

I. Stern's Diatomic Array and Beyond

II. A Weak Order Conjecture

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Dedication

Fantastically

Original

Mathematics

Is

Noteworthy

PART I

Stern's Diatomic Array and Beyond

The arithmetic triangle or Pascal's triangle

				1				
			1		1			
		1		2		1		
	1	3		3		1		
1	4		6		4		1	
1	5	10		10		5		1
			⋮					

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$$\sum_{k \geq 0} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \quad (\text{not rational})$$

Sums of cubes

$$\sum_{k \geq 0} \binom{n}{k}^3 = ??$$

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If $f(n) = \sum_{k \geq 0} \binom{n}{k}^3$ then

$$(n+2)^2 f(n+2) - (7n^2 + 21n + 16) f(n+1) - 8(n+1)^2 f(n) = 0, \quad n \geq 0$$

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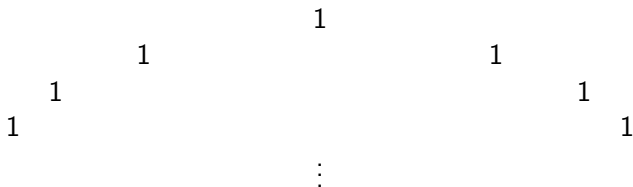
Etc.

A second triangle

Similar to Pascal's triangle, but we also “bring down” (copy) each number from one row to the next.

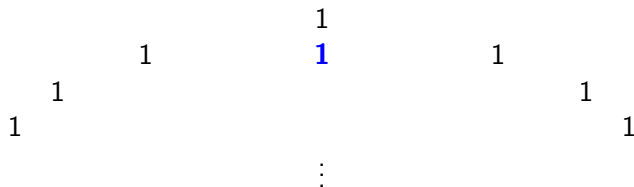
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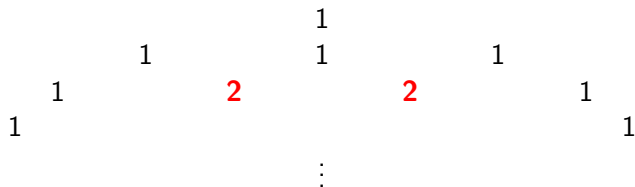
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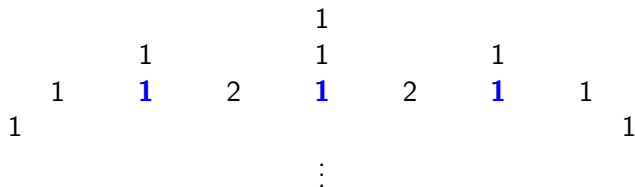
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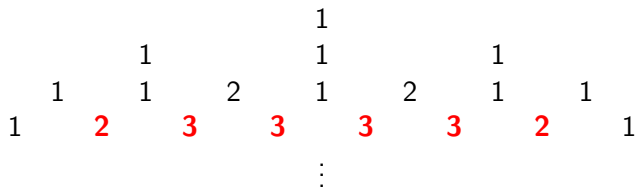
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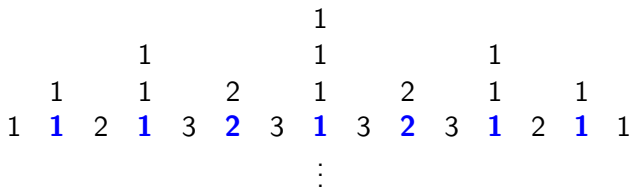
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								1						
			1					1				1		
		1	1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
								⋮						

Stern's triangle

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- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
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- Sum of entries in row n : 3^n
- Largest entry in row n : F_{n+1} (Fibonacci number)
- Let $\langle n \ k \rangle$ be the k th entry (beginning with $k = 0$) in row n .
Write

$$P_n(x) = \sum_{k \geq 0} \langle n \ k \rangle x^k.$$

Then $P_{n+1}(x) = (1 + x + x^2)P_n(x^2)$, since $x P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2)P_n(x^2)$ to summing two consecutive entries.

