On the Enumeration of Skew Young Tableaux

Richard P. Stanley[1]
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
e-mail: rstan@math.mit.edu

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1 Introduction.

A recent paper [7] of McKay, Morse, and Wilf considers the number $N(n;T)$ of standard Young tableaux (SYT) with n cells that contain a fixed standard Young tableau T of shape $\alpha \vdash k$. (For notation and terminology related to symmetric functions and tableaux, see [6] or [12, Ch. 7].) They obtain the asymptotic formula

$$N(n;T) \sim \frac{t_n f^\alpha}{k!}, \tag{1}$$

where f^α denotes the number of SYT of shape α and t_n denotes the number of involutions in the symmetric group $\mathfrak{S}_n$. Note that $N(n;T) = N(n;U)$ whenever T and U are SYT of the same shape. Hence we can write $N(n;\alpha)$ for $N(n;T)$. Moreover, it is clear that

$$N(n;\alpha) = \sum_{\lambda \vdash n} f^{\lambda/\alpha}, \tag{2}$$

where $f^{\lambda/\alpha}$ denotes the number of SYT T of skew shape λ/α.

In Section 2 we extend equation (1), using techniques from the theory of symmetric functions, to give an explicit formula for $N(n;\alpha)$ as a finite linear combination of t_{n-j}'s, from which in principle we can write down the entire asymptotic expansion of $N(n;\alpha)$. In Section 3 we apply similar techniques,

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together with asymptotic formulas for character values of $\mathfrak{S}_n$ due to Biane [2] and to Vershik and Kerov [14], to derive the asymptotic behavior of $f^{\lambda/\alpha}$ as a function of λ for fixed α.

2 A formula for $N(n;\alpha)$.

Let $\chi^\alpha(\lambda)$ denote the value of the irreducible character χ^α of $\mathfrak{S}_k$ on a permutation of cycle type $\lambda \vdash k$ (as explained e.g. in [6, §1.7] or [12, §§7.17–7.18]). Let $m_i(\mu)$ denote the number of parts of the partition μ equal to i, and write $\tilde{\mu}$ for the partition obtained from μ by replacing every even part $2i$ with the two parts i, i. For instance,

$$\mu = (6,6,5,4,2,1) \;\Rightarrow\; \tilde{\mu} = (5,3,3,3,3,2,2,1,1,1).$$

Equivalently, if w is a permutation of cycle type μ, then w^2 has cycle type $\tilde{\mu}$. Note that a permutation of cycle type $\tilde{\mu}$ is necessarily even. We will use notation such as $(\tilde{\mu}, 1^{k-j})$ to denote a partition whose parts are the parts of $\tilde{\mu}$ with $k-j$ additional parts equal to 1. Finally we let z_μ denote the number of permutations commuting with a fixed permutation of cycle type μ, so

$$z_\mu = 1^{m_1(\mu)} 2^{m_2(\mu)} \cdots m_1(\mu)!\, m_2(\mu)! \cdots.$$

The main result of this section is the following.

2.1 Theorem. *Let $\alpha \vdash k$. Then for $n \geq k$ we have*

$$N(n;\alpha) = \sum_{j=0}^{k} \frac{t_{n-j}}{(k-j)!} \sum_{\substack{\mu \vdash j \\ m_1(\mu)=m_2(\mu)=0}} z_\mu^{-1} \chi^\alpha(\tilde{\mu}, 1^{k-j}). \tag{3}$$

Proof. Let $\lambda \vdash n \geq k$, and let $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ denote the power sum symmetric function indexed by λ. Similarly $s_{\lambda/\alpha}$ denotes the skew Schur function indexed by λ/α. Since for any homogeneous symmetric function f of degree $n-k$ we have that $\langle p_1^{n-k}, f\rangle$ is the coefficient of $x_1 \cdots x_{n-k}$ in f, and since the coefficient of $x_1 \cdots x_{n-k}$ in $s_{\lambda/\alpha}$ is $f^{\lambda/\alpha}$, we have (using a basic property [6, (5.1)][12, Thm. 7.15.4] of the standard scalar product $\langle \cdot, \cdot \rangle$ on symmetric functions)

$$\begin{aligned} f^{\lambda/\alpha} &= \langle p_1^{n-k}, s_{\lambda/\alpha} \rangle \\ &= \langle p_1^{n-k} s_\alpha, s_\lambda \rangle. \end{aligned}$$

