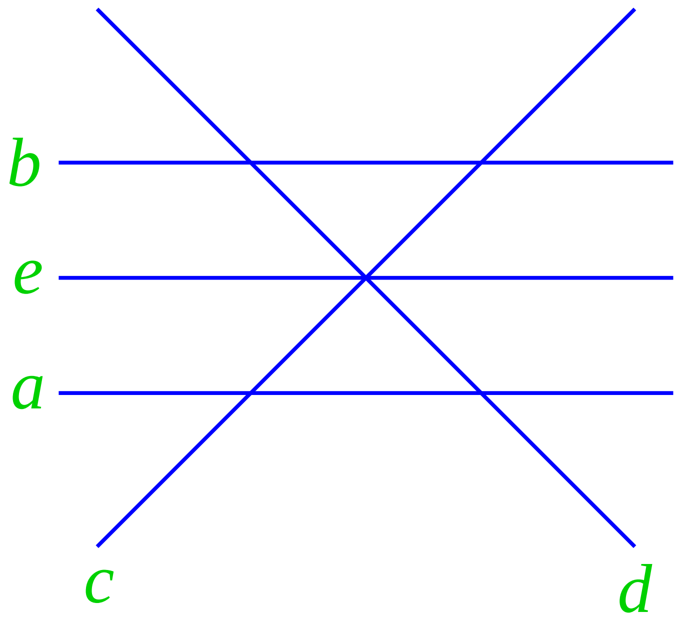
arrangement: a finite set $\mathcal{A}$ of affine hyperplanes in K^n, where K is a field

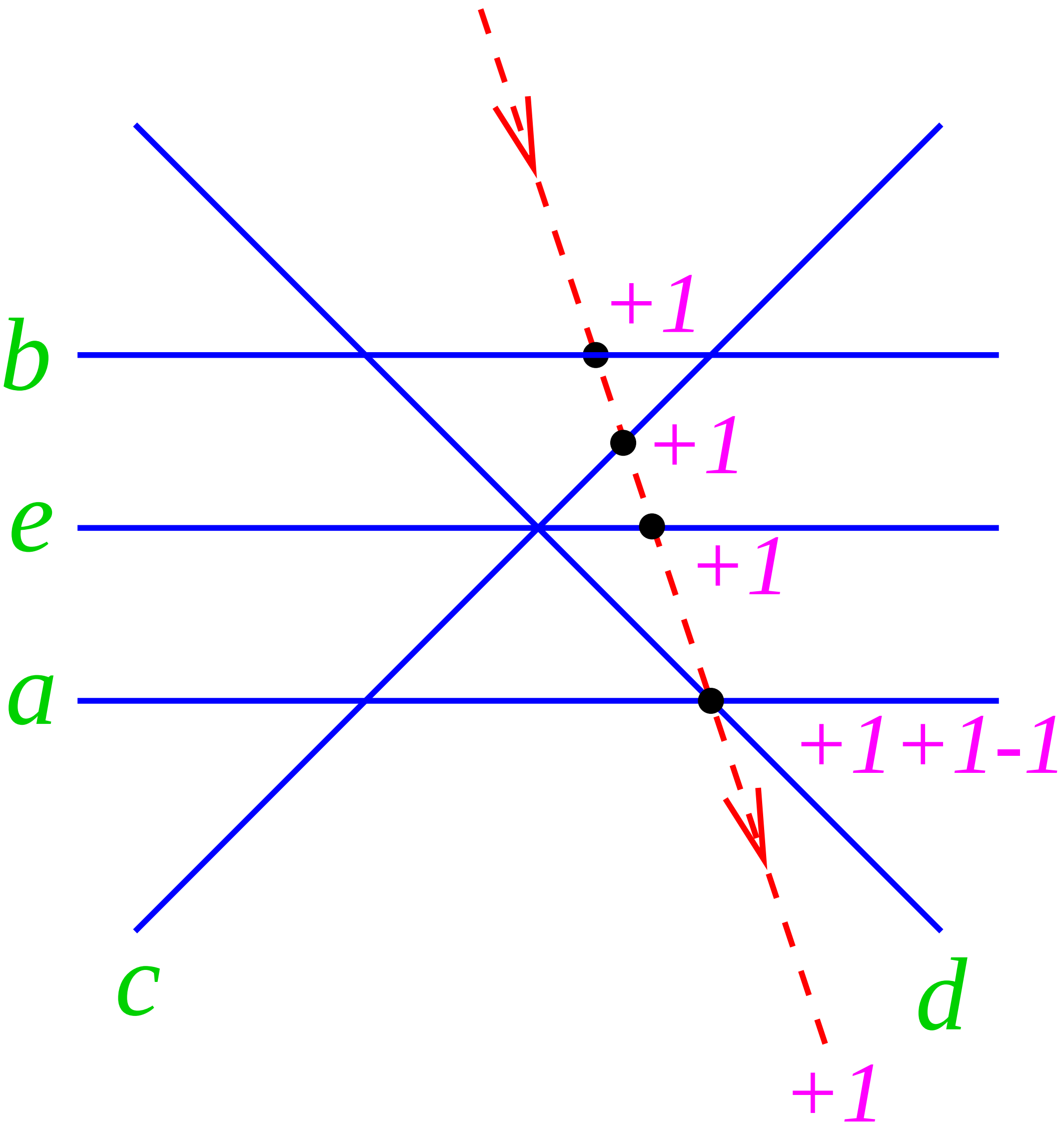


region: a connected component of $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$