Stern's diatomic sequence

- **Corollary.**
$$P_n(x) = \prod_{i=0}^{n-1} (1 + x^{2^i} + x^{2 \cdot 2^i})$$

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$$\begin{aligned} P(x) &= \prod_{i=0}^{\infty} (1 + x^{2^i} + x^{2 \cdot 2^i}) \\ &:= \sum_{n \geq 0} b_n x^n. \end{aligned}$$

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- The sequence b_0, b_1, b_2, \dots is **Stern's diatomic sequence**:

1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ...

(often prefixed with 0)

Partition interpretation

$$\sum_{n \geq 0} b_n x^n = \prod_{i \geq 0} (1 + x^{2^i} + x^{2 \cdot 2^i})$$

$\Rightarrow b_n$ is the number of partitions of n into powers of 2, where each power of 2 can appear at most twice.

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Note. If each power of 2 can appear at most once, then we obtain the (unique) binary expansion of n :

$$\frac{1}{1-x} = \prod_{i \geq 0} (1 + x^{2^i}).$$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
								⋮								

Amazing property

Theorem (Stern, 1858). Let b_0, b_1, \dots be Stern's diatomic sequence. Then every positive rational number occurs exactly once among the ratios b_i/b_{i+1} , and moreover this expression is in lowest terms.

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Can be proved inductively from

$$b_{2n} = b_n, \quad b_{2n+1} = b_n + b_{n+1},$$

but better is to use **Calkin-Wilf tree**, though following Stigler's law of eponymy was earlier introduced by **Jean Berstel** and **Aldo de Luca** as the **Raney tree**. Closely related tree by Stern, called the **Stern-Brocot tree**, and a much earlier similar tree by **Kepler** (1619).

Stigler's law of eponymy

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Note. Stigler's law of eponymy implies that Stigler's law of eponymy was not originally discovered by Stigler.

Sums of squares

								1						
			1					1				1		
	1		1		2		1		2		1		1	
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
								⋮						

$$u_2(n) := \sum_k \langle \binom{n}{k} \rangle^2 = 1, 3, 13, 59, 269, 1227, \dots$$

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$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

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$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

$$\sum_{n \geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}$$

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$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Proof for $u_2(n)$

$$\begin{aligned}u_2(n+1) &= \cdots + \langle n \rangle_k^2 + \left(\langle n \rangle_k + \langle n \rangle_{k+1} \right)^2 + \langle n \rangle_{k+1}^2 + \cdots \\ &= 3u_2(n) + 2 \sum_k \langle n \rangle_k \langle n \rangle_{k+1}\end{aligned}$$

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Thus define $u_{1,1}(n) := \sum_k \binom{n}{k} \binom{n}{k+1}$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$\begin{aligned}u_{1,1}(n+1) &= \cdots + \left(\binom{n}{k-1} + \binom{n}{k} \right) \binom{n}{k} \\ &\quad + \binom{n}{k} \left(\binom{n}{k} + \binom{n}{k+1} \right) \\ &\quad + \left(\binom{n}{k} + \binom{n}{k+1} \right) \binom{n}{k+1} + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n)\end{aligned}$$

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Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let

$$\mathbf{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then

$$\mathbf{A} \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

What about $u_3(n)$?

Now we need

$$u_{2,1}(n) := \sum_k \langle n \rangle_k^2 \langle n \rangle_{k+1}$$

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We get

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_3(n) \\ u_{2,1}(n) \end{bmatrix} = \begin{bmatrix} u_3(n+1) \\ u_{2,1}(n+1) \end{bmatrix}.$$

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Thus $u_3(n + 1) = 7u_3(n)$ and $u_{2,1}(n + 1) = 7u_{2,1}(n)$ ($n \geq 1$).

In fact, for $n \geq 1$ we have

$$\begin{aligned} u_3(n) &= 3 \cdot 7^{n-1} \\ u_{2,1}(n) &= 2 \cdot 7^{n-1}. \end{aligned}$$

What about $u_r(n)$ for general $r \geq 1$?

Get a matrix of size $\lceil (r + 1)/2 \rceil$, so expect a recurrence of this order.

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Conjecture. The least order of a homogenous linear recurrence with constant coefficients satisfied by $u_r(n)$ is $\frac{1}{3}r + O(1)$.

