$S_n$: \{permutations of 1, \ldots, n\}

$(i, j)$: the transposition $i \leftrightarrow j$ acting on positions, e.g. $(1, 2)132 = 312$.

Let $w = a_1a_2 \cdots a_n \in S_n$. Define

$$\ell(w) = \# \{(i, j) : i < j, a_i > a_j\}.$$ 

Define a partial order $\leq$ on $S_n$, called (strong) Bruhat order, to be the transitive and reflexive ($w \leq w$) closure of

$$u < (i, j)u, \quad \text{if } \ell((i, j)u) = 1 + \ell(u).$$

$$62718453 < 64718253$$

all $<$ or $>2,4$

$v \prec w$: $w$ covers $v$, i.e., $w > v$ and

$$\ell(w) = 1 + \ell(v)$$
$S_4$
$S_n$ is a graded poset, where $\text{rank}(w) = \ell(w)$. Thus the **rank-generating function** of $S_n$ is given by

$$F(S_n, q) := \sum_{w \in S_n} q^{\text{rank}(w)}$$

$$= (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}).$$
Motivation. Let $K$ be a field and $\mathcal{F}(K^n)$ the set of all (complete) flags

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = K^n$$

of subspaces of $K^n$ (so $\dim V_i = i$).

For every such flag $F$, there are unique vectors $v_1, \ldots, v_n \in K^n$ such that:

- $\{v_1, \ldots, v_i\}$ is a basis for $V_i$
• The $n \times n$ matrix with rows $v_1, \ldots, v_n$ has the form

\[
\begin{array}{cccccc}
* & * & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & * & 0 & * & * & 1 \\
0 & * & 0 & 1 & 0 & 0 \\
0 & * & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{array}
\]

The positions of the 1’s define a permutation $w_F = 316452$. The number of *’s is $\ell(w_F)$. 
For $w \in S_n$ define the **Bruhat cell**

$$\Omega_w = \{F \in \mathcal{F}(K^n) : w = w_F\}.$$ 

Thus

$$\mathcal{F}(K^n) = \bigsqcup_{w \in S_n} \Omega_w,$$

the **Bruhat decomposition** of $\mathcal{F}(K^n)$. 
\( \Omega_w \): **closed** Bruhat cell

**Theorem** (Ehresmann, 1934)

\[ \Omega_v \subseteq \Omega_w \iff v \leq w \]

(Bruhat order).
\(P\): finite poset (partially ordered set), say with a top element \(\hat{1}\)

**principal order ideal** \(\Lambda_x\) for \(x \in P\):
\[
\Lambda_x = \{y \in P : y \leq x\}
\]

**lattice**: a poset for which every two elements \(x, y\) have a greatest lower bound (meet) \(x \land y\) and least upper bound (join) \(x \lor y\)

\(L_P\): all subsets of \(P\) which are intersections of \(\Lambda_x\)'s, the **MacNeille completion** of \(P\). It is the “smallest” lattice containing \(P\) as a subposet and preserving any meets and joins existing in \(P\).

E.g., \(L_P \cong P \iff P\) is a lattice.
The diagram shows a lattice with elements labeled from 'a' to 'f'. The lattice structure is defined by the relationships between these elements. The expression \( abcd \cap abce = abc \) indicates the relationship between the sets labeled 'abcd' and 'abce', resulting in 'abc'.
**Theorem (Ehresmann)** Let \( w = a_1 \cdots a_n \in S_n \). Define

\[
T_w = \begin{pmatrix}
1 & 2 & 3 & \cdots & n \\
b_1 & b_2 & \cdots & b_{n-1} \\
c_1 & \cdots & c_{n-2} & \cdots \\
a_1
\end{pmatrix},
\]

where

\[
(b_1, \ldots, b_{n-1}) = \{a_1, \ldots, a_{n-1}\}_{\text{sorted}}
\]

\[
(c_1, \ldots, c_{n-2}) = \{a_1, \ldots, a_{n-2}\}_{\text{sorted}}
\]

etc.

