Polynomial Coefficient Enumeration

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Abstract Let $f(x_1, \dots, x_k)$ be a polynomial over a field K. This paper considers such questions as the enumeration of the number of nonzero coefficients of f or of the number of coefficients equal to $\alpha \in K^*$. For instance, if $K = \mathbb{F}_q$ then a matrix formula is obtained for the number of coefficients of f^n that are equal to $\alpha \in \mathbb{F}_q^*$, as a function of n. Many additional results are obtained related to such areas as lattice path enumeration and the enumeration of integer points in convex polytopes.

1 Introduction.

Given a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$, how many coefficients of f are nonzero? For $\alpha \in \mathbb{Z}$ and p prime, how many coefficients are congruent to α modulo p? In this paper we will investigate these and related questions. First let us review some known results that will suggest various generalizations.

The archetypal result for understanding the coefficients of a polynomial modulo p is *Lucas' theorem* [7]: if n, k are positive integers with p-ary expansions $n = \sum_{i \geq 0} a_i p^i$ and $k = \sum_{i \geq 0} b_i p^i$, then

$$\binom{n}{k} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \pmod{p}.$$

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Thus, for instance, it immediately follows that the number of odd coefficients of the polynomial $(1+x)^n$ is equal to $2^{s(n)}$, where $s(n)$ denotes the number of 1's in the 2-ary (binary) expansion of n. In particular, the number of odd coefficients of $(1+x)^{2^m-1}$ is equal to 2^m (which can also be easily proved without using Lucas' theorem). In Section 2 we will vastly generalize this result by determining the behavior of the number of coefficients equal to $\alpha \in \mathbb{F}_q$ of multivariable polynomials $f(\boldsymbol{x})^n$ as a function of n for any $f(\boldsymbol{x}) \in \mathbb{F}_q[\boldsymbol{x}]$. A few related results for special polynomials $f(\boldsymbol{x})$ will also be given.

In Sections 3–5 we turn to other multivariate cases. The following items give a sample of our results on sequences of polynomials not of the form $f(\boldsymbol{x})^n$.

1. Extensions of Lucas' theorem and related results. For instance, in Section 3 we determine the number of coefficients of the polynomial $\prod_{i=1}^n (1+x_i+x_{i+1})$ that are not divisible by a given prime p.

2. It is easy to see, as pointed out by E. Deutsch [6], that the number of nonzero coefficients of the polynomial

$$\prod_{i=1}^{n} (x_1 + x_2 + \cdots + x_i) \tag{1}$$

is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. This statement is generalized in Section 4, where we interpret the number of nonzero coefficients of more general polynomials in terms of lattice path counting.

3. A number of examples are known where the nonzero coefficients are in bijection with the lattice points in a polytope. The machinery of counting lattice points can be brought to bear. For example, this technique has been used to show that the number of nonzero coefficients of the polynomial $\prod_{1\le i<j\le n}(x_i+x_j)$ is equal to the number of forests on an n-vertex set (equivalent to the case $n=1$ of [22, Exer. 4.32(a)]). For a more complicated example of this nature, see [21]. In Section 4 we develop a connection between the nonzero coefficients of polynomials of the form $\prod_{i=1}^k (x_1+x_2+\cdots+x^{\lambda_i})$ and a polytope studied by Pitman and Stanley [17].

4. The number of odd coefficients of the polynomial $\prod_{1\le i<j\le n}(x_i+x_j)$ is

equal to $n!$. This result is most easily seen using the trick

$$\prod_{1\le i<j\le n}(x_i+x_j) \equiv \prod_{1\le i<j\le n}(x_i-x_j) \pmod 2$$
$$= \sum_{w\in S_n} \operatorname{sgn}(w) x_{w(1)}^{n-1} x_{w(2)}^{n-2} \cdots x_{w(n-1)},$$

by the expansion of the Vandermonde determinant. An alternative cancellation proof was given by Gessel [10], i.e., pairing off equal monomials among $2^{\binom{n}{2}}$ monomials appearing in the product until $n!$ distinct monomials remain. Some other results along this line appear in Section 5.

2 Powers of polynomials over finite fields

The prototype for the next result is the fact mentioned in Section 1 that the number of odd coefficients of the polynomial $(1+x)^{2^n-1}$ is equal to 2^n.

Let $q = p^r$ where p is a prime and $r \ge 1$. Let $k \ge 1$ and set $\boldsymbol{x} = (x_1, \dots, x_k)$. Fix $f(\boldsymbol{x}) \in \mathbb{F}_q[\boldsymbol{x}]$ and $\alpha \in \mathbb{F}_q^*$. Define $N_\alpha(n)$ to be the number of coefficients of the polynomial $f(\boldsymbol{x})^n$ that are equal to α. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. We use the multivariate notation $\boldsymbol{x}^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$, where $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{N}^k$.

2.1 Theorem. *There exist $\mathbb{Z}$-matrices $\Phi_0, \Phi_1, \dots, \Phi_{q-1}$ of some square size, and there exist a row vector u and a column vector v with the following property. For any integer $n \ge 1$ let $a_0 + a_1 q + \cdots + a_r q^r$ be its base q expansion, so $n = a_0 + a_1 q + \cdots + a_r q^r$ and $0 \le a_i \le q-1$. Then*

$$N_\alpha(n) = u \Phi_{a_r} \Phi_{a_{r-1}} \cdots \Phi_{a_0} v.$$

The vector v and matrices Φ_i do not depend on α.

Proof. Our proof is an adaptation of an argument of Moshe [16, Thm. 1]. Suppose that $a_0, a_1, \dots$ is an infinite sequence of integers satisfying $0 \le a_i \le q-1$. Let $\mathcal{P}'$ be the Newton polytope of f, i.e., the convex hull in $\mathbb{R}^k$ of the exponent vectors of monomials appearing in f, and let $\mathcal{P}$ be the convex hull of $\mathcal{P}'$ and the origin. If $c > 0$, then write $c\mathcal{P} = \{cv : v \in \mathcal{P}\}$. Set $S = (q-1)\mathcal{P} \cap \mathbb{N}^k$ and $r_m = \sum_{i=0}^m a_i q^i$.

Suppose that $f(\boldsymbol{x})^{r_m} = \sum_\gamma c_{m,\gamma}\boldsymbol{x}^\gamma$. We set $f(\boldsymbol{x})^{r_{-1}} = 1$. Let $\mathbb{F}_q^S$ be the set of all functions $F\colon S \to \mathbb{F}_q$. We will index our matrices and vectors by elements of $\mathbb{F}_q^S$ (in some order). Set

$$R_m = \{0, 1, \ldots, q^{m+1} - 1\}^k.$$

For $m \geq -1$, define a column vector ψ_m by letting $\psi_m(F)$ (the coordinate of ψ_m indexed by $F \in \mathbb{F}_q^S$) be the number of vectors $\gamma \in R_m$ such that for all $\delta \in S$ we have $c_{m,\gamma+q^{m+1}\delta} = F(\delta)$. Note that by the definition of S we have $c_{m,\gamma+q^{m+1}\delta} = 0$ if $\delta \notin S$. (This is the crucial finiteness condition that allows our matrices and vectors to have a fixed finite size.) Note also that given m, every point η in $\mathbb{N}^k$ can be written uniquely as $\eta = \gamma + q^{m+1}\delta$ for $\gamma \in R_{m+1}$ and δ in $\mathbb{N}^k$.

For $0 \leq i \leq q-1$ define a matrix Φ_i with rows and columns indexed by $\mathbb{F}_q^S$ as follows. Let $F, G \in \mathbb{F}_q^S$. Set

$$\begin{aligned} g(\boldsymbol{x}) &= f(\boldsymbol{x})^i \sum_{\beta \in S} G(\beta)\boldsymbol{x}^\beta \\ &= \sum_\gamma d_\gamma \boldsymbol{x}^\gamma \in \mathbb{F}_q[\boldsymbol{x}]. \end{aligned}$$

Define the (F, G)-entry $(\Phi_i)_{FG}$ of Φ_i to be the number of vectors $\gamma \in R_0 = \{0, 1, \ldots, q-1\}^k$ such that for all $\delta \in S$ we have $d_{\gamma+q\delta} = F(\delta)$. A straightforward computation shows that

$$\Phi_{a_m}\psi_{m-1} = \psi_m, \quad m \geq 0. \tag{2}$$

Let u_α be the row vector for which $u_\alpha(F)$ is the number of values of F equal to α, and let $n = a_0 + a_1 q + \cdots + a_r q^r$ as in the statement of the theorem. Then it follows from equation (2) that

$$N_\alpha(n) = u\Phi_{a_r}\Phi_{a_{r-1}} \cdots \Phi_{a_0}\psi_{-1},$$

completing the proof. □

2.2 Example. We illustrate the above proof with the simplest possible example, since any more complicated example involves much larger matrices. Take $q = 2$, $k = 1$, $f(x) = x + 1$ and (necessarily) $\alpha = 1$. Then $S = \{0, 1\}$. A function $F\colon S \to \mathbb{F}_2$ will be identified with the binary word $f(0)f(1)$, and our vectors and matrices will be indexed by the words 00, 10, 01, 11 in that order. Take $a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$. Then

$$\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1+x \\
(1+x)^{1+2^1} &= 1+x+x^2+x^3 \\
(1+x)^{1+2^1+0\cdot 2^2} &= 1+x+x^2+x^3 \\
(1+x)^{1+2^1+0\cdot 2^2+2^3} &= 1+x+x^2+x^3+x^8+x^9+x^{10}+x^{11}
\end{aligned}$$

$$\psi_{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \psi_0 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix},\ \psi_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix},\ \psi_2 = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix},\ \psi_3 = \begin{bmatrix} 8 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

$$\Phi_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \Phi_1 = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For instance, to obtain ψ_2, break up $(1+x)^{1+2^1+0\cdot 2^2}$ into $2^{2+1} = 8$ parts according to the congruence class of the exponent modulo $2^{2+1} = 8$: $1+0\cdot x^8$, $x+0\cdot x^9$, $x^2+0\cdot x^{10}$, $x^3+0\cdot x^{11}$, $0\cdot x^4+0\cdot x^{12}$, $0\cdot x^5+0\cdot x^{13}$, $0\cdot x^6+0\cdot x^{14}$, $0\cdot x^7+0\cdot x^{15}$. Four of these "sections" have coefficient sequence 00 and four have 10, so $\psi_2(00) = \psi_2(10) = 4$, $\psi_2(01) = \psi_2(11) = 0$. To get the 01-column of Φ_1 (or the third column using the usual indexing $1, 2, 3, 4$ of the rows and columns), multiply x (corresponding to 01) by $f(x) = 1+x$ to get $x+x^2$. Bisect $x+x^2$ into the two sections $0+x^2$ and $x+0\cdot x^3$. The coefficient sequences of these two sections are 01 and 10. Hence $\Phi_1(01) = \Phi_1(10) = 1$, $\Phi_1(00) = \Phi_1(11) = 0$, and $u = [0, 1, 1, 2]$.

