# Longest Alternating Subsequences of Permutations[1]

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

**Abstract**

The length $\mathrm{is}(w)$ of the longest increasing subsequence of a permutation $w$ in the symmetric group $\mathfrak{S}_n$ has been the object of much investigation. We develop comparable results for the length $\mathrm{as}(w)$ of the longest alternating subsequence of $w$, where a sequence $a, b, c, d, \ldots$ is *alternating* if $a > b < c > d < \cdots$. For instance, the expected value (mean) of $\mathrm{as}(w)$ for $w \in \mathfrak{S}_n$ is exactly $(4n+1)/6$ if $n \geq 2$.

# 1 Introduction.

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $1, 2, \ldots, n$, and let $w = w_1 \cdots w_n \in \mathfrak{S}_n$. An *increasing subsequence* of $w$ of length $k$ is a subsequence $w_{i_1} \cdots w_{i_k}$ satisfying

$$w_{i_1} < w_{i_2} < \cdots < w_{i_k}.$$

There has been much recent work on the length $\mathrm{is}_n(w)$ of the longest increasing subsequence of a permutation $w \in \mathfrak{S}_n$. A highlight is the asymptotic determination of the expectation $E(n)$ of $\mathrm{is}_n$ by Logan-Shepp [11] and Vershik-Kerov [18], viz.,

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}_n(w) \sim 2\sqrt{n}, \ n \to \infty. \tag{1}$$

Baik, Deift and Johansson [3] obtained a vast strengthening of this result, viz., the limiting distribution of $\mathrm{is}_n(w)$ as $n \to \infty$. Namely,

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for $w$ chosen uniformly from $\mathfrak{S}_n$ we have

$$\lim_{n\to\infty} \operatorname{Prob}\left(\frac{\operatorname{is}_n(w)-2\sqrt{n}}{n^{1/6}} \le t\right) = F(t), \tag{2}$$

where $F(t)$ is the Tracy-Widom distribution. The proof uses a result of Gessel [9] that gives a generating function for the quantity

$$u_k(n) = \#\{w \in \mathfrak{S}_n \,:\, \operatorname{is}(w) \le k\}.$$

Namely, define

$$U_k(x) = \sum_{n\ge 0} u_k(n)\frac{x^{2n}}{n!^2},\ \ k \ge 1$$

$$I_i(2x) = \sum_{n\ge 0} \frac{x^{2n+i}}{n!\,(n+i)!},\ \ i \in \mathbb{Z}.$$

The function $I_i$ is the *hyperbolic Bessel function* of the first kind of order $i$. Note that $I_i(2x) = I_{-i}(2x)$. Gessel then showed that

$$U_k(x) = \det\left(I_{i-j}(2x)\right)_{i,j=1}^{k}.$$

In this paper we will develop an analogous theory for *alternating subsequences*, i.e., subsequences $w_{i_1}\cdots w_{i_k}$ of $w$ satisfying

$$w_{i_1} > w_{i_2} < w_{i_3} > w_{i_4} < \cdots w_{i_k}.$$

Note that according to our definition, an alternating sequence $a, b, c, \ldots$ (of length at least two) must begin with a descent $a > b$. Let $\operatorname{as}(w) = \operatorname{as}_n(w)$ denote the length (number of terms) of the longest alternating subsequence of $w \in \mathfrak{S}_n$, and let

$$a_k(n) = \#\{w \in \mathfrak{S}_n \,:\, \operatorname{as}(w) = k\}.$$

For instance, $a_1(w) = 1$, corresponding to the permutation $12\cdots n$, while $a_n(n)$ is the total number of alternating permutations in $\mathfrak{S}_n$. This number is customarily denoted $E_n$. A celebrated result of André [1][16, §3.16] states that

$$\sum_{n\ge 0} E_n \frac{x^n}{n!} = \sec x + \tan x. \tag{3}$$

The numbers $E_n$ were first considered by Euler (using (3) as their definition) and are known as *Euler numbers*. Because of (3) $E_{2n}$ is also known as a *secant number* and $E_{2n-1}$ as a *tangent number*.