$r(\mathcal{A})$ = number of regions

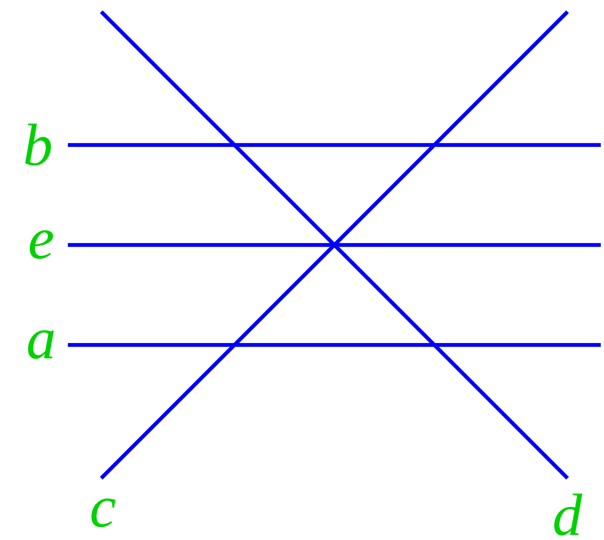
E.g, for above arrangement,

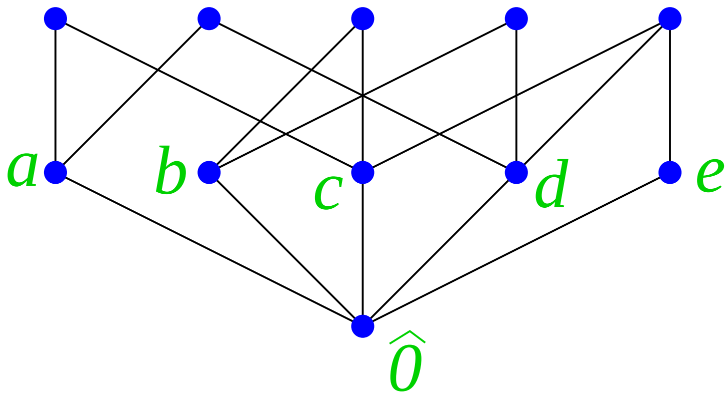
$$r(\mathcal{A}) = 12.$$



b
e
a
c
d
+1
+1
+1
+1+1-1
+1

intersection poset $P_{\mathcal{A}}$: set of **nonempty** intersections of hyperplanes in $\mathcal{A}$, ordered by reverse inclusion (including $\mathbb{R}^n$, denoted $\hat{0}$)

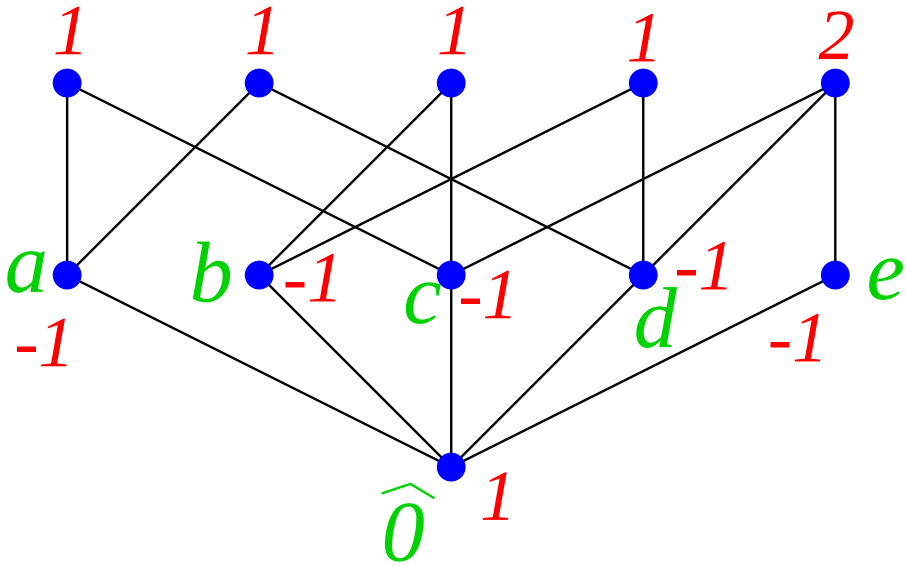


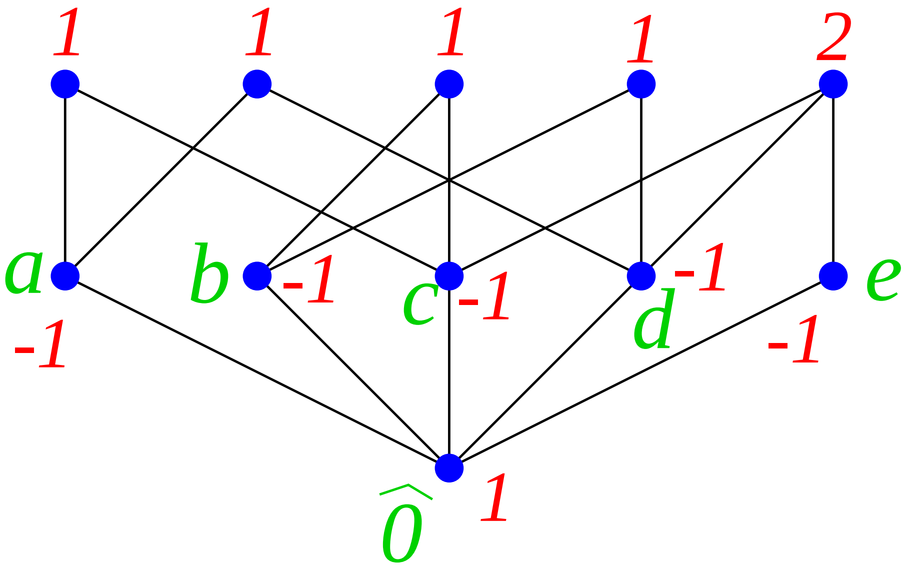


Möbius function $\mu : P_{\mathcal{A}} \to \mathbb{Z}$:

$$\mu(\hat{0}) = 1$$

$$t > 0 \Rightarrow \sum_{s \leq t} \mu(s) = 0$$





characteristic polynomial:

$$\chi_{\mathcal{A}}(q) = \sum_{t \in P_{\mathcal{A}}} \mu(t) q^{\dim t}$$

For example above,

$$\chi_{\mathcal{A}}(q) = q^2 - 5q + 6.$$

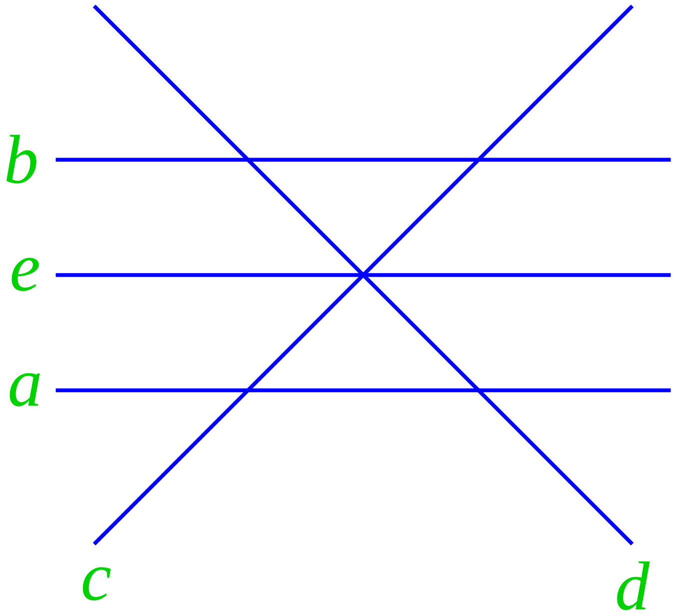
Note: $(-1)^{\operatorname{codim} t} \mu(t) > 0$, so coefficients of $\chi_{\mathcal{A}}(q)$ **alternate in sign**.

