(with A. Postnikov)

$\boldsymbol{S_n}$: symmetric group on $\{1, 2, \ldots, n\}$

$\boldsymbol{\ell(w)} = \#\{(i, j) : i < j, w(i) > w(j)\}$

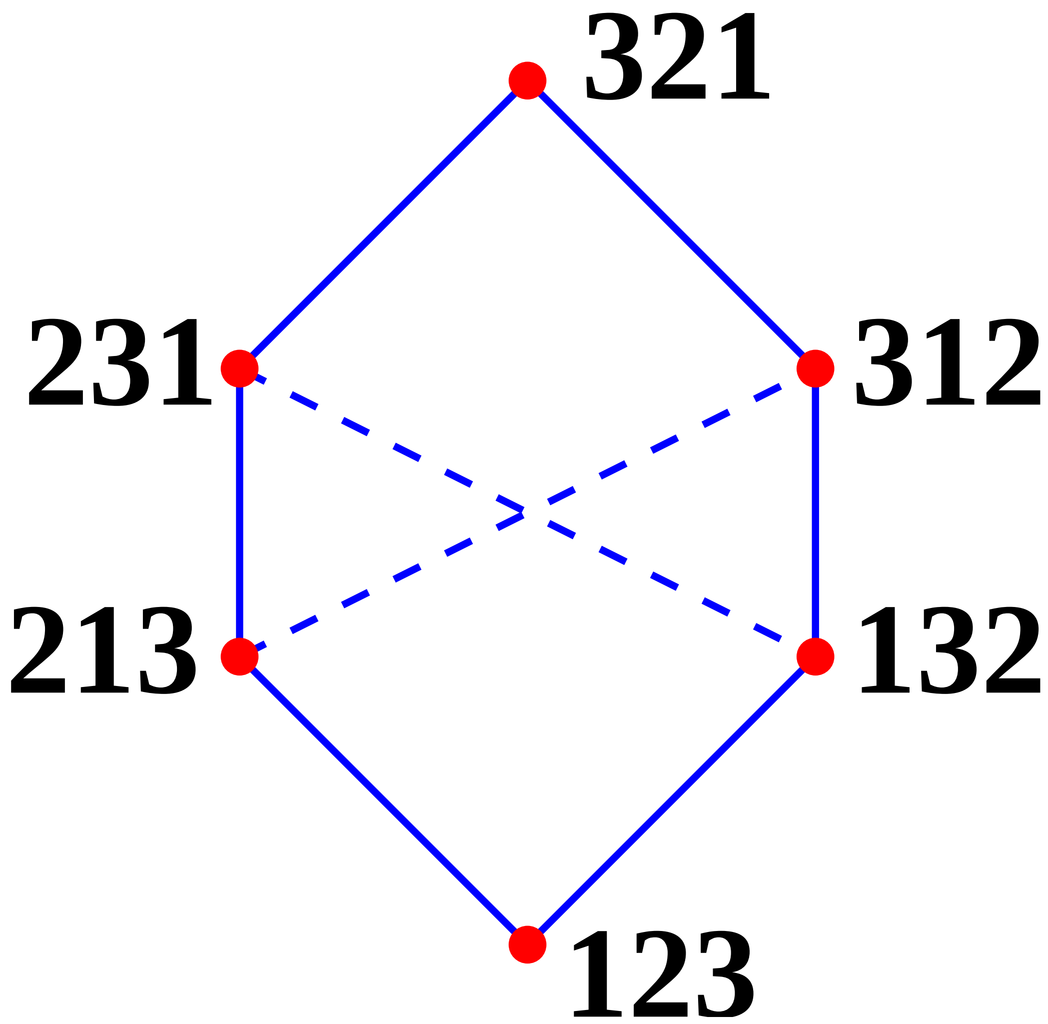
$\boldsymbol{s_i} = (i, i + 1)$ (**adjacent transposition**)

$\boldsymbol{W(S_n)}$: **weak (Bruhat) order** on S_n, with cover relations:

$$\boldsymbol{u \prec^* v} \text{ if } v = us_i,\ \ell(v) = 1 + \ell(u)$$

$\boldsymbol{S_n}$: **(strong) Bruhat order** on S_n, with cover relations:

$$\boldsymbol{u \prec v} \text{ if } v = u(i, j),\ \ell(v) = 1 + \ell(u)$$


321
231
312
213
132
123

Let $u \prec^* us_i$ in $W(S_n)$. Define

$$\boldsymbol{m}^*(u, us_i) = i.$$

If

$$C : u_0 \prec^* u_1 \prec^* u_2 \prec^* \cdots \prec^* u_k$$

in $W(S_n)$, then define

$$\boldsymbol{m}^*_{\boldsymbol{C}} = m^*(u_0, u_1)m^*(u_1, u_2) \cdots m^*(u_{k-1}, u_k).$$