Define

$$\begin{aligned} b_k(n) &= \#\{w \in \mathfrak{S}_n : \operatorname{as}(w) \leq k\} \\ &= a_1(n) + a_2(n) + \cdots + a_k(n), \end{aligned} \tag{4}$$

so for instance $b_k(n) = n!$ for $k \geq n$. Also define the generating functions

$$\begin{aligned} A(x,t) &= \sum_{k,n\geq 0} a_k(n) t^k \frac{x^n}{n!} \\ B(x,t) &= \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}. \end{aligned} \tag{5}$$

Our main result (Theorem 2.3) is the formulas

$$\begin{aligned} B(x,t) &= \frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}} \\ A(x,t) &= (1-t)B(x,t), \end{aligned} \tag{6}$$

where $\rho = \sqrt{1-t^2}$.

As a consequence of these formulas we obtain explicit formulas for $a_k(n)$ and $b_k(n)$:

$$\begin{aligned} b_k(n) &= \frac{1}{2^{k-1}} \sum_{\substack{r+2s\leq k \\ r\equiv k \,(\mathrm{mod}\, 2)}} (-2)^s \binom{k-s}{(k+r)/2} \binom{n}{s} r^n \\ a_k(n) &= b_k(n) - b_{k-1}(n). \end{aligned}$$

We also obtain from equation (6) formulas for the factorial moments

$$\nu_k(n) = \frac{1}{n!} \sum_{w\in\mathfrak{S}_n} \operatorname{as}(w)(\operatorname{as}(w)-1)\cdots(\operatorname{as}(w)-k+1).$$

For instance, the mean $\nu_1(n)$ and variance $\mathrm{var}(\mathrm{as}_n) = \nu_2(n)+\nu_1(n)-\nu_1(n)^2$ are given by

$$\begin{aligned} \nu_1(n) &= \frac{4n+1}{6},\ n \geq 2 \\ \mathrm{var}(\mathrm{as}_n) &= \frac{8}{45}n - \frac{13}{180},\ n \geq 4. \end{aligned} \tag{7}$$

The limiting distribution of $\mathrm{as}_n$ (the analogue of equation 2)) was obtained independently by Pemantle and Widom, as discussed at the end of Section 3. Rather than the Tracy-Widom distribution as in (2), this time we obtain a Gaussian distribution.

NOTE. We can give an alternative description of $b_k(n)$ in terms of pattern avoidance. If $v = v_1v_2\cdots v_k \in \mathfrak{S}_k$, then we say that a permutation $w = w_1w_2\cdots w_n \in \mathfrak{S}_n$ *avoids* $v$ if $w$ has no subsequence $w_{i_1}w_{i_2}\cdots w_{i_k}$ whose terms are in the same relative order as $v$ [6, Ch. 4.5][17, §7]. If $X \subset \mathfrak{S}_k$, then we say that $w \in \mathfrak{S}_n$ *avoids* $X$ if $w$ avoids all $v \in X$. Now note that $b_{k-1}(n)$ is the number of permutations $w \in \mathfrak{S}_n$ that avoid all $E_k$ alternating permutations in $\mathfrak{S}_k$.

After seeing the first draft of this paper Miklós Bóna pointed out that the statistic $\mathrm{as}_n$ can be expressed very simply in terms of a previously considered statistic on $\mathfrak{S}_n$, viz., the number of *alternating runs*. Hence many of our results can also be deduced from known results on alternating runs. This development is discussed further in Section 4. In particular, it follows from [20] that the polynomials $T_n(t) = \sum_k a_k(n)t^k$ have interlacing real zeros. This result can be used to give a third proof (in addition to the proofs of Pemantle and Widom) that the limiting distribution of $\mathrm{as}_n$ is Gaussian.

