conclusion

$$
\begin{aligned}
\mu(n) &= 2^{-n^2}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right) \\
&\cdot \frac{3}{2}\cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2 \\
&\cdot \prod_{p>3}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).
\end{aligned}
$$

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF $\mathrm{diag}(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$

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Theorem. $$\kappa(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$$

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Corollary.
$$\lim_{n\to\infty} \kappa(n) = \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)} \approx 0.846936\cdots.$$

Small number of generators

g: number of generators of cokernel (number of entries of SNF $\neq$ 1) as $n \to \infty$

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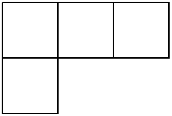
Example of SNF computation

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \dots)$, identified with its Young diagram



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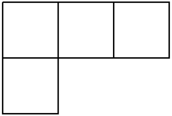

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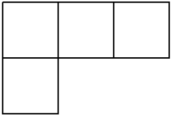

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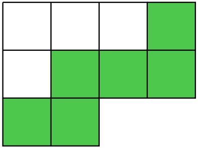
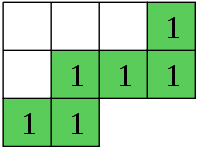


$(3,1)^* = (4,4,2)$

Initialization

Insert 1 into each square of λ^*/λ.



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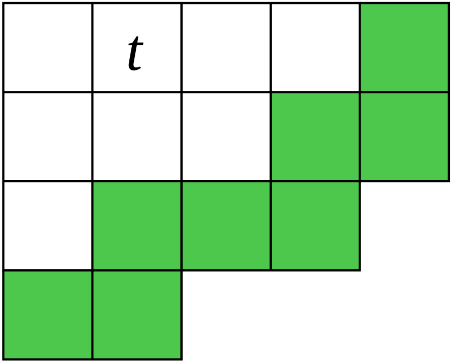

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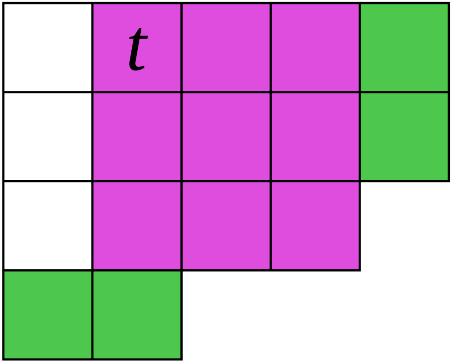

Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.

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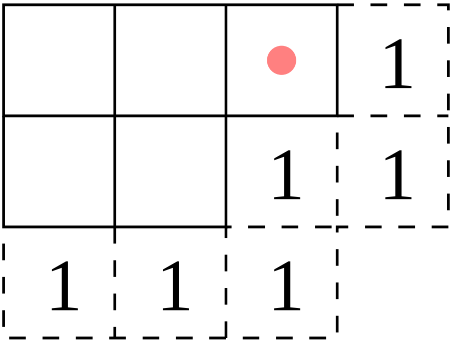

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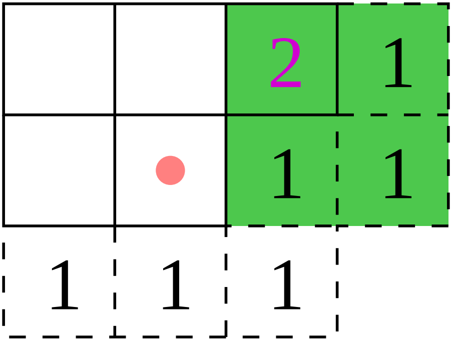

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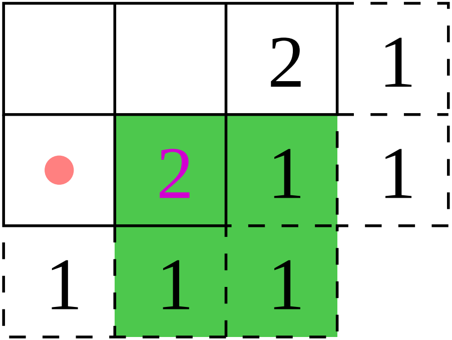

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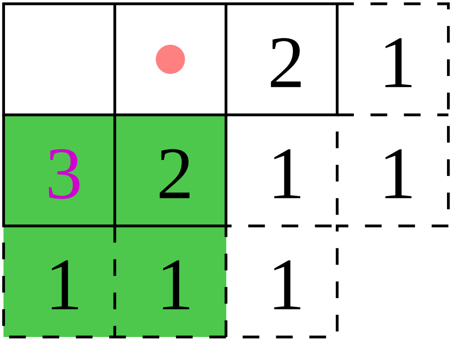

