SMITH NORMAL FORM IN COMBINATORICS

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Abstract. This paper surveys some combinatorial aspects of Smith normal form, and more generally, diagonal form. The discussion includes general algebraic properties and interpretations of Smith normal form, critical groups of graphs, and Smith normal form of random integer matrices. We then give some examples of Smith normal form and diagonal form arising from (1) symmetric functions, (2) a result of Carlitz, Roselle, and Scoville, and (3) the Varchenko matrix of a hyperplane arrangement.

1. Introduction

Let A be an $m \times n$ matrix over a field K. By means of elementary row and column operations, namely:

(1) add a multiple of a row (respectively, column) to another row (respectively, column), or
(2) multiply a row or column by a unit (nonzero element) of K,

we can transform A into a matrix that vanishes off the main diagonal (so A is a diagonal matrix if $m = n$) and whose main diagonal consists of k 1's followed by $m-k$ 0's. Moreover, k is uniquely determined by A since $k = \mathrm{rank}(A)$.

What happens if we replace K by another ring R (which we always assume to be commutative with identity 1)? We allow the same row and column operations as before. Condition (2) above is ambiguous since a unit of R is not the same as a nonzero element. We want the former interpretation, i.e., we can multiply a row or column by a unit only. Equivalently, we transform A into a matrix of the form PAQ, where P is an $m \times m$ matrix and Q is an $n \times n$ matrix, both invertible over R. In other words, $\det P$ and $\det Q$ are units in R. Now the situation becomes much more complicated.

We say that PAQ is a *diagonal form* of A if it vanishes off the main diagonal. (Do not confuse the diagonal form of a square matrix with the matrix D obtained by diagonalizing A. Here $D = XAX^{-1}$ for some invertible matrix X, and the diagonal entries are the eigenvalues of A.) If A has a diagonal form B whose main diagonal is $(\alpha_1, \dots, \alpha_r, 0, \dots, 0)$, where α_i divides α_{i+1} in R for $1 \le i \le r-1$, then we call B a *Smith normal form* (SNF) of A. If A is a nonsingular square matrix, then taking determinants of both sides of the equation $PAQ = B$ shows that $\det A = u\alpha_1 \cdots \alpha_n$ for some unit $u \in R$. Hence an SNF of A yields a factorization of $\det A$. Since there is a huge literature on determinants of combinatorially interesting matrices (e.g., [26][27]), finding an SNF of such matrices could be a fruitful endeavor.

Date: April 18, 2016.

2010 *Mathematics Subject Classification.* 05E99, 15A21.

Key words and phrases. Smith normal form, diagonal form, critical group, random matrix, Jacobi-Trudi matrix, Varchenko matrix.

Partially supported by NSF grant DMS-1068625.

In the next section we review the basic properties of SNF, including questions of existence and uniqueness, and some algebraic aspects. In Section 3 we discuss connections between SNF and the abelian sandpile or chip-firing process on a graph. The distribution of the SNF of a random integer matrix is the topic of Section 4. The remaining sections deal with some examples and open problems related to the SNF of combinatorially defined matrices.

We will state most of our results with no proof or just the hint of a proof. It would take a much longer paper to summarize all the work that has been done on computing SNF for special matrices. We therefore will sample some of this work based on our own interests and research. We will include a number of open problems which we hope will stir up some further interest in this topic.

2. Basic properties

In this section we summarize without proof the basic properties of SNF. We will use the following notation. If A is an $m \times n$ matrix over a ring R, and B is the matrix with $(\alpha_1, \dots, \alpha_m)$ on the main diagonal and 0's elsewhere then we write $A \stackrel{\text{snf}}{\to} (\alpha_1, \dots, \alpha_m)$ to indicate that B is an SNF of A.

2.1. **Existence and uniqueness.** For connections with combinatorics we are primarily interested in the ring $\mathbb{Z}$ or in polynomial rings over a field or over $\mathbb{Z}$. However, it is still interesting to ask over what rings R does a matrix always have an SNF, and how unique is the SNF when it exists. For this purpose, define an *elementary divisor ring* R to be a ring over which every matrix has an SNF. Also define a *Bézout ring* to be a commutative ring for which every finitely generated ideal is principal. Note that a noetherian Bézout ring is (by definition) a principal ideal ring, i.e., a ring (not necessarily an integral domain) for which every ideal is principal. An important example of a principal ideal ring that is not a domain is $\mathbb{Z}/k\mathbb{Z}$ (when k is not prime). Two examples of non-noetherian Bézout domains are the ring of entire functions and the ring of all algebraic integers.

Theorem 2.1. *Let R be a commutative ring with identity.*

(1) *If every rectangular matrix over R has an SNF, then R is a Bézout ring. In fact, if I is an ideal with a minimum size generating set $a_1, \dots, a_k$, then the 1×2 matrix $[a_1, a_2]$ does not have an SNF. See* [25, p. 465].

(2) *Every diagonal matrix over R has an SNF if and only if R is a Bézout ring* [29, (3.1)].

(3) *A Bézout domain R is an elementary divisor domain if and only if it satisfies:*

For all $a, b, c \in R$ with $(a, b, c) = R$, there exists $p, q \in R$ such that $(pa, pb + qc) = R$.

See [25, §5.2][20, §6.3].

(4) *Every principal ideal ring is an elementary divisor ring. This is the classical existence result (at least for principal ideal domains), going back to Smith* [40] *for the integers.*

(5) *Suppose that R is an* associate ring*, that is, if two elements a and b generate the same principal ideal there is a unit u such that $ua = b$. (Every integral domain is an associate ring.) If a matrix A has an SNF PAQ over R, then PAQ is unique (up to multiplication of each diagonal entry by a unit). This result is immediate from* [31, §IV.5, Thm. 5.1].

It is open whether every Bézout domain is an elementary divisor domain. For a recent paper on this question, see Lorenzini [32].

Let us give a simple example where SNF does not exist.

Example 2.2. Let $R = \mathbb{Z}[x]$, the polynomial ring in one variable over $\mathbb{Z}$, and let $A = \begin{bmatrix} 2 & 0 \\ 0 & x \end{bmatrix}$. Clearly A has a diagonal form (over R) since it is already a diagonal matrix. Suppose that A has an SNF $B = PAQ$. The only possible SNF (up to units ± 1) is $\mathrm{diag}(1, 2x)$, since $\det B = \pm 2x$. Setting $x = 2$ in $B = PAQ$ yields the SNF $\mathrm{diag}(1, 4)$ over $\mathbb{Z}$, but setting $x = 2$ in A yields the SNF $\mathrm{diag}(2, 2)$.