Summing on $\lambda \vdash n$ gives

$$N(n;\alpha) = \left\langle p_1^{n-k} s_\alpha, \sum_{\lambda \vdash n} s_\lambda \right\rangle. \tag{4}$$

Now [6, Exam. I.5.4, p. 76][12, Cor. 7.13.8]

$$\sum_\lambda s_\lambda = \frac{1}{\prod_i (1-x_i) \cdot \prod_{i<j} (1-x_i x_j)},$$

summed over all partitions λ of all $n \geq 0$. Since

$$\begin{aligned} \frac{1}{\prod_i (1-x_i) \cdot \prod_{i<j} (1-x_i x_j)} &= \exp \sum_{n \geq 1} \frac{1}{n} \left(\sum_i x_i^n + \sum_{i<j} x_i^n x_j^n \right) \\ &= \exp \left(\sum_{n \geq 1} \frac{p_{2n-1}}{2n-1} + \sum_{n \geq 1} \frac{p_n^2}{2n} \right), \end{aligned}$$

there follows

$$\begin{aligned} \sum_{\lambda \vdash n} s_\lambda &= \sum_{\lambda = (1^{m_1}, 2^{m_2}, \ldots) \vdash n} z_\lambda^{-1} p_1^{m_1 + 2m_2} p_2^{2m_4} p_3^{m_3 + 2m_6} p_4^{2m_8} \cdots \\ &= \sum_{\lambda \vdash n} z_\lambda^{-1} p_{\tilde{\lambda}}, \end{aligned} \tag{5}$$

where $(1^{m_1}, 2^{m_2}, \ldots)$ denotes the partition with m_i parts equal to i.

It follows from [6, p. 76][12, solution to Exer. 7.35(a)] that for any symmetric functions f and g we have

$$\langle p_1 f, g \rangle = \left\langle f, \frac{\partial}{\partial p_1} g \right\rangle,$$

where $\frac{\partial}{\partial p_1} g$ indicates that we are to expand g as a polynomial in the p_i's and then differentiate with respect to p_1. Applying this to equation (4) and using (5) yields

$$\begin{aligned} N(n;\alpha) &= \left\langle s_\alpha, \frac{\partial^{n-k}}{\partial p_1^{n-k}} \sum_{\lambda \vdash n} z_\lambda^{-1} p_{\tilde{\lambda}} \right\rangle \\ &= \left\langle s_\alpha, \sum_{\lambda \vdash n} z_\lambda^{-1} (m_1 + 2m_2)_{n-k}\, p_1^{-n+k} p_{\tilde{\lambda}} \right\rangle, \end{aligned} \tag{6}$$

where $m_i = m_i(\lambda)$ and $(a)_{n-k} = a(a-1)\cdots(a-n+k+1)$.

Fix $m_1 + 2m_2 = n - j$ in equation (6). Thus $\lambda = (\mu, 2^{m_2}, 1^{m_1})$ for some unique $\mu \vdash k$ satisfying $m_1(\mu) = m_2(\mu) = 0$. Since $n!/z_\lambda$ is the number of permutations in $\mathfrak{S}_n$ of cycle type λ, we have for fixed $\mu \vdash j$ that

$$\sum_{\lambda = (\mu, 2^{m_2}, 1^{m_1})} \frac{n!}{z_\lambda} = t_{n-j} \binom{n}{j} \frac{j!}{z_\mu}.$$

