INTRODUCTORY LECTURES IN COMBINATORIAL ANALYSIS

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From a course taught by

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Combinatorial Analysis - Table of Contents

Foreword

This is an extremely rough and pre-preliminary set of notes based on a course in Combinatorial Theory I taught at M.I.T. in the fall of 1962. They were almost entirely written by students, and they will be completely rewritten when I teach the course again, presumably in 1964. Meanwhile, I shall appreciate any suggestions for changes.

There are many errors in these notes, but these are in most cases so obvious that they do not obscure the text.

The material on Möbius functions can now be found in more complete form in my paper "On the foundations of combinatorial analysis."

G.-C. Rota, August 1963.

Introduction

Combinatorial analysis has applications to genetics, logic, automata, physics, among a variety of other areas. Problems in these areas often lend themselves to simple statment, but prove difficult to solve. Some examples of problems, solved and unsolved, will give the student an idea of the wide range of these applications.

Solved problems:

1) a) Given a string of n boxes and n distinct pearls, how many different ways can the pearls be arranged in the boxes?

b) If the ends of the string of boxes in (a) are joined, how many different necklaces can be formed?

2) Given a set of circles of different radii such that any circle can contain all the smaller circles without any of them containing each other, in how many different ways can the circles be placed in the plane without touching?

3) It has been determined that 5 queens, when placed properly, will dominate every position on a chessboard. In how many different ways can the queens be so placed?

4) The partition problem is to find how many different ways an integer may be represented as the sum of other integers, where the order in the sum is immaterial.

5) Can a chessboard from which two opposite corner squares have been removed, be covered exactly by 31 dominos?

6) Given a graph, what is the minimum number of breaks in the edges needed to isolate the vertices?

7) Mazes and games ((e.g. Nim).

Unsolved problems:

1) Find an algorithm for the minimal path that a travelling salesman must use to visit each of a set of n cities.

2) Latin squares. Given a checker board of n x n squares, in how many ways can one place the integers from 1 to n in the squares so that each row and each column contains all the integers from 1 to n only once?

3) Four color problem.

4) Given n squares, the cell growth problem seeks the number of ways they can be arranged adjacent to one another.

5) How many different graphs of prescribed shapes (e.g. with two loops) are there with n vertices?

6) A geometrical figure is cut by n straight lines. Which figures can be constructed with the resulting pieces?

7) What is the minimum number of squares, no two the same, that can cover a given square?

Analytic Preliminaries

1) Powers and factorials

The following notations will be used:

$$x! = (x)(x-1)(x-2)\ldots(2)(1)$$
$$(x)_n = (x)(x-1)(x-2)\ldots(x-n+1)$$
$$(x)^{(n)} = (x)(x+1)(x+2)\ldots(x+n-1)$$
$$\binom{x}{n} = \frac{x!}{n!(x-n)!} = (x)_n/n!$$

Using this terminology, the bionomial formula and two other relations are stated without proof.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}$$
$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}$$

2) Operators

We shall be dealing with several "operators", devices which "operate" on functions or sequences. The most common of these appear below, defined by the manner in which they operate on functions.

differential operator D: $Df(x) = f'(x)$

ex. $Dx^n = nx^{n-1}$

difference operator Δ : $\Delta f(x) = f(x+1) - f(x)$

ex. $\Delta (x)_n = (x+1)_n - (x)_n = n(x)_{n-1}$