Ordering Events in Minkowski Space

Richard P. Stanley[1]
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
e-mail: rstan@math.mit.edu

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Abstract

Let $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ be k points (events) in $(n+1)$-dimensional Minkowski space $\mathbb{R}^{1,n}$. Using the theory of hyperplane arrangments and chromatic polynomials, we obtain information on the number of different orders in which the events can occur in different reference frames if the events are sufficiently generic. We consider the question of what sets of orderings of the points are possible and show a connection with sphere orders and the allowable sequences of Goodman and Pollack.

1 Introduction.

Let $\mathbb{R}^{1,n}$ denote $(n+1)$-dimensional Minkowski space. We denote points (or *events*) in $\mathbb{R}^{1,n}$ as $\boldsymbol{p} = (t, \boldsymbol{x})$, where $t \in \mathbb{R}$ is the time coordinate and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are the space coordinates. The geometry of $\mathbb{R}^{1,n}$ is defined by the *Minkowski norm*

$$\begin{aligned} |(t, \boldsymbol{x})|^2 &= t^2 - |\boldsymbol{x}|^2 \\ &= t^2 - (x_1^2 + \cdots + x_n^2). \end{aligned}$$

An event $(t, \boldsymbol{x})$ is said to be *timelike* if $|(t, \boldsymbol{x})|^2 > 0$, *lightlike* if $|(t, \boldsymbol{x})| = 0$, and *spacelike* if $|(t, \boldsymbol{x})|^2 < 0$. Two events $(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ are *timelike separated* if their difference $(s, \boldsymbol{x}) - (t, \boldsymbol{y})$ is timelike, and similarly *lightlike separated* and *spacelike separated*. Two events are timelike separated if and only if they are *causally related*;

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a signal (traveling slower than $c = 1$, where c denotes the speed of light) can reach one event from the other.

Now suppose that F' is a second reference frame, moving with constant velocity $\boldsymbol{v} \in \mathbb{R}^n$ with respect to the original frame F. By convention an observer in the frame F' measures coordinates $(t', \boldsymbol{x'})$, synchronized so that $t = t' = 0$ when the two frames coincide. Write

$$\boldsymbol{v} = (\tanh \rho)\boldsymbol{u},$$

where $\boldsymbol{u}$ is a unit vector and $\tanh \rho = |\boldsymbol{v}| < 1$. The *Lorentz transformation* expresses the F' coordinates $(t', \boldsymbol{x'})$ in terms of the F coordinates $(t, \boldsymbol{x})$. All we need here is the formula for t':

$$t' = (\cosh \rho)t - (\sinh \rho)\boldsymbol{x} \cdot \boldsymbol{u},$$

where $\boldsymbol{x} \cdot \boldsymbol{u}$ is the ordinary dot product of two vectors in $\mathbb{R}^n$.

It is easy to verify from the Lorentz transformation that two timelike separated events occur in the same order for any observers (i.e., in any reference frame F'). (Otherwise, in fact, causality would be violated.) On the other hand, two spacelike separated events can always occur in either order in suitable reference frames. These facts for two events suggest generalizing to more events.

Main problem. Given k events in $\mathbb{R}^{1,n}$ in what different orders can they occur for different observers? How many such orders are there?

For instance, given three events $\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}$ in $\mathbb{R}^{1,1}$, we will see in Section 4 that there do not exist three observers for which the events occur in the orders $\boldsymbol{p_1} < \boldsymbol{p_2} < \boldsymbol{p_3}$, (i.e., $\boldsymbol{p_1}$ before $\boldsymbol{p_2}$ before $\boldsymbol{p_3}$ in time) $\boldsymbol{p_2} < \boldsymbol{p_3} < \boldsymbol{p_1}$, and $\boldsymbol{p_3} < \boldsymbol{p_1} < \boldsymbol{p_2}$. On the other hand, given any two permutations π_1 and π_2 of $[k] = \{1, 2, \ldots, k\}$, there exist k events $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k} \in \mathbb{R}^{1,1}$ and two observers F_1, F_2 such that for F_i the events occur in the order π_i, $1 \leq i \leq 2$. In general, we write $\mathcal{O}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ for the number of different orders in which the events $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ occur for different observers.