# 2 The main generating function.

The key result that allows us to obtain explicit formulas is the following lemma.

**Lemma 2.1.** *Let $w \in \mathfrak{S}_n$. Then there is an alternating subsequence of $w$ of maximum length that contains $n$.*

*Proof.* Let $a_1 > a_2 < \cdots a_k$ be an alternating subsequence of $w$ of maximum length $k = \mathrm{as}(w)$, and suppose that $n$ is not a term of this subsequence. If $n$ precedes $a_1$ in $w$, then we can replace $a_1$ by $n$ and obtain an alternating subsequence of length $k$ containing $n$. If $n$ appears between $a_i$ and $a_{i+1}$ in $w$, then we can similarly replace the larger of $a_i$ and $a_{i+1}$ by $n$. Finally, suppose that $n$ appears to the right of $a_k$. If $k$ is even that we can append $n$ to the end of the subsequence to obtain a longer alternating subsequence, contradicting the definition of $k$. But if $k$ is odd, then we can replace $a_k$ by $n$, again obtaining an alternating subsequence of length $k$ containing $n$. □

We can use Lemma 2.1 to obtain a recurrence for $a_k(n)$, beginning with the initial condition $a_0(0) = 1$.

**Lemma 2.2.** *Let* $1 \leq k \leq n+1$. *Then*

$$a_k(n+1) = \sum_{j=0}^{n} \binom{n}{j} \sum_{\substack{2r+s=k-1 \\ r,s\geq 0}} (a_{2r}(j) + a_{2r+1}(j)) a_s(n-j). \tag{8}$$

*Proof.* We can choose a permutation $w = a_1 \cdots a_{n+1} \in \mathfrak{S}_{n+1}$ such that $\mathrm{as}(w) = k$ as follows. First choose $0 \leq j \leq n$ such that $a_{j+1} = n+1$. Then choose in $\binom{n}{j}$ ways the set $\{a_1, \ldots, a_j\}$. For $s \geq 0$ we can choose in $a_s(n-j)$ ways a permutation $w' = a_{j+2} \cdots a_{n+1}$ satisfying $\mathrm{as}(w') = s$. Next we choose a permutation $w'' = a_1 \cdots a_j$ such that the longest *even* length of an alternating subsequence of $w''$ is $2r = k-1-s$. We can choose $w''$ to satisfy either $\mathrm{as}(w'') = 2r$ or $\mathrm{as}(w'') = 2r+1$. The concatenation $w = w''(n+1)w' \in \mathfrak{S}_{n+1}$ will then satisfy $\mathrm{as}(w) = k$, and conversely all such $w$ arise in this way. Hence equation (8) follows. □

Now write

$$F_k(x) = \sum_{n\geq 0} a_k(n) \frac{x^n}{n!}.$$

For instance, $F_0(x) = 1$ and $F_1(x) = e^x - 1$. Multiplying (8) by $x^n/n!$ and summing on $n \geq 0$ gives

$$F_k'(x) = \sum_{2r+s=k-1} (F_{2r}(x) + F_{2r+1}(x)) F_s(x). \tag{9}$$

Note that

$$A(x,t) = \sum_{k\geq 0} F_k(x)t^k,$$

where $A(x,t)$ is defined by (5). Since $k-1-s$ is even in (9), we need to work with the even part $A_e(x,t)$ and odd part $A_o(x,t)$ of $A(x,t)$, defined by

$$\begin{aligned} A_e(x,t) &= \sum_{k\geq 0} F_{2k}(x)t^{2k} \\ &= \frac{1}{2}(A(x,t) + A(x,-t)) \\ A_o(x,t) &= \sum_{k\geq 0} F_{2k+1}(x)t^{2k+1} \\ &= \frac{1}{2}(A(x,t) - A(x,-t)). \end{aligned} \tag{10}$$

Multiply equation (9) by $t^k$ and sum on $k \geq 0$. We obtain

$$\frac{\partial A(x,t)}{\partial x} = tA_e(x,t)A(x,t) + A_o(x,t)A(x,t). \tag{11}$$

