The Sperner Property

Richard P. Stanley
M.I.T. and U. Miami

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Banff, 1981

Banff, 1981

San Diego
January 26, 2003

Sperner's theorem

Theorem (**E. Sperner**, 1927). *Let* $S_1, S_2, \ldots, S_m$ *be subsets of an* n*-element set* X *such that* $S_i \not\subseteq S_j$ *for* $i \neq j$*, Then* $m \leq \binom{n}{\lfloor n/2 \rfloor}$*, achieved by taking all* $\lfloor n/2 \rfloor$*-element subsets of* X*.*

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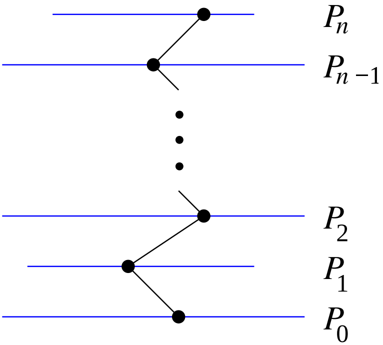
Emanuel Sperner
9 December 1905 – 31 January 1980

Graded posets

Let

$$P = P_0 \cup P_1 \cup \cdots \cup P_n$$

be a finite graded poset of rank n.



Rank-symmetry and unimodality

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rank-unimodal and rank-symmetric $\Rightarrow j = \lfloor n/2 \rfloor$

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antichain $A \subseteq P$:

$$s, t \in A, \quad s \leq t \Rightarrow s = t$$

$$\bullet\bullet\bullet\bullet$$

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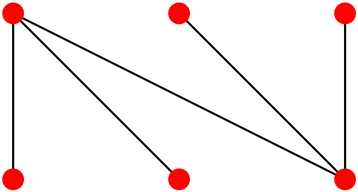
P is **Sperner** (or has the **Sperner property**) if

$$\max_A \#A = \max_i p_i$$

An example



rank-symmetric, rank-unimodal, $F_P(q) = 3 + 3q$

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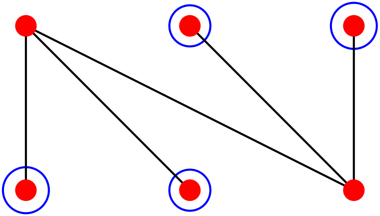

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not Sperner

The boolean algebra

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$$p_i = \binom{n}{i}, \quad F_{B_n}(q) = (1+q)^n$$

rank-symmetric, rank-unimodal

Sperner's theorem, restated

Theorem. *The boolean algebra B_n is Sperner.*

Proof (**D. Lubell**, 1966).

- B_n has $n!$ maximal chains.

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- Divide by $n!$:

$$\sum_{A \in S} \frac{1}{\binom{n}{|S|}} \leq 1.$$

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- $\Rightarrow |A| \leq \binom{n}{\lfloor n/2 \rfloor}$ $\square$

A generalization

Lubell's proof carries over to all graded posets P of rank n satisfying:

- The number of elements covered by $x \in P$ depends only on $\text{rank}(x)$.
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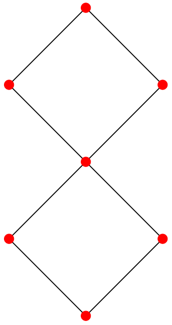
Examples

(a) The poset $B_n(q)$ of all subspaces of $\mathbb{F}_q^n$.

(b) The face poset of an n-cube (and its q-analogue).