Then

\[
v \leq w \iff T_v \leq T_w
\]

(componentwise).
Example. $v = 35124, w = 45123$

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 5 \\
1 & 3 & 5 \\
3 & 5 \\
3
\end{bmatrix}
$$

$T_v = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 \\ 3 \end{bmatrix}$

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 4 & 5 \\
1 & 4 & 5 \\
4 & 5 \\
4
\end{bmatrix}
$$

$\leq T_w = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 5 \\ 4 \end{bmatrix}$

$T_w$ is a (special) monotone triangle.
(general) monotone triangle:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 3 & 4 & 5 \\
2 & 3 & 5 \\
2 & 4 \\
4 \\
\end{array}
\]

monotone triangle $\leftrightarrow$ ASM

$T_w \leftrightarrow$ permutation matrix
\( \mathcal{M}_n \): set of all monotone triangles of length \( n \), ordered componentwise

Thus \( S_n \) is a subposet of \( \mathcal{M}_n \).

**Theorem** (Lascoux-Schützenberger, 1996)

\( \mathcal{M}_n \) is the MacNeille completion of \( S_n \).
Topology of the Bruhat order

\( P \): finite poset

Define the Möbius function

\[ \mu : P \times P \to \mathbb{Z} \]

recursively by:

\[ \mu(x, y) = \begin{cases} 
0, & \text{unless } x \leq y \\
1, & x = y \\
- \sum_{x \leq z < y} \mu(x, z), & x < y 
\end{cases} \]

Thus

\[ x < y \Rightarrow \sum_{x \leq z \leq y} \mu(x, z) = 0. \]
**Theorem** (Verma, 1971) For $v \leq w$ in $S_n$ we have

$$
\mu(v, w) = (-1)^{\ell(w) - \ell(v)}.
$$
For $x \leq y$ in any finite poset $P$, let $c_i$ be the number of chains
\[ x < x_0 < x_1 < \cdots < x_i < y, \]
with $c_{-1} = 1$.

**Theorem** (P. Hall, 1936)
\[ \mu(x, y) = -c_{-1} + c_0 - c_1 + c_2 - \cdots \]

**order complex** $\Delta(x, y)$: the abstract simplicial complex on the set
\[ (x, y) = \{ z \in P : x < z < y \} \]
whose faces (simplices) are the chains in $(x, y)$.

**P. Hall’s theorem restated:**
\[ \mu(x, y) = \tilde{\chi}(\Delta(x, y)), \]
the reduced Euler characteristic of $\Delta(x, y)$. 

$\Delta(x, y)$
Verma’s theorem on $\mu$ for $S_n$ suggests:

**Conjecture.** For all $v \leq w$ in $S_n$, $
\Delta(x, y)$ is a triangulation of a sphere.

**Note.** Given an abstract simplicial complex $\Delta$, it is **undecidable** whether $\Delta$ triangulates a sphere.
Basic tool: lexicographic shellability (Björner, Wachs). Let $P$ be a finite graded poset with $\hat{0}$ and $\hat{1}$, with 
\[ \mu(x, y) = (-1)^{\text{rank}(y) - \text{rank}(x)} \quad \forall x \leq y \]
(i.e., $P$ is Eulerian). Let 
\[ \lambda : \mathcal{E}_P \to \{1, 2, \ldots\} \]
be a labeling of the edges of the (Hasse) diagram of $P$ satisfying 

- For all $x < y$, \exists a unique saturated increasing chain
  \[ C : x_0 < x_1 < \cdots < x_r = y, \text{ i.e.,} \]
  \[ \lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{r-1}, x_r). \]
- The label sequence of $C$ lexicographically precedes that of all other saturated chains from $x$ to $y$.