Let us remark that there is a large literature on the computation of SNF over a PID (or sometimes more general rings) which we will not discuss. We are unaware of any literature on deciding whether a given matrix over a more general ring, such as $\mathbb{Q}[x_1, \ldots, x_n]$ or $\mathbb{Z}[x_1, \ldots, x_n]$, has an SNF.

2.2. **Algebraic interpretation.** Smith normal form, or more generally diagonal form, has a simple algebraic interpretation. Suppose that the $m \times n$ matrix A over the ring R has a diagonal form with diagonal entries $\alpha_1, \ldots, \alpha_m$. The rows $v_1, \ldots, v_m$ of A may be regarded as elements of the free R-module R^n.

Theorem 2.3. *We have*

$$R^n/(v_1, \ldots, v_m) \cong (R/\alpha_1 R) \oplus \cdots \oplus (R/\alpha_m R).$$

Proof. It is easily seen that the allowed row and column operations do not change the isomorphism class of the quotient of R^n by the rows of the matrix. Since the conclusion is tautological for diagonal matrices, the proof follows. $\square$

The quotient module $R^n/(v_1, \ldots, v_m)$ is called the *cokernel* (or sometimes the *Kasteleyn cokernel*) of the matrix A, denoted $\mathrm{coker}(A)$

Recall the basic result from algebra that a finitely-generated module M over a PID R is a (finite) direct sum of cyclic modules $R/\alpha_i R$. Moreover, we can choose the α_i's so that $\alpha_i | \alpha_{i+1}$ (where $\alpha | 0$ for all $\alpha \in R$). In this case the α_i's are unique up to multiplication by units. In the case $R = \mathbb{Z}$, this result is the "fundamental theorem for finitely-generated abelian groups." For a general PID R, this result is equivalent to the PID case of Theorem 2.1(4).

2.3. **A formula for SNF.** Recall that a *minor* of a matrix A is the determinant of some square submatrix.

Theorem 2.4. *Let R be a unique factorization domain (e.g., a PID), so that any two elements have a greatest common divisor (gcd). Suppose that the $m \times n$ matrix M over R satisfies $M \stackrel{\mathrm{snf}}{\to} (\alpha_1, \ldots, \alpha_m)$. Then for $1 \leq k \leq m$ we have that $\alpha_1 \alpha_2 \cdots \alpha_k$ is equal to the gcd of all $k \times k$ minors of A, with the convention that if all $k \times k$ minors are 0, then their gcd is 0.*

Sketch of proof. The assertion is easy to check if M is already in Smith normal form, so we have to show that the allowed row and column operations preserve the gcd of the $k \times k$ minors. For $k = 1$ this is easy. For $k > 1$ we can apply the $k = 1$ case to the matrix $\wedge^k M$, the kth exterior power of M. For details, see [34, Prop. 8.1].

3. The critical group of a graph

Let G be a finite graph on the vertex set V. We allow multiple edges but not loops (edges from a vertex to itself). (We could allow loops, but they turn out to be irrelevant.) Write $\mu(u,v)$ for the number of edges between vertices u and v, and $\deg v$ for the degree (number of incident edges) of vertex v. The *Laplacian matrix* $\boldsymbol{L} = \boldsymbol{L}(G)$ is the matrix with rows and columns indexed by the elements of V (in some order), with

$$\boldsymbol{L_{uv}} = \begin{cases} -\mu(u,v), & \text{if } u \neq v \\ \deg(v), & \text{if } u = v. \end{cases}$$

The matrix $\boldsymbol{L}(G)$ is always singular since its rows sum to 0. Let $\boldsymbol{L_0} = \boldsymbol{L_0}(G)$ be $\boldsymbol{L}$ with the last row and column removed. (We can just as well remove any row and column.) The well-known *Matrix-Tree Theorem* (e.g., [42, Thm. 5.6.8]) asserts that $\det \boldsymbol{L_0} = \kappa(G)$, the number of spanning trees of G. Equivalently, if $\#V = n$ and $\boldsymbol{L}$ has eigenvalues $\theta_1, \dots, \theta_n$, where $\theta_n = 0$, then $\kappa(G) = \theta_1 \cdots \theta_{n-1}/n$. We are regarding $\boldsymbol{L}$ and $\boldsymbol{L_0}$ as matrices over $\mathbb{Z}$, so they both have an SNF. It is easy to see that $\boldsymbol{L_0} \overset{\text{snf}}{\to} (\alpha_1, \dots, \alpha_{n-1})$ if and only if $\boldsymbol{L} \overset{\text{snf}}{\to} (\alpha_1, \dots, \alpha_{n-1}, 0)$.

Let G be connected. The group $\operatorname{coker}(\boldsymbol{L_0})$ has an interesting interpretation in terms of chip-firing, which we explain below. For this reason there has been a lot of work on finding the SNF of Laplacian matrices $\boldsymbol{L}(G)$.

A *configuration* is a finite collection σ of indistinguishable chips distributed among the vertices of the graph G. Equivalently, we may regard σ as a function $\sigma\colon V \to \mathbb{N} = \{0, 1, 2, \dots\}$. Suppose that for some vertex v we have $\sigma(v) \geq \deg(v)$. The *toppling* or *firing* τ of vertex v is the configuration obtained by sending a chip from v along each incident edge to the vertex at the other end of the edge. Thus

$$\tau(u) = \begin{cases} \sigma(v) - \deg(v), & u = v \\ \sigma(u) + \mu(u,v), & u \neq v. \end{cases}$$

Now choose a vertex w of G to be a *sink*, and ignore chips falling into the sink. (We never topple the sink.) This dynamical system is called the *abelian sandpile* model. A *stable* configuration is one for which no vertex can topple, i.e., $\sigma(v) < \deg(v)$ for all vertices $v \neq w$. It is easy to see that after finitely many topples a stable configuration will be reached, which is independent of the order of topples. (This independence of order accounts for the word "abelian" in "abelian sandpile.")

Let M denote the set of all stable configurations. Define a binary operation $\oplus$ on M by vertex-wise addition followed by stabilization. An *ideal* of M is a subset $J \subseteq M$ satisfying $\sigma \oplus J \subseteq J$ for all $\sigma \in M$. The *sandpile group* or *critical group* $K(G)$ is the minimal ideal of M, i.e., the intersection of all ideals. (Following the survey [30] of Levine and Propp, the reader is encouraged to prove that the minimal ideal of any finite commutative monoid is a group.) The group $K(G)$ is independent of the choice of sink up to isomorphism.