2 Generic spacelike separated events.

In this section we assume that all the events $\boldsymbol{p_i}$ are spacelike separated. It is easy to construct such events, e.g., choose any events with different space coordinates and dilate their space coordinates by a sufficiently large factor. Let $(t_i, \boldsymbol{x_i})$ be the coordinates of $\boldsymbol{p_i}$ with respect to a fixed observer. An observer moving at velocity $\boldsymbol{v} = (\tanh\rho)\boldsymbol{u}$ sees $\boldsymbol{p_i}$ occur at time

$$t_i' = (\cosh\rho)t_i - (\sinh\rho)\boldsymbol{x_i}\cdot\boldsymbol{u}.$$

Hence $\boldsymbol{p_i}$ occurs simultaneously to $\boldsymbol{p_j}$ for this observer if $t_i' = t_j'$, i.e.,

$$(\cosh\rho)t_i - (\sinh\rho)\boldsymbol{x_i}\cdot\boldsymbol{u} = (\cosh\rho)t_j - (\sinh\rho)\boldsymbol{x_j}\cdot\boldsymbol{u}.$$

Equivalently,

$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j})\cdot\boldsymbol{v}.$$

The set of such velocities $\boldsymbol{v}$ forms a hyperplane in $\mathbb{R}^n$. The different sides of this hyperplane determine whether $\boldsymbol{p_i}$ occurs before or after $\boldsymbol{p_j}$. In order for the ordering of points determined by a region R to correspond to an actual observer, we must have $|\boldsymbol{v}| < 1$ for some $\boldsymbol{v} \in R$. Thus we obtain the following result.

2.1 Theorem. *The number $\mathcal{O}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ is equal to the number of regions R of the hyperplane arrangement $\mathcal{A} = \mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ with hyperplanes H_{ij} given by*

$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j})\cdot\boldsymbol{v}, \quad 1 \le i < j \le k, \tag{1}$$

such that $|\boldsymbol{v}| < 1$ for some $\boldsymbol{v} \in R$.

For basic results about hyperplane arrangements and their number of regions, see e.g. [10][13]. In general we denote the number of regions of an arrangement $\mathcal{A}$ by $r(\mathcal{A})$.

In general, there is no simple formula for the number of regions of the arrangements $\mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$. There are general formulas from the theory of arrangements for $r(\mathcal{A})$ for any (finite) arrangement $\mathcal{A}$ (e.g., equation (2)), but such formulas do not shed much further light per se on the main problem. If, however, we assume that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$

are generic (in a sense to be made precise), then more can be said. Moreover, for fixed k the quantity $r(\mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}))$ is maximized when $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are generic.

We now review a fundamental result of Zaslavsky [10, Thm. 2.3.21] [13, Thm. 2.5][17] for computing the number of regions of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$. The *intersection poset* $L_{\mathcal{A}}$ of $\mathcal{A}$ is the set of all *nonempty* intersections of hyperplanes in $\mathcal{A}$, ordered by reverse inclusion. We always include the ambient space $\mathbb{R}^n$ as the bottom element $\hat{0}$ of $L_{\mathcal{A}}$. The *Möbius function* $\mu : L_{\mathcal{A}} \to \mathbb{Z}$ of $L_{\mathcal{A}}$ is defined recursively by $\mu(\hat{0}) = 1$, and for all $y > \hat{0}$ in $L_{\mathcal{A}}$,

$$\sum_{x \leq y} \mu(x) = 0.$$

(Usually μ is defined on *intervals* of $L_{\mathcal{A}}$, not elements, so our $\mu(x)$ corresponds to $\mu(\hat{0}, x)$.) The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ is defined by

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) t^{\dim(x)},$$

where $\dim(x)$ refers to the dimension of x as an affine subspace of $\mathbb{R}^n$. Zaslavsky's theorem then states that

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1). \tag{2}$$

Since it is known that $(-1)^{n-\dim x}\mu(x) > 0$, equation (2) can be restated as

$$r(\mathcal{A}) = c_0 + c_1 + \cdots + c_n, \tag{3}$$

where $\chi_{\mathcal{A}}(t) = c_0 t^n - c_1 t^{n-1} + \cdots + (-1)^n c_n$.

2.2 Example. An example relevant to our results is the *braid arrangement* $\mathcal{B}_k$ in $\mathbb{R}^k$, with hyperplanes

$$z_i = z_j, \quad 1 \leq i < j \leq k.$$

The intersection poset $L_{\mathcal{B}_k}$ of $\mathcal{B}_k$ is just the lattice of partitions of the set $[k] = \{1, 2, \ldots, k\}$, ordered by refinement. (See [12, Exam. 3.10.4] for a discussion of this lattice.) A partition such as 134–26–5 (i.e., the partition with blocks $\{1,3,4\}$, $\{2,6\}$, $\{5\}$) corresponds to the intersection $z_1 = z_3 = z_4$, $z_2 = z_6$. The characteristic polynomial of $\mathcal{B}_k$ is given by

$$\begin{aligned}\chi_{\mathcal{B}_k}(t) &= t(t-1)\cdots(t-k+1) \\ &= \sum_{i=1}^{k}(-1)^{k-i}c(k,i)t^i, \qquad (4)\end{aligned}$$

where $c(k,i)$ denotes a *signless Stirling number of the first kind* (the number of permutations of $[k]$ with i cycles) [12, §1.3]. We set $c(k,i) = 0$ if $i < 1$ or $i > k$.