Call $\lambda$ an **EL-labeling**.
Theorem (Björner) Let $P$ be a finite Eulerian poset with an EL-labeling. Then for all $x < y$ in $P$, $\Delta(x, y)$ triangulates a sphere.
First EL-labeling of $S_n$ due to Edelman (1981): Let $\tau_1, \tau_2, \ldots, \tau_{n\choose 2}$ be the transpositions in $S_n$ in lexicographic order. E.g., $n = 4$:

$\tau_1 = (1, 2), \tau_2 = (1, 3), \tau_3 = (1, 4)$

$\tau_4 = (2, 3), \tau_5 = (2, 4), \tau_6 = (3, 4)$

Let $w \triangleright v$ in $S_n$. Define

$$\lambda(v, w) = j \text{ if } \tau_j v = w.$$

**Theorem** (Edelman). $\lambda$ is an EL-labeling of $S_n$, so $\forall v < w$ in $S_n$, $\Delta(v, w)$ triangulates a sphere.
Counting maximal chains in $S_n$

weak (Bruhat) order $\mathcal{W}S_n$ on $S_n$: transitive, reflexive closure of

$u < (i, i + 1)u$, if $\ell((i, i+1)u) = 1 + \ell(u)$.

Compare ordinary (strong) order:

$u < (i, j)u$, if $\ell((i, j)u) = 1 + \ell(u)$. 

![Diagram showing the weak and strong order for $S_3$]
**Theorem** (RS, 1984). The number $M_n$ of maximal chains of $W\mathcal{S}_n$ (i.e., the number of ways to move from $1, 2, \ldots, n$ to $n, n-1, \ldots, 1$ by $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ adjacent transpositions) is given by

$$M_n = \# \text{ SYT of shape } (n - 1, n - 2, \ldots, 1)$$

$$= \frac{\left(\begin{array}{c} n \\ 2 \end{array}\right)!}{1^{n-1} 3^{n-2} 5^{n-3} \cdots (2n - 3)^1}$$

Is there something analogous for $S_n$ (strong order)?
If \((i, j)\nu = w \succeq \nu\) in \(S_n\), define the weight
\[
\text{wt}(\nu, w) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}.
\]
If \(C: \text{id} = \nu_0 \prec \nu_1 \prec \cdots \prec \nu_{\binom{n}{2}} = w_0\)
is a maximal chain in \(S_n\), define
\[
\text{wt}(C) = \text{wt}(\nu_0, \nu_1) \text{wt}(\nu_1, \nu_2) \cdots \\
\text{wt}(\nu_{\binom{n}{2} - 1}, \nu_{\binom{n}{2}}).
\]

**Theorem** ( Stembridge, 2001) *We have*
\[
\sum_C \text{wt}(C) = \frac{\binom{n}{2}!}{1^{n-1}2^{n-2}\cdots(n-1)!} \\
\cdot \prod_{1 \leq i < j \leq n} (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}).
\]
(extends to any Weyl group)
\\[ \sum_{C} \text{wt}(C) = \alpha^2 \beta + \alpha \beta^2 + 2 \alpha \beta (\alpha + \beta) = 3 \alpha \beta (\alpha + \beta). \]
Alternative proof (sketch). Based on Schubert polynomials. Let \( s_i \) be the adjacent transposition (or simple reflection) \((i, i + 1)\). Let \( w \in S_n \) and \( \ell = \ell(w) \).

**reduced decomposition** of \( w \): a sequence \((a_1, \ldots, a_\ell)\), \(1 \leq a_j \leq n - 1\), such that

\[
w = s_{a_1} s_{a_2} \cdots s_{a_\ell}.
\]

**divided difference operator** \( \partial_i \):

\[
\partial_i f = \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}
\]

Define \( w_0 = n, n - 1, \ldots, 1 \in S_n \) (the longest permutation in \( S_n \), of length \( \binom{n}{2} \)).
Let \((a_1, \ldots, a_\ell)\) be a reduced decomposition of \(v \in S_n\). Define
\[
\partial_v = \partial_{a_1} \cdots \partial_{a_\ell}
\]
(independent of choice of reduced decomposition).