2.3 Corollary. *Preserve the notation of Theorem 2.1, and set*

$$F_{f,\alpha}(t) = \sum_{m\geq 0} N_\alpha(1+q+\cdots+q^{m-1})t^m.$$

Then $F_{f,\alpha}(t)$ is a rational function of t.

Proof. By Theorem 2.1 we have

$$N_\alpha(1+q+\cdots+q^{m-1}) = u\Phi_1^m v.$$

The proof now follows from standard arguments (e.g., [22, Thm. 4.7.2]) from linear algebra. □

Note. It is clear from the proof of Theorem 2.1 that Corollary 2.3 can be considerably generalized. For instance, for any $f(\boldsymbol{x}), g(\boldsymbol{x}) \in \mathbb{F}_q[\boldsymbol{x}] = \mathbb{F}_q[x_1, \ldots, x_k]$, $r \geq 1$, and $\alpha \in \mathbb{F}_q^*$, let $L(m)$ be the number of coefficients of the polynomial $g(\boldsymbol{x})f(\boldsymbol{x})^{(q^{rm}-1)/(q^r-1)}$ equal to α. Then $\sum_{m\geq 0} L(m)t^m$ is a rational function of t.

The examples provided below (working in $\mathbb{F}_2[\boldsymbol{x}]$) demonstrate the conclusion promised by Corollary 2.3. In some cases we give an independent argument to arrive at the generating function.

2.4 Example. Suppose $\boldsymbol{x} = (x_1, \ldots, x_k)$ and let

$$f_n(\boldsymbol{x}) = \left(1 + \sum_{i=1}^{k} x_i + x_1 \sum_{i=2}^{k} x_i^2\right)^{2^n-1}.$$

Then the number of odd coefficients is

$$N_1(f_n(\boldsymbol{x})) = k\cdot(k+1)^n - (k-1)\cdot k^n.$$

Proof. We work in $\mathbb{F}_2[\boldsymbol{x}]$. In particular, using $(a+b)^2 = a^2+b^2$, it is evident that

$$\begin{aligned}
f_n(\boldsymbol{x}) &= ((1+x_2+\cdots+x_k) + x_1(1+x_2^2+\cdots+x_k^2))^{2^n-1} \\
&= \sum_{j=0}^{2^n-1} x_1^j \binom{2^n-1}{j} (1+x_2+\cdots+x_k)^{2^n-1-j}(1+x_2^2+\cdots+x_k^2)^j \\
&= \sum_{j=0}^{2^n-1} x_1^j (1+x_2+\cdots+x_k)^{2^n-1-j}(1+x_2^2+\cdots+x_k^2)^j \\
&= \sum_{j=0}^{2^n-1} x_1^j (1+x_2+\cdots+x_k)^{2^n-1+j}.
\end{aligned}$$

Therefore our enumeration translates to

$$N(f_n(\boldsymbol{x})) = \sum_{j=0}^{2^n-1} N((1+x_2+\cdots+x_k)^{2^n-1+j}). \tag{3}$$

We prove the assertion of this example by inducting on n. The base case $n = 0$ is obvious. Assume its validity for $n-1$.

An argument verifying $N((1+x)^m) = 2^{s(m)}$ (see Introduction) also shows that $N((1+x_2+\cdots+x_k)^m) = k^{s(m)}$, where $s(m)$ denotes the number of 1's in the binary expansion of m. Thus equation (3) becomes

$$N(f_n(\boldsymbol{x})) = \sum_{j=0}^{2^n-1} k^{s(2^n-1+j)}. \tag{4}$$

Applying the following simple facts

$$s(2^n-1+j) = \begin{cases} n, & j = 0 \\ s(2^{n-1}-1+j), & 1 \le j \le 2^{n-1}-1 \\ 1+s(2^{n-1}-1+j'), & 0 \le j' := j-2^{n-1} \le 2^{n-1}-1 \end{cases}$$

in equation (4) produces

$$\begin{aligned} N(f_n(\boldsymbol{x})) &= \sum_{j=0}^{2^{n-1}-1} k^{s(2^n-1+j)} + \sum_{j=2^{n-1}}^{2^n-1} k^{s(2^n-1+j)} \\ &= k^n + \sum_{j=1}^{2^{n-1}-1} k^{s(2^{n-1}-1+j)} + \sum_{j=0}^{2^{n-1}-1} k^{1+s(2^{n-1}-1+j)} \\ &= k^n - k^{n-1} + N(f_{n-1}(\boldsymbol{x})) + kN(f_{n-1}(\boldsymbol{x})). \end{aligned}$$

Then by the induction assumption, we get

$$N(f_n(\boldsymbol{x})) = k^n - k^{n-1} + (k+1)(k(k+1)^{n-1} - (k-1)k^{n-1}) = k(k+1)^n - (k-1)k^n.$$

□

2.5 Example. Denote the forward shift by $E(a_n) = a_{n+1}$.

(i) If $k = 2$ and $f_n = (1 + x_1 + x_2 + x_1 x_2^2)^{2^n-1}$ then

$$N(f_n) = 2 \cdot 3^n - 2^n.$$

The same formula holds for $g_n = (1 + x_1 + x_2^2 + x_1 x_2)^{2^n-1}$.

(ii) When $k = 3$ so $f_n = (1+x_1+x_2+x_3+x_1x_2^2+x_1x_3^2)^{2^n-1}$, then $a_n = N(f_n)$ satisfies

$$(E^2 - 7E + 12)a_n = (E-4)(E-3)a_n = 0.$$

We get $a_n = 3 \cdot 4^n - 2 \cdot 3^n$.

(iii) If $g_n = (1 + x_1 + x_2 + x_3 + x_1x_2^2 + x_2x_3^2)^{2^n-1}$ then $a_n = N(g_n)$ satisfies

$$(E^2 - 7E + 10)a_n = (E-5)(E-2)a_n = 0.$$

We get $a_n = \frac{1}{3}(4 \cdot 5^n - 2^n)$.

(iv) Let $g_n = (1 + x_1 + x_2 + x_3 + x_1x_2^2)^{2^n-1}$. Then $a_n = N(g_n)$ satisfies $(E^2 - 6E + 7)a_n = 0$. Here the eigenvalues are not even rational; i.e. if $c = 3 + \sqrt{2}$ and $b = 1 + \sqrt{2}$ and we write $\overline{\mu}$ for quadratic conjugation in $\mathbb{Z}[\sqrt{2}]$, then

$$a_n = \frac{bc^n + \overline{b}\overline{c}^n}{2}.$$

(v) Let $g_n = (1+x_1+x_2+x_3+x_1x_2+x_1x_3^2)^{2^n-1}$. Now $a_n = N(g_n)$ satisfies $(E^2-7E+8)a_n = 0$. Again the eigenvalues are not even rational; i.e. if $c = \frac{7+\sqrt{17}}{2}$ and $b = 17+5\sqrt{17}$ and we write $\overline{\mu}$ for quadratic conjugation in $\mathbb{Q}[\sqrt{17}]$, then

$$a_n = \frac{bc^n + \overline{b}\overline{c}^n}{34}.$$

(vi) Let $g_n = (1 + x_1 + x_2^2 + x_1x_2^3)^{2^n-1}$, $c = 2 + \sqrt{3}$, and $\overline{c} = 2 - \sqrt{3}$. We have

$$N(g_n) = \frac{c^{n+1} + \overline{c}^{n+1} - 2^n}{3}.$$

(vii) Let $g_n = (1+x_1+x_2+x_1^2x_2^2)^{2^n-1}$. Then $a_n = N(g_n)$ satisfies $(E^4-5E^3+6E^2 - 2E - 4)a_n = 0$ whose eigenvalues are complex. Put $c = \frac{3+\sqrt{17}}{2}$ and $b = 17^2 + 73\sqrt{17}$ with conjugates in $\mathbb{Q}[\sqrt{17}]$. Also let $i = \sqrt{-1}$, then

$$N(g_n) = \frac{bc^n + \overline{b}\overline{c}^n}{442} - \frac{1}{13}(-2+3i)(1+i)^n - \frac{1}{13}(2+3i)(1-i)^n.$$

(viii) Let $g_n = (1 + x_1^2 + x_2^2 + x_1 x_2^3)^{2^n - 1}$. Then

$$\sum_{n \geq 0} N(g_n) z^n = \frac{1 - 2z + 4z^2}{1 - 6z + 12z^2 - 12z^3}.$$

(ix) Let $g_n = (1 + x_1 + x_2 + x_2^2)^{2^n - 1}$. Then

$$\sum_{n \geq 0} N(g_n) z^n = \frac{1 + 2z}{1 - 2z - 4z^2}.$$

In fact $N(g_n) = 2^n F(n+2)$ where $F(n)$ is the Fibonacci sequence; that is, $F(1) = 1, F(2) = 1$, $F(m) = F(m-1) + F(m-2)$ for $m \geq 3$.

(x) The following three statements are immediate from example (ix).

(a) Given $k \in \mathbb{P}$, break up the binary digits of k into maximal strings of consecutive 1's and let the lengths of these strings be the multiset $\boldsymbol{k} = \{k_1, k_2, \ldots\}$. Then we obtain the averaging value

$$\frac{1}{2^n} \sum_{k=0}^{2^n - 1} \prod_{\boldsymbol{k}} \frac{2^{k_i + 2} + (-1)^{k_i + 1}}{3} = F(n+2).$$

(b) The probability of not landing two consecutive heads in a fair toss of n coins is equal to that of finding a 1 in the triangle of coefficients formed by

$$(1 + x_2 + x_2^2)^m \mod 2, \qquad 0 \leq m \leq 2^n - 1,$$

which in fact is $F(n+2)/2^n$.

(c) The generating function

$$\Lambda(z) = \sum_{m=0}^{\infty} N((1 + x_2 + x_2^2)^m) z^m$$

satisfies $\Lambda(z) = (1+z)\Lambda(z^2) + 2z(1+z^2)\Lambda(z^4)$.