We now define what we mean for the spacelike separated events $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ in $\mathbb{R}^{1,n}$ to be generic. We state the condition in terms of the hyperplanes H_{ij} given by (1). These hyperplanes are defined for $i < j$, but we extend this definition to all $i \neq j$, so $H_{ij} = H_{ji}$. Namely, $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ are *generic* if (1) no $n+1$ of the hyperplanes H_{ij} intersect (i.e., have nonempty intersection), and (2) the minimal subsets of the H_{ij}'s with $m \leq n$ elements that do intersect have the form $C = \{H_{i_1,i_2}, H_{i_2,i_3}, \dots, H_{i_{m-1},i_m}, H_{i_m,i_1}\}$, where $i_1, i_2, \dots, i_m$ are distinct. Note that such sets C do indeed intersect. It is easy to see that "almost all" k-element sequences $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ of spacelike separated points are generic, i.e., those that aren't form a set of measure 0 in the space of all k-tuples of spacelike separated points in $\mathbb{R}^{1,n}$.

We come to the main result of this section. If $\boldsymbol{p} = (t, \boldsymbol{x})$ and $a \in \mathbb{R}$, then write $\boldsymbol{p}(a) = (t, a\boldsymbol{x})$.

2.3 Theorem. *Let $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ be k spacelike separated events in $\mathbb{R}^{1,n}$. Then*

$$\mathcal{O}(\boldsymbol{p_1}, \dots, \boldsymbol{p_k}) \leq c(k,k) + c(k,k-1) + \cdots + c(k,k-n). \qquad (5)$$

Moreover, we have

$$\mathcal{O}(\boldsymbol{p_1}(a), \dots, \boldsymbol{p_k}(a)) = c(k,k) + c(k,k-1) + \cdots + c(k,k-n) \qquad (6)$$

for sufficiently large a if and only if $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ are generic. In particular, if $n \geq k-1$ and $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ are generic, then

$$\mathcal{O}(\boldsymbol{p_1}(a), \dots, \boldsymbol{p_k}(a)) = k!$$

for sufficiently large a, i.e., the events can occur in any order.

Proof. Suppose that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are generic. Consider the intersection poset $L_{\mathcal{A}}$ of the arrangement $\mathcal{A} = \mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$. By genericity, no $n+1$ of the hyperplanes intersect, and the minimal subsets that do intersect have the form $\{H_{i_1,i_2}, H_{i_2,i_3}, \ldots, H_{i_{m-1},i_m}, H_{i_m,i_1}\}$, where $m \leq n$. This is exactly the same as for the braid arrangement $\mathcal{B}_k$, where H_{ij} corresponds to the hyperplane $z_i = z_j$, except there is no condition that $m \leq n$ for $\mathcal{B}_k$. It follows that the elements of $L_{\mathcal{A}}$ correspond to partitions of $[k]$ with at least $k-n$ blocks, with the partial ordering on $L_{\mathcal{A}}$ corresponding to refinement of partitions. Hence $L_{\mathcal{A}}$ is isomorphic to the rank n *truncation* of Π_k, i.e., the subposet of Π_k consisting of all elements of rank at most n (where the rank of a partition π with j blocks is $k-j$). By (4) we have

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c(k, k-i) t^{n-i}.$$

Hence it follows from Theorem 2.1 and (3) that

$$\mathcal{O}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}) \leq r(\mathcal{A}) = c(k,k) + c(k,k-1) + \cdots + c(k,k-n).$$

Now replacing each point $\boldsymbol{p_i}$ with $\boldsymbol{p_i}(a)$ for $a > 0$ dilates the arrangement $\mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ by a factor of $1/a$ and maintains genericity. Hence for a sufficiently large every region of the dilated arrangement $\mathcal{A}(\boldsymbol{p_1}(a), \ldots, \boldsymbol{p_k}(a))$ intersects the open unit ball $|\boldsymbol{v}| < 1$, showing that equation (6) holds for a sufficiently large.