Substituting $-t$ for $t$ yields

$$\frac{\partial A(x,-t)}{\partial x} = -tA_e(x,t)A(x,-t) - A_o(x,t)A(x,-t). \tag{12}$$

Adding and subtracting equations (11) and (12) gives the following system of differential equations for $A_e = A_e(x,t)$ and $A_o = A_o(x,t)$:

$$\frac{\partial A_e}{\partial x} = tA_eA_o + A_o^2 \tag{13}$$

$$\frac{\partial A_o}{\partial x} = tA_e^2 + A_eA_o. \tag{14}$$

Thus we need to solve this system of equations in order to find $A(x,t) = A_e(x,t) + A_o(x,t)$.

**Theorem 2.3.** *We have*

$$B(x,t) = \frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}} \tag{15}$$

$$A(x,t) = (1-t)B(x,t) \tag{16}$$

$$= (1-t)\frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}}, \tag{17}$$

*where* $\rho = \sqrt{1-t^2}$.

*Proof.* We can simply verify that the stated expression (17) for $A(x,t)$ satisfies (13) and (14) with the initial condition $A(0,t) = 1$, a routine computation (especially with the use of a computer). The relationship (16) between $A(x,t)$ and $B(x,t)$ is then an immediate consequence of (4), which is equivalent to $a_k(n) = b_k(n) - b_k(n-1)$.

It might be of interest, however, to explain how the formula (17) for $A(x,t)$ can be derived if the answer is not known in advance. If we divide equation (13) by (14), then we obtain

$$\frac{\partial A_e/\partial x}{\partial A_o/\partial x} = \frac{A_o}{A_e}.$$

Hence $\frac{\partial}{\partial x}(A_e^2 - A_o^2) = 0$, so $A_e^2 - A_o^2$ is independent of $x$. This observation suggests computing the generating function in $t$ for $A_e^2 - A_o^2$, which the computer shows is equal to $1+O(t^N)$ for a large value of $N$. Assuming then that $A_e^2 - A_o^2 = 1$ (or even proving it combinatorially), we can substitute $\sqrt{1-A_e^2}$ for $A_o$ in (13) to obtain

$$\frac{\partial A_e}{\partial x} = tA_e\sqrt{A_e^2-1} + A_e^2 - 1,$$

a single differential equation for $A_e$. This equation can routinely be solved by separation of variables (though some care must be taken to choose the correct branch of the resulting integral, including the correct sign of $\sqrt{A_e^2-1}$); we will spare the reader the details. A similar argument yields $A_o$, so we obtain $A = A_e + A_o$. ☐

Note. Ira Gessel has pointed out the following simplified expression for $B(x,t)$:

$$B(x,t) = \frac{2/\rho}{1 - \frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\sqrt{1-t^2}}. \tag{18}$$

# 3 Consequences.

A number of corollaries follow from Theorem 2.3. The first is the explicit expressions for $a_k(n)$ and $b_k(n)$ stated in the introduction. I am grateful to Ira Gessel for providing the proof given below.

**Corollary 3.1.** *For all $k, n \geq 1$ we have*

$$b_k(n) = \frac{1}{2^{k-1}} \sum_{\substack{r+2s\leq k \\ r\equiv k \pmod 2}} (-2)^s \binom{k-s}{(k+r)/2}\binom{n}{s} r^n \tag{19}$$

$$a_k(n) = b_k(n) - b_{k-1}(n). \tag{20}$$

*Proof.* Define $b'_k(n)$ to be the right-hand side of (19), and set

$$B'(x,t) = \sum_{k,n\geq 0} b'_k(n) t^k \frac{x^n}{n!}.$$

Set $n = s + m$ and $k = r + 2s + 2l$, so

$$\begin{aligned} B'(x,t) &= \sum_{r,s,l,m} (-1)^s 2^{1-r-s-2l} \binom{r+s+2l}{r+s+l}\binom{s+m}{s} r^{s+m} t^{r+2s+2l} \frac{x^{s+m}}{(s+m)!} \\ &= 2\sum_{r,s\geq 0} \left(\frac{t}{2}\right)^r \frac{(-rt^2x/2)^s}{s!} \left[\sum_l \binom{r+s+2l}{l}\left(\frac{t^2}{4}\right)^l\right]\left[\sum_m \frac{(rx)^m}{m!}\right]. \end{aligned}$$