Moreover,

$$p_1^{-n+k} p_{\tilde{\lambda}} = p_1^{k-j} p_{\tilde{\mu}}.$$

It follows that

$$\begin{aligned} N(n;\alpha) &= \left\langle s_\alpha, \sum_{j=0}^{k} j! \binom{n}{j} (n-j)_{n-k} \frac{t_{n-j}}{n!} \sum_{\substack{\mu \vdash j \\ m_1(\mu) = m_2(\mu) = 0}} z_\mu^{-1} p_1^{k-j} p_{\tilde{\mu}} \right\rangle \\ &= \left\langle s_\alpha, \sum_{j=0}^{k} \frac{t_{n-j}}{(k-j)!} \sum_{\substack{\mu \vdash j \\ m_1(\mu) = m_2(\mu) = 0}} z_\mu^{-1} p_1^{k-j} p_{\tilde{\mu}} \right\rangle. \end{aligned}$$

Since [6, (7.7)][12, p. 348]

$$\langle s_\alpha, p_1^{k-j} p_{\tilde{\mu}} \rangle = \chi^\alpha(\tilde{\mu}, 1^{k-j}),$$

the proof follows. $\square$

Note that the restriction $n \geq k$ in Theorem 2.1 is insignificant since $N(n;\alpha) = 0$ for $n < k$.

Theorem 2.1 expresses $N(n;\alpha)$ as a linear combination of the functions t_{n-j}, $0 \le j \le k$. Since $t_{n-j-1} = o(t_{n-j})$, this formula for $N(n;\alpha)$ is actually an asymptotic expansion. The first few terms are

$$\begin{aligned} N(n;\alpha) &= \frac{1}{k!} f^\alpha t_n + \frac{1}{3(k-3)!}\chi^\alpha(3,1^{k-3})t_{n-3} \\ &\quad + \frac{1}{4(k-4)!}\chi^\alpha(2,2,1^{k-4})t_{n-4} + \frac{1}{5(k-5)!}\chi^\alpha(5,1^{k-5})t_{n-5} \\ &\quad + \frac{2}{9(k-6)!}\chi^\alpha(3,3,1^{k-6})t_{n-6} + O(t_{n-7}). \end{aligned} \tag{7}$$

Note that by symmetry it is clear that if α' is the conjugate partition to α then $N(n;\alpha) = N(n;\alpha')$. Indeed, since a permutation of cycle type $\tilde{\mu}$ is even we have $\chi^\alpha(\tilde{\mu},1^{k-j}) = \chi^{\alpha'}(\tilde{\mu},1^{k-j})$. The exact formulas for $N(n;\alpha)$ when $|\alpha| \le 5$ and $|\alpha| \le n$ are given as follows (where we write e.g. $N(n;21)$ for $N(n;(2,1))$):

$$\begin{aligned}
N(n;1) &= t_n \\
N(n;2) = N(n;11) &= \frac{1}{2}t_n \\
N(n;3) = N(n;111) &= \frac{1}{6}(t_n + 2t_{n-3}) \\
N(n;21) &= \frac{1}{3}(t_n - t_{n-3}) \\
N(n;4) = N(n;1111) &= \frac{1}{24}(t_n + 8t_{n-3} + 6t_{n-4}) \\
N(n;31) = N(n;211) &= \frac{1}{8}(t_n - 2t_{n-4}) \\
N(n;22) &= \frac{1}{12}(t_n - 4t_{n-3} + 6t_{n-4}) \\
N(n;5) = N(n;11111) &= \frac{1}{120}(t_n + 20t_{n-3} + 30t_{n-4} + 24t_{n-5}) \\
N(n;41) = N(n;2111) &= \frac{1}{30}(t_n + 5t_{n-3} - 6t_{n-5}) \\
N(n;32) = N(n;221) &= \frac{1}{24}(t_n - 4t_{n-3} + 6t_{n-4}) \\
N(n;311) &= \frac{1}{20}(t_n - 10t_{n-4} + 4t_{n-5}).
\end{aligned}$$