(c)



(not unimodal)

Order-matchings

Order matching: $\mu\colon P_i \to P_{i+1}$: injective and $\mu(t) > t$

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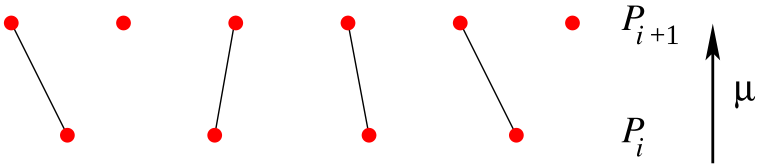

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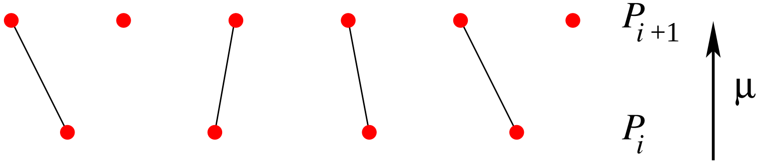

Similarly $\mu\colon P_i \to P_{i-1}$: injective and $\mu(t) < t$

A combinatorial condition for Spernicity

Theorem. *Let P be graded of rank n. Suppose that for some j there exist order-matchings*

$$P_0 \to P_1 \to \cdots \to P_j \leftarrow P_{j+1} \leftarrow \cdots \leftarrow P_n.$$

Then P is rank-unimodal and Sperner.

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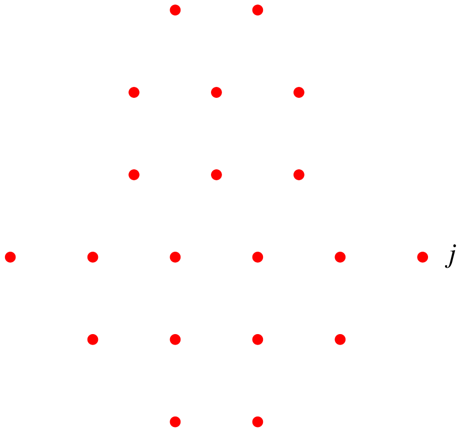
"Glue together" the order-matchings.

Gluing example



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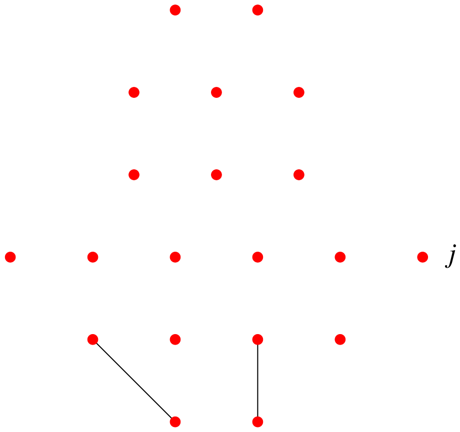

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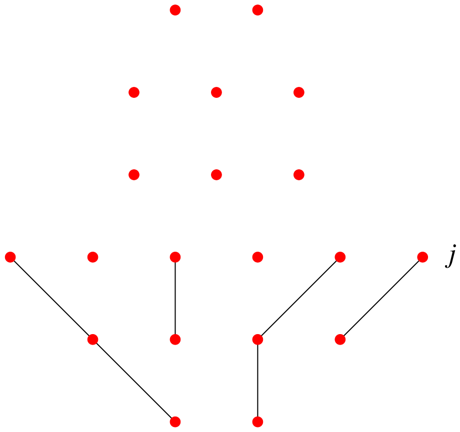

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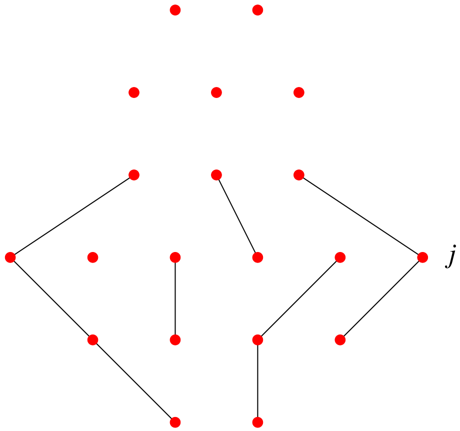