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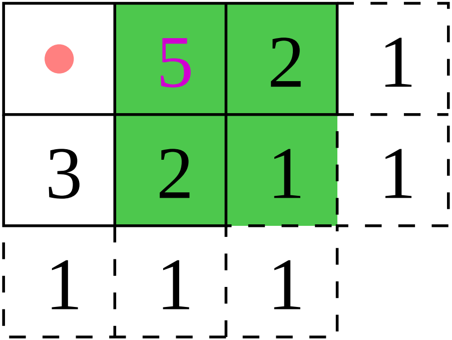

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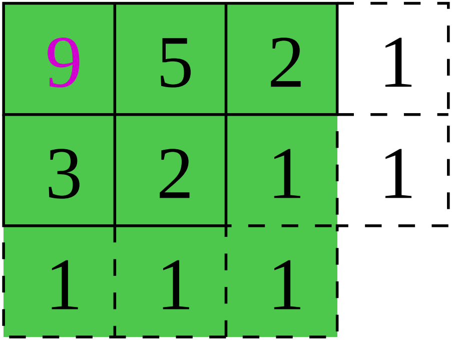

Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

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Why? Expand $\det M_t$ by the first row. The coefficient of n_t is 1 by induction.

$\lambda(t)$

If $t \in \lambda$, let $\boldsymbol{\lambda(t)}$ consist of all squares of λ to the southeast of t.

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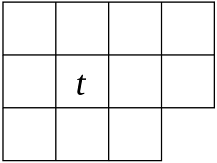

$\lambda = (4,4,3)$

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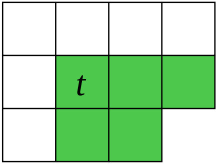

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$\lambda(t) = (3,2)$

u_λ

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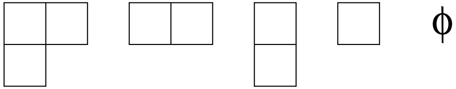
Example. $u_{(2,1)} = 5$:



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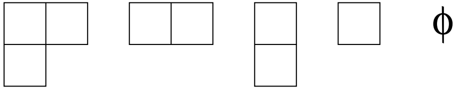

There is a determinantal formula for u_λ, due essentially to **MacMahon** and later **Kreweras** (not needed here).

Carlitz-Scoville-Roselle theorem

- **Berlekamp** (1963) first asked for $n_t \pmod 2$ in connection with a coding theory problem.
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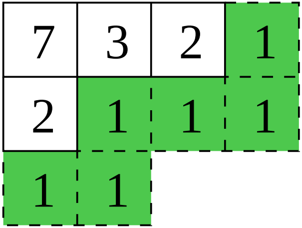
Proofs. 1. Induction (row and column operations).

2. Nonintersecting lattice paths.

An example



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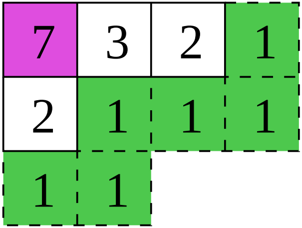

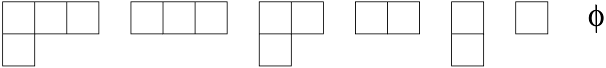

A q-analogue

Weight each $\mu \subseteq \lambda$ by $q^{|\lambda/\mu|}$.

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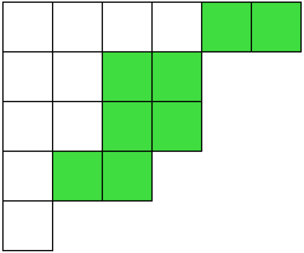
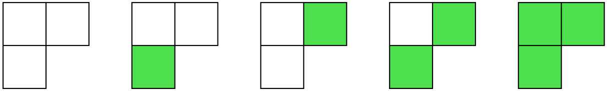


$$\lambda = 64431, \quad \mu = 42211, \quad q^{\lambda/\mu} = q^8$$

$u_\lambda(q)$

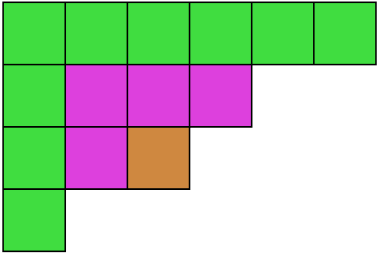
$$u_\lambda(q) = \sum_{\mu \subseteq \lambda} q^{|\lambda/\mu|}$$

$u_{(2,1)}(q) = 1 + 2q + q^2 + q^3$:



Diagonal hooks

$$d_i(\lambda) = \lambda_i + \lambda_i' - 2i + 1$$



$$d_1 = 9, \quad d_2 = 4, \; d_3 = 1$$

Main result (with C. Bessenrodt)