The complete asymptotic expansion of t_n beginning

$$t_n \approx \frac{1}{\sqrt{2}} n^{n/2} e^{-\frac{n}{2}+\sqrt{n}-\frac{1}{4}} \left(1 + \frac{7}{24\sqrt{n}} - \frac{119}{1152n} + \cdots\right)$$

was obtained by Moser and Wyman [8, 3.39]. In principle this can be used to obtain the asymptotic expansion of $N(n;\alpha)$ in terms of more "familiar" functions than t_{n-j}. The first few terms can be obtained from the formula

$$t_{n-j} = \frac{1}{\sqrt{2}} n^{\frac{n-j}{2}} e^{-\frac{n}{2}+\sqrt{n}-\frac{1}{4}} \left(1 + \left(\frac{7}{24} - \frac{j}{2}\right)\frac{1}{\sqrt{n}} - \left(\frac{119}{1152} + \frac{7}{48}j - \frac{3}{8}j^2\right)\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right)\right),$$

though we omit the details.

Instead of counting the number $N(n;\alpha)$ of SYT with n cells containing a fixed SYT T of shape α, we can ask (as also done in [7]) for the probability $P(n;\alpha)$ that a random SYT with n cells (chosen from the uniform distribution on all SYT with n cells) contains T as a subtableau. Since the total number of SYT with n cells is t_n, we have

$$P(n;\alpha) = N(n;\alpha)/t_n.$$

Let $e_j(\alpha)$ denote the coefficient of t_{n-j} in the right-hand side of (3), viz.,

$$e_j(\alpha) = \frac{1}{(k-j)!} \sum_{\substack{\mu \vdash j \\ m_1(\mu)=m_2(\mu)=0}} z_\mu^{-1} \chi^\alpha(\tilde{\mu}, 1^{k-j}). \tag{8}$$

It follows from Theorem 2.1, using the fact that $e_0(\alpha) = f^\alpha/k!$ and $e_1(\alpha) = e_2(\alpha) = 0$, that

$$P(n;\alpha) = \frac{f^\alpha}{k!} + \frac{e_3(\alpha)}{n^{3/2}} - \frac{3e_3(\alpha) - 2e_4(\alpha)}{n^2} + O\left(n^{-5/2}\right).$$

The leading term of this expansion was obtained in [7, Thm. 1].

There is an alternative formula for $N(n;\alpha)$ which, though not as convenient for asymptotics, is more combinatorial than equation (3) because it avoids using the characters of $\mathfrak{S}_n$. This formula could be derived directly from Theorem 2.1, but we give an alternative proof which is implicitly bijective (since the formulas on which it is based have bijective proofs).

2.2 Theorem. *Let $\alpha \vdash k$. Then for all $n \geq 0$ we have*

$$N(n+k;\alpha) = \sum_{j=0}^{k} \binom{n}{j} \left(\sum_{\mu \vdash k-j} f^{\alpha/\mu} \right) t_{n-j}. \tag{9}$$

Proof. We begin with the following Schur function identity, proved independently by Lascoux, Macdonald, Towber, Stanley, Zelevinsky, and perhaps others. This identity appears in [6, Exam. I.5.27(a), p. 93][12, Exer. 7.27(e)] and was given a bijective proof by Sagan and Stanley [11, Cor. 6.4]:

$$\sum_{\lambda} s_{\lambda/\alpha} = \frac{1}{\prod_i (1-x_i) \cdot \prod_{i<j}(1-x_i x_j)} \sum_{\mu} s_{\alpha/\mu}.$$

Apply the homomorphism ex that takes the power sum symmetric function p_n to $\delta_{1n} u$, where u is an indeterminate. This homomorphism is the exponential specialization discussed in [12, pp. 304–305]. Two basic properties of ex are the following:

$$\mathrm{ex}(f) = \sum_{n \geq 0} [x_1 x_2 \cdots x_n] f \frac{u^n}{n!}$$

$$\mathrm{ex} \frac{1}{\prod_i (1-x_i) \cdot \prod_{i<j}(1-x_i x_j)} = e^{u + \frac{1}{2}u^2},$$

where $[x_1 x_2 \cdots x_n] f$ denotes the coefficient of $x_1 x_2 \cdots x_n$ in f. Since

$$[x_1 x_2 \cdots x_n] s_{\lambda/\alpha} = f^{\lambda/\alpha}, \ \text{when } |\lambda/\alpha| = n,$$

we obtain

$$\sum_{n \geq 0} \frac{u^n}{n!} \sum_{\lambda \vdash n+k} f^{\lambda/\alpha} = e^{u + \frac{1}{2}u^2} \sum_{j=0}^{k} \frac{u^j}{j!} \sum_{\mu \vdash k-j} f^{\alpha/\mu}. \tag{10}$$