2.6 Example. (i) Let $g_n = (1 + x_1 + x_2 + x_2^3)^{2^n - 1}$. Then

$$\sum_{n \geq 0} N(g_n) z^n = \frac{1 + z - 2z^3}{1 - 3z - 2z^2 + 2z^3 + 4z^4}.$$

(ii) Let $g_n = (1 + x_1 + x_2 + x_2^4)^{2^n-1}$. Then

$$\sum_{n\geq 0} N(g_n)z^n = \frac{1 + z + 4z^2 + 2z^3 - 4z^4}{1 - 3z - 2z^3 - 8z^4 + 8z^5}.$$

(iii) If $g_n = (1 + x_1 + x_2 + x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2)^{2^n-1}$, then

$$\sum_{n\geq 0} N(g_n)z^n = \frac{(1-z)^2}{1 - 9z + 23z^2 - 19z^3}.$$

(iv) If $g_n = (1 + x_1 + x_2 + x_3 + x_1x_2^2 + x_2x_1^2)^{2^n-1}$, then

$$\sum_{n\geq 0} N(g_n)z^n = \frac{1 - z + 2z^2 - 4z^3}{1 - 7z + 12z^2 - 12z^3 + 8z^4}.$$

2.7 Example. *Symmetric polynomials.* Consider the Vandermonde-type polynomials in $\mathbb{F}_2[\boldsymbol{x}]$ given by

$$V(k,n) := \prod_{1\leq i<j\leq k} (x_i - x_j)^{2^n-1} \qquad \text{and}$$

$$V'(k,n) := \left(1 + \prod_{1\leq i<j\leq k} (x_i - x_j)\right)^{2^n-1}.$$

We find that

(i) $N(V(2,n)) = 2^n$ and $N(V'(2,n)) = 3^n$.

(ii) If $c = \frac{5+\sqrt{33}}{2}$, $b = 11 + 3\sqrt{33}$ with conjugation in $\mathbb{Q}[\sqrt{33}]$, then

$$N(V(3,n)) = 6 \cdot 4^{n-1}, \qquad N(V'(3,n)) = \frac{bc^n + \overline{b}\overline{c}^n}{22}.$$

(iii) $N(V(4,n)) = 5 \cdot 8^n - 8 \cdot 2^n$.

Consider the special case of Corollary 2.3 that $k = 1$ and $f(x)$ has the form $g(x)^{q-1}$. Thus $f(x)^{(q^n-1)/(q-1)} = g(x)^{q^n-1}$. For $\alpha \in \mathbb{F}_q^*$ let $M_\alpha(m)$ be the number of coefficients of $g(x)^{q^m-1}$ equal to α. We can give more precise information about the linear recurrence satisfied by $M_\alpha(m)$. To give a slightly more general result, we also fix $c \in \mathbb{P}$. Without loss of generality we may assume $g(0) \neq 0$. For $m \in \mathbb{P}$ such that $q^m \geq c$, let $N_\alpha(m)$ denote the number of coefficients of the polynomial $g(x)^{q^m-c}$ that are equal to α.

2.8 Theorem. *(a) There exist periodic functions $u(m)$ and $v(m)$ depending on $g(x), c$, and α, such that*

$$N_\alpha(m) = u(n)q^m + v(m) \tag{5}$$

for m sufficiently large.

(b) Let d be the least positive integer for which $g(x)$ divides $x^{q^m(q^d-1)} - 1$ for some $m \geq 0$. In other words, d is the degree of the extension field of $\mathbb{F}_q$ obtained by adjoining all zeros of $g(x)$. Then the functions $u(m)$ and $v(m)$ have period d (and possibly smaller periods, necessarily dividing d).

(c) Let μ be the largest multiplicity of any irreducible factor (or any zero) of $g(x)$. Then equation (5) holds for all $m \geq \lceil \log_q \mu c \rceil$. In particular, if $g(x)$ is squarefree and $c = 1$, then (5) holds for all $m \geq 0$.

(d) If $g(x)$ is primitive over $\mathbb{F}_q$ and $c = 1$, then $d = \deg f$ and $u(m) = dq^{d-1}/(q^d - 1)$, a constant.

Proof. We have $g(x)^{q^m - c} = g(x^{q^m})/g(x)^c$. Let $g(x) = a_0 + a_1 x + \cdots + a_\delta x^\delta$, and for $0 \leq i < \delta$ set

$$G_{in}(x) = \frac{(a_0 + a_1 x + \cdots + a_i x^i)^{q^m}}{g(x)^c} \tag{6}$$

$$= \frac{a_0 + a_1 x^{q^m} + a_2 x^{2q^m} + \cdots + a_i x^{iq^m}}{g(x)^c} \tag{7}$$

$$= g_{im}(x) + \frac{h_{im}(x)}{g(x)^c}, \tag{8}$$

where $g_{im}(x), h_{im}(x) \in \mathbb{F}_q[x]$ and $\deg h_{im}(x) < c \deg g(x)$. Thus $h_{im}(x)$ is the remainder upon dividing $(a_0 + a_1 x + \cdots + a_i x^i)^{q^m}$ by $g(x)^c$. Hence $h_{im}(x)$ determines the coefficient of x^j in $g(x)^{q^m - c}$ for $iq^m \leq j < (i+1)q^m$. These coefficients will be periodic, with period of the form $\pi = q^s(q^d - 1)$ for some $s \geq 0$. If α occurs k times within each period, then the number of times α occurs as a coefficient of x^j in $g(x)^{q^m - c}$ for $iq^m \leq j < (i+1)q^m$ has the form $kq^m/\pi + v_i(m)$, where $v_i(m)$ depends only on q^m modulo π.

Suppose that $0 \leq l < m$ and $l \equiv m \pmod{d}$. Then

$$G_{im}(x) - G_{il}(x) = \frac{a_1(x^{q^m} - x^{q^l}) + \cdots + a_i(x^{iq^m} - x^{iq^l})}{g(x)^c} \in \mathbb{F}_q[x].$$

Hence if l is large enough so that $g(x)^c$ divides $x^{q^l(q^d-1)}$, then it follows that $h_{il}(x) = h_{im}(x)$. Thus the polynomial $h_{im}(x)$ depends only on the congruence class of m modulo d for $m \geq l$. We can take l to be the least integer such that $g(x)^c$ divides $x^{q^l(q^d-1)}$. Thus l is the least integer for which $\mu c \leq q^l$, i.e., $l = \lceil \log_q \mu c \rceil$.

From the above discussion it follows that the coefficients of $g(x)^{q^m-c}$ are periodic between 1 and x^{q^m-1} (i.e., for the coefficients of $x^0 = 1, x, x^2, \ldots, x^{q^m-1}$), then periodic between x^{q^m} and x^{2q^m-1}, etc. The lengths of these periods can all be taken to be d. (Of course d may not be the length of the *minimal* period). Moreover, the coefficients themselves within each period depend only on m modulo d. If the number of times $\alpha \in \mathbb{F}_q$ appears within each period between x^{iq^m} and $x^{(i+1)q^m-1}$ is k, then the total number of coefficients between x^{iq^m} and $x^{(i+1)q^m-1}$ that are equal to α is kq^m/d plus an error that is periodic with period d. It follows that $N_\alpha(m) = u(m)q^m + v(m)$ for some periodic functions u and v of period d.

Suppose that $g(x)$ is primitive, so we can take $d = q^d - 1$. Let $g(x)$ and $h(x)$ be polynomials of degree less than d. If

$$\frac{g(x)}{g(x)} = \frac{b_0 + b_1 x + \cdots + b_{d-1}x^{d-1}}{1 - x^d},$$

then for some $0 \leq j \leq d-1$ we have

$$\frac{h(x)}{g(x)} = \frac{b_j + b_{j+1}x + \cdots + b_{d-1}x^{d-j-1} + b_0 x^{d-j} + \cdots + b_{j-1}x^{d-1}}{1 - x^d}.$$

Moreover, all elements of $\mathbb{F}_q^*$ occur equally often among $b_0, b_1, \ldots, b_{d-1}$, while 0 occurs one fewer times. Hence each h_{im} has q^{d-1} coefficients equal to $\alpha \in \mathbb{F}_q^*$ (and $q^{d-1} - 1$ coefficients equal to 0). Thus the number of coefficients of h_{im} equal to α has the form $q^{m+d}/d = q^{m+d}/(q^d - 1)$ plus a periodic term. Summing over $0 \leq i \leq d-1$ gives

$$N_\alpha(m) = \frac{dq^{m+d}}{q^d - 1} + \text{periodic term},$$

and the proof follows. □

2.9 Example. (a) Write $[a_0, a_1, \ldots, a_{k-1}]$ for the periodic function $p(m)$ on $\mathbb{Z}$ satisfying $p(m) = a_i$ for $m \equiv i \pmod{k}$. Let $q = 2$, $g(x) =$

$1+x+x^2+x^3+x^4$, and $c=1$. The polynomial $g(x)$ is irreducible over $\mathbb{Z}_2$ but not primitive. It can then be computed that

$$\begin{aligned} N_1(m) &= 2^{m+1} - \frac{2}{5}(-2)^m + \frac{1}{5}[-3,1,3,-1]. \\ &= \frac{1}{5}[8,12]2^m + \frac{1}{5}[-3,1,3,-1]. \end{aligned} \tag{9}$$

In fact, in Theorem 2.8 we can take $d=4$. For m even, the Taylor series expansion (at $x=0$) of each $h_{im}(x)/g(x)$ has two coefficients within each period equal to 1. Hence in this case

$$N_1(m) = 4 \cdot \frac{2}{5} \cdot 2^m + \cdots = \frac{8}{5}2^m + \cdots .$$

If m is odd, then the expansions of $h_{0m}(x)/g(x)$ and $h_{3m}(x)/g(x)$ have two coefficients within each period equal to 1, while $h_{1m}(x)/g(x)$ and $h_{2m}(x)/g(x)$ have four such coefficients equal to 1. Hence in this case

$$N_1(m) = \left(\frac{2}{5} + \frac{4}{5} + \frac{4}{5} + \frac{2}{5}\right) 2^m + \cdots = \frac{12}{5}2^m + \cdots ,$$

agreeing with equation (9).

(b) Let $q=2$, $g(x) = 1+x^2+x^5, c=1$, and $\alpha=1$. Then $g(x)$ is primitive, and we have

$$N_1(m) = \frac{80}{31}2^m + \frac{1}{31}[-49,-67,-41,11,-9].$$

(c) Let $q=2$, $g(x) = 1+x+x^3+x^4+x^5, c=1$, and $\alpha=1$. Then $g(x)$ is primitive, and we have

$$N_1(m) = \frac{80}{31}2^m + \frac{1}{31}[-49,-5,-41,11,-9].$$

Note that $u(m) = 80/31$ for both (b) and (c), as guaranteed by part (d) of the theorem, but the periodic terms $v(m)$ differ (though only for $n \equiv 1 \pmod{5}$).