The sum on $m$ is $e^{rx}$. Using the formula

$$\sum_k \binom{2k+a}{k} u^k = \frac{C(u)^a}{\sqrt{1-4u}},$$

where

$$C(u) = \sum_{n\geq 0} C_n u^n = \frac{1-\sqrt{1-4x}}{2x},$$

the generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, we find that the sum on $l$ is

$$\frac{C(t^2/4)^{r+s}}{\sqrt{1-t^2}} = \frac{1}{\rho}\left(\frac{2-2\rho}{t^2}\right)^{r+s}.$$

Thus

$$\begin{aligned}
B'(x,t) &= \frac{2}{\rho}\sum_{r,s\geq 0}\left(\frac{t}{2}\right)^r \frac{(-rt^2x/2)^s}{s!} e^{rx}\left(\frac{2-2\rho}{t^2}\right)^{r+s} \\
&= \frac{2}{\rho}\sum_r \left(\frac{1-\rho}{t}e^x\right)^r \sum_s \frac{(-r(1-\rho)x)^s}{s!} \\
&= \frac{2}{\rho}\sum_r \left(\frac{1-\rho}{t}e^x\right)^r e^{-r(1-\rho)x} \\
&= \frac{2}{\rho}\frac{1}{1-\dfrac{1-\rho}{t}e^{\rho x}},
\end{aligned}$$

and the proof of (19) follows from (18). Equation (20) is then an immediate consequence of (4). □

By Corollary 3.1, when $k$ is fixed $b_k(n)$ is a linear combination of $k^n$, $(k-2)^n$, $(k-4)^n, \ldots$ with coefficients that are polynomials in $n$. For $k \leq 6$ we have

$$\begin{aligned}
b_2(n) &= 2^{n-1} \\
b_3(n) &= \frac{1}{4}(3^n - 2n + 3) \\
b_4(n) &= \frac{1}{8}(4^n - 2(n-2)2^n) \\
b_5(n) &= \frac{1}{16}(5^n - (2n-5)3^n + 2(n^2 - 5n + 5)) \\
b_6(n) &= \frac{1}{32}(6^n - 2(n-3)4^n + (2n^2 - 12n + 15)2^n).
\end{aligned}$$

As a further application of Theorem 2.3 we can obtain the factorial moment generating function

$$F(x,t) = \sum_{s,n\geq 0} \nu_j(n) x^n \frac{t^j}{j!},$$

where

$$\nu_j(n) = \frac{1}{n!} \sum_{w\in\mathfrak{S}_n} (\mathrm{as}(w))_j = \frac{1}{n!} \sum_k a_k(n)(k)_j.$$

and

$$(h)_j = h(h-1)\cdots(h-j+1).$$

Namely, we have

$$\begin{aligned} \left.\frac{\partial^j A(x,t)}{\partial t^j}\right|_{t=1} &= \sum_{n\geq 0} \frac{1}{n!} \sum_{k\geq 0} a_k(n)(k)_j\, x^n \\ &= \sum_{n\geq 0} \nu_j(n) x^n. \end{aligned}$$

On the other hand, by Taylor's theorem we have

$$A(x,t) = \sum_{j\geq 0} \left.\frac{\partial^j A(x,t)}{\partial t^j}\right|_{t=1} \frac{(t-1)^j}{j!}.$$