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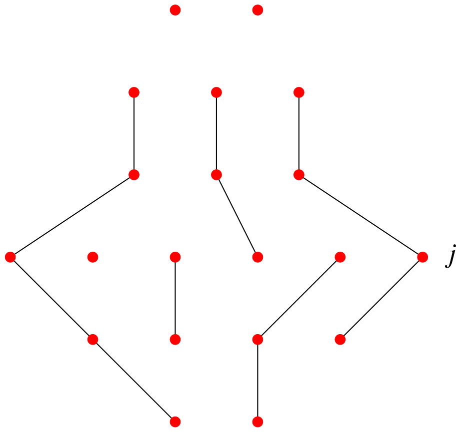

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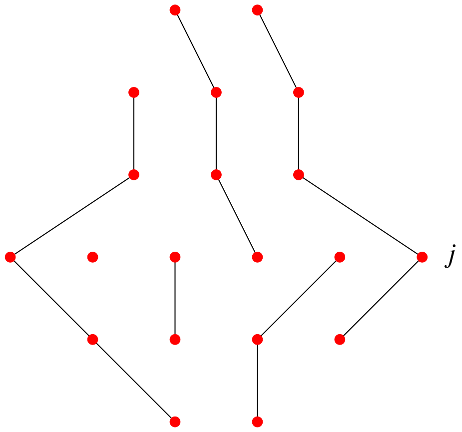

A chain decomposition

$$P = C_1 \cup \cdots \cup C_{p_j} \quad \text{(chains)}$$

$$A = \text{ antichain}, C = \text{ chain} \Rightarrow \#(A \cap C) \leq 1$$

$$\Rightarrow \#A \leq p_j. \qquad \square$$

Back to B_n

Explicit order matching $(B_n)_i \to (B_n)_{i+1}$ for $i < n/2$:

Example. $S = \{1, 4, 6, 7, 11\} \in (B_{13})_5$

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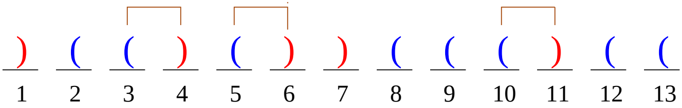

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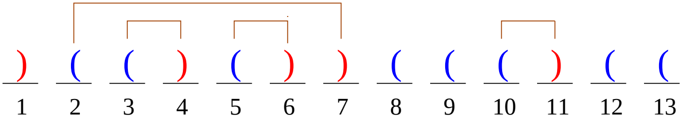

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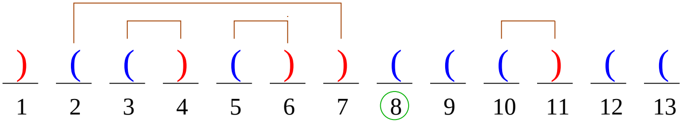
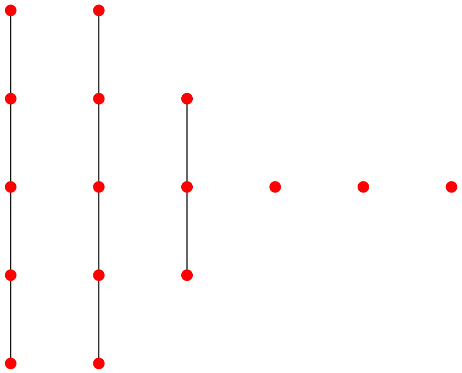


Stronger properties

Symmetric chain decomposition: a partition into saturated chains about the middle. Implies rank-symmetry and rank-unimodality. Includes B_n.



Stronger properties (cont.)

Symmetric boolean decomposition: a partition into boolean algebras symmetric about the middle. Implies rank-symmetry, **γ-positivity** (stronger than rank-unimodality), and symmetric chain decomposition. Trivial for B_n.