(d) To illustrate that equation (5) need not hold for all $m \geq 0$, let $q = 2, g(x) = (1+x^2+x^5)^3$, and $\alpha = 1$. Then

$$N_1(m) = \begin{cases} 1, & m=0 \\ 9, & m=1 \\ \frac{168}{31}2^m + \frac{1}{31}[297,-243,-393,-507,-177], & m \geq 2. \end{cases}$$

(e) Some examples for $q = 2$ and $c = 3$: first let $g(x) = 1+x+x^2+x^3+x^4$. Then

$$N_1(m) = 2^{m+1} - \frac{1}{4}(-2)^m + \frac{1}{5}[11, 3, -11, -3], \qquad m \geq 2.$$

If $g(x) = 1 + x^2 + x^5$, then

$$N_1(m) = \frac{60}{31}2^m + \frac{1}{31}[33, -27, -147, -201, -123], \qquad m \geq 2.$$

If $g(x) = 1 + x + x^2 + x^3 + x^4 + x^5$, then

$$N_1(m) = \frac{60}{31}2^m + \frac{1}{31}[-153, 35, -85, -77, -61], \qquad m \geq 2.$$

(f) Two examples for $q = 3$ and $c = 1$. Let $g(x) = 2 + x + x^2$, a primitive polynomial. Then for $m \geq 0$,

$$N_1(m) = \frac{3}{4}3^m + \frac{1}{2} - \frac{1}{4}(-1)^m \tag{10}$$

$$N_2(m) = \frac{3}{4}3^m - \frac{1}{2} - \frac{1}{4}(-1)^m. \tag{11}$$

Let $g(x) = 2 + x^2 + x^3$, an irreducible but not primitive polynomial. Then for $m \geq 0$,

$$N_1(m) = \frac{18}{13}3^m + \frac{1}{13}[-5, 11, 7] \tag{12}$$

$$N_2(m) = \frac{9}{13}3^m - \frac{1}{13}[9, 14, 3]. \tag{13}$$

(g) A class of function for which $u(m)$ is independent of m. Let $q = 2, c = 1, \alpha = 1$ and $g(x) = 1 + x^{k-1} + x^k$.

If $k = 2^h$, then

$$u(n) = \frac{k(3^h - 1)}{k^2 - 1}.$$

If $k = 2^h + 1$, then

$$u(n) = \frac{k(k-2)(3^h + 1)}{2^{3h} - 1}.$$

If $k = 2^h - 1$ and $d_1, \ldots, d_r$ are the degrees of the irreducible factors of $1 + x^{h-1} + x^h \in \mathbb{Z}_2[x]$, and we set $\delta_h = \operatorname{lcm}\{2^{d_1} - 1, \ldots, 2^{d_r} - 1\}$, then

$$u(n) = \frac{k 2^{\delta_h - 1}}{2^{\delta_h} - 1}.$$

Let us consider some examples involving arbitrary powers $g(x)^n$ of the polynomial $g(x)$.

Let $N_\alpha(g)$ denote the number of coefficients of $g(x)$ that are equal to α and $N(g)$ the total number of nonzero coefficients of $g(x)$. Note that $N(g) + N_0(g) = 1 + \deg g$. Write the p-ary digits of $m \in \mathbb{N}$ as $\langle m_0, m_1, \ldots, m_s \rangle$ so that $m = m_0 + m_1 p + \cdots + m_s p^s$.

2.10 Proposition. (a) *Let* $g(x) = (1 + x + \cdots + x^{p-1})^n \in \mathbb{Z}_p[x]$ *for a prime* p *and* $n \in \mathbb{P}$. *Then the coefficient of* x^k *in* $g(x)$ *is given by*

$$[x^k]g(x) = (-1)^k \binom{pn - n}{k}, \qquad 0 \le k \le pn - n.$$

(b) *If* $\langle b_0, b_1, \ldots, b_m \rangle$ *are the* p*-ary digits of* $(p-1)n$*, then the number of coefficients of* $g(x)$ *not divisible by* p *is*

$$N(g) = \prod_{i=0}^{m} (1 + b_i). \tag{14}$$

Proof. Use the trick $1 + x + \cdots + x^{p-1} = (1 - x)^{p-1}$ in $\mathbb{Z}_p[x]$. Applying the binomial expansion to $(1 - x)^{(p-1)n}$ proves (a). On the other hand, Lucas' theorem implies the congruence

$$\binom{pn - n}{k} \equiv \prod_{i=0}^{m} \binom{b_i}{k_i},$$

where $\langle k_0, k_1, \ldots, k_m \rangle$ are the p-ary digits of k. A simple counting argument leads to (b). □

2.11 Example. (a) Let $p = 3$, $g(x) = (1 + x + x^2)^n$ and $\langle b_0, \ldots, b_m \rangle$ be the ternary digits of $2n$. Then by equation (14),

$$N(g) = N_1(g) + N_2(g) = \prod_{i=0}^{m} (1 + b_i).$$

Observe that $(-1)^{k_i}\binom{2}{k_i} = 1$ in $\mathbb{F}_3$ for any $k_i = 0,1,2$, but $(-1)^{k_i}\binom{1}{k_i} = 1,2$ or 0 depending on $k_i = 0,1$ or 2. In the latter case both 1 and 2 are equally attainable. Obviously $N(g) + N_0(g) = 1 + \deg g = 2n+1$. It follows that $N_1(g) = 3^{m+1}, N_2(g) = 0, N_0(g) = 2n+1-3^{m+1}$ when no $b_i = 1$. In the case that there is some $b_i = 1$ then $N_1(g) = N_2(g) = \frac{1}{2}\prod_{i=0}^{m}(1+b_i)$ (and hence $N_0(g) = 2n+1-2N_1(g)$).

2.12 Example. The above proposition does *not* apply to the function $g(x) = (1+x+x^2)^n \in \mathbb{F}_2[x]$. Still we are able to determine the number of odd coefficients of $g(x)$. Polynomials of the form $(1+x^{k-1}+x^k)^n \in \mathbb{F}_2[x]$ should be prone to the technique outlined here. Write $\omega(n) = N(g)$.

Suppose $n = 2^j(2^k-1)$. Since $(1+x+x^2)^{2^j} = 1 + x^{2^j} + x^{2^{j+1}}$ in $\mathbb{F}_2[\boldsymbol{x}]$, we have $\omega(n) = \omega(2^k-1)$. Now

$$(1+x+x^2)^{2^k-1} = \frac{1+x^{2^k}+x^{2^{k+1}}}{1+x+x^2}.$$

It is easy to check that for k odd we have (writing $i \equiv t(3)$ for $i \equiv t \pmod 3$)

$$(1+x+x^2)^{2^k-1} = \frac{1+x^{2^k}+x^{2^{k+1}}}{1+x+x^2} = \sum_{\substack{i=0\\ i\equiv 0(3)}}^{2^k-2} x^i + \sum_{\substack{i=1\\ i\equiv 1(3)}}^{2^{k+1}-3} x^i + \sum_{\substack{i=2^k\\ i\equiv 2(3)}}^{2^{k+1}-2} x^i.$$

It follows that $\omega(2^k-1) = (2^{k+2}+1)/3$. Similarly, when k is even we have

$$(1+x+x^2)^{2^k-1} \equiv \frac{1+x^{2^k}+x^{2^{k+1}}}{1+x+x^2} \equiv \sum_{\substack{i=0\\ i\equiv 0(3)}}^{2^{k+1}-2} x^i + \sum_{\substack{i=1\\ i\equiv 1(3)}}^{2^k-2} x^i + \sum_{\substack{i=2^k+1\\ i\equiv 2(3)}}^{2^{k+1}-3} x^i \pmod 2.$$

Hence in this case $\omega(2^k-1) = (2^{k+2}-1)/3$. Now any positive integer n can be written uniquely as $n = \sum_{i=1}^{r} 2^{j_i}(2^{k_i}-1)$, where $k_i \geq 1, j_1 \geq 0$, and $j_{i+1} > j_i + k_i$. We are simply breaking up the binary expansion of n into the maximal strings of consecutive 1's. The lengths of these strings are $k_1, \ldots, k_r$. Thus

$$(1+x+x^2)^n \equiv \prod_{i=1}^{r}(1+x^{2^{j_i}}+x^{2^{j_i+1}})^{2^{k_i}-1} \pmod 2.$$

The key observation is: there are *no cancellations* among the coefficients when we expand this product since $j_{i+1} > j_i + 1$. Hence

$$\omega(n) = \prod_{i=1}^{r}\omega(2^{k_i}-1) = \prod_{i=1}^{r}\frac{2^{k_i+2}+(-1)^{k_i+1}}{3}.$$

Take for instance $n = 6039$ with binary expansion 1011110010111. The maximal strings of consecutive 1's have lengths $1, 4, 1$ and 3. Hence $\omega(6039) = \omega(1)\omega(15)\omega(1)\omega(7) = 3 \cdot 21 \cdot 3 \cdot 11 = 2079$.

3 Other multivariate polynomials over $\mathbb{F}_p[\boldsymbol{x}]$

For the remainder of this paper we make no attempt to be systematic, but rather confine ourselves to some interesting examples.

Theorem 2.1 deals with coefficients of $f(\boldsymbol{x})^n$ over $\mathbb{F}_q$. We discuss certain cases for which we can be more explicit. We also give some examples of counting coefficients over $\mathbb{F}_q$ of class of polynomials not of the form $f(\boldsymbol{x})^n$ for fixed $f(\boldsymbol{x})$.

3.1 Example. The Pascal triangle modulo an integer d (in the present context, the coefficients of $(1+x)^n \pmod{d}$) receives a good discussion and further references by Allouche and Shallit [1, Chapter 14, Section 14.6].

3.2 Theorem. *Suppose* $H_n(\boldsymbol{x}) = \prod_{i=1}^{n}(1 + x_i + x_{i+1}) \in \mathbb{F}_p(\boldsymbol{x})$ *with* $\boldsymbol{x} = (x_1, \ldots, x_{n+1})$ *and* p *a prime. Then we have*

$$\sum_{n=0}^{\infty} N(H_n) z^n = \frac{1 - z^p}{(1-z)^2 - z(1-z^p)}.$$

Proof. The generating function can be simplified to $(1 + z + \cdots + z^{p-1})/(1 - 2z - z^2 - \cdots - z^p)$. Thus we need to verify that the sequence $N(H_n)$ satisfies the recurrence relation

$$t_{n+p} - 2t_{n+p-1} - t_{n+p-2} - \cdots - t_{n+1} - t_n = 0,$$

with $t_k = N(H_k)$. To this end write $H_{n+p} = H_p \cdot \prod_{i=1}^{n}(1 + x_{i+p} + x_{i+p+1})$. Clearly it is enough to prove the case $n = 0$; that is, $t_p = 2t_{p-1} + t_{p-2} + \cdots + t_1 + t_0$. Note that $t_0 = 1$. We have

$$\begin{aligned}
H_p &= (1 + x_p + x_{p+1})H_{p-1} \\
&= x_{p+1}H_{p-1} + (1 + x_p)H_{p-1} \\
&= x_{p+1}H_{p-1} + (1 + x_{p-1} + x_p^2)H_{p-2} + x_p(2 + x_{p-1})H_{p-2} \\
&= x_{p+1}H_{p-1} + \widetilde{H}_{p-1} + x_p(2 + x_{p-1})H_{p-2},
\end{aligned}$$

where $\widetilde{H}_{p-1} = (1 + x_{p-1} + x_p^2)H_{p-2}$ and $N(x_{p+1}H_{p-1}) = N(\widetilde{H}_{p-1}) = N(H_{p-1})$ (just replace $x_p \to x_p^2$).