Similarly let $u \prec u(i, j)$ in S_n, and define

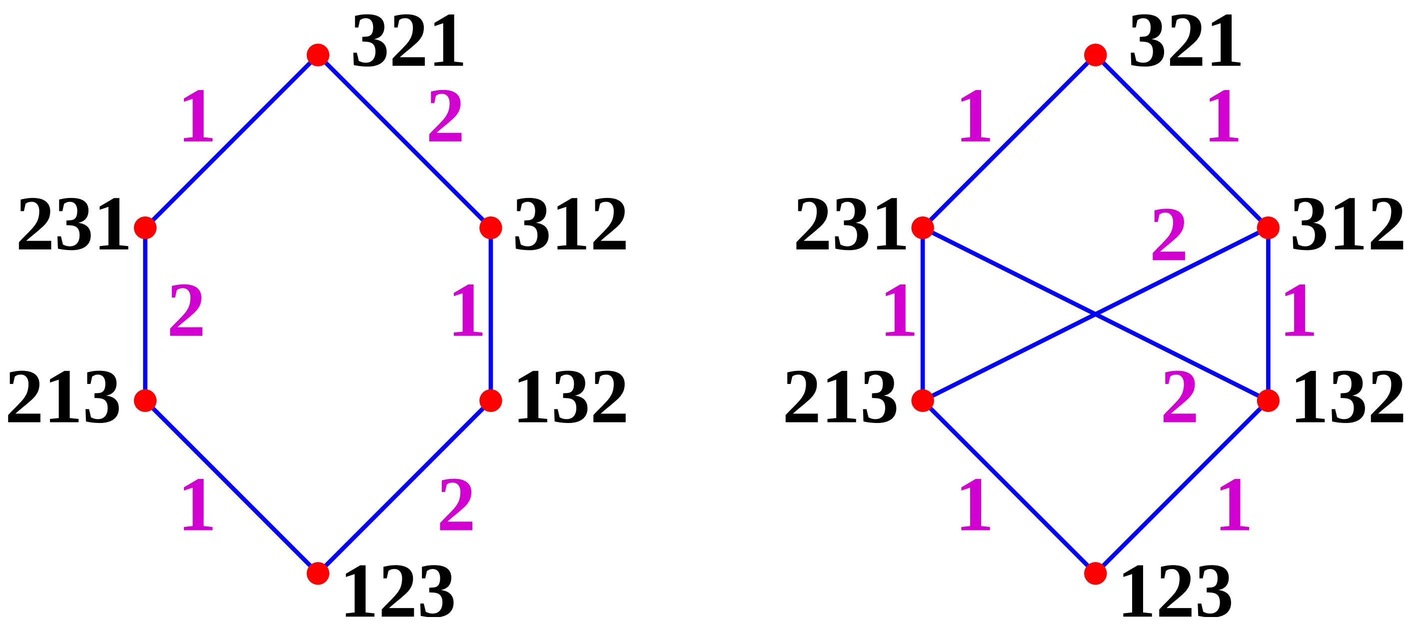
$$\boldsymbol{m}(u, u(i, j)) = j - i.$$

If

$$C : u_0 \prec u_1 \prec u_2 \prec \cdots \prec u_k$$

in S_n, then define

$$\boldsymbol{m}_{\boldsymbol{C}} = m(u_0, u_1)m(u_1, u_2) \cdots m(u_{k-1}, u_k).$$



Let $\mathcal{M}(P)$ denote the set of maximal chains of the poset P. Thus

$$\sum_{C \in \mathcal{M}(W(S_3))} m_C^* = 1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2$$
$$= 6$$

$$\sum_{C \in \mathcal{M}(S_3)} m_C = 1 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1$$
$$+ 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 1$$
$$= 6.$$

Theorem. (a) (Macdonald; Fomin & RS)

$$\sum_{C \in \mathcal{M}(W(S_n))} m_C^* = \binom{n}{2}!$$

(b) (Stembridge (explicitly))

$$\sum_{C \in \mathcal{M}(S_n)} m_C = \binom{n}{2}!$$

Open. A bijective proof of (a) or (b), or a bijective proof that (a) = (b).