An equivalent but somewhat less abstract definition of $K(G)$ is the following. A configuration u is called *recurrent* if, for all configurations v, there is a configuration y such that $v \oplus y = u$. A configuration that is both stable and recurrent is called *critical.* Given critical configurations C_1 and C_2, define $C_1 + C_2$ to be the unique critical configuration reachable from the vertex-wise sum of C_1 and C_2. This operation turns the set of critical configurations into an abelian group isomorphic to the critical group $K(G)$.

The basic result on $K(G)$ [4][15] is the following.

Theorem 3.1. *We have $K(G) \cong \operatorname{coker}(\boldsymbol{L_0}(G))$. Equivalently, if $\boldsymbol{L_0}(G) \stackrel{\text{snf}}{\to} (\alpha_1, \dots, \alpha_{n-1})$, then*

$$K(G) \cong \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_{n-1}\mathbb{Z}.$$

Note that by the Matrix-Tree Theorem we have $\#K(G) = \det \boldsymbol{L_0(G)} = \kappa(G)$. Thus the critical group $K(G)$ gives a canonical factorization of $\kappa(G)$. When $\kappa(G)$ has a "nice" factorization, it is especially interesting to determine $K(G)$. The simplest case is $G = K_n$, the complete graph on n vertices. We have $\kappa(K_n) = n^{n-2}$, a classic result going back to Sylvester and Borchardt. There is a simple trick for computing $K(K_n)$ based on Theorem 2.4. Let $\boldsymbol{L_0}(K_n) \stackrel{\text{snf}}{\to} (\alpha_1, \dots, \alpha_{n-1})$. Since $\boldsymbol{L_0}(K_n)$ has an entry equal to -1, it follows from Theorem 2.4 that $\alpha_1 = 1$. Now the 2×2 submatrices (up to row and column permutations) of $\boldsymbol{L_0}(K_n)$ are given by

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \quad \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $\alpha_2 = n$ by Theorem 2.4. Since $\prod \alpha_i = \pm n^{n-2}$ and $\alpha_i | \alpha_{i+1}$, we get $K(G) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$.

Note. A similar trick works for the matrix $M = \left[\binom{2(i+j)}{i+j}\right]_{i,j=0}^{n-1}$, once it is known that $\det M = 2^{n-1}$ (e.g., [18, Thm. 9]). Every entry of M is even except for M_{00}, so $2|\alpha_2$, yielding $M \stackrel{\text{snf}}{\to} (1, 2, 2, \dots, 2)$. The matrix $\left[\binom{3(i+j)}{i+j}\right]_{i,j=0}^{n-1}$ is much more complicated. For instance, when $n = 8$ the diagonal elements of the SNF are

$$1,\ 3,\ 3,\ 3,\ 3,\ 6,\ 2 \cdot 3 \cdot 29 \cdot 31,\ 2 \cdot 3^2 \cdot 11 \cdot 29 \cdot 31 \cdot 37 \cdot 41.$$

It seems that if d_n denotes the number of diagonal entries of the SNF that are equal to 3, then d_n is close to $\frac{2}{3}n$. The least n for which $|d_n - \lfloor \frac{2}{3}n \rfloor| > 1$ is $n = 224$. For the determinant of M, see [22, (10)]. If $M = \left[\binom{a(i+j)}{i+j}\right]_{i,j=0}^{n-1}$ for $a \geq 4$, then $\det M$ does not seem "nice" (it doesn't factor into small factors).

The critical groups of many classes of graphs have been computed. As a couple of nice examples, we mention threshold graphs (work of B. Jacobson [24]) and Paley graphs (D. B. Chandler, P. Sin, and Q. Xiang [9]). Critical groups have been generalized in various ways. In particular, A. M. Duval, C. J. Klivans, and J. L. Martin [16] consider the critical group of a simplicial complex.

4. Random matrices

There is a huge literature on the distribution of eigenvalues and eigenvectors of a random matrix. Much less has been done on the distribution of the SNF of a random matrix. We will restrict our attention to the situation where $k \geq 0$ and M is an $m \times n$ integer matrix with independent entries uniformly distributed in the interval $[-k, k]$, in the limit as $k \to \infty$. We write $P_k^{(m,n)}(\mathcal{E})$ for the probability of some event under this model (for fixed k). To illustrate that the distribution of SNF in such a model might be interesting, suppose that $M \stackrel{\text{snf}}{\to} (\alpha_1, \dots, \alpha_m)$. Let $j \geq 1$. The probability $P_k^{(m,n)}(\alpha_1 = j)$ that $\alpha_1 = j$ is equal to the probability that mn integers between $-k$ and k have gcd equal to j. It is then a well-known, elementary result that when $mn > 1$,

$$\lim_{k\to\infty} P_k^{(m,n)}(\alpha_1 = j) = \frac{1}{j^{mn}\zeta(mn)}, \tag{4.1}$$

where ζ denotes the Riemann zeta function. This suggests looking, for instance, at such numbers as

$$\lim_{k\to\infty} P_k^{(m,n)}(\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 12).$$

In fact, it turns out that if $m < n$ and we specify the values $\alpha_1, \dots, \alpha_m$ (subject of course to $\alpha_1|\alpha_2|\cdots|\alpha_{m-1}$), then the probability as $k \to \infty$ exists and is strictly between 0 and 1. For $m = n$ the same is true for specifying $\alpha_1, \dots, \alpha_{n-1}$. However, for any $j \geq 1$, we have $\lim_{k\to\infty} P_k^{(n,n)}(\alpha_n = j) = 0$.

The first significant result of this nature is due to Ekedahl [19, §3], namely, let

$$\sigma(n) = \lim_{k\to\infty} P_k^{(n,n)}(\alpha_{n-1} = 1).$$

Note that this number is just the probability (as $k \to \infty$) that the cokernel of the $n \times n$ matrix M is cyclic (has one generator). Then

$$\sigma(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}, \tag{4.2}$$

where p ranges over all primes. It is not hard to deduce that

$$\begin{aligned} \lim_{n\to\infty} \sigma(n) &= \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)} \\ &= 0.84693590173\cdots. \end{aligned} \tag{4.3}$$

At first sight it seems surprising that this latter probability is not 1. It is the probability (as $k \to \infty$, $n \to \infty$) that the n^2 $(n-1)\times(n-1)$ minors of M are relatively prime. Thus the $(n-1)\times(n-1)$ minors do not behave at all like n^2 independent random integers.