Theorem (Zaslavsky, 1975)

(a) $r(\mathcal{A}) = \sum_{t \in P_{\mathcal{A}}} |\mu(t)| = (-1)^n \chi_{\mathcal{A}}(-1)$

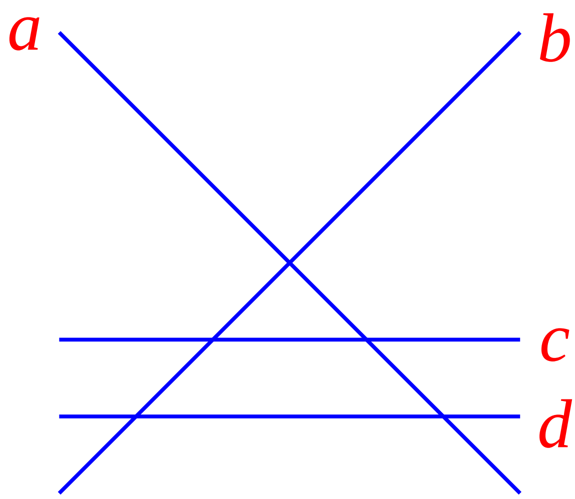
(b) *The number* $b(\mathcal{A})$ *of (relatively)* **bounded** *regions of* $\mathcal{A}$ *is equal to*

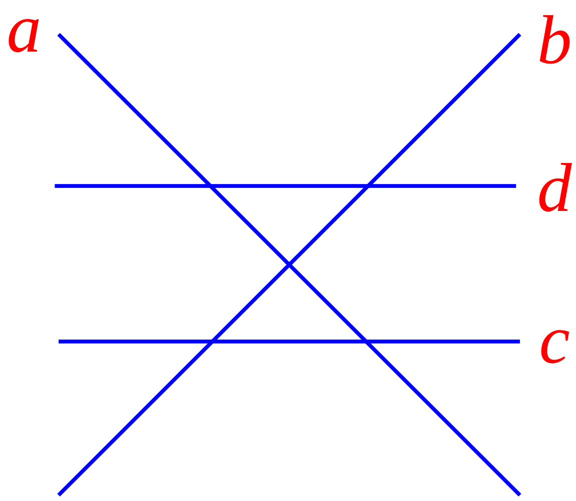
$$b(\mathcal{A}) = \left| \sum_{t \in P_{\mathcal{A}}} \mu(t) \right| = \pm \chi_{\mathcal{A}}(1).$$



$$\chi_{\mathcal{A}}(q) = q^2 - 5q + 6$$

Corollary. $r(\mathcal{A})$ and $b(\mathcal{A})$ depend only on $P_{\mathcal{A}}$.





OTHER APPEARANCES OF $\chi_{\mathcal{A}}(q)$

(1) Let $\mathcal{A}_{\mathbb{C}}$ denote the complexification of $\mathcal{A}$. Let

$$X = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}_{\mathbb{C}}} H$$

$H_i(X;\mathbb{Z}) = i$th homology group of X.

Theorem (Orlik-Solomon, 1980)

$$\sum_i (\operatorname{rank} H_i(X;\mathbb{Z}))\, q^i = (-q)^n \chi_{\mathcal{A}}(-1/q).$$

(2) Suppose that $\mathcal{A}$ is **central**, i.e., $0 \in H$ for all $H \in \mathcal{A}$. Let

$$x = (x_1, \dots, x_n).$$

Define the **Terao module**

$$T(\mathcal{A}) =$$

$$\{p(x) = (p_1(x), \dots, p_n(x)) \in \mathbb{R}[x]^n : p(\alpha) \in H \text{ for all } \alpha \in H,\ H \in \mathcal{A}\}.$$

Theorem (Terao, 1980) *If $T(\mathcal{A})$ is a* **free** *$\mathbb{R}[x]$-module with homogeneous generators of degrees $d_1, \dots, d_n$, then*

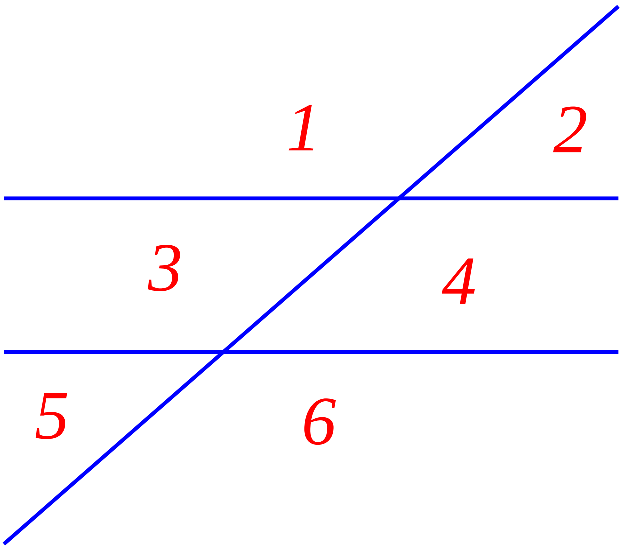
$$\chi_{\mathcal{A}}(q) = \prod_{i=1}^{n} (q - d_i).$$

Open: Does the freeness of $T(\mathcal{A})$ depend only on $P_{\mathcal{A}}$? (probably not)

(3) Given two regions R, R' of $\mathcal{A}$, let

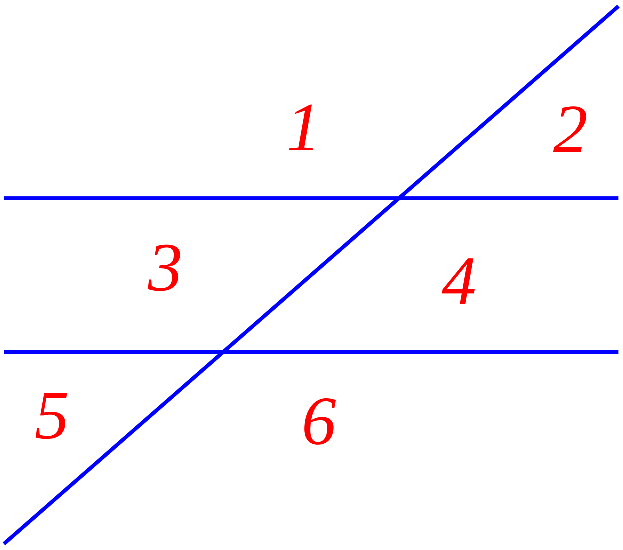
$$d(R,R') = \#\, H \in \mathcal{A} \text{ separating } R \text{ and } R'.$$

$$D(\mathcal{A}) = \left[q^{d(R,R')} \right]$$



Write $\mathbf{i} = q^i$.

$$D(\mathcal{A}) = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$



Smith normal form of $D(\mathcal{A})$:

$$\operatorname{diag}\Big(1, q^2-1, q^2-1, q^2-1, (q^2-1)^2, (q^2-1)^2\Big).$$

Let a_i = # entries of SNF exactly divisible by $(q-1)^i$.

$$a_0 = 1, \quad a_1 = 3, \quad a_2 = 2$$

Theorem (Varchenko, 1993). *Entries of SNF are products of cyclotomic polynomials.*

Theorem (Denham-Hanlon, 1997).

$$\chi_{\mathcal{A}}(q) = \sum_i (-1)^{n-i} a_{n-i} q^i.$$

OPEN: SNF of $D(\mathcal{A})$

(4) A **face** of $\mathcal{A}$ is a (nonempty) face of some region of $\mathcal{A}$.