Taking the coefficient of $u^n/n!$ on both sides yields (9). $\square$

2.3 Corollary. *We have*

$$\sum_{n\geq 0}\sum_{\alpha} N(n+|\alpha|;\alpha)s_\alpha \frac{u^n}{n!} = \left(\sum_{\mu} s_\mu\right) e^{(p_1+1)u+\frac{1}{2}u^2}.$$

Proof. Multiply (10) by s_α and sum on α to get

$$\begin{aligned}
\sum_{n\geq 0}\sum_{\alpha} N(n+|\alpha|;\alpha)s_\alpha \frac{u^n}{n!} &= e^{u+\frac{1}{2}u^2}\sum_{j\geq 0}\frac{u^j}{j!}\sum_{|\alpha/\mu|=j} f^{\alpha/\mu}s_\alpha \\
&= e^{u+\frac{1}{2}u^2}\sum_{j\geq 0}\frac{u^j}{j!}\sum_{|\alpha/\mu|=j} \langle p_1^j, s_{\alpha/\mu}\rangle s_\alpha \\
&= e^{u+\frac{1}{2}u^2}\sum_{j\geq 0}\frac{u^j}{j!}\sum_{|\alpha/\mu|=j} \langle p_1^j s_\mu, s_\alpha\rangle s_\alpha \\
&= e^{u+\frac{1}{2}u^2}\sum_{j\geq 0}\frac{u^j}{j!}\sum_{\mu} p_1^j s_\mu \\
&= \left(\sum_{\mu} s_\mu\right) e^{(p_1+1)u+\frac{1}{2}u^2}. \ \square
\end{aligned}$$

The case when α consists of a single row (or column) is particularly simple, since then each $\chi^\alpha(\tilde{\mu}, 1^{k-j}) = 1$ in (3). We will then write $N(n;k)$ as short for $N(n;(k))$. The coefficient $e_j(\alpha)$ becomes simply $e_j(k) = q_j/(k-j)!$, where $j!q_j$ is the number of permutations $w \in \mathfrak{S}_n$ with no cycles of length one or two. By standard enumerative reasoning (see e.g. [12, Exam. 5.2.10]) we have

$$\sum_{j\geq 0} q_j x^j = \frac{e^{-x-\frac{1}{2}x^2}}{1-x}. \qquad (11)$$

From this and Theorems 2.1 and 2.2 it is easy to deduce the following results, which we simply state without proof.

2.4 Corollary. (a) *We have*

$$N(n+k;k) = \sum_{j=0}^{k}\binom{n}{j} t_{n-j} = \sum_{j=0}^{k}\frac{q_j}{(k-j)!}t_{n+k-j},$$

where q_j is given by (11).

(b) *Define polynomials $A_n(x)$ by $A_0(x) = 1$ and*

$$A_{n+1}(x) = A_n'(x) + (x+1)A_n(x), \quad n \geq 0.$$

Then

$$\sum_{k \geq 0} N(n+k;k)x^k = \frac{A_n(x)}{1-x}.$$

(c) *Let*

$$e^{\frac{1}{2}u^2+2u} = \sum_{n \geq 0} b_n \frac{u^n}{n!}.$$

Then $N(n+k;k) = b_n$ if $n \leq k$.

The stability property of Corollary 2.4(c) is easy to see by direct combinatorial reasoning. If $n \leq k$, then a skew SYT of shape λ/α, where $\lambda \vdash n+k$ and $\alpha \vdash k$, consists of a first row containing some j-element subset of $1, 2, \ldots, n$, together with some disjoint SYT U on the remaining $n-j$ letters. There are t_{n-j} possibilities for U, so

$$N(n+k;k) = \sum_{j=0}^{n} \binom{n}{j} t_{n-j},$$

which is equivalent to Corollary 2.4(c).