Theorem. *M_t has an SNF over $\mathbb{Z}[q]$. Write $d_i = d_i(\lambda_t)$. If M_t is a $(k+1)\times(k+1)$ matrix then M_t has SNF*

$$\mathrm{diag}(1, q^{d_k}, q^{d_{k-1}+d_k}, \ldots, q^{d_1+d_2+\cdots+d_k}).$$

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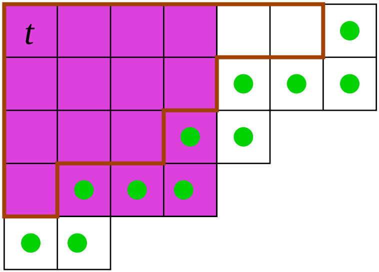
Note. There is a multivariate generalization.

An example



$$\lambda = 6431, \quad d_1 = 9, \quad d_2 = 4, \quad d_3 = 1$$

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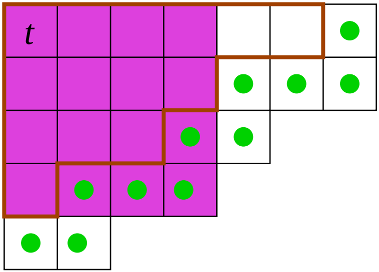

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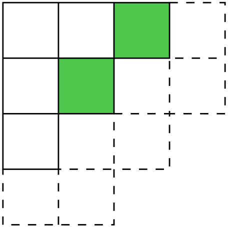
$$\text{SNF of } M_t : \ (1, q, q^5, q^{14})$$

A special case

Let λ be the **staircase** $\delta_n = (n-1, n-2, \ldots, 1)$.



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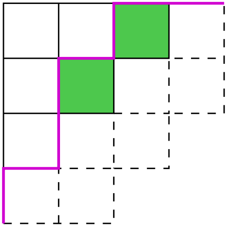
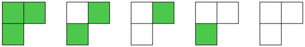


$u_{\delta_{n-1}}(q)$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known q-analogue $\boldsymbol{C_n(q)}$ of the Catalan number C_n.

A q-Catalan example



$$C_3(q) = q^3 + q^2 + 2q + 1$$

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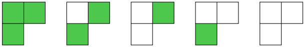

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$$\left|\begin{matrix} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{matrix}\right| \overset{\text{SNF}}{\sim} \text{diag}(1, q, q^6)$$

since $d_1(3,2,1) = 1,\ d_2(3,2,1) = 5$.

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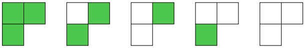

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- q-Catalan determinant previously known
- SNF is new

Ramanujan

$$\begin{aligned} F(q,x) &:= \sum_{n\geq 0} C_n(q)x^n \\ &= \cfrac{1}{1-\cfrac{x}{1-\cfrac{qx}{1-\cfrac{q^2x}{1-\cdots}}}} \end{aligned}$$

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$$

$$
e^{-2\pi/5}F(e^{-2\pi},-e^{-2\pi}) = \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{1+\sqrt{5}}{2}}.
$$

An open problem

$\ell(w)$: length (number of inversions) of $w = a_1 \cdots a_n \in \mathfrak{S}_n$, i.e.,

$$\ell(w) = \#\{(i,j)\ :\ i<j,\ w_i > w_j\}.$$

$\boldsymbol{V(n)}$: the $n! \times n!$ matrix with rows and columns indexed by $w \in \mathfrak{S}_n$, and

$$V(n)_{uv} = q^{\ell(uv^{-1})}.$$

$n = 3$

$$\det \begin{bmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q & q^3 & q^2 \\ q & q^2 & 1 & q^3 & q & q^2 \\ q^2 & q & q^3 & 1 & q^2 & q \\ q^2 & q^3 & q & q^2 & 1 & q \\ q^3 & q^2 & q^2 & q & q & 1 \end{bmatrix} = (1 - q^2)^6 (1 - q^6)$$

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special case of ***q*-Varchenko matrix** of a real hyperplane arrangment

Zagier's theorem

Theorem (**D. Zagier**, 1992)

$$\det V(n) = \prod_{j=2}^{n} \left(1 - q^{j(j-1)}\right)^{\binom{n}{j}(j-2)!\,(n-j+1)!}$$

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SNF is open. Partial result:

Theorem (**Denham-Hanlon**, 1997) *Let*

$$V(n) \overset{\text{snf}}{\to} \mathrm{diag}(e_1, e_2, \ldots, e_{n!}).$$

The number of e_i's exactly divisible by $(q-1)^j$ (or by $(q^2-1)^j$) is the number $c(n, n-j)$ of $w \in \mathfrak{S}_n$ with $n-j$ cycles (signless Stirling number of the first kind).

The last slide

The last slide 

The last slide