Further work on the SNF of random integer matrices appears in [49] and the references cited there. These papers are concerned with powers of a fixed prime p dividing the α_i's. Equivalently, they are working (at least implicitly) over the p-adic integers $\mathbb{Z}_p$. The first paper to treat systematically SNF over $\mathbb{Z}$ is by Wang and Stanley [47]. One would expect that the behavior of the prime power divisors to be independent for different primes as $k \to \infty$. This is indeed the case, though it takes some work to prove. In particular, for any positive integers $h \leq m \leq n$ and $a_1|a_2|\cdots|a_h$ Wang and Stanley determine

$$\lim_{k\to\infty} P_k^{(m,n)}(\alpha_1 = a_1, \dots, \alpha_h = a_h).$$

A typical result is the following:

$$\begin{aligned}\lim_{k\to\infty} P_k^{(n,n)}(\alpha_1=2,\alpha_2=6) &= 2^{-n^2}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} 2^{-i}+\sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right)\\ &\quad \cdot\frac{3}{2}\cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2\\ &\quad \cdot\prod_{p>3}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} p^{-i}+\sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).\end{aligned}$$

A further result in [47] is an extension of Ekedahl's formula (4.2). The authors obtain explicit formulas for

$$\rho_j(n) := \lim_{k\to\infty} P_k^{(n,n)}(\alpha_{n-j}=1),$$

i.e., the probability (as $k\to\infty$) that the cokernel of M has at most j generators. Thus (4.2) is the case $j=1$. Write $\rho_j=\lim_{n\to\infty}\rho_j(n)$. Numerically we have

$$\begin{aligned}\rho_1 &= 0.846935901735\\ \rho_2 &= 0.994626883543\\ \rho_3 &= 0.999953295075\\ \rho_4 &= 0.999999903035\\ \rho_5 &= 0.999999999951.\end{aligned}$$

The convergence $\rho_n\to 1$ looks very rapid. In fact [47, (4.38)],

$$\rho_n = 1-c\,2^{-(n+1)^2}(1-2^{-n}+O(4^{-n})),$$

where

$$c=\frac{1}{(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{8})\cdots}=3.46275\cdots.$$

A major current topic related to eigenvalues and eigenvectors of random matrices is *universality* (e.g., [45]). A certain distribution of eigenvalues (say) occurs for a large class of probability distributions on the matrices, not just for a special distribution like the GUE model on the space of $n\times n$ Hermitian matrices. Universality of SNF over the rings $\mathbb{Z}_p$ of p-adic integers and over $\mathbb{Z}/n\mathbb{Z}$ was considered by Kenneth Maples [33]. On the other hand, Clancy, Leake, and Payne [12] make some conjectures for the SNF distribution of the Laplacian matrix of an Erdős-Rényi random graph that differs from the distribution obtained in [47]. (It is clear, for instance, that $\alpha_1=1$ for Laplacian matrices, in contradistinction to equation (4.1), but conceivably equation (4.3) could carry over.) Considerable progress on these conjectures was made by Wood [48].

5. Symmetric functions

5.1. **An up-down linear transformation.** Many interesting matrices arise in the theory of symmetric functions. We will adhere to notation and terminology on this subject from [42, Chap. 7]. For our first example, let $\Lambda^n_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space of homogeneous symmetric functions of degree n in the variables $x=(x_1,x_2,\dots)$ with rational coefficients.

One basis for $\Lambda^n_{\mathbb{Q}}$ consists of the *Schur functions* s_λ for $\lambda \vdash n$. Define a linear transformation $\psi_n \colon \Lambda^n_{\mathbb{Q}} \to \Lambda^n_{\mathbb{Q}}$ by

$$\psi_n(f) = \frac{\partial}{\partial p_1} p_1 f.$$

Here $p_1 = s_1 = \sum x_i$, the first power sum symmetric function. The notation $\frac{\partial}{\partial p_1}$ indicates that we differentiate with respect to p_1 after writing the argument as a polynomial in the p_k's, where $p_k = \sum x_i^k$. It is a standard result [42, Thm. 7.15.7, Cor. 7.15.9, Exer. 7.35] that for $\lambda \vdash n$,

$$\begin{aligned} p_1 s_\lambda &= \sum_{\substack{\mu \vdash n+1 \\ \mu \supset \lambda}} s_\mu \\ \frac{\partial}{\partial p_1} s_\lambda &= s_{\lambda/1} = \sum_{\substack{\mu \vdash n-1 \\ \mu \subset \lambda}} s_\mu. \end{aligned}$$

Note that the power sum p_λ, $\lambda \vdash n$, is an eigenvector for ψ_n with eigenvalue $m_1(\lambda)+1$, where $m_1(\lambda)$ is the number of 1's in λ. Hence

$$\det \psi_n = \prod_{\lambda \vdash n} (m_1(\lambda)+1).$$

The factorization of $\det \psi_n$ suggests looking at the SNF of ψ_n with respect to the basis $\{s_\lambda\}$. We denote this matrix by $[\psi_n]$. Since the matrix transforming the s_λ's to the p_μ's is not invertible over $\mathbb{Z}$, we cannot simply convert the diagonal matrix with entries $m_1(\lambda)+1$ to SNF. As a special case of a more general conjecture Miller and Reiner [34] conjectured the SNF of $[\psi_n]$, which was then proved by Cai and Stanley [7]. Subsequently Nie [36] and Shah [38] made some further progress on the conjecture of Miller and Reiner. We state two equivalent forms of the result of Cai and Stanley.

Theorem 5.1. *Let* $[\psi_n] \stackrel{\text{snf}}{\to} (\alpha_1, \ldots, \alpha_{p(n)})$*, where* $p(n)$ *denotes the number of partitions of* n*.*

(a) *The* α_i*'s are as follows:*
- $(n+1)(n-1)!$*, with multiplicity* 1
- $(n-k)!$*, with multiplicity* $p(k+1) - 2p(k) + p(k-1)$*,* $3 \le k \le n-2$
- 1*, with multiplicity* $p(n) - p(n-1) + p(n-2)$*.*

(b) *Let* $\mathcal{M}_1(n)$ *be the multiset of all numbers* $m_1(\lambda)+1$*, for* $\lambda \vdash n$*. Then* $\alpha_{p(n)}$ *is the product of the* distinct *elements of* $\mathcal{M}_1(n)$*;* $\alpha_{p(n)-1}$ *is the product of the remaining* distinct *elements of* $\mathcal{M}_1(n)$*, etc.*

In fact, the following stronger result than Theorem 5.1 is actually proved.

Theorem 5.2. *Let* t *be an indeterminate. Then the matrix* $[\psi_n + tI]$ *has an SNF over* $\mathbb{Z}[t]$*.*

To see that Theorem 5.2 implies Theorem 5.1, use the fact that $[\psi_n]$ is a symmetric matrix (and therefore semisimple), and for each eigenvalue λ of ψ_n consider the rank of the matrices obtained by substituting $t = -\lambda$ in $[\psi_n + tI]$ and its SNF over $\mathbb{Z}[t]$. For details and further aspects, see [34, §8.2].