Continuing the above process we arrive at

$$H_p = x_{p+1}H_{p-1} + \widetilde{H}_{p-1} + \widetilde{H}_{p-2} + \cdots + \widetilde{H}_1 + \widetilde{H}_0, \tag{15}$$

where

$$\widetilde{H}_0 := x_p \cdots x_2(p + x_1)H_0 \equiv x_p \cdots x_1 H_0 \pmod{p},$$

and

$$\widetilde{H}_{p-k} := x_p \cdots x_{p-k+2}(k + kx_{p-k} + x_{p-k+1}^2)H_{p-k-1}$$

for $1 \le k < p$.

In $\mathbb{F}_p(\boldsymbol{x})$ the map $\beta \to \beta^2/k$ is bijective and hence $N(\widetilde{H}_{p-k}) = N(H_{p-k})$. Since the terms in equation (15) are mutually exclusive, a straightforward counting completes the argument. □

Note. When $p = 2$, it follows from [22, Exam. 4.1.2] that $N(H_n)$ is the number of self-avoiding lattice paths of length n from $(0, 0)$ with steps $(1, 0)$, $(-1, 0)$, or $(0, 1)$.

3.3 Corollary. *If $p \ge 3$ and $h_n(x) = (1 + x + x^p)^{(p^n-1)/(p-1)} \in \mathbb{F}_p[x]$, then*

$$\sum_{n=0}^{\infty} N(h_n)z^n = \frac{1 - z^p}{(1 - z)^2 - z(1 - z^p)}.$$

Proof. In Theorem 3.2, reindex x_i as x_{i-1} so that $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$; put $x_i = x^{p^i}$ and apply $a^p + b^p = (a + b)^p \pmod{p}$. The map between the new $H_n(\boldsymbol{x})$ and $h_n(x)$ defined by

$$x_{i_1}x_{i_2} \cdots x_{i_r} \to x^{p^{i_1}+p^{i_2}+\cdots+p^{i_r}}$$

is easily checked to be a bijection due to the uniqueness (up to permutation of the terms) of p-ary digits in the field $\mathbb{F}_p$. The result follows from Theorem 3.2. □

4 Counting integer coefficients in shifting products

The motivation for the next discussion comes from the simple formula

$$N((x_1 + x_2 + \cdots + x_r)^s) = \binom{r + s - 1}{s}, \tag{16}$$

the number of s-combinations of an r-element set allowing repetition. Denote $\binom{r+s-1}{s}$ by $\left(\!\binom{r}{s}\!\right)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an integer partition, so $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Set $\lambda_{n+1} = 0$. We will identify λ with its *Young diagram* [22, p. 29] consisting of λ_i left-justified squares in the ith row.

Notation. For $m \in \mathbb{N}$, write $\boldsymbol{x}^{(m)} = \sum_{j=1}^{m} x_j$, with $\boldsymbol{x}^{(0)} = 0$. If $\{\alpha_1, \alpha_2, \ldots\}$ is a sequence, then the *forward difference* is $\Delta\alpha_i = \alpha_{i+1} - \alpha_i$.

We now find a generalization of equation (16).

4.1 Lemma. *(a) The number of distinct monomials in the product* $\Omega_n(\lambda) = \prod_{i=1}^{n} \boldsymbol{x}^{(\lambda_i)}$ *is equal to*

$$N(\Omega_n(\lambda)) = \sum_{\boldsymbol{k} \in K_n} \prod_{i=1}^{n} \left(\!\binom{-\Delta\lambda_i}{k_i}\!\right),$$

where $K_n = \{\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N} : k_1 + \cdots + k_i \leq i; \sum k_j = n\}$.

(b) There holds the recurrence

$$N(\Omega_n(\ldots, \lambda_i + 1, \ldots)) = N(\Omega_n(\ldots, \lambda_i, \ldots)) +$$

$$N(\Omega_i(\lambda_1 - \lambda_i, \ldots, \lambda_{i-1} - \lambda_i, 1)) \cdot N(\Omega_{n-i}(\lambda_{i+1}, \ldots)).$$

Proof. Divide up the Young diagram into compartments by drawing vertical lines alongside the edges λ_i. The vertices then generate the set K_n while side-lengths take the form $\lambda_i - \lambda_{i+1} = -\Delta\lambda_i$. Now apply equation (16) to each of the disjoint rectangular blocks. □

REMARK. The set K_n is easily seen to be in bijection with Dyck paths of length $2n$ [23, Cor. 6.2.3(iv)] and hence has cardinality C_n, a Catalan number. The elements of K_n are called *G-draconian sequences* by Postnikov [18].

A very special case involves the ubiquitous Catalan number C_n.

4.2 Corollary. *Let* $\lambda = (n, n-1, \ldots, 1)$. *Then*

(a) $N(\Omega_n(\lambda)) = C_n$.

(b) $C_n = \sum_{j \geq 1} (-1)^{j+1} \binom{n+2-j}{j} C_{n-j}$.

Proof. (a) Since $\Delta\lambda_i = -1$ each product weight equals 1, therefore $N(\Omega_n(\lambda)) = N(K_n) = C_n$.

(b) Consider $\Omega_n(\lambda)$ as lattice boxes in the region $\{(x, y) : y \geq x \geq 0\}$, and let $a(n, j)$ be the number of lattice paths from $(0, 0)$ to (n, j). This results in the system

$$a(n,j) = \begin{cases} a(n, j-1) & \text{if } j = n \\ a(n, j-1) + a(n-1, j) & \text{if } 1 \leq j < n, \end{cases}$$

with the conditions $a(0, 0) = 1$ and $a(n, j) = 0$ whenever $j > n$ or $j < 0$. By construction and part (a), we have $a(n, n) = a(n, n-1) = C_{n-1}$. Combining with the relation $a(n, n-(j+1)) = a(n, n-j) - a(n-1, (n-1)-(j-1))$ and induction, one can show that

$$a(n, n-j) = \sum_{i \geq 1} (-1)^{i+1} \binom{j+2-i}{i} C_{n-i}.$$

Using $\sum_{j=0}^{n-1} a(n, j) = C_n$ and simple manipulations completes the proof. □

Next we list some interesting connections between the result in Lemma 4.1 and several other enumerations arising in recent work by different authors.

4.1 Lattice paths under cyclically shifting boundaries

Chapman *et al.* [4] and Irving-Rattan [13] have enumerated lattice paths under *cyclically shifting* boundaries. For notations and terminology refer to [13, Thm. 15].

4.3 Theorem. (Irving-Rattan) *Let $s, t, n \in \mathbb{P}$. Let U and R denote up and right steps, respectively. Then there are $\frac{1}{n}\binom{(s+t)n-2}{n-1}$ lattice paths from $(0, 0)$ to $(sn-1, tn-1)$ with steps U and R lying weakly beneath $U^{t-1}(R^sU^t)^{n-1}R^{s-1}$.*

Setting $s = t$ yields a result of Bonin-Mier-Noy [3, Thm. 8.3], as follows.

4.4 Corollary. *Let n and t be positive integers. Then there are tC_{nt-1} lattice paths from $(0, 0)$ to $(nt-1, nt-1)$ with steps U and R lying weakly beneath $U^{t-1}(R^tU^t)^{n-1}R^{t-1}$.*

Applying Lemma 4.1 it is possible to give a generating function reformulation of Theorem 4.3.

4.5 Corollary. *Let $Z_{n,s,t}$ be the polynomial*

$$Z_{n,s,t} = \left(\sum_{i=1}^{sn} x_i\right)^{t-1} \prod_{j=1}^{n-1} \left(\sum_{i=1}^{sj} x_i\right)^{t}.$$

Then the set of distinct monomials in $Z_{n,s,t}$ is equinumerous with lattice paths from $(0,0)$ to $(sn-1, tn-1)$ lying weakly beneath $U^{t-1}(R^sU^t)^{n-1}R^{s-1}$. Moreover,

$$N(Z_{n,s,t}) = \sum_{\boldsymbol{k}\in L_{n,t}} \prod_{i=1}^{n} \binom{k_i+s-1}{k_i} = \frac{1}{n}\binom{(s+t)n-2}{n-1},$$

where $L_{n,t} := \{\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{N} : k_1 + \cdots + k_j \leq tj-1; \sum k_j = tn-1\}$.

Proof. For convenience, write each factor $x_1 + \cdots + x_\mu$ in decreasing order $x_\mu + \cdots + x_1$. Treat this as a horizontal box of μ unit squares. Then encode the factors $Z_{n,s,t}$ by way of stacking up smaller boxes on top of larger ones, so that everything is right-justified. There will be n rectangular blocks with the bottom one of size $sn \times (t-1)$ while the topmost has dimensions $s \times t$. Now the bijection with the lattice paths weakly beneath $U^{t-1}(R^sU^t)^{n-1}R^{s-1}$ is most natural and apparent.

The second assertion results from applying Lemma 4.1 and Theorem 4.3. There is only a slight alteration (simplification) done to the left-hand side. Namely, for $1 \leq j \leq n-1$ let $I_j := \{jt, \dots, (j+1)t-1\}$ and choose λ to be

$$\lambda_i := \begin{cases} s(n-j), & i \in I_j \\ sn, & 1 \leq i \leq t-1. \end{cases}$$

According to Lemma 4.1 the underlying set will be K_{nt-1} with elements denoted by $\boldsymbol{k} = (k_1, \dots, k_{nt-1})$. By direct calculation we find $\Delta\lambda_i = -s$ for $i = t-1, 2t-1, \dots, nt-1$ and $\Delta\lambda_i = 0$ otherwise. In the latter case, if the corresponding $k_i \neq 0$ then the related binomial term vanishes. On the other hand, if such $k_i = 0$ then the binomial contribution is 1. Dropping off the elements $\boldsymbol{k}$ with zero outputs (hence redundant) offers a large reduction on the set K_{nt-1}. Hence the relevant set to sum over becomes $L_{n,t}$. □

Write $\lambda \vdash n$ to denote that $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of $n \geq 0$. We also write $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$ to denote that λ has m_i parts equal to i. Certain specialized values in Corollary 4.5 produce the following identities.