It remains to show that equality cannot hold in (5) if $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are not generic. By Theorem 2.1 it suffices to show that in this case,

$$r(\mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})) < c(k,k) + c(k,k-1) + \cdots + c(k,k-n).$$

Assume that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are any k spacelike separated events in $\mathbb{R}^{1,n}$. Let $\kappa\mathcal{A} = \kappa\mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ denote the *cone* over $\mathcal{A}$ [10, §1.2][13, p. 7], i.e., introduce a new coordinate u and define the hyperplanes of $\kappa\mathcal{A}$ by

$$\begin{aligned} (t_i - t_j)u &= (\boldsymbol{x_i} - \boldsymbol{x_j}) \cdot \boldsymbol{v}, \quad 1 \leq i < j \leq k \\ u &= 0. \end{aligned}$$

It is not hard to see [13, end of §4.2] that $r(\kappa\mathcal{A}) = 2r(\mathcal{A})$. Let L be a linear ordering of the hyperplanes of $\kappa\mathcal{A}$. A *circuit* of $\kappa\mathcal{A}$ is a minimal set of linearly dependent hyperplanes. (A set of hyperplanes is defined to be linearly independent if their normals are linearly independent.) A *broken circuit* is a circuit with its largest hyperplane (in the order L) deleted. It is an immediate consequence of the broken circuit theorem [1, (6.73)][13, Thm. 4.12] and equation (3) that

$$r(\kappa\mathcal{A}) = \#\{S \subseteq \kappa\mathcal{A} \,:\, S \text{ contains no broken circuit}\}. \tag{7}$$

Now suppose that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are any k spacelike separated points in $\mathbb{R}^{1,n}$. Let $\boldsymbol{p'_1}, \ldots, \boldsymbol{p'_k}$ be k generic spacelike separated points in $\mathbb{R}^{1,n}$. Denote the hyperplanes of $\mathcal{A} = \mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ by H_{ij} as in Theorem 2.1, and the corresponding hyperplanes in $\mathcal{A}' = \mathcal{A}(\boldsymbol{p'_1}, \ldots, \boldsymbol{p'_k})$ by H'_{ij}. Let J denote the hyperplane $u = 0$ of $\kappa\mathcal{A}$ and J' the corresponding hyperplane of $\kappa\mathcal{A}'$. If $S \subseteq \kappa\mathcal{A}$, then let $S' = \{H' \,:\, H \in S\}$.

Now let C' be a circuit of $\kappa\mathcal{A}'$. Then C is a linearly dependent subset of $\kappa\mathcal{A}$. Hence if B is a broken circuit of $\kappa\mathcal{A}$, then B' is contained in a broken circuit of $\kappa\mathcal{A}'$. It follows from Theorem 2.3 and equation (7) that

$$\begin{aligned} 2r(\mathcal{A}) &= r(\kappa\mathcal{A}) \\ &\leq r(\kappa\mathcal{A}') \\ &= 2r(\mathcal{A}') \\ &= c(k,k) + c(k,k-1) + \cdots + c(k,k-n). \end{aligned}$$

It remains to show that if $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are not generic, then the above inequality is strict. This is equivalent to showing that there exists a broken circuit B of $\kappa\mathcal{A}$ such that B' contains no broken circuit of $\kappa\mathcal{A}'$. We are free to choose any linear ordering L of $\kappa\mathcal{A}$ that is convenient, and the corresponding linear ordering L' of $\kappa\mathcal{A}'$ (i.e., if $H < K$ in L, then $H' < K'$ in L'). Let C be a circuit of $\kappa\mathcal{A}$ such that C' is not a circuit of $\kappa\mathcal{A}'$. Such a circuit C exists since $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are not generic. Thus C' is linearly independent. Let

$$X' = \{H' \in \kappa\mathcal{A}' \,:\, C' \cup \{H'\} \text{ contains a circuit}\}.$$

Choose L' so that all elements of X' come before all elements of C'. Then C' contains no broken circuit of $\kappa\mathcal{A}'$ with respect to L'. Hence if D is a broken circuit of $\kappa\mathcal{A}$ contained in C (e.g., we can always take D to be C minus its largest element), then D' contains no broken circuit of $\kappa\mathcal{A}'$, completing the proof. $\square$

NOTE. The above argument extends to any matroid and shows the following. (For matroid theory terminology, see e.g. [15].) Let M be a (finite) matroid with characteristic polynomial $t^m - a_1t^{m-1} + \cdots + (-1)^m a_m$. Let N be a weak map image of M with characteristic polynomial $t^n - b_1t^{n-1} + \cdots + (-1)^n b_n$. Then $b_i \le a_i$ for all i. This result is essentially known [8, Props. 7.3, 7.4], but since it is only given in the case $\mathrm{rank}(M) = \mathrm{rank}(N)$ (i.e., $m = n$) we have provided the above proof.

3 Timelike separated events.

We consider in this section the case where some of the events are timelike separated. Recall that if $(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ are timelike separated, then they occur in the same order in all reference frames. In that case, solutions $\boldsymbol{v}$ to $s - t = (\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{v}$ satisfy $|v| > 1$, so these hyperplanes can be ignored since they are physically meaningless. Thus let $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ be *any* k events in $\mathbb{R}^{1,n}$. For convenience assume no two are lightlike separated. Define the *separation graph* $G = G(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ to be the (undirected) graph on the vertex set $V(G) = [k]$ with edge set

$$E(G) = \{ij \, : \, \boldsymbol{p_i}, \boldsymbol{p_j} \text{ are spacelike separated}\}.$$

The following extension of Theorem 2.1 is clear.