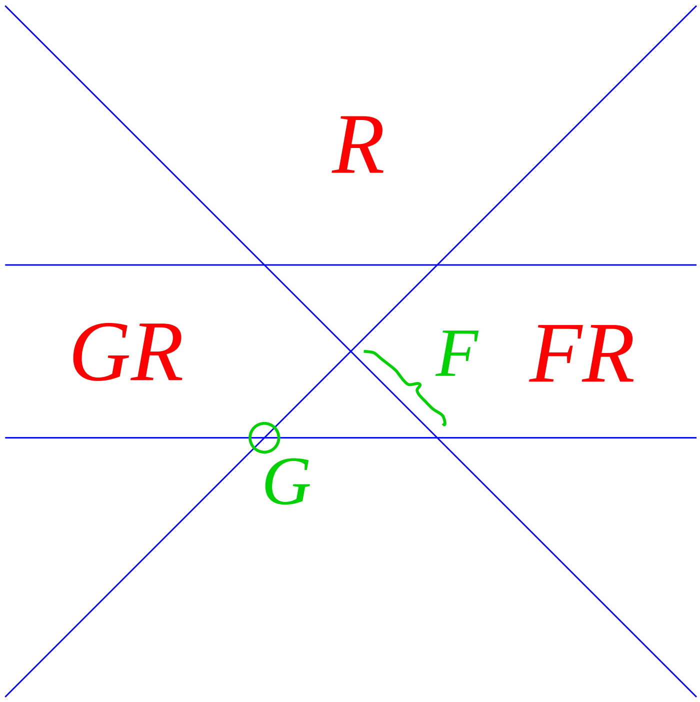
$$\begin{aligned} \mathcal{F} &= \text{set of faces of } \mathcal{A} \\ \mathcal{R} &= \text{set of regions of } \mathcal{A} \end{aligned}$$

If $F \in \mathcal{F}$ and $R \in \mathcal{R}$, define

$$\mathbf{FR} = \text{nearest region to } R$$
$$\text{with } F \text{ as a face.}$$

$$p = \text{probability measure on } \mathcal{F}$$

Define a random walk on $\mathcal{R}$: from $R \in \mathcal{R}$, choose F from p and move to FR.



Transition matrix:

$$K(R, R') = \sum_{FR=R'} p(F)$$

Theorem (Bridigare-Hanlon-Rockmore, 1997) *For each* $x \in P_{\mathcal{A}}$ *there is an eigenvalue*

$$\lambda_x = \sum_{\substack{F \in \mathcal{F} \\ F \subseteq x}} p(F)$$

of K *with multiplicity* $|\mu(x)|$*.*

(5) Theorem (Crapo-Rota 1971, Orlik-Terao 1992, Athanasiadis 1996) *Let* $\mathcal{A}$ *be defined over* $\mathbb{Z}$. *For* $q > 0$, *let*

$$\mathcal{A}_{\mathbf{q}} = \mathcal{A} \text{ reduced modulo } q.$$

Then for q *prime,* $q >> 0$,

$$\chi_{\mathcal{A}}(q) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right).$$

* Second method for computing $\chi_{\mathcal{A}}(q)$.

Example. G = graph with vertices $1, 2, \ldots, n$, edge set E.

$$\mathcal{A}_{\mathbf{G}} : \ x_i = x_j, \quad ij \in E$$

(graphical arrangement)

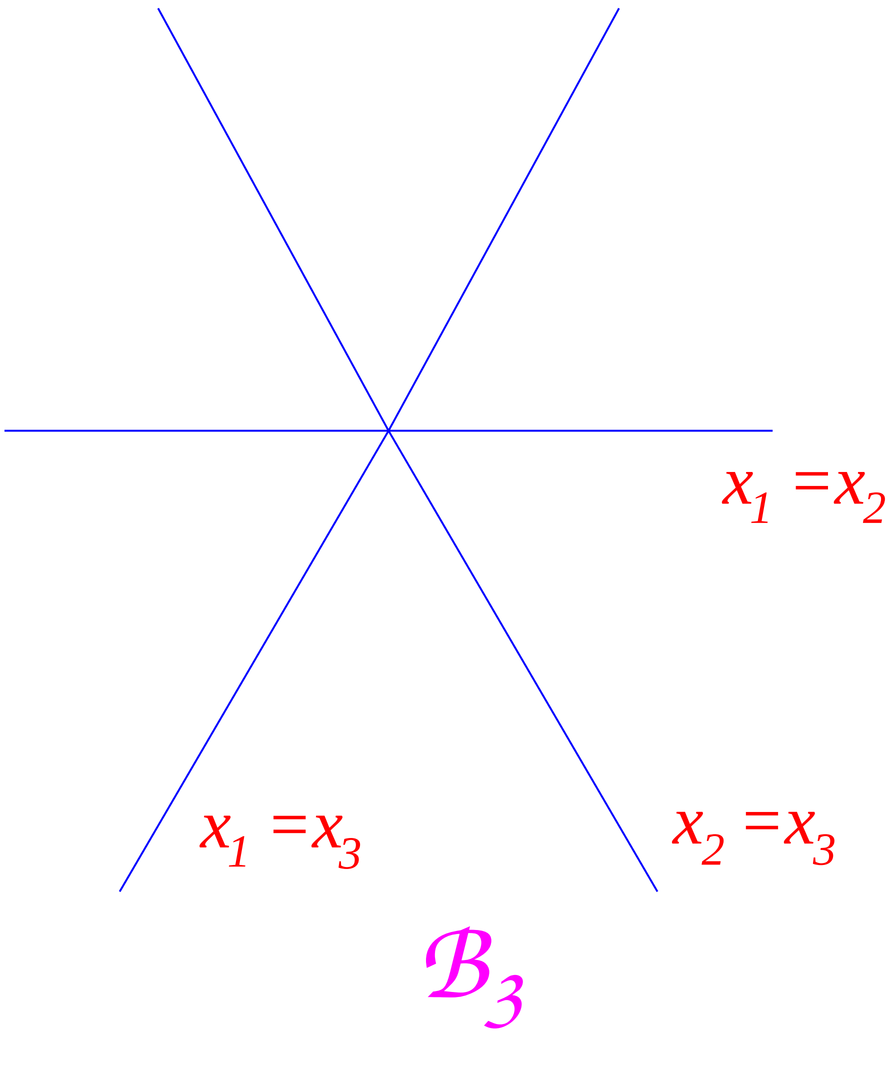
$$\begin{aligned}\chi_G(q) &= \mathbb{F}_q^n - \#\{(x_1, \ldots, x_n) : \\ &\qquad x_i = x_j \text{ for some } ij \in E\} \\ &= \#\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n : \\ &\qquad x_i \neq x_j \text{ if } ij \in E\} \\ &= \# \text{ proper } q\text{-colorings of } G\end{aligned}$$