A more accurate conjecture

Write $[a_0, \dots, a_{m-1}]_m$ for the periodic function $f: \mathbb{N} \rightarrow \mathbb{R}$ satisfying $f(n) = a_i$ if $n \equiv i \pmod{m}$.

A_r : matrix arising from $u_r(n)$

$e_i(r)$: # eigenvalues of A_r equal to i

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Conjecture. We have

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T. Amdeberhan: $e_0(2k-1) > 0$

Even d

Conjecture. We have

$$\begin{aligned}e_1(2k) &= \frac{1}{6}k + \left[-1, -\frac{1}{6}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}, \frac{1}{6} \right]_6 \\e_{-1}(2k) &= e_1(2k + 6).\end{aligned}$$

The eigenvalues 1 and -1 are semisimple, and there are no other multiple eigenvalues.

Minimum order of recurrence

mo(r): minimum order of recurrence satisfied by $u_r(n)$

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Conjecture. We have $\text{mo}(2) = 2$, $\text{mo}(6) = 4$, and otherwise

$$\begin{aligned}\text{mo}(2s) &= 2 \left\lfloor \frac{s}{3} \right\rfloor + 3 \quad (s \neq 1, 3) \\ \text{mo}(6s + 1) &= 2s + 1, \quad s \geq 0 \\ \text{mo}(6s + 3) &= 2s + 1, \quad s \geq 0 \\ \text{mo}(6s + 5) &= 2s + 2, \quad s \geq 0.\end{aligned}$$

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True for $r \leq 125$.

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True for $r \leq 125$.

$$\sum_{r \geq 0} \text{mo}(r)x^r = \frac{\text{irred. deg } 13}{(1-x)(1-x^6)}$$

General α

$$\alpha = (\alpha_0, \dots, \alpha_{m-1})$$

$$u_\alpha(n) := \sum_k \binom{n}{k}^{\alpha_0} \binom{n}{k+1}^{\alpha_1} \cdots \binom{n}{k+m-1}^{\alpha_{m-1}}$$

A closer look at $\alpha = (1, 1, 1, 1)$

$$u_{1,1,1,1}(n) = \sum_k \binom{n}{k} \binom{n}{k+1} \binom{n}{k+2} \binom{n}{k+3}$$

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$$u_{1,1,1,1}(n+1) =$$

$$\begin{aligned} & \sum_k \left(\binom{n}{k} + \binom{n}{k+1} \right) \binom{n}{k+1} \left(\binom{n}{k+1} + \binom{n}{k+2} \right) \binom{n}{k+2} \\ & + \sum_k \binom{n}{k} \left(\binom{n}{k} + \binom{n}{k+1} \right) \binom{n}{k+1} \left(\binom{n}{k+1} + \binom{n}{k+2} \right) \end{aligned}$$

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$$A_{(1,1,1,1)} = \begin{bmatrix} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix} \begin{matrix} 4 \\ 3, 1 \\ 2, 2 \\ 1, 2, 1 \\ 2, 1, 1 \\ 1, 1, 1, 1 \end{matrix}$$

A closer look at $\alpha = (1, 1, 1, 1)$

$$u_{1,1,1,1}(n) = \sum_k \binom{n}{k} \binom{n}{k+1} \binom{n}{k+2} \binom{n}{k+3}$$

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Reduction to $\alpha = (r)$

min. polynomial for $\alpha = (4)$: $(x + 1)(2x^2 - 11x + 1)$

min. polynomial for $\alpha = (1, 1, 1, 1)$: $(x - 1)^2(x + 1)(2x^2 - 11x + 1)$

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mp(α): minimum polynomial of A_α

Theorem. Let $\alpha \in \mathbb{N}^m$ and $\sum \alpha_i = r$. Then $\text{mp}(\alpha)$ has the form $x^{w_\alpha}(x - 1)^{z_\alpha} \text{mp}(r)$ for some $w_\alpha, z_\alpha \in \mathbb{N}$.

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No conjecture for value of w_α, z_α .