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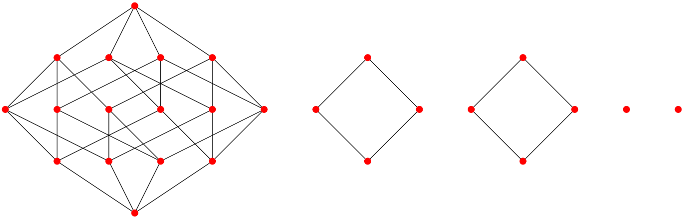

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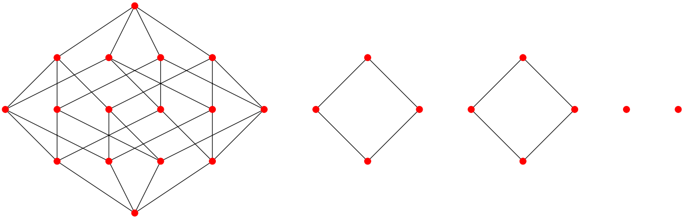

$$F_P(q) = 2q^2 + 2q(1+q)^2 + (1+q)^4$$

$$\gamma - \textbf{vector} : (2, 2, 1)$$

Explicit order-matchings

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Known for:

- products of chains (includes B_n)

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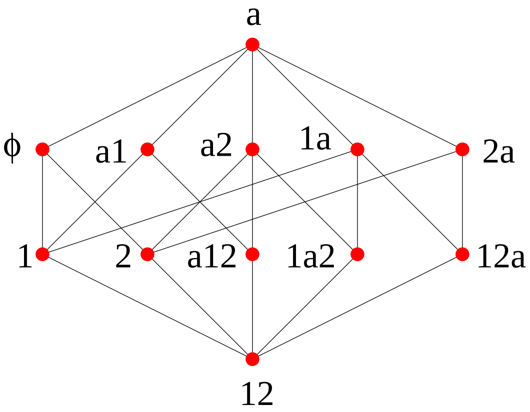
Known for:

- products of chains (includes B_n)
- noncrossing partition lattice $\mathbf{NC}_n$
  Simion-Ullman (1991):
  explicit symmetric boolean decomposition.
  Generalized by **Mühle**, 2015.

Explicit order-matchings (cont.)

- Shuffle posets (**Hersh**, 1999): symmetric chain decomposition



Normalized matchings

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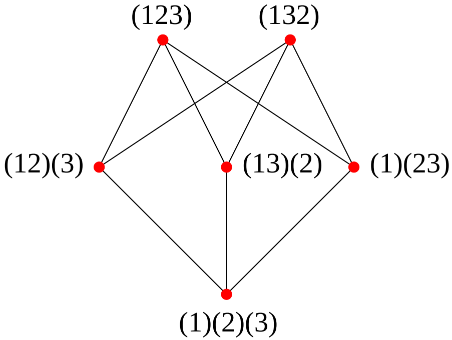
Absolute order. On $\mathfrak{S}_n$,

$$F_P(q) = (1+q)(1+2q)\cdots(1+(n-1)q)$$

$$w \text{ maximal } \Rightarrow [\mathrm{id}, w] \cong \mathrm{NC}_n$$

]

Absolute order (continued)



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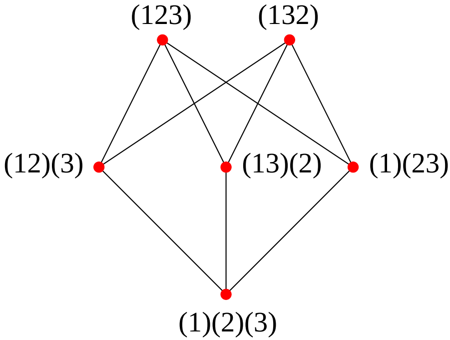

Spernicity for $\mathfrak{S}_n$ and the hyperoctahedral group $\mathfrak{H}_n$: **Harper, Kim, Livesay**, 2019

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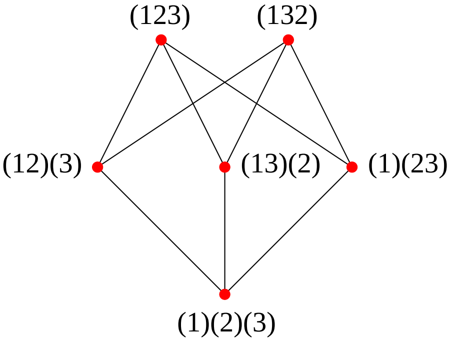

Spernicity for $\mathfrak{S}_n$ and the hyperoctahedral group $\mathfrak{H}_n$: **Harper, Kim, Livesay**, 2019

Some more general Coxeter groups: **Gaetz, Gao**, 2019.