(chromatic polynomial)

braid arrangement $\mathcal{B}_n$:

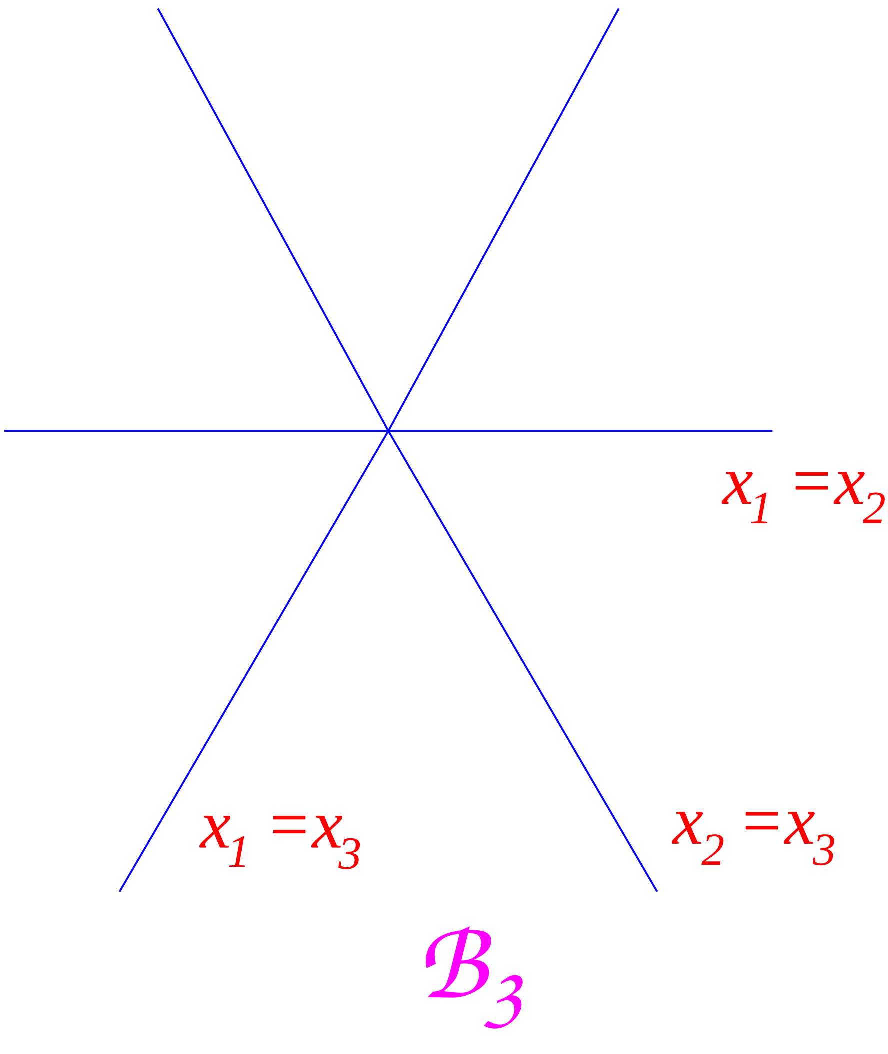
$$x_i - x_j = 0, \quad 1 \leq i < j \leq n$$

$$\mathcal{B}_n = \mathcal{A}_{K_n}$$



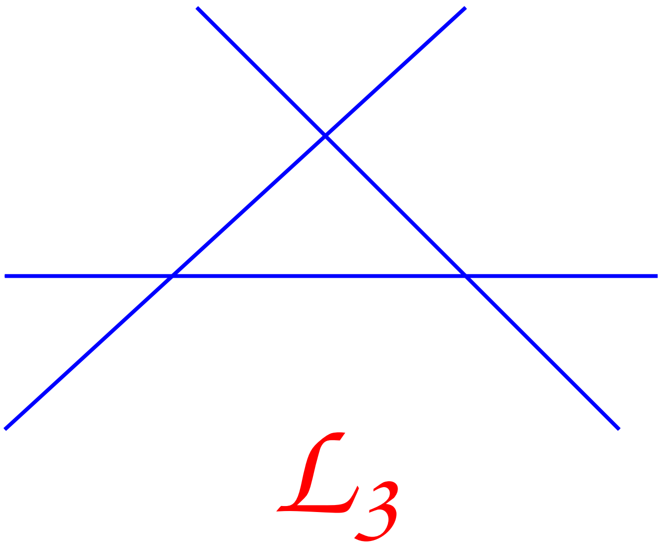
$$\chi_{\mathcal{B}_n} = q(q-1)\cdots(q-n+1)$$