4.6 Corollary. *With notation as in Corollary 4.5, if* $t = 1$ *then we have*

$$\sum_{\lambda=\langle 1^{m_1},2^{m_2},\ldots\rangle \vdash n-1} \binom{n+1}{m_1+\cdots+m_j}\binom{m_1+\cdots+m_j}{m_1,\cdots,m_j}\prod_{i\geq 1}\binom{i+s-1}{i}^{m_i}$$

$$= \binom{(s+1)n-2}{n-1}.$$

If instead $s = 1$*, then* $L_{n,t} := \{(a_1,\ldots,a_n) \in \mathbb{P}^n_{\geq 0} : a_1+\cdots+a_k \leq tk - 1; \sum a_i = tn-1\}$ *is of cardinality*

$$\#L_{n,t} = \frac{1}{n}\binom{(t+1)n-2}{n-1}.$$

4.2 The PS-polytope

Let $t_1,\ldots,t_n \geq 0$. In [17] Pitman and Stanley discuss the n-dimensional polytope

$$\Pi_n(t_1,\ldots,t_n) = \{\boldsymbol{y} \in \mathbb{R}^n : y_i \geq 0 \text{ and } y_1+\cdots+y_i \leq t_n+\cdots+t_{n-i+1} \text{ for all } 1 \leq i \leq n\}.$$

We call this a *PS-polytope.* One of the results in [17] concerns the total number $\#\Pi_n$ of lattice points in Π_n when each $t_i \in \mathbb{N}$:

$$\#\Pi_n(t_1,\ldots,t_n) = \sum_{\boldsymbol{k}\in K_n}\left(\!\!\binom{t_n+1}{k_n}\!\!\right)\prod_{i=1}^{n-1}\left(\!\!\binom{t_i}{k_i}\!\!\right), \tag{17}$$

where K_n is as in Lemma 4.1.

It turns out that the enumeration in Lemma 4.1 coincides with that of [17]. The next result makes this assertion precise; the proof is immediate.

4.7 Corollary. *Let* $\boldsymbol{t} = (t_1,\ldots,t_n) \in \mathbb{P}^n$ *and* $\lambda_i = t_i + t_{i+1} + \cdots + t_n$*. Then the number of monomials in* $\Omega_n(\lambda) := \prod_{i=1}^n \sum_{j=1}^{\lambda_i} x_j$ *equals the number of lattice points* $\#\Pi_n(t_1,t_2,\ldots,t_n-1)$ *in the PS-polytope* $\Pi(t_1,\ldots,t_{n-1},t_n-1)$.

The term $t_n - 1$ can be symmetrized as follows. Consider the so-called *trimmed generalized permutohedron,* introduced by Postnikov [18] as the Minkowski sum

$$P_G^-(\boldsymbol{t}) = t_1\Delta_{[n+1]} + t_2\Delta_{[n]} + \cdots + t_n\Delta_{[2]}$$

of the standard *simplices* $\Delta_{[i]}$ =conv$(e_1, \ldots, e_i)$, where $[i] = \{1, \ldots, i\}$, the e_k's are the coordinate vectors in $\mathbb{R}^i$, and conv denotes convex hull. With these objects defined, then the two polytopes are related as

$$P_G^-(t_1, \ldots, t_n) + \Delta_{[n+1]} = \Pi_n(t_1, \ldots, t_n).$$

Hence $N(P_G^-(\boldsymbol{t}))$ is exactly the count on the distinct monomials of $\Omega_n(\lambda)$.

When $t_1 = t_2 = \cdots = t_n + 1 = t$, a direct proof of Pitman-Stanley's result (17) can be given.

4.8 Corollary. *Suppose* $\boldsymbol{t} = (t, t, \ldots, t-1)$. *Then*

$$\#\Pi_n(\boldsymbol{t}) = \sum_{\boldsymbol{k} \in K_n} \prod_{i=1}^{n} \binom{k_i + t - 1}{k_i} = \frac{1}{n}\binom{(t+1)n - 2}{n-1},$$

the number of lattice paths from $(0,0)$ *to* $(n-1, nt-1)$ *with steps* U *and* R *lying weakly beneath* $U^{t-1}(RU^t)^{n-1}$ *(or equivalently* $(R^tU)^{n-1}R^{t-1}$*).*

Proof. Combining Lemma 4.1 together with Corollary 4.5 we write

$$\sum_{\boldsymbol{k} \in L_{n,t}} \prod_{i=1}^{n} \binom{k_i + s - 1}{k_i} = \frac{1}{n}\binom{(s+t)n - 2}{n-1},$$

where $L_{n,t} = \{\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n | k_1 + \cdots + k_j \leq tj - 1; \sum k_i = tn - 1\}$.

Taking note of the symmetry in s and t (evident from the right side), compute first at $(t, s) = (1, t)$ and then at $(t, s) = (t, 1)$. The outcome is

$$\sum_{L_{n,1}} \prod_{i=1}^{n} \binom{k_i + t - 1}{k_i} = \sum_{L_{n,t}} \prod_{i=1}^{n} \binom{k_i + 1 - 1}{k_i} = \frac{1}{n}\binom{(t+1)n - 2}{n-1}. \tag{18}$$

Observe that the middle term in equation (18) is simply the cardinality of $L_{n,t}$ and by Rado's result [19] on permutohedrons [18, Proposition 2.5], we have $L_{n,t} = \Pi_n(\boldsymbol{t})$. Also note that $\#L_{n,1} = \#K_{n-1} = C_{n-1}$. The proof follows. □

4.3 Noncrossing matchings

The discussion below has its roots in the following scenario: if $2n$ people are seated around a circular table, in how many ways can all of them be simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other? Answer: the Catalan number C_n.

It is convenient to formulate the above question for a circularly arranged points $OXOX\cdots OX = (OX)^n$ of O's and X's where only opposite symbols can be connected. Call the desired goal *noncrossing matchings*. Mahlburg-Rattan-Smyth-Kemp [14] have extended the concept in a more general setting, i.e., for a string of the type $O^{m_1}X^{m_1}\cdots O^{m_n}X^{m_n}$. The next statement captures a seemingly nonobvious coincidence.

4.9 Corollary. *Let $0 \le m_1 \le \cdots \le m_n$ be integers and $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_i = m_n + \cdots + m_i$. Then the number of noncrossing matchings for a string of type $O^{m_1}X^{m_1}\cdots O^{m_n-1}X^{m_n-1}$ equals the number of monomials in $\Omega_n(\lambda)$. Equivalently,*

$$\sum_{\boldsymbol{k}\in K_n} \binom{m_n}{k_n} \prod_{i=1}^{n-1} \binom{m_i+1}{k_i} = \sum_{\boldsymbol{k}\in K_n} \prod_{i=1}^{n} \binom{m_i+k_i-1}{k_i}. \tag{19}$$

Proof. The left-hand side of the equation above is precisely the enumeration found in [14] while, the right-hand side is from Lemma 4.2. Thus we only need to prove the identity (19).

Given the polytope $\Pi_n(\boldsymbol{m}) = \{\boldsymbol{y} \in \mathbb{N}^n : y_1 + \cdots + y_i \le m_n + \cdots + m_{n+1-i}; 1 \le i \le n\}$, we know that

$$\#\Pi_n(\boldsymbol{m}) = \sum_{k\in\boldsymbol{k_n}} \binom{m_n+k_n}{k_n} \prod_{i=1}^{n-1} \binom{m_i+k_i-1}{k_i}.$$

Denoting the interior of $\Pi_n(\boldsymbol{m})$ by $\Pi_n^o(\boldsymbol{m})$ and applying the *Reciprocity Law* for polytopes [2, Chapter 4] yields

$$\begin{aligned}
\#\Pi_n^o(\boldsymbol{m}) &= (-1)^n \sum_{k\in\boldsymbol{k}_n} \binom{-m_n+k_n}{k_n} \prod_{i=1}^{n-1} \binom{-m_i+k_i-1}{k_i} \\
&= (-1)^n(-1)^{\sum k_i} \sum_{k\in\boldsymbol{k}_n} \binom{m_n-1}{k_n} \prod_{i=1}^{n-1} \binom{m_i}{k_i} \\
&= \sum_{k\in\boldsymbol{k}_n} \binom{m_n-1}{k_n} \prod_{i=1}^{n-1} \binom{m_i}{k_i};
\end{aligned}$$

where we used the binomial identity $\binom{-a+b}{b} = (-1)^b\binom{a-1}{b}$ and $\sum_{i=1}^n k_i = n$.

On the other hand, $\Pi_n^o(\boldsymbol{m})$ is characterized by the conditions $y_i > 0$ and $y_1 + \cdots + y_i < m_n + \cdots + m_{n+1-i}$; equivalently, $y_i' := y_i - 1 \geq 0$ and $y_1' + \cdots + y_i' < (m_n - 1) + \cdots + (m_{n+1-i} - 1)$ or

$$y_i' := y_i - 1 \geq 0, \qquad y_1' + \cdots + y_i' \leq (m_n - 2) + (m_{n-1} - 1) + \cdots + (m_{n+1-i} - 1).$$

Therefore

$$\begin{aligned} \#\Pi_n^o(\boldsymbol{m}) &= \#\Pi_n(m_n - 2, m_{n-1} - 1, \ldots, m_1 - 1) \\ &= \sum_{k \in \boldsymbol{k_n}} \binom{m_n + k_n - 2}{k_n} \prod_{i=1}^{n-1} \binom{m_i + k_i - 2}{k_i}. \end{aligned}$$

Now substituting $m_i \to m_i + 1$ and equating the last two formulas justifies the assertion. □

Remark. Although the corollary above has been stated for weakly increasing sequences $(m_1, \ldots, m_n)$, in fact the assertion is valid as a polynomial identity for the n-tuple $\boldsymbol{m}$ of indeterminates. In such generality, however, the interpretation in terms of noncrossing matchings will be lost.

4.10 Example. We consider particular shapes λ for which we find relations with some known enumerative results.

(a) For $n \geq m$, define the multivariate polynomial $\phi_{n,m} = \prod_{j=1}^{n-1} \sum_{i=0}^{j+m} x_i$. Then

$$N(\phi_{n,m}) = \frac{m+2}{2n+m}\binom{2n+m}{n+m+1},$$

the number of standard Young tableaux of shape $(n+m, n-1)$ as well as the number of lattice points $\#\Pi_{n-1}(0, 1, 1, \ldots, 1, m+2)$.

(b) Let $T_{n,k} = \prod_{j=1}^{n-1} \left(\sum_{i=0}^{j} x_i\right)^k$ for $n, k \in \mathbb{P}$. Then $N(T_{n,k}) = \frac{1}{kn+1}\binom{(k+1)n}{n}$, a *Fuss-Catalan number* [11]. See also Sloane [20] for a host of other combinatorial interpretations.