Linear algebra

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_m : \quad \text{graded poset}$$

$$\mathbb{Q}\boldsymbol{P_i} : \quad \text{vector space with basis } \mathbb{Q}$$

$U : \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ is **order-raising** if for all $s \in P_i$,

$$U(s) \in \mathrm{span}_{\mathbb{Q}}\{t \in P_{i+1} \ : \ s < t\}$$

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U is a "**quantum**" order-matching.

A criterion for Spernicity

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_n : \text{ finite graded poset}$$

Proposition. *If for some j there exist order-raising operators*

$$\mathbb{Q}P_0 \stackrel{\text{inj.}}{\to} \mathbb{Q}P_1 \stackrel{\text{inj.}}{\to} \cdots \stackrel{\text{inj.}}{\to} \mathbb{Q}P_j \stackrel{\text{surj.}}{\to} \mathbb{Q}P_{j+1} \stackrel{\text{surj.}}{\to} \cdots \stackrel{\text{surj.}}{\to} \mathbb{Q}P_n,$$

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Order-raising and order-matchings

Key Lemma. *If* $U \colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ *is injective and order-raising, then there exists an order-matching* $\mu \colon P_i \to P_{i+1}$.

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Key Lemma. *If* $U: \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ *is injective and order-raising, then there exists an order-matching* $\mu: P_i \to P_{i+1}$.

Proof. Consider the matrix of U with respect to the bases P_i and P_{i+1}.

Key lemma proof

$$
\begin{array}{rl}
 & \overbrace{\begin{array}{ccccc} t_1 & \cdots & t_m & \cdots & t_n \end{array}}^{P_{i+1}} \\
P_i \left\{ \begin{array}{c} s_1 \\ \vdots \\ s_m \end{array} \right. &
\left[\begin{array}{ccc|c}
\neq 0 & & & * \\
 & \ddots & & * \\
 & & \neq 0 & *
\end{array} \right]
\end{array}
$$

$$\mathbf{det \neq 0}$$

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$$\mathbf{det \neq 0}$$

$$\Rightarrow s_1 < t_1, \ldots, s_m < t_m \qquad \square$$

Dual version

Similarly if there exists **surjective** order-raising $U\colon \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$, then there exists an order-matching $\mu\colon P_{i+1} \to P_i$.

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Implies Spernicity criterion

$$\mathbb{Q}P_0 \overset{\text{inj.}}{\to} \mathbb{Q}P_1 \overset{\text{inj.}}{\to} \cdots \overset{\text{inj.}}{\to} \mathbb{Q}P_j \overset{\text{surj.}}{\to} \mathbb{Q}P_{j+1} \overset{\text{surj.}}{\to} \cdots \overset{\text{surj.}}{\to} \mathbb{Q}P_n,$$

Order-raising for B_n

Define

$$\boldsymbol{U}: \mathbb{Q}(B_n)_i \to \mathbb{Q}(B_n)_{i+1}$$

by

$$U(S) = \sum_{\substack{\#T=i+1 \\ S\subset T}} T, \quad S \in (B_n)_i.$$

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Similarly define $\boldsymbol{D} : \mathbb{Q}(B_n)_{i+1} \to \mathbb{Q}(B_n)_i$ by

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Note. UD is positive semidefinite, and hence has nonnegative real eigenvalues, since the matrices of U and D with respect to the bases $(B_n)_i$ and $(B_n)_{i+1}$ are *transposes*.