$$r(\mathcal{B}_n) = n!$$



THE LINIAL ARRANGEMENT

$$\mathcal{L}_{\mathbf{n}}: \; x_i = x_j + 1, \; 1 \leq i < j \leq n$$



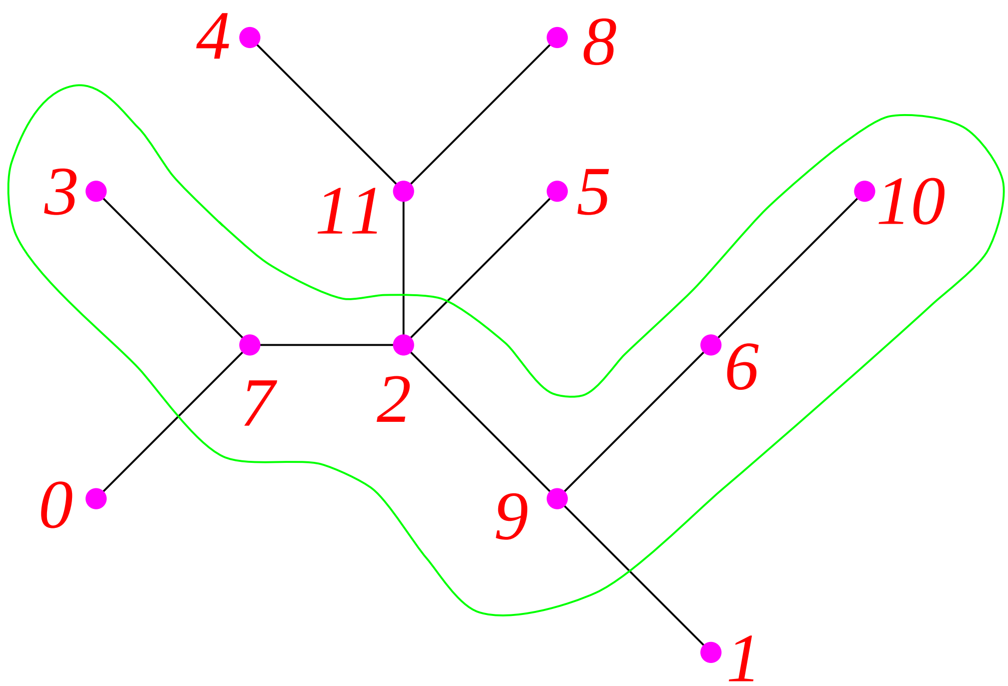
$$\chi_{\mathcal{L}_3}(q) = q^3 - 3q^2 + 3q$$

An **alternating** (or **intransitive**) **tree** on $0, 1, \ldots, n$ is a tree with vertices $0, 1, \ldots, n$ such that every path has the form

$$a_1 < a_2 > a_3 < a_4 > \cdots \text{ or}$$

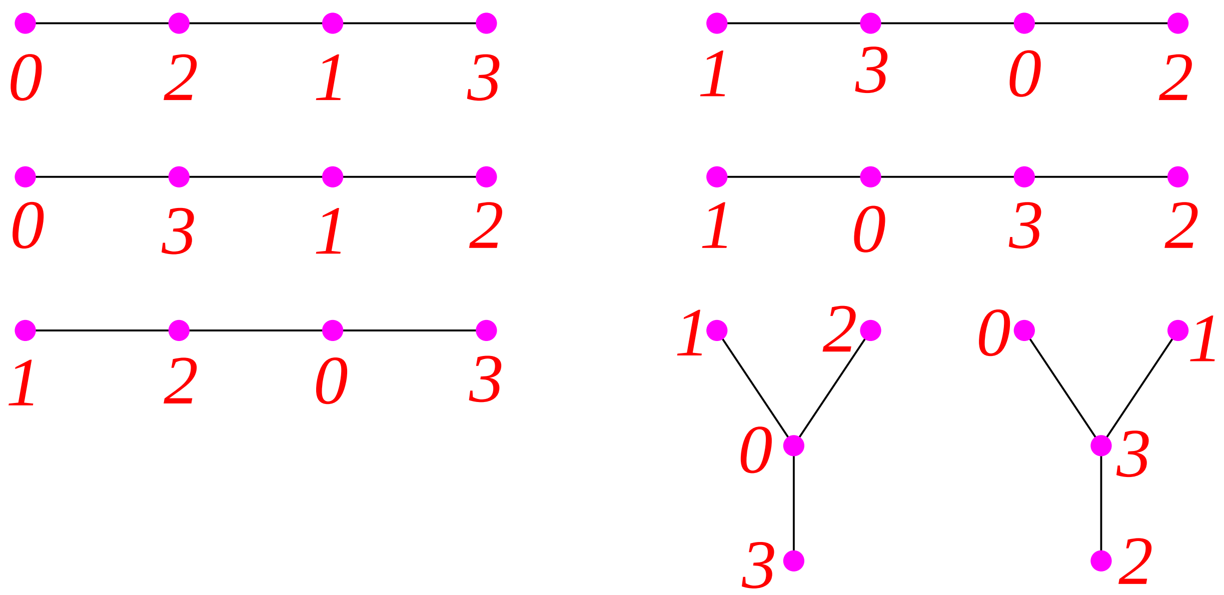
$$a_1 > a_2 < a_3 > a_4 < \cdots .$$

Equivalently, every vertex is either less than all its neighbors or greater than all its neighbors.



$$\mathbf{f_n} = \#\ \text{alt. trees on } 0, 1, \ldots, n$$

n	1	2	3	4	5	6
f(n)	1	2	7	36	246	2104



$$y = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

$$\Rightarrow y = e^{\frac{x}{2}(1+y)}$$

Lagrange inversion $\Rightarrow$

$$f_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$$

$$= \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-1}{k-1} k^{n-2}.$$

No bijective proof known.

Note. Let e_n be the number of alternating permutations of $1, 2, \ldots, n$, i.e.,

$$a_1 > a_2 < a_3 > a_4 < \cdots a_n.$$

$e_4 = 5:$ 2143 3142 3241 4132 4231

Theorem (D. André, 1879)