(c) Let M and N be two infinite triangular matrices with entries given for $0 \leq j \leq i$ by

$$\begin{aligned} M(i,j) &= \binom{\binom{i}{2} - \binom{j}{2} + \binom{i}{1} - \binom{j}{1} + 3}{i - j} \\ N(i,j) &= \binom{\binom{i}{2} - \binom{j}{2} + \binom{i}{1} - \binom{j}{1} + 2}{i - j}. \end{aligned}$$

Also let $R := MN^{-1}$, and denote its elements by $R(n,k)$. Define the polynomial

$$S_{n,k} = \prod_{j=1}^{n-1}\left(\sum_{i=0}^{j} x_i\right)^{j+k}.$$

Suppose $\boldsymbol{t} = (t_1 - 1, t_2, \ldots, t_{n-1})$ with $t_i = k + n - i$. Then $N(S_{n,k}) = \sum_{\boldsymbol{y}\in\Pi_{n-1}(\boldsymbol{t})}(1 + y_{n-1}) = R(n,k)$.

5 Traveling polynomials in $\mathbb{F}[\boldsymbol{x}]$

Continuing in the same spirit as in the previous section we consider further multinomials with shifting terms, given by

$$W_{j,k,n} = \prod_{i=1}^{n}(x_{(i-1)j+1} + x_{(i-1)j+2} + \cdots + x_{(i-1)j+k}). \tag{20}$$

We call $W_{j,k,n}$ a *traveling polynomial.*

5.1 Theorem. *For fixed $j, k \geq 1$ we have*

$$\sum_{n\geq 0} N(W_{j,k,n})z^n = \frac{1}{\sum_{h\geq 0}(-1)^h\binom{k-j(h-1)}{h}z^h}. \tag{21}$$

Proof. The proof is by the Principle of Inclusion-Exclusion. First note that if $x_{m_1}x_{m_2}\cdots x_{m_r}$ is a monomial appearing in the expansion of $W_{j,k,n}$, where $m_1 \leq m_2 \leq \cdots \leq m_r$, then we can obtain this monomial by choosing x_{m_s} from the sth factor (i.e., the factor indexed by $i = s$) from the right-hand side of equation (20). Suppose $x_{m_s} = x_{(s-1)j+b_s}$. Then $b_1b_2\cdots b_n$ is a sequence satisfying

$$1 \leq b_s \leq k, \quad b_i \geq b_{i-1} - j, \tag{22}$$

and conversely any sequence satisfying equation (22) yields a different monomial in the expansion of $W_{j,k,n}$. Hence we want to count the number $f(n)$ of sequences (22). Let us call such a sequence a *valid n-sequence.*

For $h \geq 0$ let $\mathcal{A}_h$ be the set of all sequences $b_1b_2\cdots b_n$ satisfying the conditions:

- $1 \leq b_i \leq k$ for $1 \leq i \leq n$
- $b_i \geq b_{i-1} - j$ for $2 \leq i \leq n - h$

- $b_i < b_{i-1} - j$ for $n - h + 2 \leq i \leq n$.

In particular, $\mathcal{A}_0$ is the set of valid n-sequences, so $f(n) = \#\mathcal{A}_0$, and $\mathcal{A}_1$ consists of valid $(n-1)$-sequences followed by any number $1 \leq b_n \leq k$. For a sequence $\beta = b_1 b_2 \cdots b_n$ let

$$\chi(\beta \in \mathcal{A}_h) = \begin{cases} 1, & \beta \in \mathcal{A}_h \\ 0, & \beta \notin \mathcal{A}_h. \end{cases}$$

It is easy to see that

$$\sum_{h \geq 0} (-1)^h \chi(\beta \in \mathcal{A}_h) = \begin{cases} 1, & \beta \in \mathcal{A}_0 \\ 0, & \beta \notin \mathcal{A}_0. \end{cases}$$

It follows that

$$\#\mathcal{A}_0 = \#\mathcal{A}_1 - \#\mathcal{A}_2 + \#\mathcal{A}_3 - \cdots .$$

By elementary and well-known enumerative combinatorics (e.g., [22, Exer. 1.13]) we have

$$\#\mathcal{A}_h = \binom{k - j(h-1)}{h} f(n-h).$$

Hence

$$f(n) = \sum_{h \geq 1} (-1)^h \binom{k - j(h-1)}{h} f(n-h).$$

This reasoning is valid for all $n \geq 1$ provided we set $f(-1) = f(-2) = \cdots = f(-(h-1)) = 0$, from which equation (21) follows immediately. □

5.2 Example. It is rather interesting that even a minor alteration in the traveling polynomial $W_{j,k,n}$ unveils curious structures. To wit, take the special case $Q_n := W_{1,3,n}$. From Theorem 5.1 we gather that

$$\sum_{n=0}^{\infty} N(Q_n) z^n = \frac{1}{1 - 3z + z^2}, \tag{23}$$

or more explicitly $N(Q_n) = F(2n+2)$, with $F(m)$ the Fibonacci sequence.

Let us now introduce

$$\begin{aligned} G_n &= \prod_{i=1}^{n} (x_i + x_{i+2} + x_{i+4}) \\ B_{n,t} &= \prod_{i=1}^{n} (1 - x_i + x_{i+t}). \end{aligned}$$

(a) We have $N(G_n) = F(n+2)^2 - \eta(n)$, where $\eta(n) = \frac{1-(-1)^n}{2}$.

Proof. Denote $n' = \lfloor n/2 \rfloor$, $n'' = \lfloor (n+1)/2 \rfloor$. Writing G_n in the form

$$G_n = \prod_{i \text{ odd}} (x_i + x_{i+2} + x_{i+4}) \prod_{i \text{ even}} (x_i + x_{i+2} + x_{i+4})$$

suggests that (if necessary rename $y_i = x_{2i}$ and $y_i = x_{2i-1}$, respectively) $N(G_n) = N(Q_{n'}) \cdot N(Q_{n''})$. From equation (23) we have $N(G_n) = F(n+2)^2$ if n is even; $N(G_n) = F(n+1)F(n+3)$ if n is odd. Now invoke *Cassini's formula* $F(m+1)F(m+3) = F(m+2)^2 + (-1)^m$. We also obtain

$$\sum_{n \geq 0} N(G_n) z^n = \frac{1}{(1-z^2)(1-3z+z^2)}.$$

(b) Recall the fact that the polynomial $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ consists of $n!$ nonzero coefficients half of which are $+1$'s, the other half -1's. In fact, this gives the easiest way to prove that the number of odd coefficients of $\prod_{1 \leq i < j \leq n} (x_i + x_j)$ is also $n!$.

A similar phenomenon occurs with $B_{n,t}$: the nonzero coefficients of $B_{n,t}$ are all ± 1 with one extra $+1$ than -1. We also have

$$\sum_{n \geq 0} N(B_{n,1}) z^n = \frac{1+z}{1-2z-z^2}$$

$$\sum_{n \geq 0} N(B_{n,2}) z^n = \frac{1}{1-z} \left\{ \frac{z^2}{1+z^2} + \frac{1}{1-2z-z^2} \right\}.$$

Proof. We will only consider $B_{n,1}$; the general case is analogous. The assertions are trivial when $n = 0, 1$. Proceed by induction. Start with the simplification

$$(1-x_n+x_{n+1})(1-x_{n+1}+x_{n+2}) = x_{n+1}(1-x_n+x_{n+1}) + (1-x_n+x_n x_{n+1}) - x_{n+1}^2.$$

It follows that

$$B_{n+1,1} = x_{n+2} B_{n,1} + (1 - x_n + x_n x_{n+1}) B_{n,1} - x_{n+1}^2 B_{n-1,1}. \qquad (24)$$

Now, the three terms on the right-hand are independent and

$$N((1 - x_n + x_n x_{n+1}) B_{n-1,1}) = N(B_{n,1}).$$

As a consequence $N(B_{n+1,1}) = 2N(B_{n,1}) + N(B_{n-1,1})$ and thereby we validate the generating function.

By induction assumption, the polynomials $x_{n+2}B_{n,1}$ and $(1-x_n+x_nx_{n+1})B_{n,1}$ each contain one extra $+1$ than a -1; while $-x_{n+1}^2 B_{n-1,1}$ has one extra -1. By equation (24) the proof is complete. □

We include the next result (Theorem 5.4) as a generalization of the case $j = 1$ of Theorem 5.1 and as an indication of alternative method of proof towards possible extensions of our work on traveling polynomials. In particular, it should be possible to find common generalization of Theorem 5.1 and Theorem 5.4, though we have not pursued this question.

Define

$$V_{n,k,m} = \prod_{i=1}^{n} (x_i + x_{i+1} + \cdots + x_{i+k})^m, \qquad n, k, m \in \mathbb{N}.$$

We need a preliminary result before stating Theorem 5.4.

5.3 Lemma. *Let k, m be nonnegative integers and $A^{(k,m)}$ be the $(k+1) \times (k+1)$ matrix given by*

$$A_{i,j}^{(k,m)} = \begin{cases} \binom{m-1+i}{m-1} & \text{if } j = 0, \\ \binom{m+i-j}{m-1} & \text{if } j \neq 0; \end{cases} \qquad 0 \leq i, j \leq k.$$

Then $A^{(k,m)}$ has the characteristic polynomial

$$\Theta_{k,m}(\rho) = \sum_{\tau \geq 0} (-1)^\tau \binom{1 + (k+1-\tau)m}{\tau} \rho^{k+1-\tau}.$$

Proof. Fix m and induct on k. We want to compute $\Theta_{k,m}(\rho) = \det(A^{(k,m)} - \rho I)$. For brevity write Θ_k instead of $\Theta_{k,m}$. Observe that apart from the first column, the matrix $A^{(k,m)}$ is almost Toeplitz with one super-diagonal. Taking advantage of this fact, extract the determinant using a repeated application of Laplace expansion along the last column. We determine the recurrence

$$\Theta_k = (m-\rho)\Theta_{k-1} + (-1)^k \binom{m+k-1}{m-1} + \sum_{i=2}^{k} (-1)^{i+1} \binom{m-1+i}{m-1} \Theta_{k-i}.$$

Absorb Θ_k and $m\Theta_{k-1}$ into the sum:

$$0 = -z\Theta_{k-1} + (-1)^k \binom{m-1+k}{m-1} + \sum_{i=0}^{k} (-1)^{i+1} \binom{m-1+i}{m-1} \Theta_{k-i}.$$

The statement of the lemma is then equivalent to

$$\begin{aligned}0 &= \sum_{\tau=0}^{k} (-1)^{k+1-\tau} \binom{1+(k-\tau)m}{\tau} \rho^{k+1-\tau} + (-1)^k \binom{m-1+k}{m-1} \\ &+ (-1)^k \binom{m+k}{m-1} \\ &+ \sum_{i=0}^{k+1} (-1)^{i+1} \binom{m-1+i}{m-1} \sum_{j=0}^{k+1-i} (-1)^{k+1-i-j} \binom{1+(k+1-i-j)m}{j} \rho^{k+1-i-j}.\end{aligned}$$