A generalization

Let $p(x), q(x) \in \mathbb{C}[x]$, $\alpha = (\alpha_0, \dots, \alpha_{m-1}) \in \mathbb{N}^m$, and $b \geq 2$. Set

$$q(x) \prod_{i=0}^{n-1} p(x^{b^i}) = \sum_k \langle n \rangle_{p,q,\alpha,b} x^k = \sum_k \langle n \rangle x^k$$

and

$$u_{p,q,\alpha,b}(n) = \sum_k \langle n \rangle_k^{\alpha_0} \langle n \rangle_{k+1}^{\alpha_1} \cdots \langle n \rangle_{k+m-1}^{\alpha_{m-1}}.$$

Main theorem

Theorem. For fixed p, q, α, b , the function $u_{p,q,\alpha,b}(n)$ satisfies a linear recurrence with constant coefficients ($n \gg 0$). Equivalently, $\sum_n u_{p,q,\alpha,b}(n)x^n$ is a rational function of x .

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Note. \exists multivariate generalization.

A special case

Let $p(x) = (1 + x)^2 = 1 + 2x + x^2$, $b = 2$, $\alpha = (r)$, so $u_{p,\alpha,b}(n)$ is the sum of the r th powers of the coefficients of

$$\prod_{i=0}^{n-1} (1 + x^{2^i})^2.$$

Theorem. $u_{(1+x)^2,(2),2}(n)$ has the form $\sum c_i 2^{2in}$ or $\sum c_i 2^{(2i+1)n}$.

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$$u_{(1+x)^2,(2),2}(n) = \frac{1}{3} (2 \cdot 2^{3n} + 2^n)$$

$$u_{(1+x)^2,(3),2}(n) = \frac{1}{2} (2^{4n} + 2^{2n})$$

$$u_{(1+x)^2,(4),2}(n) = \frac{1}{15} (6 \cdot 2^{5n} + 10 \cdot 2^{3n} - 2^n).$$

A special case (cont.)

Key to proof for $p(x) = (1 + x)^2$:

$$\prod_{i=0}^{n-1} (1 + x^{2^i})^2 = \left(\frac{1 - x^{2^n}}{1 - x} \right)^2.$$

A special case (cont.)

Key to proof for $p(x) = (1 + x)^2$:

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Generalizes to $u_{(1+x+x^2+\dots+x^{c-1})^d, \alpha, b}(n)$, $c|b$.

PART II

A Weak Order Conjecture

Graded posets

P : finite poset

chain : $u_1 < u_2 < \cdots < u_k$.

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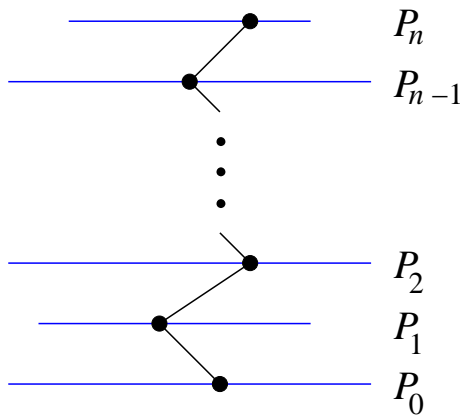
Assume P is **finite**. P is **graded of rank n** if

$$P = P_0 \cup P_1 \cup \cdots \cup P_n,$$

such that every maximal chain has the form

$$t_0 < t_1 < \cdots < t_n, \quad t_i \in P_i.$$

Diagram of a graded poset



Rank-symmetry and unimodality

Let $p_i = \#P_i$.

Rank-generating function: $F_P(q) = \sum_{i=0}^n p_i q^i$

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rank-unimodal and rank-symmetric $\Rightarrow j = \lfloor n/2 \rfloor$

The Sperner property

antichain $A \subseteq P$:

$$s, t \in A, \quad s \leq t \Rightarrow s = t$$

••••

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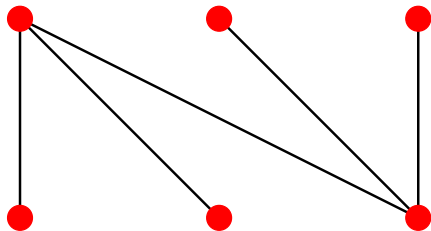