Define the **Schubert polynomial** \(\mathcal{S}_w\) by
\[
\mathcal{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}).
\]
\(\mathcal{S}_w\) is homogeneous of degree \(\ell(w)\) in \(x_1, \ldots, x_{n-1}\).

\[
\mathcal{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}
\]
Example. \( w = 4132 = s_2s_3s_2s_1 \)

\[
\begin{align*}
  w_0 &= s_2s_3s_2s_1s_3s_4 \\
  w^{-1}w_0 &= s_3s_4 \\
  \mathcal{S}_{4132} &= \partial_3 \partial_4 x_1^3 x_2^2 x_3 \\
  &= \partial_3 x_1^3 x_2^2 \\
  &= x_1^2 x_2 + x_1^3 x_3.
\end{align*}
\]
Note. $\mathcal{S}_{s_i} = x_1 + x_2 + \cdots + x_i$.

Monk’s rule. $\mathcal{S}_{s_r} \mathcal{S}_w = \sum \mathcal{S}_{(j,k)w}$, summed over all transpositions $(j, k)$ such that

$$1 \leq j \leq r < k$$

$\ell((j, k)w) = \ell(w) + 1$ (i.e., $(j, k)w \succ w$).

For $w \in S_n$ let

$$N(w) = \sum_C \text{wt}(C),$$

where $C$ ranges over all saturated chains

$$\text{id} = v_0 \prec v_1 \prec \cdots \prec v_\ell = w.$$
Iteration of Monk’s rule gives:

\[
\left( \alpha_1 \mathcal{G}_{s_1} + \alpha_2 \mathcal{G}_{s_2} + \cdots + \alpha_{n-1} \mathcal{G}_{s_{n-1}} \right)^\ell
= \sum_{w \in S_n, \ell(w) = \ell} N(w) \mathcal{G}_w.
\]

Let

\[
\beta_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1},
\]

so

\[
\alpha_1 \mathcal{G}_{s_1} + \alpha_2 \mathcal{G}_{s_2} + \cdots + \alpha_{n-1} \mathcal{G}_{s_{n-1}}
= \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{n-1} x_{n-1}
\]
Lemma. Fix \( v \in S_n \). Then the nonzero polynomials \( \partial_v \mathfrak{S}_w \) are linearly independent.

Let \( \Psi_i f = f(\beta_i \leftarrow \beta_{i+1}) \). Then e.g.
\[
0 = \partial_i \Psi_i (\beta_1 x_1 + \cdots + \beta_{n-1} x_{n-1})^{(n)}_{(2)}
= \sum_{\ell(w) = \binom{n}{2}} \Psi_i N(w) \partial_i \mathfrak{S}_w.
\]
But \( \partial_i \mathfrak{S}_w_0 \neq 0 \), so
\[
\Psi_i N(w_0) = 0.
\]
Thus \( (\beta_i - \beta_{i+1}) | N(w_0) \). Similarly,
\[
(\beta_i - \beta_j) | N(w_0) \quad \forall \ 1 \leq i < j \leq n
\]
Since \( \text{deg} \ N(w_0) = \binom{n}{2} \), we get
\[
N(w_0) = C_n \prod_{1 \leq i < j \leq n} (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}).
\]
To show:

\[ C_n = 1^{n-1}2^{n-2}\cdots(n - 1)^1. \]

Follows from: for every permutation \( b_1 \cdots b_{\binom{n}{2}} \) of \( \{1^{n-1}, 2^{n-2}, \ldots, (n-1)^1\} \), there is a unique maximal chain

\[ \text{id} = v_0 \prec v_1 \prec \cdots \prec v_{\binom{n}{2}} = w_0 \]

in \( S_n \) such that \( \forall i, \)

\[ v_i = (b_i, c_i)v_{i-1}, \text{ for some } c_i > b_i. \]
Example.  \( n = 4, \ (b_1, \ldots, b_6) = 211312 \)

\[
\begin{align*}
1 & \ 2 \ 3 \ 4 \\
1 & \ 3 \ 2 \ 4 \\
2 & \ 3 \ 1 \ 4 \\
3 & \ 2 \ 1 \ 4 \\
3 & \ 2 \ 4 \ 1 \\
4 & \ 2 \ 3 \ 1 \\
4 & \ 3 \ 2 \ 1
\end{align*}
\]