Generalize the definition $m(u, u(i,j))$ to

$$\boldsymbol{m}(u, u(i,j)) = \lambda_i - \lambda_j.$$

(Original definition corresponds to $\lambda_i = -i$.)
As before, if

$$C : u_0 \prec u_1 \prec u_2 \prec \cdots \prec u_k$$

in S_n, then define

$$\boldsymbol{m_C}(\lambda) = m(u_0, u_1)m(u_1, u_2) \cdots m(u_{k-1}, u_k).$$

If $u \leq v$ in S_n, define

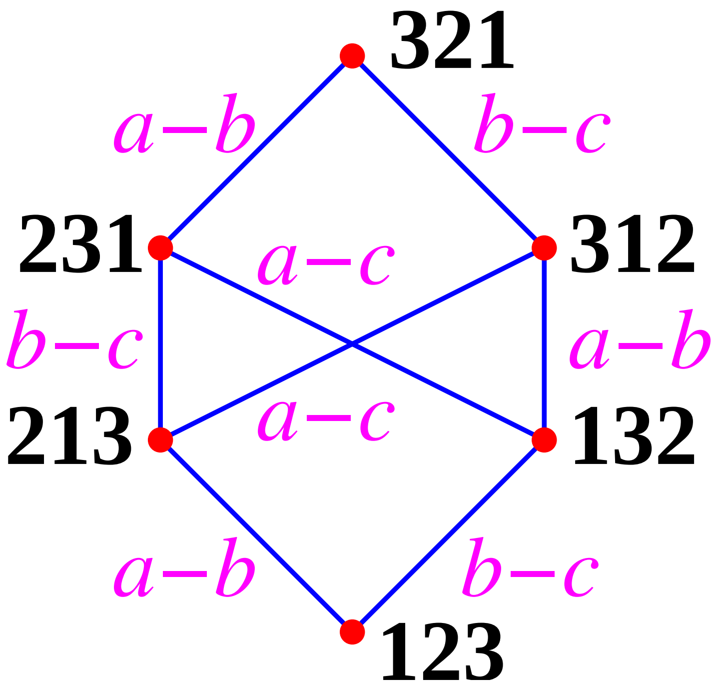
$$\boldsymbol{\mathcal{D}_{u,v}}(\lambda) = \frac{1}{(\ell(v) - \ell(u))!} \sum_C m_C(\lambda),$$

where C ranges over all saturated chains from u to v in S_n.

Set

$$\boldsymbol{\mathcal{D}_w}(\lambda) = \mathcal{D}_{\mathrm{id},w}(\lambda).$$

Write $a = \lambda_1$, $b = \lambda_2$, $c = \lambda_3$.



$$u = 123 = \mathrm{id}, \quad v = 321 = w_0$$

$$\begin{aligned}
\mathcal{D}_{321}(\lambda) &= \frac{1}{3!}((a-b)(b-c)(a-b) \\
&\quad +(a-b)(a-c)(b-c) \\
&\quad +(b-c)(a-c)(a-b) \\
&\quad +(b-c)(a-b)(b-c)) \\
&= \frac{1}{2}(a-b)(a-c)(b-c)
\end{aligned}$$

Schubert polynomials. For $w \in S_n$ let

$$\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots, x_{n-1}]$$

denote the **Schubert polynomial** indexed by w. Regard $S_n \subset S_{n+1}$ via $w(n+1) = n+1$ for $w \in S_n$. Let

$$S_\infty = \bigcup S_n,$$

the permutations of $\{1, 2, \ldots\}$ moving finitely many letters. Then $\{\mathfrak{S}_w : w \in S_\infty\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, x_2, \ldots]$.

Monk's rule for $(x_1 + x_2 + \cdots + x_i)\mathfrak{S}_w$ gives:

$$(\lambda_1 x_1 + \lambda_2 x_2 + \cdots)^k \mathfrak{S}_u = k! \sum_{\ell(v)=k+\ell(u)} \mathcal{D}_{u,v}(\lambda)\mathfrak{S}_v.$$

Geometric interpretation of $\mathcal{D}_w(\lambda)$.