A commutation relation

Lemma. *On* $\mathbb{Q}(B_n)_i$ *we have*

$$DU - UD = (n - 2i)I,$$

where $\boldsymbol{I}$ *is the identity operator.*

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$$DU - UD = (n - 2i)I,$$

where I *is the identity operator.*

Corollary. *If* $i < n/2$ *then* U *is injective.*

Proof. UD has eigenvalues $\theta \geq 0$, and eigenvalues of DU are $\theta + n - 2i > 0$. Hence DU is invertible, so U is injective. $\square$

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Corollary. B_n *is Sperner.*

What's the point?

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The symmetric group $\mathfrak{S}_n$ acts on B_n by

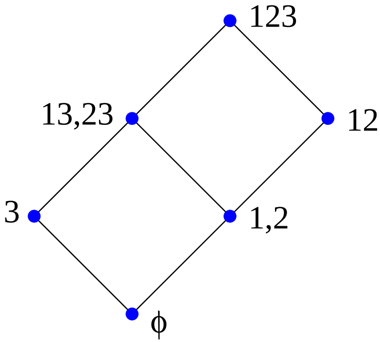
$$w \cdot \{a_1, \ldots, a_k\} = \{w \cdot a_1, \ldots, w \cdot a_k\}.$$

If G is a subgroup of $\mathfrak{S}_n$, define the **quotient poset** B_n/G to be the poset on the orbits of G (acting on B_n), with

$$\mathfrak{o} \leq \mathfrak{o}' \quad \Leftrightarrow \quad \exists S \in \mathfrak{o}, T \in \mathfrak{o}', \quad S \subseteq T.$$

An example

$$n = 3, \quad G = \{(1)(2)(3), (1,2)(3)\}$$



Spernicity of B_n/G

Easy: B_n/G is graded of rank n and rank-symmetric.

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Crux of proof. The action of $w \in G$ on B_n commutes with U, so we can "transfer" U to B_n/G, preserving injectivity on the bottom half.

An interesting example

R: set of squares of an $m \times n$ rectangle of squares.

$\boldsymbol{G_{mn}} \subset \mathfrak{S}_R$: can permute elements in each row, and permute rows among themselves, so $\#G_{mn} = n!^m m!$.

$$G_{mn} \cong \mathfrak{S}_n \wr \mathfrak{S}_m \quad (\textbf{wreath product})$$

$L(m, n)$

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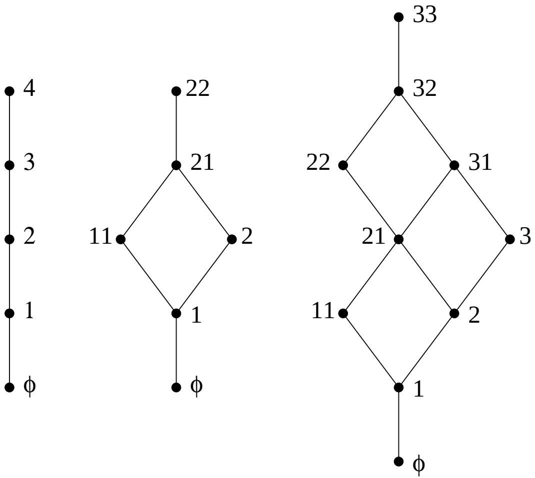
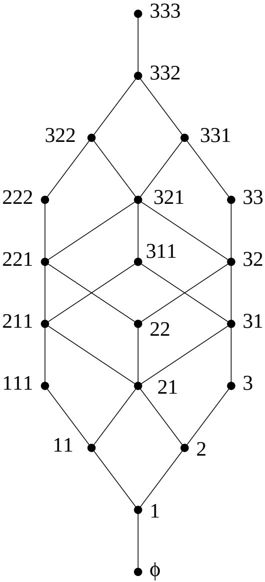
$\boldsymbol{L(m, n)}$: poset of Young diagrams in an $m \times n$ rectangle

Corollary. $B_R / G_{mn} \cong L(m, n)$

Examples of $L(m, n)$



$L(3,3)$



q-binomial coefficients

$$F(L(m,n),q) = \binom{\boldsymbol{m+n}}{\boldsymbol{m}} \quad (q\text{-binomial coefficient})$$

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- Not an order-matching. Still open to find an explicit order-matching $L(m,n)_i \to L(m,n)_{i+1}$.