3.1 Theorem. *Let $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ be events in $\mathbb{R}^{1,n}$ with separation graph G. Then $\mathcal{O}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ is equal to the number of regions R of the hyperplane arrangement $\mathcal{A} = \mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ with hyperplanes H_{ij} given by*

$$t_i - t_j = (\boldsymbol{x_i} - \boldsymbol{x_j}) \cdot \boldsymbol{v}, \quad ij \in E(G),$$

such that $|\boldsymbol{v}| < 1$ for some $\boldsymbol{v} \in R$.

Let G be a graph with $V(G) = [k]$. The *graphical arrangement* $\mathcal{B}_G$ is the hyperplane arrangement in $\mathbb{R}^k$ with hyperplanes $z_i = z_j$ for $ij \in E(G)$. For instance, if $G = K_k$ (the complete graph on $[k]$), then $\mathcal{B}_{K_k} = \mathcal{B}_k$, the braid arrangement. Let $\chi_G(t)$ denote the *chromatic polynomial* of G, i.e., for $m \in \mathbb{P}$, $\chi_G(m)$ is the number of ways to color the vertices of G from a set of m colors such that adjacent vertices have different colors. It is well-known [10, Thm. 2.4.19][13, Thm. 2.7] that the characteristic polynomial of $\mathcal{B}_G$ is given by

$$\chi_{\mathcal{B}_G}(t) = \chi_G(t).$$

Fix a graph G on the vertex set $[k]$, and assume that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ satisfy $G = G(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ but are otherwise generic. Then just as for the case $G = K_k$, we have that $L_{\mathcal{A}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})}$ is the rank n truncation of $L_{\mathcal{B}_G}$. We obtain just as for Theorem 2.3 (the special case $G = K_k$) the following result.

3.2 Theorem. *Let $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k} \in \mathbb{R}^{1,n}$ be k events in $\mathbb{R}^{1,n}$ with separation graph G. Let $\chi_G(t) = t^k - a_1 t^{k-1} + \cdots + (-1)^{k-1} a_{k-1} t$. Set $a_i = 0$ if $i \geq k$. Then*

$$\mathcal{O}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}) \leq 1 + a_1 + a_2 + \cdots + a_n, \tag{8}$$

with equality only if $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ are generic (with respect to having separation graph G).

Unlike Theorem 2.3 we don't necessarily have equality holding in (8) for generic $\boldsymbol{p_i}(a)$ and a sufficiently large, because the transformation $\boldsymbol{p} \mapsto \boldsymbol{p}(a)$ for large a will not preserve the separation graph. Timelike separated points will become spacelike separated. The following problem is therefore suggested.

Problem. Characterize those graphs G for which there exist events $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ in $\mathbb{R}^{1,n}$ with separation graph G such that

$$\mathcal{O}(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}) = 1 + a_1 + a_2 + \cdots + a_n,$$

where $\chi_G(t) = t^k - a_1 t^{k-1} + \cdots + (-1)^{k-1} a_{k-1} t$.

Theorems 3.1 and 3.2 and the above problem suggest the problem of characterizing separation graphs of subsets of $\mathbb{R}^{1,n}$. This problem has been considered previously, and we briefly summarize known results. Given $\boldsymbol{p} = (s, \boldsymbol{x}) \in \mathbb{R}^{1,n}$, define the (open) *future light cone* $C(\boldsymbol{p})$ to consist of all points $\boldsymbol{q} = (t, \boldsymbol{y}) \in \mathbb{R}^{1,n}$ such that (1) $t > s$ and (2) $\boldsymbol{p}$ and $\boldsymbol{q}$ are timelike separated. Equivalently,

$$C(\boldsymbol{p}) = \{(t, \boldsymbol{y}) \in \mathbb{R}^{1,n} \,:\, t - s > |\boldsymbol{y} - \boldsymbol{x}|\},$$

a half-cone with apex $\boldsymbol{p}$, slope 45°, and opening in the t-direction. Note that if $\boldsymbol{q} \in C(\boldsymbol{p})$ then $C(\boldsymbol{q}) \subset C(\boldsymbol{p})$. Define $(s, \boldsymbol{x}) < (t, \boldsymbol{y})$ if $s < t$ and if $(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ are timelike separated. It follows that the reflexive closure of the relation $<$ (i,e., define $\boldsymbol{p} \leq \boldsymbol{q}$ if $\boldsymbol{p} < \boldsymbol{q}$ or $\boldsymbol{p} = \boldsymbol{q}$) is a partial order P_n. Any induced subposet of P_n is called a *timelike poset* or *causal poset*. Thus if G is the separation graph of a finite subset S of $\mathbb{R}^{1,n}$, then G is the *incomparability graph* of the restriction of P_n to S. In other words, the vertices of G are the elements of S, with an edge between two vertices if they are incomparable in P_n.