3 Asymptotics of $f^{\lambda/\alpha}$.

Rather than considering the sum $\sum_{\lambda \vdash n} f^{\lambda/\alpha}$, we could investigate instead the individual terms $f^{\lambda/\alpha}$. The analogue of Theorem 2.1 is the following.

3.1 Theorem. *Let $\alpha \vdash k$ and $n \geq k$. Then for any partition $\lambda \vdash n$ we have*

$$f^{\lambda/\alpha} = \sum_{\nu \vdash k} z_\nu^{-1} \chi^\lambda(\nu, 1^{n-k}) \chi^\alpha(\nu). \tag{12}$$

Proof. The proof parallels that of Theorem 2.1. Instead of the power sum expansion of $\sum_{\lambda\vdash n} s_\lambda$, we need the expansion of s_λ (where $\lambda \vdash n$), given by [6, p. 114][12, Cor. 7.17.5]

$$s_\lambda = \sum_{\mu\vdash n} z_\mu^{-1}\chi^\lambda(\mu)p_\mu.$$

We therefore have

$$\begin{aligned}
f^{\lambda/\alpha} &= \langle p_1^{n-k}, s_{\lambda/\alpha}\rangle \\
&= \left\langle s_\alpha p_1^{n-k}, \sum_{\mu\vdash k} z_\mu^{-1}\chi^\lambda(\mu)p_\mu \right\rangle \\
&= \left\langle s_\alpha, \frac{\partial^{n-k}}{\partial p_1^{n-k}} \sum_{\mu\vdash n} z_\mu^{-1}\chi^\lambda(\mu)p_\mu \right\rangle \\
&= \left\langle s_\alpha, \sum_{\nu\vdash k} z_{(\nu,1^{n-k})}^{-1}\chi^\lambda(\nu,1^{n-k})(n-k+m_1(\nu))_n\, p_\nu \right\rangle \\
&= \sum_{\nu\vdash k} z_{(\nu,1^{n-k})}^{-1}(n-k+m_1(\nu))_n\, \chi^\lambda(\nu,1^{n-k})\chi^\alpha(\nu).
\end{aligned}$$

But

$$z_{(\nu,1^{n-k})}^{-1}(n-k+m_1(\nu))_n = z_\nu^{-1},$$

and the proof follows. $\square$

Theorem 3.1 can also be proved by inverting the formula given in [12, Exer. 7.62].

We would like to regard equation (12) as an asymptotic formula for $f^{\lambda/\alpha}$ when α is fixed and λ is "large." For this we need an asymptotic formula for $\chi^\lambda(\nu,1^{n-k})$ when ν is fixed. Such a formula will depend on the way in which the partitions λ increase. The first condition considered here is the following. Let $\lambda^1, \lambda^2, \ldots$ be a sequence of partitions such that $\lambda^n \vdash n$, and such that the diagrams of the λ^n's, rescaled by a factor $n^{-1/2}$ (so that they all have area one) converge uniformly to some limit ω. (See [2] for a more precise statement.) We will denote this convergence by $\lambda^n \to \omega$. The following result is due to Biane [2], building on work of Vershik and Kerov.

3.2 Theorem. *Suppose that $\lambda^n \to \omega$. Then for $i \geq 2$ there exist constants (defined explicitly in [2]) $C_i(\omega)$, with $C_2(\omega) = 1$, such that for any fixed partition $\nu \vdash k$ of length $\ell(\nu)$ we have*

$$\chi^{\lambda^n}(\nu, 1^{n-k}) = f^{\lambda^n} \left(\prod_{i=0}^{\ell(\nu)} C_{\nu_i+1}(\omega) \right) n^{-\frac{1}{2}(k-\ell(\nu))} \left(1 + O(1/n)\right),$$

as $n \to \infty$.