Note. P_i is an antichain

P is **Sperner** (or has the **Sperner property**) if

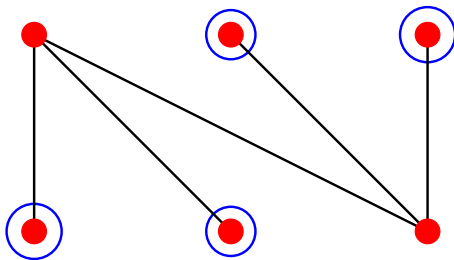
$$\max_A \#A = \max_i p_i$$

An example



rank-symmetric, rank-unimodal, $F_P(q) = 3 + 3q$

An example



rank-symmetric, rank-unimodal, $F_P(q) = 3 + 3q$ not Sperner

The boolean algebra

B_n : subsets of $\{1, 2, \dots, n\}$, ordered by inclusion

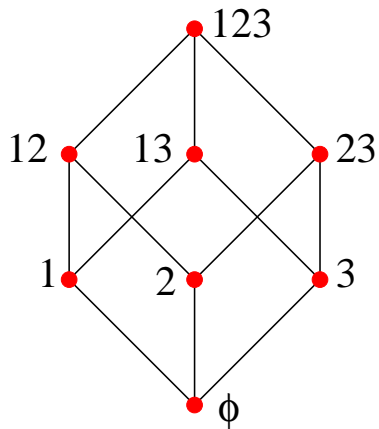
The boolean algebra

B_n : subsets of $\{1, 2, \dots, n\}$, ordered by inclusion

$$p_i = \binom{n}{i}, \quad F_{B_n}(q) = (1 + q)^n$$

rank-symmetric, rank-unimodal

Diagram of B_3



Sperner's theorem, 1927

Theorem. B_n is Sperner.

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Theorem. B_n is Sperner.

Emanuel Sperner

9 December 1905 – 31 January 1980



Linear algebra to the rescue!

$P = P_0 \cup \dots \cup P_m$: graded poset

$\mathbb{Q}P_i$: vector space with basis P_i

$U: \mathbb{Q}P_i \rightarrow \mathbb{Q}P_{i+1}$ is **order-raising** if

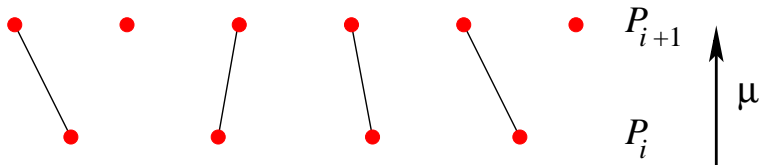
$$U(s) \in \text{span}_{\mathbb{Q}}\{t \in P_{i+1} : s < t\}$$

Order-matchings

Order matching: $\mu: P_i \rightarrow P_{i+1}$: injective and $\mu(t) > t$

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Order-raising and order-matchings

Key Lemma. *If $U: \mathbb{Q}P_i \rightarrow \mathbb{Q}P_{i+1}$ is injective and order-raising, then there exists an order-matching $\mu: P_i \rightarrow P_{i+1}$.*

Order-raising and order-matchings

Key Lemma. *If $U: \mathbb{Q}P_i \rightarrow \mathbb{Q}P_{i+1}$ is injective and order-raising, then there exists an order-matching $\mu: P_i \rightarrow P_{i+1}$.*

Proof. Consider the matrix of U with respect to the bases P_i and P_{i+1} .

Key lemma proof

$$P_i \left\{ \begin{array}{l} s_1 \\ \vdots \\ s_m \end{array} \right. \left[\begin{array}{cccc|c} \overbrace{t_1 \cdots t_m \cdots t_n}^{P_{i+1}} & & & & \\ \neq 0 & & & & * \\ & \ddots & & & * \\ & & & \neq 0 & * \end{array} \right]$$

$\det \neq 0$

Key lemma proof

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P_{i+1}

$t_1 \quad \cdots \quad t_m \quad \cdots \quad t_n$

det $\neq 0$

$$\Rightarrow s_1 < t_1, \dots, s_m < t_m$$



Minor variant

Similarly if there exists **surjective** order-raising $U: \mathbb{Q}P_i \rightarrow \mathbb{Q}P_{i+1}$, then there exists an order-matching $\mu: P_{i+1} \rightarrow P_i$.