It is a strong restriction on separation graphs to be incomparability graphs. See e.g. [14, §3.2] for some characterizations of incomparability graphs. We may further ask what other conditions are satisfied by separation graphs. Suppose $C(\boldsymbol{p})$ and $C(\boldsymbol{q})$ are future light cones. Intersect them with a hyperplane $t = t_0$ for t_0 large. The intersections are just balls $B(\boldsymbol{p})$ and $B(\boldsymbol{q})$. Moreover, $C(\boldsymbol{p}) \subset C(\boldsymbol{q})$ if and only if $B(\boldsymbol{p}) \subset B(\boldsymbol{q})$. It follows that a finite poset P is a timelike poset in $\mathbb{R}^{1,n}$ if and only if it is an n-dimensional *sphere order*, i.e., isomorphic to a set of spheres in $\mathbb{R}^n$, ordered by inclusion of their interiors. In fact, the concept of sphere orders originally arose in the above context of special relativity [9]. The paper [3] solves a long-standing problem by showing that not all finite posets are sphere orders. In particular, the poset $\boldsymbol{n}^3$ is not a sphere order for n sufficiently large, where $\boldsymbol{n}$ denotes an n-element chain.

For $n = 1$ the situation is much simpler. The timelike posets for $n = 1$ (i.e., events in $\mathbb{R}^{1,1}$) are just the posets of dimension 2, i.e., posets that are an intersection of two chains [9, Prop. 2]. Equivalently, they are subposets of $\mathbb{Z} \times \mathbb{Z}$ (with the usual product ordering). For characterizations of posets of dimension 2, see e.g. [14, §3.3].

4 What permutations of the events are possible?

Let $f(n,k) = c(k,k) + c(k,k-1) + \cdots + c(k,k-n)$. We know from Theorem 2.3 that there exist generic spacelike separated events $\boldsymbol{p_1},\ldots,\boldsymbol{p_k}$ in $\mathbb{R}^{1,n}$ such that $\mathcal{O}(\boldsymbol{p_1},\ldots,\boldsymbol{p_k}) = f(n,k)$. Regard an ordering of these events as a permutation $\pi \in \mathfrak{S}_k$, the symmetric group of all permutations of $[k]$. Thus the k events determine a subset of $\mathfrak{S}_k$ of cardinality $f(n,k)$. We may further ask what subsets of $\mathfrak{S}_k$ of cardinality $f(n,k)$ are possible, and how many such subsets are there? In general this seems to be a difficult question, so we will restrict our attention to the case $n=1$.

Assume then $n=1$, so $\boldsymbol{v} = v \in \mathbb{R}$. Note that $f(1,k) = 1 + \binom{k}{2}$ by Theorem 2.3, since $c(k,k-1) = \binom{k}{2}$. For the remainder of this section we continue to assume that

$$\mathcal{O}(\boldsymbol{p_1},\ldots,\boldsymbol{p_k}) = 1 + \binom{k}{2}. \tag{9}$$

As v increases from -1 to 1, the order of the events will change (as seen from a reference frame moving at velocity v) when v passes through the value

$$v_{ij} = \frac{t_i - t_j}{x_i - x_j},$$

where $\boldsymbol{p_i} = (t_i, x_i)$. We thus get a sequence

$$\Lambda = \left(\pi_0, \pi_1, \ldots, \pi_{\binom{k}{2}}\right),$$

of permutations of $1,2,\ldots,k$ (in agreement with (9) Theorem 2.3 in the case $n=1$). Assume without loss of generality that for $v=0$ the permutation is $12\cdots k$, i.e., $t_1 < t_2 < \cdots < t_k$. Varying the $\boldsymbol{p_i}$'s, the sequence Λ will change when crossing the surface

$$\frac{t_i - t_j}{x_i - x_j} = \frac{t_r - t_s}{x_r - x_s}.$$

Hence the number of different Λ is governed by the arrangement

$$\frac{t_i - t_j}{x_i - x_j} = \frac{t_r - t_s}{x_r - x_s}, \tag{10}$$

$1 \leq i < j \leq k$, $1 \leq r < s \leq k$, of quadric hypersurfaces in $\mathbb{R}^n \times \mathbb{R}^n$. In particular, the number of regions of this arrangement is an upper bound for the number of Λ. We can't be sure that equality holds since we could have two different regions lying on the same side of all the hypersurfaces. Note, however, that if we fix the times $t_1, \ldots, t_k$ (or the points $x_1, \ldots, x_k$), then (10) defines a hyperplane arrangement $\mathcal{D} = \mathcal{D}(t_1, \ldots, t_k)$. Thus in this situation the number of different Λ is just $r(\mathcal{D})$. In general it seems difficult to compute this number. A special case was considered in a different context in [7]; see the paragraph after Theorem 5.2 below.