Algebraic geometry

$\boldsymbol{X}$: smooth complex projective variety of dimension n

$\boldsymbol{H^*(X;\mathbb{C})} = H^0(X;\mathbb{C}) \oplus H^1(X;\mathbb{C}) \oplus \cdots \oplus H^{2n}(X;\mathbb{C})$: cohomology ring, so $H^i \cong H^{2n-i}$.

Hard Lefschetz Theorem. *There exists* $\boldsymbol{\omega} \in H^2$ *(the class of a generic hyperplane section) such that for* $0 \leq i \leq n$, *the map*

$$\omega^{n-2i} \colon H^i \to H^{2n-i}$$

is a bijection. Thus $\omega \colon H^i \to H^{i+1}$ *is injective for* $i \leq n$ *and surjective for* $i \geq n$.

Cellular decompositions

X has a **cellular decomposition** if $X = \sqcup C_i$, each $C_i \cong \mathbb{C}^{d_i}$ (as affine varieties), and each $\bar{C}_i$ is a union of C_j's.

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Fact. If X has a cellular decomposition and $[C_i] \in H^{2(n-d_i)}$ denotes the corresponding cohomology classes, then the $[C_i]$'s form a $\mathbb{C}$-basis for H^*.

The cellular decomposition poset

Let $X = \sqcup C_i$ be a cellular decomposition. Define a poset $\boldsymbol{P_X} = \{C_i\}$, by

$$C_i \leq C_j \text{ if } C_i \subseteq \bar{C}_j$$

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$\Rightarrow$ **Theorem.** P_X has the Sperner property.

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rational canonical form $\Rightarrow P_{\mathrm{Gr}(m+n,m)} \cong L(m,n)$

Another special case

$G = \mathrm{SO}(2n+1, \mathbb{C})$, $Q =$ "spin" maximal parabolic subgroup

$\boldsymbol{M(n)} \coloneqq P_{G/Q} \cong \mathfrak{B}_n/\mathfrak{S}_n$, where $\boldsymbol{\mathfrak{B}_n}$ is the hyperoctahedral group (symmetries of n-cube) of order $2^n n!$, so $\#M(n) = 2^n$

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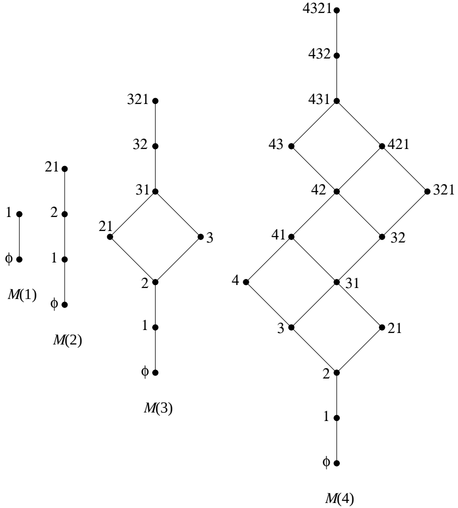
$M(n)$ is isomorphic to the set of all subsets of $\{1, 2, \ldots, n\}$ with the ordering

$$\{a_1 > a_2 > \cdots > a_r\} \le \{b_1 > b_2 > \cdots > b_s\},$$

if $r \le s$ and $a_i \le b_i$ for $1 \le i \le r$.

Examples of $M(n)$



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No combinatorial proof known, though can be done with just elementary linear algebra (**Proctor**).