The proof of Theorem 5.2 begins by working with the basis $\{h_\lambda\}$ of complete symmetric functions rather than with the Schur functions, which we can do since the transition matrix between these bases is an integer unimodular matrix. The proof then consists basically of describing the row and column operations to achieve SNF.

The paper [7] contains a conjectured generalization of Theorem 5.2 to the operator $\psi_{n,k} := k\frac{\partial}{\partial p_k}p_k \colon \Lambda^n_{\mathbb{Q}} \to \Lambda^n_{\mathbb{Q}}$ for any $k \geq 1$. Namely, the matrix $[\psi_{n,k} + tI]$ with respect to the basis $\{s_\lambda\}$ has an SNF over $\mathbb{Z}[t]$. This implies that if $[\psi_{n,k}] \overset{\text{snf}}{\to} (\alpha_1, \ldots, \alpha_{p(n)})$ and $\mathcal{M}_k(n)$ denotes the multiset of all numbers $k(m_k(\lambda)+1)$, for $\lambda \vdash n$, then $\alpha_{p(n)}$ is the product of the *distinct* elements of $\mathcal{M}_k(n)$; $\alpha_{p(n)-1}$ is the product of the remaining *distinct* elements of $\mathcal{M}_k(n)$, etc. This conjecture was proved in 2015 by Zipei Nie (private communication).

There is a natural generalization of the SNF of $\psi_{n,k}$, namely, we can look at operators like $(\prod \lambda_i)\frac{\partial^\ell}{\partial p_\lambda}p_\lambda$. Here λ is a partition of n with ℓ parts and

$$\frac{\partial^\ell}{\partial p_\lambda} = \frac{\partial^\ell}{\partial p_1^{m_1}\partial p_2^{m_2}\cdots},$$

where λ has m_i parts equal to i. Even more generally, if $\lambda, \mu \vdash n$ where λ has ℓ parts, then we could consider $(\prod \lambda_i)\frac{\partial^\ell}{\partial p_\lambda}p_\mu$. No conjecture is known for the SNF (with respect to an integral basis), even when $\lambda = \mu$.

5.2. **A specialized Jacobi-Trudi matrix.** A fundamental identity in the theory of symmetric functions is the *Jacobi-Trudi identity.* Namely, if λ is a partition with at most t parts, then the *Jacobi-Trudi matrix* JT_λ is defined by

$$\mathrm{JT}_\lambda = [h_{\lambda_i+j-i}]_{i,j=1}^t,$$

where h_i denotes the complete symmetric function of degree i (with $h_0 = 1$ and $h_{-i} = 0$ for $i \geq 1$). The *Jacobi-Trudi identity* [42, §7.16] asserts that $\det \mathrm{JT}_\lambda = s_\lambda$, the Schur function indexed by λ.

For a symmetric function f, let $\varphi_n f$ denote the specialization $f(1^n)$, that is, set $x_1 = \cdots = x_n = 1$ and all other $x_i = 0$ in f. It is easy to see [42, Prop. 7.8.3] that

$$\varphi_n h_i = \binom{n+i-1}{i}, \tag{5.1}$$

a polynomial in n of degree i. Identify λ with its (Young) diagram, so the squares of λ are indexed by pairs (i,j), $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$. The *content* $c(u)$ of the square $u = (i,j)$ is defined to be $c(u) = j - i$. A standard result [42, Cor. 7.21.4] in the theory of symmetric functions states that

$$\varphi_n s_\lambda = \frac{1}{H_\lambda}\prod_{u\in\lambda}(n + c(u)), \tag{5.2}$$

where H_λ is a positive integer whose value is irrelevant here (since it is a unit in $\mathbb{Q}[n]$). Since this polynomial factors a lot (in fact, into linear factors) over $\mathbb{Q}[n]$, we are motivated to consider the SNF of the matrix

$$\varphi_n \mathrm{JT}_\lambda = \left[\binom{n+\lambda_i+j-i-1}{\lambda_i+j-i}\right]_{i,j=1}^t.$$

Let D_k denote the kth *diagonal hook* of λ, i.e., all squares $(i,j) \in \lambda$ such that either $i = k$ and $j \geq k$, or $j = k$ and $i \geq k$. Note that λ is a disjoint union of its diagonal hooks. If

0	1	2	3	4	5	6
−1	0	1	2	3		
−2	−1	0	1	2		
−3	−2					

FIGURE 1. The contents of the partition $(7,5,5,2)$

$r = \operatorname{rank}(\lambda) := \max\{i : \lambda_i \geq i\}$, then note also that $D_k = \emptyset$ for $k > r$. The following result was proved in [44].

Theorem 5.3. *Let* $\varphi_n \mathrm{JT}_\lambda \overset{\mathrm{snf}}{\to} (\alpha_1, \alpha_2, \ldots, \alpha_t)$, *where* $t \geq \ell(\lambda)$. *Then we can take*

$$\alpha_i = \prod_{u \in D_{t-i+1}} (n + c(u)).$$

An equivalent statement to Theorem 5.3 is that the α_i's are squarefree (as polynomials in n), since α_t is the largest squarefree factor of $\varphi_n s_\lambda$, α_{t-1} is the largest squarefree factor of $(\varphi_n s_\lambda)/\alpha_t$, etc.

Example 5.4. Let $\lambda = (7,5,5,2)$. Figure 1 shows the diagram of λ with the content of each square. Let $t = \ell(\lambda) = 4$. We see that

$$\begin{aligned} \alpha_4 &= (n-3)(n-2)\cdots(n+6) \\ \alpha_3 &= (n-2)(n-1)n(n+1)(n+2)(n+3) \\ \alpha_2 &= n(n+1)(n+2) \\ \alpha_1 &= 1. \end{aligned}$$

The problem of computing the SNF of a suitably specialized Jacobi-Trudi matrix was raised by Kuperberg [28]. His Theorem 14 has some overlap with our Theorem 5.3. Propp [37, Problem 5] mentions a two-part question of Kuperberg. The first part is equivalent to our Theorem 5.3 for rectangular shapes. (The second part asks for an interpretation in terms of tilings, which we do not consider.)