4.1 Example. Let $(\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}, \boldsymbol{p_4}) = ((0,1),(1,6),(2,4),(3,11))$. Then for instance

$$v_{12} = \frac{1-0}{6-1} = \frac{1}{5},$$

and we obtain

$$v_{23} < 0 < v_{34} < v_{12} < v_{14} < v_{24} < v_{13}.$$

Hence $\Lambda = (1324, 1234, 1243, 2143, 2413, 4213, 4231)$.

If $\Lambda = \left(\pi_0, \pi_1, \ldots, \pi_{\binom{k}{2}}\right)$, then π_{i+1} differs from π_i by an adjacent transposition. Hence

$$\Lambda = \pi_0 \cdot \left(\rho_0, \rho_1, \ldots, \rho_{\binom{k}{2}}\right) = \left(\pi_0\rho_0, \pi_0\rho_1, \ldots, \pi_0\rho_{\binom{k}{2}}\right),$$

where some $\rho_i = \pi_0^{-1}$, and $\left(\rho_0, \rho_1, \ldots, \rho_{\binom{k}{2}}\right)$ is a maximal chain in the *weak (Bruhat) order* of $\mathfrak{S}_k$ (see e.g. [2][4][11]). This means that $\rho_0 = 123\cdots k$ (the identity permutation), $\rho_{\binom{k}{2}} = k\cdots 21$ (the permutation $w_0 \in \mathfrak{S}_k$ of longest length, i.e., with the most number of pairs out of order), and for all $1 \leq i \leq \binom{k}{2}$ we have $\rho_i = s_{a_i}\rho_{i-1}$ for some adjacent transposition $s_{a_i} = (a_i, a_i + 1)$. It is well-known (see the previous three references) that the number of maximal chains in the weak order of $\mathfrak{S}_k$ is equal to the number of standard Young tableaux $f^{(k-1,k-2,\ldots,1)}$ of shape $(k-1, k-2, \ldots, 1)$, given by

$$f^{(k-1,k-2,\ldots,1)} = \frac{\binom{k}{2}!}{1^{k-1}\, 3^{k-2}\, 5^{k-3} \cdots (2k-3)^1}.$$

Since $\pi_0 = \rho^{-1}$ for some i, this gives an upper bound of

$$\left(1 + \binom{k}{2}\right) f^{(k-1,k-2,\ldots,1)}$$

for the number of possible $\Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$. Note that if the chain $\left(\rho_0, \ldots, \rho_{\binom{k}{2}}\right)$ in the weak order is achievable, then so is $\sigma \cdot \left(\rho_0, \ldots, \rho_{\binom{k}{2}}\right)$ whenever $\sigma = \rho_i^{-1}$ for some i, since σ simply specifies which reference frame (or velocity v) we regard as the rest frame ($v = 0$). Thus for the problem of characterizing the possible sequences $\Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}) = \left(\pi_0, \ldots, \pi_{\binom{k}{2}}\right)$, we may assume that $\pi_0 = 12 \cdots k$ or equivalently, $\Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ is a maximal chain in the weak order of $\mathfrak{S}_k$.

When $k = 3$, it is easy to find examples of all eight sequences $\sigma \cdot (\rho_0, \rho_1, \rho_2, \rho_3)$ (or of the two maximal chains in the weak order of $\mathfrak{S}_3$). To be concrete, these sequences are

$$\begin{gathered}(123, 132, 312, 321),\ (213, 123, 132, 312),\ (231, 213, 123, 132)\\ (321, 231, 213, 123),\ (321, 312, 132, 123),\ (312, 132, 123, 213\\ (132, 123, 213, 231),\ (123, 213, 231, 321).\end{gathered}$$

In particular, none of these sequences contain all three of 123, 231, 312, thereby justifying the assertion made at the end of Section 1.