Let $c_\nu = z_\nu^{-1} \chi^\lambda(\nu, 1^{n-k}) \chi^\alpha(\nu)$. It follows from Theorem 3.2 that $c_{(21^{k-2})} = O\left(c_{(1^k)} n^{-1/2}\right)$, while $c_\nu = O\left(c_{(1^k)} n^{-1}\right)$ and $c_\nu = O\left(c_{(21^{k-2})} n^{-1/2}\right)$ for $\ell(\nu) \leq k-2$. Hence if $\lambda^n \to \omega$ then

$$\begin{aligned} f^{\lambda^n/\alpha} &= \left(z_{(1^k)}^{-1} \chi^{\lambda^n}(1^n) \chi^\alpha(1^k) + z_{(21^{k-2})}^{-1} \chi^{\lambda^n}(21^{n-2}) \chi^\alpha(21^{k-2}) \right) \left(1 + O(1/n)\right) \\ &= f^{\lambda^n} \left(\frac{1}{k!} f^\alpha + \frac{1}{2(k-2)!} C_3(\omega) \chi^\alpha(21^{k-2}) \frac{1}{\sqrt{n}} + O(1/n) \right). \end{aligned} \tag{13}$$

Let us note that by [6, p. 118][12, Exer. 7.51] the integer $\chi^\alpha(21^{k-2})$ appearing in (13) has the explicit value

$$\chi^\alpha(21^{k-2}) = f^\alpha \frac{\sum \binom{\alpha_i}{2} - \sum \binom{\alpha_i'}{2}}{\binom{k}{2}}.$$

The leading term of the right-hand side of (13) is independent of ω, and in fact it follows from [2] that $f^{\lambda^n/\alpha} \sim \frac{1}{k!} f^\alpha f^{\lambda^n}$ holds under the weaker hypothesis that there exists a constant $A > 0$ for which $\lambda_1^n < A\sqrt{n}$ and $\ell(\lambda^n) < A\sqrt{n}$ for all $n \geq 1$.

Given $\epsilon > 0$, let

$$\begin{aligned} \mathrm{Par}_\epsilon(n) = \{\lambda \vdash n \,:\, & (2-\epsilon)\sqrt{n} < \lambda_1 < (2+\epsilon)\sqrt{n} \\ & \text{and } (2-\epsilon)\sqrt{n} < \ell(\lambda) < (2+\epsilon)\sqrt{n}\}. \end{aligned}$$

It is a consequence of the work of Logan and Shepp [5] or Vershik and Kerov [13] (see e.g. [1] for much stronger results) that for any $\epsilon > 0$,

$$\sum_{\lambda \in \mathrm{Par}_\epsilon(n)} f^\lambda \sim t_n, \quad n \to \infty.$$

Thus not only is the sum $N(n;\alpha) = \sum_{\lambda \vdash n} f^{\lambda/\alpha}$ asymptotic to $f^\alpha t_n/k!$ as $n \to \infty$ (as follows from (7)), but the terms $f^{\lambda/\alpha}$ contributing to "most" of the sum are "close" to $f^\alpha f^\lambda/k!$.

Another way of letting λ become large was considered by Vershik and Kerov in [14] and in many subsequent papers (after first being introduced by Thoma). Let $\lambda^1, \lambda^2, \ldots$ be a sequence of partitions such that $\lambda^n \vdash n$ and such that for all $i > 0$, there exist real numbers $a_i \geq 0$ and $b_i \geq 0$ satisfying $\sum_i (a_i + b_i) = 1$ and

$$\begin{aligned} \lim_{n\to\infty} \frac{\lambda_i^n}{n} &= a_i \\ \lim_{n\to\infty} \frac{(\lambda^n)_i'}{n} &= b_i, \end{aligned}$$

where $(\lambda^n)_i'$ denotes the ith part of the conjugate partition to λ^n (i.e., the length of the ith column of the diagram of λ^n). We denote this situation by $\lambda^n \xrightarrow{\text{TVK}} (a;b)$, where $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$. For instance, if $\lambda^{2n} = (n,n)$ and $\lambda^{2n-1} = (n, n-1)$, then $\lambda^n \xrightarrow{\text{TVK}} ((1/2, 1/2, 0, \ldots); (0, 0, \ldots))$.

The following result is immediate from [14].