The weak order on $\mathfrak{S}_n$

$s_i :== (i, i+1) \in \mathfrak{S}_n$, $1 \le i \le n-1$ (**adjacent transposition**)

For $w \in \mathfrak{S}_n$,

$$\begin{aligned} \ell(w) &:= \#\{(i,j) : i < j, w(i) > w(j)\} \\ &= \min\{p : w = s_{i_1} \cdots s_{i_p}\}. \end{aligned}$$

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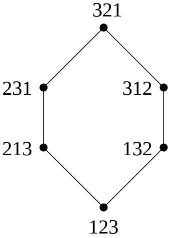
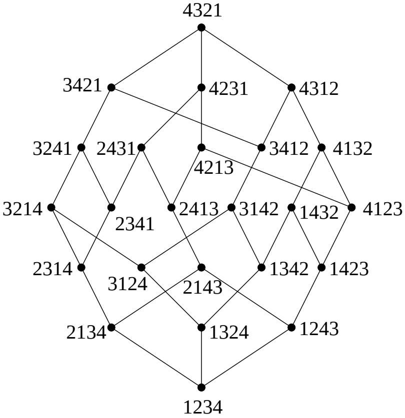
A. Björner (1984): does W_n have the Sperner property?

Examples of weak order



W_3



W_4

An order-raising operator

theory of Schubert polynomials suggests:

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Fact (**Macdonald**, **Fomin-S**, **Billey-Hoylroyd-Young**). Let $u < v$ in W_n, $\ell(v) - \ell(u) = p$. The coefficient of v in $U^p(u)$ is

$$p!\,\mathfrak{S}_{vu^{-1}}(1, 1, \ldots, 1),$$

where $\mathfrak{S}_w(x_1, \ldots, x_{n-1})$ is a **Schubert polynomial**.

A down operator

C. Gaetz and **Y. Gao** (2018): constructed $D\colon \mathbb{Q}(W_n)_i \to \mathbb{Q}(W_n)_{i-1}$ such that

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Suffices for Spernicity.

Note. D is order-lowering on the **strong** Bruhat order. Leads to duality between weak and strong order.

Another method

Z. Hamaker, **O. Pechenik**, **D. Speyer**, and **A. Weigandt** (2018): for $k < \frac{1}{2}\binom{n}{2}$, let

$$\boldsymbol{D(n,k)} = \text{matrix of } U^{\binom{n}{2}-2k} \colon \mathbb{Q}(W_n)_k \to \mathbb{Q}(W_n)_{\binom{n}{2}-k}$$

with respect to the bases $(W_n)_k$ and $(W_n)_{\binom{n}{2}-k}$ (in some order). Then (conjectured by **RS**):

$$\det D(n,k) = \pm \left(\binom{n}{2} - 2k \right)!^{\#(W_n)_k} \prod_{i=0}^{k-1} \left(\frac{\binom{n}{2} - (k+i)}{k-i} \right)^{\#(W_n)_i} .$$

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Gaetz-Gao: Smith normal form of $D(n,k)$

An open problem

The weak order $W(G)$ can be defined for any (finite) Coxeter group G. Is $W(G)$ Sperner?

Infinite posets

Exercise. If P is a poset for which every chain and antichain is finite, then P is finite.

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Sample result. There is a poset of cardinality continuum in which all chains and all antichains are countable.

Proof. Let $<$ be usual ordering of $\mathbb{R}$, and $\prec$ a well-ordering of $\mathbb{R}$. For $x, y \in \mathbb{R}$ define $x \ll y$ if $x < y$ and $x \prec y$. In $(\mathbb{R}, \ll)$, every chain is a well-ordered subset of $(\mathbb{R}, <)$ (since on a chain $<$ and $\prec$ are the same), and every antichain is a well-ordered subset of $(\mathbb{R}, <^*)$. It is easy to see that every well-ordered subset of $(\mathbb{R}, <)$ is countable, so the proof follows. $\square$

The final slide

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