$$\sum_{n \geq 0} e_n \frac{x^n}{n!} = \sec x + \tan x.$$

$\longrightarrow$ **Combinatorial Trigonometry**

Exercise. Prove **combinatorially** that

$$1 + \tan^2 x = \sec^2 x.$$

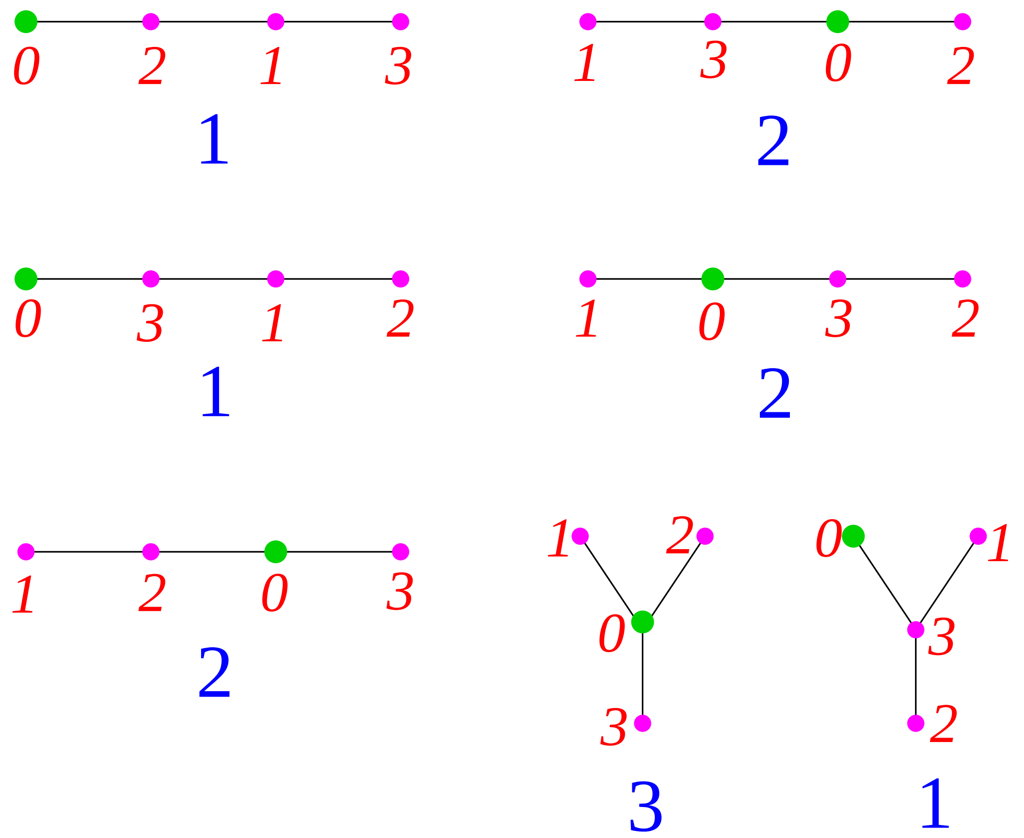
Theorem (Athanasiadis 1996, Postnikov 1996)

$$r(\mathcal{L}_n) = f_n.$$

(No bijective proof known.)

What about $\chi_{\mathcal{L}_n}(q)$?

Let $\mathbf{f_n(q)} = \sum_T q^{\deg(0)}$, summed over all alternating trees on $0, 1, \ldots, n$.



$$f_3(q) = q^3 + 3q^2 + 3q$$

$$\sum_{n\geq 0} f_n(q)\frac{x^n}{n!} = \left(\sum_{n\geq 0} f_n \frac{x^n}{n!}\right)^q$$

$$f_n(q) = \frac{q}{2^n}\sum_{k=0}^{n}\binom{n}{k}(q+k)^{n-1}$$

Theorem (Athanasiadis, Postnikov 1996)

$$\chi_{\mathcal{L}_n}(q) = (-1)^n f_n(-q)$$

Theorem (Riemann hypothesis for $\mathcal{L}_n$) *Every zero of* $\chi_{\mathcal{L}_n}(q)$*, except* $q = 0$*, has real part* $n/2$*.*

Corollary (functional equation for $\mathcal{L}_n$)

$$\frac{\chi_{\mathcal{L}_n}(q)}{q} = \frac{(-1)^n \chi_{\mathcal{L}_n}(-q+n)}{-q+n}$$

Proof of theorem. Let $Ef(q) = f(q-1)$. Then

$$\chi_{\mathcal{L}_n}(q) = \frac{q}{2^n}(E-1)^n q^{n-1}.$$

Lemma. *Let $f(q) \in \mathbb{C}[q]$, such that every zero of f has real part m. Let $|s| = 1$ and*

$$h(q) = (E-s)f(q) = f(q-1) - sf(q).$$

Then every zero of $h(q)$ has real part $m + \frac{1}{2}$.

Theorem (Postnikov). *Let*

$$\psi_n(q) = (-2i)^{n-1} \frac{\chi_{\mathcal{L}_n}((iq+n)/2)}{(iq+n)/2},$$

so all zeros are real. Then

$$\lim_{m\to\infty} \frac{\psi_{2m}(q)}{\psi'_{2m}(1)} \to \frac{\sin(hq)}{q},$$

where

$$\cosh(h) = \frac{h}{\sqrt{h^2-1}}, \quad h > 1,$$

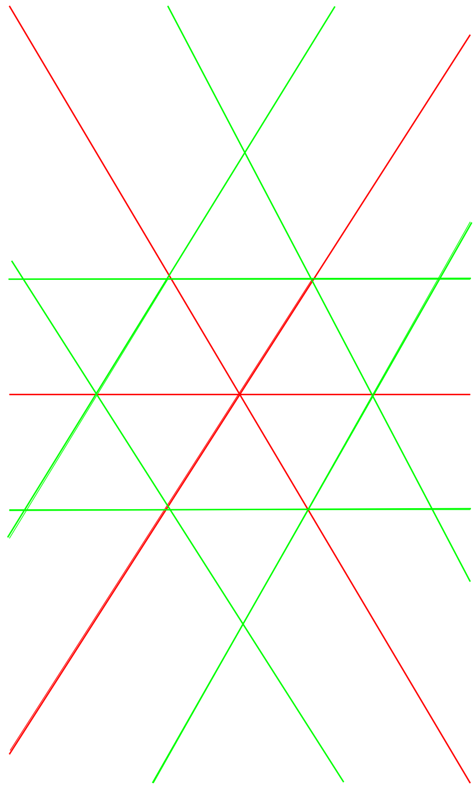
$$h \approx 1.199678640 \cdots .$$

Corollary. *Every zero of* $\sin z$ *is real.*

SOME OTHER ARRANGEMENTS

Catalan arrangment $\mathcal{C}_n$:

$$x_i - x_j = -1, 0, 1, \text{ for } 1 \le i < j \le n$$

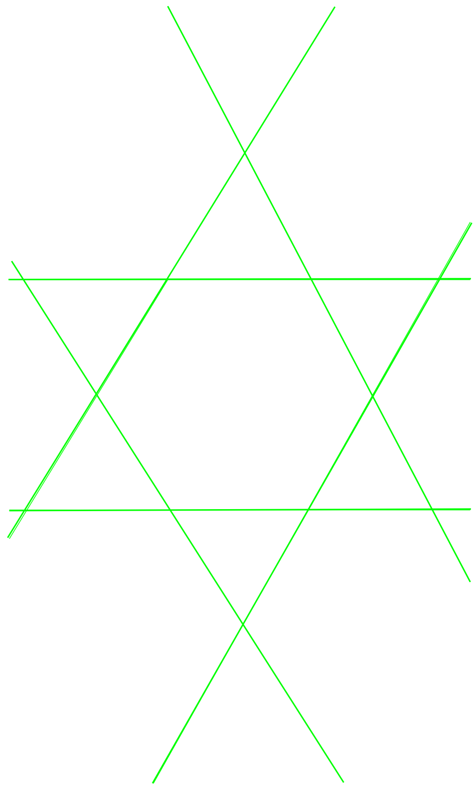


$$r(\mathcal{C}_n) = n!\, C_n = n! \frac{1}{n+1}\binom{2n}{n}$$

$$\chi_{\mathcal{C}_n}(q) =$$
$$q(q-n-1)(q-n-2)\cdots(q-2n+1)$$

semiorder arrangement $\mathcal{I}_n$:

$$x_i - x_j = \pm 1, \text{ for } 1 \le i < j \le n$$



A **semiorder** is a partial ordering $\prec$ of a set S obtained by choosing $f : S \to \mathbb{R}$ and defining

$$i \prec j \ \text{ if } \ f(i) + 1 < f(j).$$

$r(\mathcal{I}_n)$ = number of semiorders on an n-set

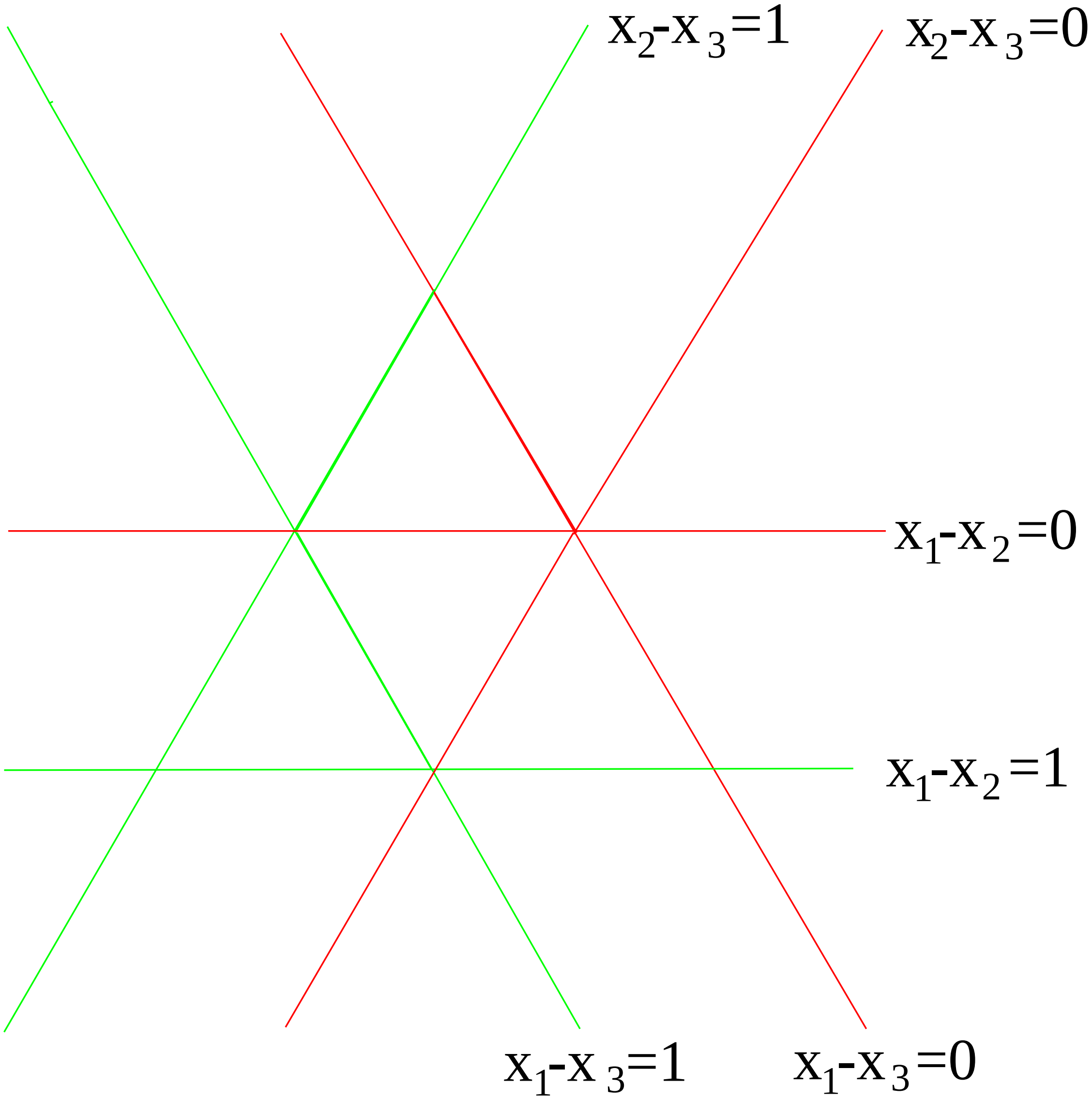
$$\sum_{n \geq 0} r(\mathcal{I}_n) \frac{x^n}{n!} = C(1 - e^{-x}),$$

where

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Shi arrangement $\mathcal{S}_n$:

$$x_i - x_j = 0, 1, \text{ for } 1 \leq i < j \leq n$$



$$
\begin{aligned}
r(\mathcal{S}_n) &= (n+1)^{n-1} \\
\chi_{\mathcal{S}_n}(q) &= q(q-n)^{n-1}
\end{aligned}
$$

threshold arrangement $\mathcal{T}_n$:

$$x_i + x_j = 0, \text{for } 1 \le i < j \le n$$

$$\sum_{n \ge 0} r(\mathcal{T}_n)\frac{x^n}{n!} = \frac{e^x(1-x)}{2-e^x}$$

$$\sum_{m \ge 0} (-1)^n \chi_{\mathcal{T}_n}(-q)\frac{x^n}{n!} = (1-x)\left(\frac{e^x}{2-e^x}\right)^{\frac{q+1}{2}}$$

REFERENCES (incomplete)

1. C. A. Athanasiadis, Algebraic combinatorics of graph spectra, subspace arrangements, and Tutte polynomials, Ph.D. thesis, M.I.T., 1996.

2. C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Advances in Math.* **122** (1996), 193–233.

3. ?? Bridigare, P. Hanlon, and D. Rockmore,

4. T. Brylawski and A. Varchenko, The determinant formula for a matroid bilinear form, *Advances in Math.* **129** (1997), 1–24.

5. H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, preliminary edition, MIT Press, Cambridge, MA, 1970.

6. G. Denham and P. Hanlon, On the Smith normal form of the Varchenko bilinear form of a hyperplane arrangement, *Pacific J. Math.*, Special Issue (1997), 123–146.

7. P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplanes, *Invent. math.* **56** (1980), 167–189.

8. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, Berlin/Heidelberg/New York, 1992.

9. A. Postnikov, Intransitive trees, *J. Combinatorial Theory (A)* **79** (1997), 360–366.

10. A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint dated 14 April 1997.

11. R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci. U.S.A.* **93** (1996), 2620–2625.

12. H. Terao,

13. A. Varchenko, Bilinear form of real configuration of hyperplanes, *Advances in Math.* **97** (1993), 110–144.

14. T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.*, vol. 1, no. 154, 1975.