3.3 Theorem. *Let $\lambda^n \xrightarrow{\text{TVK}} (a;b)$. Then for any fixed partition $\nu \vdash k$,*

$$\chi^{\lambda^n}(\nu, 1^{n-k}) = f^{\lambda^n} \prod_{j=1}^{\ell(\nu)} \left(\sum_i \alpha_i^{\nu_j} + (-1)^{\nu_j - 1} \sum_i \beta_i^{\nu_j} \right) (1 + O(1/n)).$$

It follows that from Theorems 3.1 and 3.3 we have for fixed $\alpha \vdash k$ the asymptotic formula

$$f^{\lambda^n/\alpha} = f^{\lambda^n} \left[\sum_{\nu \vdash k} z_\nu^{-1} \prod_{j=1}^{\ell(\nu)} \left(\sum_i \alpha_i^{\nu_j} + (-1)^{\nu_j - 1} \sum_i \beta_i^{\nu_j} \right) \right] (1 + O(1/n)). \quad (14)$$

Now let $s_\lambda(x \,/\, y)$ denote the super-Schur function indexed by $\lambda \vdash n$ in the variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ [6, Exam. I.23–I.24], defined by $s_\lambda(x \,/ -y) = \omega_y s_\lambda(x, y)$ (where ω_y denotes the standard involution ω acting on the y-variables only). (Note that our $s_\lambda(x \,/\, y)$ corresponds to $s_\lambda(-y \,/\, x)$ in [6].) It follows that the expansion of $s_\lambda(x \,/\, y)$ in terms of power sums is given by

$$s_\lambda(x \,/\, y) = \sum_{\nu \vdash n} z_\lambda^{-1} \chi^\lambda(\nu) \left(p_\nu(x) - p_\nu(y) \right).$$

Hence from equation (14) we obtain the following result.

3.4 Theorem. *Let* $\lambda^n \xrightarrow{\text{TVK}} (a; b)$. *Then for a fixed partition* α *we have*

$$f^{\lambda^n/\alpha} = f^{\lambda^n} s_\alpha(a \,/ -b)(1 + O(1/n)).$$

An explicit statement of Theorem 3.4 does not seem to have been published before. However, it was known by Vershik and Kerov and appears in the unpublished doctoral thesis of Kerov. It is also a simple consequence of Okounkov's formula [9, Thm. 8.1] for $f^{\lambda/\alpha}$ in terms of shifted Schur functions. The asymptotics of shifted Schur functions is carried out (in slightly greater generality) in [3, Thm. 8.1 and Cor. 8.1]. A special case of Theorem 3.4 appears in [10, Thm. 1.3].

Theorem 3.4 can be made more explicit in certain cases for which the super-Schur function $s_\alpha(a \,/ -b)$ can be explicitly evaluated. In particular, suppose that α consists of an $i \times j$ rectangle with a shape $\mu = (\mu_1, \ldots, \mu_i)$ attached at the right and the conjugate ν' of a shape $\nu = (\nu_1, \ldots, \nu_j)$ attached at the bottom. Thus

$$\alpha = (\mu_1 + j, \ldots, \mu_i + j, \nu_1', \nu_2', \ldots).$$

Then (e.g., [4, pp. 115–118][6, (4) on p. 59])

$$s_\alpha(a_1,\ldots,a_i\,/\,-b_1,\ldots,-b_j) = s_\mu(a_1,\ldots,a_i)s_\nu(b_1,\ldots,b_j)\prod_{i,j}(a_i+b_j).$$

In certain cases we can explicitly evaluate $s_\mu(a_1,\ldots,a_i)$ or $s_\nu(b_1,\ldots,b_j)$, e.g., when $a_1=\cdots=a_i$ or $b_1=\cdots=b_j$. See [12, Thm. 7.21.2 and Exer. 7.32]. Note also that when $\mu=\nu=\emptyset$ (so $\alpha=(j^i)$) we have simply

$$s_{(j^i)}(a_1,\ldots,a_i\,/\,-b_1,\ldots,-b_j) = \prod_{i,j}(a_i+b_j).$$

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