A criterion for Spernicity

$P = P_0 \cup \dots \cup P_n$: finite graded poset

Proposition. *If for some j there exist order-raising operators*

$$\mathbb{Q}P_0 \xrightarrow{\text{inj.}} \mathbb{Q}P_1 \xrightarrow{\text{inj.}} \dots \xrightarrow{\text{inj.}} \mathbb{Q}P_j \xrightarrow{\text{surj.}} \mathbb{Q}P_{j+1} \xrightarrow{\text{surj.}} \dots \xrightarrow{\text{surj.}} \mathbb{Q}P_n,$$

then P is rank-unimodal and Sperner.

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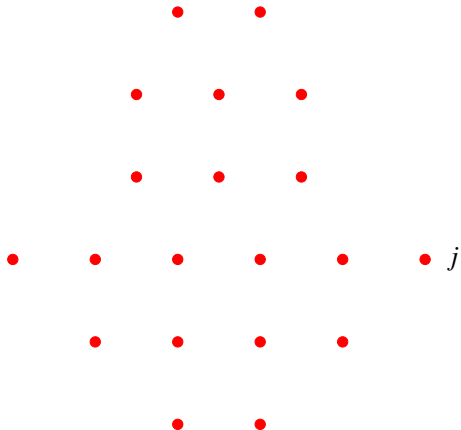
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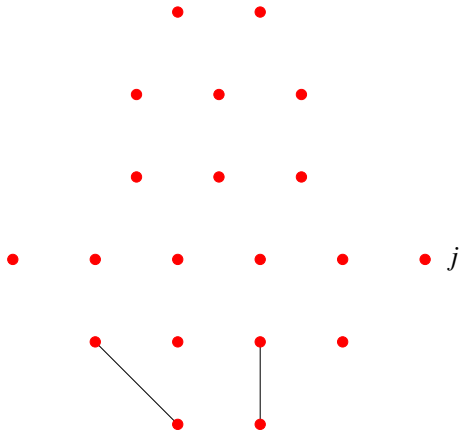
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Proof. “Glue together” the order-matchings.

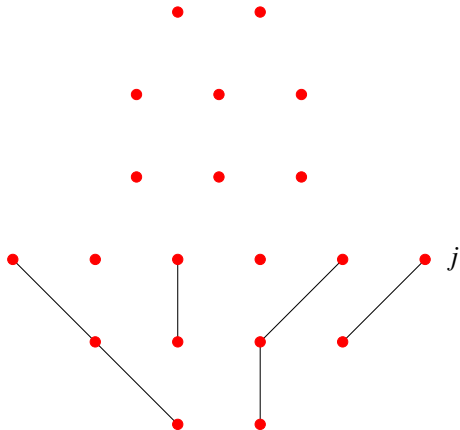
Gluing example



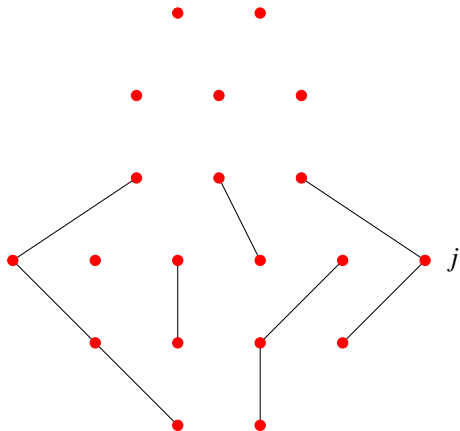
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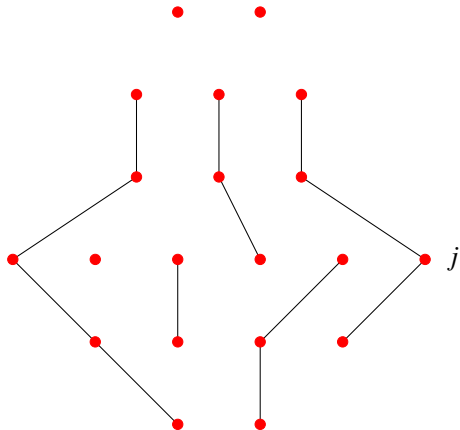
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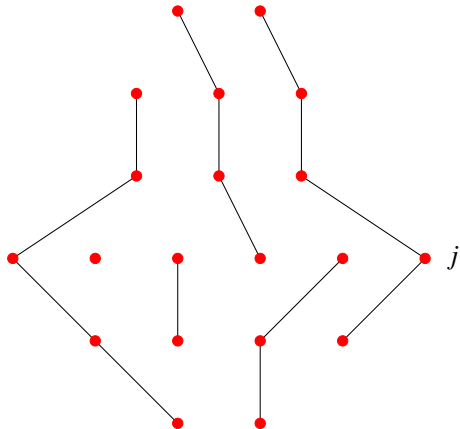
Gluing example