Replace $i+j \to \tau$ and rearrange the double summation

$$\begin{aligned}0 &= \sum_{\tau=0}^{k} (-1)^{k+1-\tau} \rho^{k+1-\tau} \binom{1+(k-\tau)m}{\tau} \\ &+ (-1)^k \binom{m-1+k}{m-1} + (-1)^k \binom{m+k}{m-1} \\ &+ \sum_{\tau=0}^{k+1} (-1)^{k+1-\tau} \rho^{k+1-\tau} \sum_{j=0}^{\tau} (-1)^{\tau-j+1} \binom{m-1+\tau-j}{m-1} \binom{1+(k+1-\tau)m}{j}.\end{aligned}$$

After comparing coefficients in these polynomials, the problem is tantamount to proving

$$\sum_{j=0}^{\tau} (-1)^{\tau-j} \binom{m-1+\tau-j}{m-1} \binom{1+(k+1-\tau)m}{j} = \binom{1+(k-\tau)m}{\tau}.$$

But this formula is a consequence of the Vandermonde-Chu identity

$$\sum_{j=0}^{\tau} (-1)^{\tau-j} \binom{m-1+\tau-j}{m-1} \binom{x+m}{j} = \binom{x}{\tau}.$$

□

5.4 Theorem. *With A as in Lemma 5.3 and Φ_ξ being the sum of the first column of A^ξ, we have*

$$\sum_{n\geq 0} N(V_{n,k,m})z^n = \frac{\sum_{\nu=0}^{k-1} z^\nu \sum_{i=0}^{\nu}(-1)^i \binom{1+(k+1-i)m}{i}\Phi_{\nu-i}}{\sum_{i\geq 0}(-1)^i\binom{1+(k+1-i)m}{i}z^i}.$$

In fact, $\Phi_\xi = N(V_{\xi,k,m})$ *for* $0 \leq \xi \leq k-1$*.*

Proof. Given k, m construct the list $\{a^{(k,m)}(n,0), \ldots, a^{(k,m)}(n,k)\}$ the same way as in the proof of Corollary 4.2. Suppress k, m and define the column vector $\Gamma_n = [a(n,0), \ldots, a(n,k)]^T$. Reintroduce the *connectivity matrix* $A^{(k,m)}$, or simply $A = (A_{i,j})$, from Lemma 5.3, and apply this result to obtain

$$\sum_{\tau\geq 0}(-1)^\tau \binom{1+(k+1-\tau)m}{\tau} A^{k+1-\tau} = \mathbf{0}.$$

On the other hand,

$$\begin{aligned} a(n,i) &= \sum_{j=0}^{i}\binom{m-2+i-j}{m-2}\sum_{c=0}^{j+1} a(n-1,c) \\ &= \sum_{c=0}^{i+1} a(n-1,c)\sum_{j=c-1}^{i}\binom{m-2+i-j}{m-2} \\ &= \binom{m-1+i}{m-1}a(n-1,0) + \sum_{c=1}^{i+1}\binom{m+i-c}{m-1}a(n-1,c), \end{aligned}$$

where we set $a(\cdot, c) = 0$ if $c < 0$ or $c > k$. Therefore $A\Gamma_{n-1} = \Gamma_n$ and hence

$$\sum_{\tau\geq 0}(-1)^\tau\binom{1+(k+1-\tau)m}{\tau}\Gamma_{n+k+2-\tau} = \mathbf{0}.$$

This component-wise sum together with $N(V_{n,k,m}) = \sum_{i=0}^{k} a(n,i)$ implies that

$$\sum_{\tau\geq 0}(-1)^\tau\binom{1+(k+1-\tau)m}{\tau}N(V_{n+k+1-\tau,k,m}) = 0.$$

The denominator of the generating function has been justified and the proof is complete. □

5.5 Example. (a) The case $k = 2, m = 2$ recovers certain *Kekule numbers* κ_n [5, page 302] whose molecular graphs posssess remarkable combinatorial properties often related to the Fibonacci sequence.

(b) Let $k = 2$. Recall the *Narayana numbers* $Y(i,j) = \frac{1}{j}\binom{i}{j-1}\binom{i-1}{j-1}$, which enumerate Dyck paths on $2i$ steps with exactly j peaks. Then we have

$$\sum_{n\geq 0} N(V_{n,2,m})z^n = \frac{1 + Y(m,2)z}{1 - \binom{2m+1}{1}z + \binom{m+1}{2}z^2}.$$

(c) If $k = 3$ then we have

$$\sum_{n\geq 0} N(V_{n,3,m})z^n = \frac{1 + \left\{2\binom{m}{2} + \binom{m+1}{3}\right\}z + Y(m,3)z^2}{1 - \binom{3m+1}{1}z + \binom{2m+1}{2}z^2 - \binom{m+1}{3}z^3}.$$

(d) If $k = 4$ then we have

$$\sum_{n\geq 0} N(V_{n,3,m})z^n = \frac{1 + a(m)z + b(m)z^2 + Y(m,4)z^3}{1 - \binom{4m+1}{1}z + \binom{3m+1}{2}z^2 - \binom{2m+1}{3}z^3 + \binom{m+1}{4}z^4},$$

where $a(m) = 3\binom{m}{2} + 2\binom{m+1}{3} + \binom{m+2}{4}$ and $b(m) = 10\binom{m}{3} + 23\binom{m}{4} + 10\binom{m}{5}$.

(e) In sharp contrast to the above, the one-variable counterpart (setting $x_i = x^i$) to Theorem 5.4 is a lot simpler. Namely, if

$$J_{n,k,m} = \prod_{i=1}^{n}(1 + x^i + x^{i+1} + \cdots + x^{i+k})^m$$

then

$$N(J_{n,k,m}) = 1 + \left\{nk + \binom{n+1}{2}\right\}m.$$

5.6 Example. Let $D_{n,k}$ be the polynomial

$$D_{n,k}(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{n}(y_1 + \cdots + y_{i-1} + x_i + \cdots + x_{i+k}).$$

Notice that $N(D_{n+1,-1})$ is the polynomial of equation (1), so its number of terms is the Catalan number C_n. We calculate $N(E_{n,k})$ for $k = 0$ and $k = 2$.

(i) The case $k = 0$. We will be referring to the discussion and notations from Section 4. Let $[n] = \{1, 2, \ldots, n\}$ be the integer interval. Given a subset $A \subset [n]$ associate the multivariate polynomial $\Omega(A) = \prod_{a\in A}(y_1 + \cdots + y_a)$.

Consider the triangular product $\Omega([n]) = \prod_{i=1}^{n}(y_1 + \cdots + y_i)$. By the remark following Lemma 4.1, the monomials in $\Omega([n])$ can be regarded as Dyck paths of length $2n$. From Corollary 4.2 we already know that $N(\Omega([n]) = N(\Omega_n(\lambda)) = C_n$, corresponding to the partition $\lambda = (n, n-1, \ldots, 1)$.

In the polynomials $D_{n+1,0}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{n+1}(y_1 + \cdots + y_{i-1} + x_i)$ if we replace $x_i \rightarrow y_i$, then clearly $D_{n+1,0}(\boldsymbol{x}, \boldsymbol{y}) = \Omega([n+1])$. Therefore it is natural to identify the terms in $D_{n+1,0}$ as Dyck paths with ascents (the edges x_i) that are *two-colored*. Such an interpretation and viewing the Narayana numbers $Y(n, j)$ as the number of Dyck paths with exactly j *peaks* [24] produces the enumeration

$$N(D_{n+1,0}) = \sum_{j=0}^{n} 2^j Y(n, j). \tag{25}$$

The sum $\sum_{j=1}^{n} 2^j \frac{1}{n}\binom{n}{j}\binom{n}{j-1}$ on the right-hand side bears the name *large Schröder numbers*.

The same analysis actually translates as breaking up the product $\Omega_n(\lambda)$ into monomials that are cataloged according to how many x_i's appear in them. We may thus find

$$N(D_{n+1,0}) = \sum_{A \subseteq [n]} N(\Omega(A)).$$

Finally, if we employ Lemma 4.1, equation (25) and the formula for the Schröder numbers then the outcome

$$\sum_{A \subseteq [n]} \sum_{\boldsymbol{k} \in K_{\#A}} \prod_{i=1}^{\#A} \left(\!\!\binom{j_i - j_{i+1}}{k_i}\!\!\right) = \sum_{j=1}^{n} 2^j \frac{1}{n}\binom{n}{j}\binom{n}{j-1}$$

is an explicit expression where the sum runs through the power set of $[n]$.

Convention: when A is the empty set, take the product to be 1.

(ii) The case $k = 2$. Interestingly, the values $\gamma_n = N(D_{n-2,2})$ correspond exactly to the enumeration of *tandem (unrooted) duplication trees* in gene replication [9].

Claim: For $n \geq 3$, we have $\gamma_n = \frac{1}{2}\nu_n$ where $\nu_n = \sum_{j \geq 1}(-1)^{j+1}\binom{n+1-2j}{j}\nu_{n-i}$ with $\nu_2 = 1$.

Proof. Proceed with the first argument in the proof of Theorem 5.4 and put the list $\{a(n, 0), \ldots, a(n, n-1)\}$, so that $\nu_n = \sum_{i=0}^{n-1} a(n, i)$. It is easy to check that this doubly-indexed sequence $a(n, i)$ satisfies the partial-difference boundary value problem (BVP)

$$\begin{cases} a(n,i) = a(n-1,i+1) + a(n,i-1), \\ a(n,0) = a(n-1,0) + a(n-1,1), \\ a(n,n-1) = a(n,n-2) = a(n,n-3) = \nu_{n-1}. \end{cases}$$

If $\widetilde{\nu}$ and $\widetilde{a}$ are the column vectors

$$\widetilde{\nu} = [\nu_{n-1}, \nu_{n-1}, \nu_{n-1}, \nu_{n-2}, \nu_{n-3}, \ldots, \nu_2]^T,$$
$$\widetilde{a} = [a(n,n-1), a(n,n-2), \ldots, a(n,0)]^T$$

and $M_{i,j} = (-1)^{i+1}\binom{j+1-2i}{i-1}$ is a matrix, then the BVP is solved by $M\widetilde{\nu} = \widetilde{a}$. That means

$$a(n, n-j) = \sum_{i\geq 1} (-1)^{i+1} \binom{j-2i}{i-1} \nu_{n-i}.$$

The claim follows from $\nu_n = \sum_{i=1}^{n} a(n, n-j)$ and the initial conditions $a(3,0) = a(3,1) = a(3,2) = 1$. □

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