Theorem 5.3 is proved not by the more usual method of row and column operations. Rather, the gcd of the $k \times k$ minors is computed explicitly so that Theorem 2.4 can be applied. Let M_k be the bottom-left $k \times k$ submatrix of JT_λ. Then M_k is itself the Jacobi-Trudi matrix of a certain partition μ^k, so $\varphi_n M_k$ can be explicitly evaluated. One then shows using the Littlewood-Richardson rule that every $k \times k$ minor of $\varphi_n \mathrm{JT}_\lambda$ is divisible by $\varphi_n M_k$. Hence $\varphi_n M_k$ is the gcd of the $k \times k$ minors of $\varphi_n \mathrm{JT}_\lambda$, after which the proof is a routine computation.

There is a natural q-analogue of the specialization $f(x) \to f(1^n)$, namely, $f(x) \to f(1, q, q^2, \ldots, q^{n-1})$. Thus we can ask for a q-analogue of Theorem 5.3. This can be done using the same proof technique, but some care must be taken in order to get a q-analogue that reduces directly to Theorem 5.3 by setting $q = 1$. When this is done we get the following result [44, Thm. 3.2].

abcde+bcde +bce+cde+ ce+de+c+e +1	bce+ce+c +e+1	c+1	1
de+e+1	e+1	1	1
1	1	1	

FIGURE 2. The polynomials P_{rs} for $\lambda = (3, 2)$

Theorem 5.5. *For $k \geq 1$ let*

$$f(k) = \frac{n(n + \boldsymbol{(1)})(n + \boldsymbol{(2)}) \cdots (n + \boldsymbol{(k-1)})}{\boldsymbol{(1)}\boldsymbol{(2)} \cdots \boldsymbol{(k)}},$$

where $\boldsymbol{(j)} = (1 - q^j)/(1 - q)$ for any $j \in \mathbb{Z}$. Set $f(0) = 1$ and $f(k) = 0$ for $k < 0$. Define

$$\mathrm{JT}_\lambda(q) = [f(\lambda_i - i + j)]_{i,j=1}^{t},$$

where $\ell(\lambda) \leq t$. Let $\mathrm{JT}_\lambda(q) \overset{\text{snf}}{\to} (\gamma_1, \gamma_2, \ldots, \gamma_t)$ over the ring $\mathbb{Q}(q)[n]$. Then we can take

$$\gamma_i = \prod_{u \in D_{t-i+1}} (n + \boldsymbol{c(u)}).$$

6. A MULTIVARIATE EXAMPLE

In this section we give an example where the SNF exists over a multivariate polynomial ring over $\mathbb{Z}$. Let λ be a partition, identified with its Young diagram regarded as a set of squares; we fix λ for all that follows. Adjoin to λ a border strip extending from the end of the first row to the end of the first column of λ, yielding an *extended partition* λ^*. Let (r, s) denote the square in the rth row and sth column of λ^*. If $(r, s) \in \lambda^*$, then let $\lambda(r, s)$ be the partition whose diagram consists of all squares (u, v) of λ satisfying $u \geq r$ and $v \geq s$. Thus $\lambda(1, 1) = \lambda$, while $\lambda(r, s) = \emptyset$ (the empty partition) if $(r, s) \in \lambda^* \setminus \lambda$. Associate with the square (i, j) of λ an indeterminate x_{ij}. Now for each square (r, s) of λ^*, associate a polynomial P_{rs} in the variables x_{ij}, defined as follows:

$$P_{rs} = \sum_{\mu \subseteq \lambda(r,s)} \prod_{(i,j) \in \lambda(r,s) \setminus \mu} x_{ij}, \tag{6.1}$$

where μ runs over all partitions contained in $\lambda(r, s)$. In particular, if $(r, s) \in \lambda^* \setminus \lambda$ then $P_{rs} = 1$. Thus for $(r, s) \in \lambda$, P_{rs} may be regarded as a generating function for the squares of all skew diagrams $\lambda(r, s) \setminus \mu$. For instance, if $\lambda = (3, 2)$ and we set $x_{11} = a$, $x_{12} = b$, $x_{13} = c$, $x_{21} = d$, and $x_{22} = e$, then Figure 2 shows the extended diagram λ^* with the polynomial P_{rs} placed in the square (r, s).

Write

$$A_{rs} = \prod_{(i,j)\in\lambda(r,s)} x_{ij}.$$

Note that A_{rs} is simply the leading term of P_{rs}. Thus for $\lambda = (3,2)$ as in Figure 2 we have $A_{11} = abcde$, $A_{12} = bce$, $A_{13} = c$, $A_{21} = de$, and $A_{22} = e$.

For each square $(i,j) \in \lambda^*$ there will be a unique subset of the squares of λ^* forming an $m \times m$ square $S(i,j)$ for some $m \geq 1$, such that the upper left-hand corner of $S(i,j)$ is (i,j), and the lower right-hand corner of $S(i,j)$ lies in $\lambda^* \setminus \lambda$. In fact, if ρ_{ij} denotes the *rank* of $\lambda(i,j)$ (the number of squares on the main diagonal, or equivalently, the largest k for which $\lambda(i,j)_k \geq k$), then $m = \rho_{ij} + 1$. Let $M(i,j)$ denote the matrix obtained by inserting in each square (r,s) of $S(i,j)$ the polynomial P_{rs}. For instance, for the partition $\lambda = (3,2)$ of Figure 2, the matrix $M(1,1)$ is given by

$$M(1,1) = \begin{bmatrix} P_{11} & bce+ce+c+e+1 & c+1 \\ de+e+1 & e+1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where $P_{11} = abcde + bcde + bce + cde + ce + de + c + e + 1$. Note that for this example we have

$$\det M(1,1) = A_{11}A_{22}A_{33} = abcde \cdot e \cdot 1 = abcde^2.$$

The main result on the matrices $M(i,j)$ is the following. For convenience we state it only for $M(1,1)$, but it applies to any $M(i,j)$ by replacing λ with $\lambda(i,j)$.

Theorem 6.1. *Let $\rho = \operatorname{rank}(\lambda)$. The matrix $M(1,1)$ has an SNF over $\mathbb{Z}[x_{ij}]$, given explicitly by*

$$M(1,1) \overset{\text{snf}}{\to} (A_{11}, A_{22}, \ldots, A_{\rho+1,\rho+1}).$$

Hence $\det M(1,1) = A_{11}A_{22}\cdots A_{\rho\rho}$ *(since $A_{\rho+1,\rho+1} = 1$).*

Theorem 6.1 is proved by finding row and column operations converting $M(1,1)$ to SNF. In [3] this is done in two ways: an explicit description of the row and column operations, and a proof by induction that such operations exist without stating them explicitly.