Gluing example



Gluing example



A chain decomposition

$$P = C_1 \cup \cdots \cup C_{p_j} \quad (\text{chains})$$

$$A = \text{antichain}, C = \text{chain} \Rightarrow \#(A \cap C) \leq 1$$

$$\Rightarrow \#A \leq p_j. \quad \square$$

The weak order $W(S_n)$ on S_n

$$s_i = (i, i + 1), \quad 1 \leq i \leq n - 1$$

$$w \in S_n, \quad \ell(w) = \#\{1 \leq i < j \leq n : w(i) > w(j)\}$$

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$W(S_n)$ is graded of rank $\binom{n}{2}$, rank-symmetric, and rank-unimodal, with

$$\begin{aligned} F_{W(S_n)}(q) &:= \sum_{k=0}^{\binom{n}{2}} \#W(S_n)_k q^k \\ &= (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}). \end{aligned}$$

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How to define $U_k: \mathbb{Q}W(S_n)_k \rightarrow \mathbb{Q}W(S_n)_{k+1}$?

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Theorem (Macdonald 1991, Fomin-S. 1994). \mathfrak{S}_w : Schubert polynomial indexed by $w \in S_n$. Let $k = \ell(w)$.

$$k! \mathfrak{S}_w(1, 1, \dots, 1) = \sum_{(a_1, \dots, a_k) \in R(w)} a_1 \cdots a_k,$$

where $R(w)$ is the set of reduced decompositions of w , i.e.,

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Example. $321 = s_1 s_2 s_1 = s_2 s_1 s_2$, and

$$1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 6 = \ell(321)!.$$

An equivalent formulation

Define

$$U(w) = U_k(w) = \sum_{i: s_i w > w} i \cdot s_i w.$$

If $u < v$ and $\ell(v) - \ell(u) = r$, then

$$[v]U^r(u) = r! \mathfrak{S}_{u^{-1}v}(1, 1, \dots, 1).$$

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Thus U is a “natural” order-raising operator for $W(S_n)$.

A matrix

$D(n, k)$: matrix of

$$U^{\binom{n}{2}-2k} : \mathbb{Q}W(S_n)_k \rightarrow \mathbb{Q}W(S_n)_{\binom{n}{2}-k}$$

with respect to the bases $W(S_n)_k$ and $W(S_n)_{\binom{n}{2}-k}$.

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If $u \in W(S_n)_k$ and $v \in W(S_n)_{\binom{n}{2}-k}$, then

$$D(n, k)_{uv} = \begin{cases} (\cdots) \mathfrak{S}_{u^{-1}v}(1, \dots, 1), & u \leq v \\ 0, & u \not\leq v. \end{cases}$$

A determinant

To show: $\det D(n, k) \neq 0$ (implies $W(S_n)$ is Sperner).

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Conjecture. Write $W_n = W(S_n)$. Then

$$\det D(n, k) = \pm \left(\binom{n}{2} - 2k \right)!^{\#(W_n)_k} \prod_{i=0}^{k-1} \left(\frac{\binom{n}{2} - (k+i)}{k-i} \right)^{\#(W_n)_i} .$$

Evidence

- True for (n, k) where both $n \leq 12$ and $k \leq 5$, and a few more cases.

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- True for $k = 0$ (trivial) and $k = 1$.

Open problems

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Is there a nice q -analogue?

- Other types, i.e., the weak order of other Coxeter groups?

The final slide



The final slide