When $k = 4$ it can also be checked that all 16 maximal chains in the weak order of $\mathfrak{S}_4$ are achievable. However, for $k = 5$ not all maximal chains occur (see equation (11)). This can be seen by rephrasing the question of characterizing $\Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$ in terms of earlier work of Goodman and Pollack. Regard the events $\boldsymbol{p_i} = (t_i, x_i)$ as vectors in $\mathbb{R}^2$. The order of the events will be the order they appear when orthogonally projected to a line $x = C$. We regard this line as having slope 0. The velocity $v_{ij} = (t_i - t_j)/(x_i - x_j)$ at which the order of the events $\boldsymbol{p_i}$ and $\boldsymbol{p_j}$ changes is just the reciprocal of the slope of the line through $\boldsymbol{p_i}$ and $\boldsymbol{p_j}$. It follows that the order of the events in the reference frame moving at velocity v is just the order in which they appear when projected to a line of slope $-v$. In other words, as we rotate the line $t = -x$ (slope -1) counterclockwise through an angle of 90° (so it becomes the line $t = x$ of slope 1), the order of the projections of $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ on this line will change each time the line becomes perpendicular to a line through two of the points. Regard an ordering of the points $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k}$ as a permutation $\pi \in \mathfrak{S}_k$. It follows that the sequences $\left(\pi_0, \ldots, \pi_{\binom{k}{2}}\right)$ of permutations $\pi_i \in \mathfrak{S}_k$ obtained in this way from planar configurations of points are exactly the sequences $\Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$. Such sequences were considered by Goodman and Pollack [6] in connection with some problems of discrete geometry. (They considered lines of all slopes σ, not just $-1 < \sigma < 1$, but this does not produce any greater generality because we can replace (t, x) with (t, ax) for $a \gg 0$.) They showed (Theorem 3.3) that all maximal chains in the weak order of $\mathfrak{S}_k$ can occur for $k \leq 4$, but that for $k = 5$ the sequence $(\pi_0, \ldots, \pi_{10})$ is not achievable, where $\pi_i = s_{a_i}\pi_{i-1}$ and

$$(a_1, \ldots, a_{10}) = (1, 3, 4, 2, 1, 3, 4, 2, 1, 3). \tag{11}$$

Hence the same is true for the sequences $\Lambda(\boldsymbol{p_1}, \ldots, \boldsymbol{p_k})$.

5 A classical analogue.

A result of Good and Tideman [5] may be regarded as a special case of Theorem 2.3. We state their result in a form involving classical physics so that it is more analogous to Theorem 2.3, though it really has nothing to do with physics. Suppose that $\boldsymbol{p_1}, \ldots, \boldsymbol{p_k} \in \mathbb{R}^n$ (Euclidean space). At time $t = 0$ each point $\boldsymbol{p_i}$ emits a flash of light. In how many orders can these events be observed from different points $\boldsymbol{x} \in \mathbb{R}^n$? First note the fundamental difference between this question and the situation of Theorem 2.3, viz., now we are concerned with the order in which events are *observed*, not in which they *occur*. (Of course in classical physics, the order in which events occur is the same in all reference frames.)

The events $\boldsymbol{p}$ and $\boldsymbol{q}$ are observed simultaneously at points $\boldsymbol{x}$ on the perpendicular bisector of $\boldsymbol{p}$ and $\boldsymbol{q}$, with equation

$$(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{x} = \frac{1}{2}\left(|\boldsymbol{p}|^2 - |\boldsymbol{q}|^2\right).$$

Hence in analogy with Theorem 2.1 we obtain the following result.

5.1 Theorem. *The number of different orders in which $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ can be observed is the number $r(\mathcal{C})$ of regions of the arrangement $\mathcal{C}$ with hyperplanes*

$$(\boldsymbol{p_i} - \boldsymbol{p_j}) \cdot \boldsymbol{x} = \frac{1}{2}\left(|\boldsymbol{p_i}|^2 - |\boldsymbol{p_j}|^2\right), \quad 1 \le i < j \le n. \tag{12}$$

This arrangement (12) is a special case of equation (1). Moreover, the genericity of $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ in (12) is sufficient for genericity in the sense of Theorem 2.3. We therefore obtain the next result.

5.2 Theorem. *Let $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$ be generic events in $\mathbb{R}^n$ occuring at $t = 0$. Then the number of different orders in which these events can be observed at points $\boldsymbol{x} \in \mathbb{R}^n$ is given by*

$$r(\mathcal{C}) = c(k, k) + c(k, k-1) + \cdots + c(k, k-n).$$

Theorem 5.2 may be restated as determining the number of regions into which $\mathbb{R}^n$ is divided by the perpendicular bisectors of k generic points. This problem was first considered by Good and Tideman [5] in connection with voting theory. They obtained our Theorem 5.2 by a rather complicated induction argument. Zaslavsky [18] corrected an oversight in the proof of Good and Tideman and reproved their result by using standard techniques from the theory of arrangements. Zaslavsky's proof is more complicated than ours, but he works in a more general context. Recently Kamiya, Orlik, Takemura, and Terao [7] considered additional aspects of Theorem 5.2 in an analysis of ranking patterns, in particular, enumerating the number of sets of orders that can occur by varying the points $\boldsymbol{p_1}, \dots, \boldsymbol{p_k}$.

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