Another way to describe the SNF of $M(1,1)$ is to replace its nondiagonal entries with 0 and a diagonal entry with its leading term (unique monomial of highest degree). Is there some conceptual reason why the SNF has this simple description?

If we set each $x_{ij} = 1$ in $M(1,1)$ then we get $\det M(1,1) = 1$. This formula is equivalent to result of Carlitz, Roselle, and Scoville [8] which answers a question posed by Berlekamp [1][2]. If we set each $x_{ij} = q$ in $M(1,1)$ and take $\lambda = (m-1, m-2, \ldots, 1)$, then the entries of $M(1,1)$ are certain q-Catalan numbers, and $\det M(1,1)$ was determined by Cigler [10][11]. This determinant (and some related ones) was a primary motivation for [3]. Miller and Stanton [34] have generalized the q-Catalan result to Hankel matrices of moments of orthogonal polynomials and some other similar matrices.

Di Francesco [17] shows that the polynomials P_{rs} satisfy the "octahedron recurrence" and are related to cluster algebras, integrable systems, dimer models, and other topics.

7. The Varchenko matrix

Let $\mathcal{A}$ be a finite arrangement (set) of affine hyperplanes in $\mathbb{R}^n$. The complement $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$ consists of a disjoint union of finitely many open *regions*. Let $\mathcal{R}(\mathcal{A})$ denote the set of all regions. For each hyperplane $H \in \mathcal{A}$ associate an indeterminate a_H. If $R, R' \in \mathcal{R}(\mathcal{A})$ then let $\mathrm{sep}(R, R')$ denote the set of $H \in \mathcal{A}$ separating R from R', that is, R and R' lie on different sides of H. Now define a matrix $V(\mathcal{A})$ as follows. The rows and columns are indexed by $\mathcal{R}(\mathcal{A})$ (in some order). The (R, R')-entry is given by

$$V_{RR'} = \prod_{H\in \mathrm{sep}(R,R')} a_H.$$

If x is any nonempty intersection of a set of hyperplanes in $\mathcal{A}$, then define $a_x = \prod_{H\supseteq x} a_H$. Varchenko [46] showed that

$$\det V(\mathcal{A}) = \prod_x (1 - a_x^2)^{n(x)p(x)}, \tag{7.1}$$

for certain nonnegative integers $n(x), p(x)$ which we will not define here.

Note. We include the intersection x over the empty set of hyperplanes, which is the ambient space $\mathbb{R}^n$. This gives an irrelevant factor of 1 in the determinant above, but it also accounts for an essential diagonal entry of 1 in Theorem 7.1 below.

Since $\det V(\mathcal{A})$ has such a nice factorization, it is natural to ask about its diagonal form or SNF. Since we are working over the polynomial ring $\mathbb{Z}[a_H : H \in \mathcal{A}]$ or $\mathbb{Q}[a_H : H \in \mathcal{A}]$, there is no reason for a diagonal form to exist. Gao and Zhang [21] found the condition for this property to hold. We say that $\mathcal{A}$ is *semigeneric* or *in semigeneral form* if for any k hyperplanes $H_1, \dots, H_k \in \mathcal{A}$ with intersection $x = \bigcap_{i=1}^k H_i$, either $\mathrm{codim}(x) = k$ or $x = \emptyset$. (Note that x is an affine subspace of $\mathbb{R}^n$ so has a well-defined codimension.) In particular, $x = \emptyset$ if $k > n$.

Theorem 7.1. *The matrix $V(\mathcal{A})$ has a diagonal form if and only if $\mathcal{A}$ is semigeneric. In this case, the diagonal entries of $\mathcal{A}$ are given by $\prod_{H\supseteq x}(1 - a_H^2)$, where x is a nonempty intersection of the hyperplanes in some subset of $\mathcal{A}$.*

Gao and Zhang actually prove their result for pseudosphere arrangements, which are a generalization of hyperplane arrangements. Pseudosphere arrangements correspond to oriented matroids.

Example 7.2. Let $\mathcal{A}$ be the arrangement of three lines in $\mathbb{R}^2$ shown in Figure 3, with the hyperplane variables a, b, c as in the figure. This arrangement is semigeneric. The diagonal entries of the diagonal form of $V(\mathcal{A})$ are

$$1,\ 1 - a^2,\ 1 - b^2,\ 1 - c^2, (1 - a^2)(1 - c^2),\ (1 - b^2)(1 - c^2).$$

Now define the *q-Varchenko matrix* $V_q(\mathcal{A})$ of $\mathcal{A}$ to be the result of substituting $a_H = q$ for all $H \in \mathcal{A}$. Equivalently, $V_q(\mathcal{A})_{RR'} = q^{\#\mathrm{sep}(R,R')}$. The SNF of $V_q(\mathcal{A})$ exists over the PID $\mathbb{Q}[q]$, and it seems to be a very interesting and little studied problem to determine this SNF. Some special cases were determined by Cai and Mu [7]. A generalization related to distance matrices of graphs was considered by Shiu [39]. Note that by equation (7.1) the diagonal entries of the SNF of $V_q(\mathcal{A})$ will be products of cyclotomic polynomials $\Phi_d(q)$.

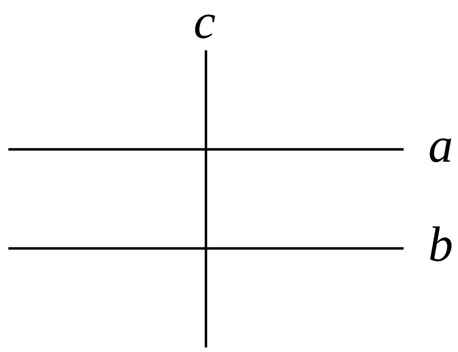

Figure 3. An arrangement of three lines

The main paper to date on the SNF of $V_q(\mathcal{A})$ is by Denham and Hanlon [13]. In particular, let

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^i c_i t^{n-i}$$

be the *characteristic polynomial* of $\mathcal{A}$, as defined for instance in [43, §1.3][41, §3.11.2]. Denham and Hanlon show the following in their Theorem 3.1.

Theorem 7.3. *Let $N_{d,i}$ be the number of diagonal entries of the SNF of $V_q(\mathcal{A})$ that are exactly divisible by $\Phi_d(q)^i$. Then $N_{1,i} = c_i$.*

It is easy to see that $N_{1,i} = N_{2,i}$. Thus the next step would be to determine $N_{3,i}$ and $N_{4,i}$.