Let $\mathbf{\Phi} \subset \mathfrak{h}^*$ denote the root lattice for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of $G = \mathrm{SL}(n, \mathbb{C})$. Let

$$\mathbf{\Lambda} = \{\lambda \in \mathfrak{h}* \, : \, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\},$$

the weight lattice of $\mathfrak{g}$. Let $\lambda \in \Lambda^+$ be a dominant weight. Let

$$\boldsymbol{V_\lambda} = \lambda\text{-weight space}$$

$$\boldsymbol{v_\lambda} \in V_\lambda : \text{ highest weight vector}$$

$$\mathbb{P}(\boldsymbol{V_\lambda}) = \text{projectivization of } V_\lambda$$

$$\boldsymbol{e} : G/B \to \mathbb{P}(V_\lambda)$$

$$e(gB) = g(v_\lambda)$$

$$\boldsymbol{X_w} \subset G/B (\text{Schubert variety})$$

Thus e is a projective embedding $G/B \hookrightarrow \mathbb{P}(V_\lambda)$. Define the **$\boldsymbol{\lambda}$-degree** of X_w by:

$$\mathbf{deg}_{\boldsymbol{\lambda}}(X_w) = \#(e(X_w) \cap L),$$

where $\boldsymbol{L}$ is a generic linear subspace of $\mathbb{P}(V_\lambda)$ of complex codimension $\ell(w)$.

Theorem. $\deg_\lambda(X_w) = \ell(w)!\, \mathcal{D}_w(\lambda)$

An expression for $\mathcal{D}_{u,v}(\lambda)$.

Theorem. *Let* $w \in S_n$ *and*

$$\boldsymbol{V_n} = \frac{1}{1!\,2!\,\cdots(n-1)!} \prod_{1\le i<j\le n} (\lambda_i - \lambda_j).$$

Then

$$\mathcal{D}_{u,v} =$$

$$\mathfrak{S}_u\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \dots\right) \mathfrak{S}_{w_0 v}\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \dots\right) \cdot V_n.$$

In particular,

$$\mathcal{D}_w = \mathcal{D}_{\mathrm{id},w} = \mathfrak{S}_{w_0 w}\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \dots\right) \cdot V_n.$$

$$\mathcal{D}_{w_0} = V_n \quad (\dots, \text{Stembridge})$$

Corollary. *$\{\mathcal{D}_w : w \in S_\infty\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\lambda_1, \lambda_2, \ldots]$. Let*

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w \boldsymbol{c^w_{u,v}} \mathfrak{S}_w.$$

Then

$$\mathcal{D}_{u,w} = \sum_v c^w_{u,v} \mathcal{D}_v.$$

Note. (1) $\{\mathcal{D}_w : w \in S_n\}$ is a $\mathbb{Z}$-basis for Har_n, the harmonic polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$.

(2) $\langle \mathfrak{S}_u, \mathcal{D}_v \rangle = \delta_{uv}$ under the "Fock pairing"

$$\langle f, g \rangle = f\left(\frac{\partial}{\partial x_1}, \ldots\right) g(x_1, x_2, \ldots)\Big|_{x_i = 0}.$$

Corollary. *Let $w \in S_n$ be 312-avoiding, i.e.,*

$$a < b < c \Rightarrow \textbf{not } w(b) < w(c) < w(a).$$

Let **code**$(w_0 w) = (c_1, c_2, \ldots)$, *where*

$$c_i = \#\{j \,:\, i < j,\ w(i) > w(j)\}.$$

Then

$$\mathcal{D}_w = \det \left(\frac{\lambda_i^{n-c_i-j}}{(n-c_i-j)!} \right)_{i,j=1}^{n},$$

where $\alpha^k/k! = 0$ if $k < 0$.

Idea of proof.

w 312-avoiding $\Rightarrow$ $w_0 w$ 132-avoiding **(dominant)**

$\Rightarrow \mathfrak{S}_{w_0 w} = x_1^{c_1} x_2^{c_2} \cdots,$ etc.

Other special values are interesting, e.g.,

$$\mathcal{D}_{41532} = \frac{1}{12}\big(f(5,4,2) - f(5,4,1) - f(5,3,2) \\ +f(5,3,1) + f(4,3,2) - f(4,3,1)\big),$$

where

$$\boldsymbol{f(i,j,k)} = (x_i - x_j)(x_i - x_k)(x_j - x_k).$$

Note that code$(4,1,5,3,2) = (3,0,2,1,0)$.

Further connections:

- Demazure characters (key polynomials)
- Gelfand-Tsetlin polytopes
- inverse "Schubert Kostka" matrix