An especially interesting hyperplane arrangement is the *braid arrangement* $\mathcal{B}_n$ in $\mathbb{R}^n$, with hyperplanes $x_i = x_j$ for $1 \le i < j \le n$. The determinant of $V_q(\mathcal{B}_n)$, originally due to Zagier [51], is given by

$$\det V_q(\mathcal{B}_n) = \prod_{j=2}^{n} \left(1 - q^{j(j-1)}\right)^{\binom{n}{j}(j-2)!\,(n-j+1)!}.$$

An equivalent description of $V_q(\mathcal{B}_n)$ is the following. Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $1, 2, \ldots, n$, and let $\mathrm{inv}(w)$ denote the number of inversions of $w \in \mathfrak{S}_n$, i.e., $\mathrm{inv}(w) = \#\{(i,j) : 1 \le i < j \le n,\ w(i) > w(j)\}$. Define $\Gamma_n(q) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} w$, an element of the group algebra $\mathbb{Q}[q]\mathfrak{S}_n$. The element $\Gamma_n(q)$ acts on $\mathbb{Q}[q]\mathfrak{S}_n$ by left multiplication, and $V_q(\mathcal{B}_n)$ is the matrix of this linear transformation (with a suitable indexing of rows and columns) with respect to the basis $\mathfrak{S}_n$. The SNF of $V_q(\mathcal{B}_n)$ (over the PID $\mathbb{Q}[q]$) is not known. Denham and Hanlon [13, §5] compute it for $n \le 6$.

Some simple representation theory allows us to refine the SNF of $V_q(\mathcal{B}_n)$. The complex irreducible representations φ_λ of $\mathfrak{S}_n$ are indexed by partitions $\lambda \vdash n$. Let $f^\lambda = \dim \varphi_\lambda$. The action of $\mathfrak{S}_n$ on $\mathbb{Q}\mathfrak{S}_n$ by right multiplication commutes with the action of $\Gamma_n(q)$. It follows (since every irreducible representation of $\mathfrak{S}_n$ can be defined over $\mathbb{Z}$) that by a unimodular change of basis we can write

$$V_q(\mathcal{B}_n) = \bigoplus_{\lambda \vdash n} f^\lambda V_\lambda,$$

for some integral matrices V_λ of size $f^\lambda \times f^\lambda$. Thus computing $\det V_\lambda$ and the SNF of V_λ is a refinement of computing $\det V_q(\mathcal{B}_n)$ and the SNF of $V_q(\mathcal{B}_n)$. (Computing the SNF of each V_λ would give a diagonal form of $V_q(\mathcal{B}_n)$, from which it is easy to determine the SNF.) The problem of computing $\det V_\lambda$ was solved by Hanlon and Stanley [23, Conj. 3.7]. Of course the SNF of V_λ remains open since the same is true for $V_q(\mathcal{B}_n)$. Denham and Hanlon have computed the SNF of V_λ for $\lambda \vdash n \le 6$ and published the results for $n \le 4$ in [13, §5]. For instance, for the partitions $\lambda \vdash 4$ we have the following diagonal elements of the SNF of V_λ:

$$\begin{aligned}
(4):&\quad \Phi_2^2\Phi_3\Phi_4\\
(3,1):&\quad \Phi_1\Phi_2,\ \Phi_1^2\Phi_2^2\Phi_3,\ \Phi_1^3\Phi_2^3\Phi_3^2\\
(2,2):&\quad \Phi_1^2\Phi_2^2,\ \Phi_1^2\Phi_2^2\Phi_{12}\\
(2,1,1):&\quad \Phi_1\Phi_2,\ \Phi_1^2\Phi_2^2\Phi_6,\ \Phi_1^3\Phi_2^3\Phi_6^2\\
(1,1,1,1):&\quad \Phi_1\Phi_2\Phi_4\Phi_6,
\end{aligned}$$

where Φ_d denotes the cyclotomic polynomial whose zeros are the primitive dth roots of unity. For a nice survey of this topic see Denham and Hanlon [14].

The discussion above of Γ_n suggests that it might be interesting to consider the SNF of other elements of $R\mathfrak{S}_n$ for suitable rings R (or possibly RG for other finite groups G). One intriguing example is the *Jucys-Murphy element* (though it first appears in the work of Alfred Young [50, §19]) $X_k \in \mathbb{Q}\mathfrak{S}_n$, $1 \le k \le n$. It is defined by $X_1 = 0$ and

$$X_k = (1,k) + (2,k) + \cdots + (k-1,k), \quad 2 \le k \le n,$$

where (i,k) denotes the transposition interchanging i and k. Just as for $\Gamma_n(q)$, we can choose an integral basis for $\mathbb{Q}\mathfrak{S}_n$ (that is, a $\mathbb{Z}$-basis for $\mathbb{Z}\mathfrak{S}_n$) so that the action of X_k on $\mathbb{Q}\mathfrak{S}_n$ with respect to this basis has a matrix of the form $\bigoplus_{\lambda\vdash n} f^\lambda W_{\lambda,k}$. The eigenvalues of $W_{\lambda,k}$ are known to be the contents of the positions occupied by k in all standard Young tableaux of shape λ. For instance, when $\lambda = (5,1)$ the standard Young tableaux are

$$\begin{array}{lllll} 12345 & 12346 & 12356 & 12456 & 13456\\ 6 & 5 & 4 & 3 & 2 \end{array}.$$

The positions occupied by 5 are $(1,5)$, $(2,1)$, $(1,4)$, $(1,4)$, $(1,4)$. Hence the eigenvalues of $W_{(5,1),5}$ are $5-1=4$, $1-2=-1$, and $4-1=3$ (three times). Darij Grinberg (private communication) computed the SNF of the matrices $W_{\lambda,k}$ for $\lambda \vdash n \le 7$. On the basis of this data we make the following conjecture.

Conjecture 7.4. *Let $\lambda \vdash n$, $1 \le k \le n$, and $W_{\lambda,k} \stackrel{\text{snf}}{\to} (\alpha_1, \dots, \alpha_{f^\lambda})$. Fix $1 \le r \le f^\lambda$. Let S_r be the set of positions (i,j) that k occupies in at least r of the SYT's of shape λ. Then $\alpha_{f^\lambda - r+1} = \pm \prod_{(i,j)\in S_r}(j-i)$.*

Note in particular that every SNF diagonal entry is (conjecturally) a product of some of the eigenvalues of $W_{\lambda,k}$.

For example, when $\lambda = (5,1)$ and $k=5$ we have $f^{(5,1)} = 5$ and $S_1 = \{(1,5),(2,1),(1,4)\}$, $S_2 = S_3 = \{(1,4)\}$, $S_4 = S_5 = \emptyset$. Hence $W_{(5,1),5} \stackrel{\text{snf}}{\to} (1,1,3,3,12)$.

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