It follows that

$$F(x,t) = A(x,t+1). \tag{21}$$

(Note that it is not at all a priori obvious from the form of $A(x,t+1)$ obtained by substituting $t+1$ for $t$ in (17) that it even has a Taylor series expansion at $t=0$.) From equations (17) and (21) it is easy to compute (using a computer) the generating functions

$$M_j(x) = \sum_{n\geq 0} \nu_j(n) x^n$$

for small $j$. For $1 \leq j \leq 4$ we get

$$M_1(x) = \frac{6x - 3x^2 + x^3}{6(1-x)^2}$$

$$M_2(x) = \frac{90x^2 - 15x^4 + 6x^5 - x^6}{90(1-x)^3}$$

$$M_3(x) = \frac{2520x^3 - 315x^4 + 189x^5 - 231x^6 + 93x^7 - 18x^8 + 2x^9}{1260(1-x)^4}$$

$$M_4(x) = \frac{N_4(x)}{9450(1-x)^5},$$

where

$$N_4(x) = 47250x^4 - 3780x^6 + 2880x^7 - 2385x^8 + 1060x^9 - 258x^{10} + 36x^{11} - 3x^{12}.$$

It is not difficult to see that in general $M_j(x)$ is a rational function of $x$ with denominator $(1-x)^{j+1}$. It follows from standard properties of rational generating functions [15, §4.3] that for fixed $j$ we have that $\nu_j(n)$ is a polynomial in $n$ of degree $j$ for $n$ sufficiently large. In particular, we have

$$\nu_1(n) = \frac{4n+1}{6}, \quad n \geq 2 \tag{22}$$

$$\nu_2(n) = \frac{40n^2 - 24n - 19}{90}, \quad n \geq 4$$

$$\nu_3(n) = \frac{1120n^3 - 2856n^2 + 440n + 1581}{3780}, \quad n \geq 6.$$

Note in particular that $\nu_1(n)$ is just the expectation (mean) of $\mathrm{as}_n$. The simple formula $(4n+1)/6$ for this quantity should be contrasted with the situation for the length $\mathrm{is}_n(w)$ of the longest increasing subsequence of $w \in \mathfrak{S}_n$, where even the asymptotic formula $E(n) \sim 2\sqrt{n}$ for the expectation is a highly nontrivial result [17, §3]. A simple proof of (22) follows from (27) and an argument of Knuth [10, Exer. 5.1.3.15].

From the formulas for $\nu_1(n)$ and $\nu_2(n)$ we easily compute the variance $\mathrm{var}(\mathrm{as}_n)$ of $\mathrm{as}_n$, namely,

$$\mathrm{var}(\mathrm{as}_n) = \nu_2(n) + \nu_1(n) - \nu_1(n)^2 = \frac{32n-13}{180},\ n \geq 4. \tag{23}$$

We now consider a further application of Theorem 2.3. Let

$$T_n(t) = \sum_{k=0}^{n} a_k(n)t^k. \tag{24}$$

For instance,

$$\begin{aligned}
T_1(t) &= t\\
T_2(t) &= t + t^2\\
T_3(t) &= t + 3t^2 + 2t^3\\
T_4(t) &= t + 7t^2 + 11t^3 + 5t^4\\
T_5(t) &= t + 15t^2 + 43t^3 + 45t^4 + 16t^5\\
T_6(t) &= t + 31t^2 + 148t^3 + 268t^4 + 211t^5 + 61t^6\\
T_7(t) &= t + 63t^2 + 480t^3 + 1344t^4 + 1767t^5 + 1113t^6 + 272t^7.
\end{aligned}$$

**Corollary 3.2.** *The polynomial $T_n(t)$ is divisible by $(1+t)^{\lfloor n/2 \rfloor}$. Moreover, if $U_n(t) = T_n(t)/(1+t)^{\lfloor n/2 \rfloor}$, then*

$$U_{2n}(-1) = -U_{2n+1}(-1) = \frac{(-1)^n E_{2n+1}}{2^n},$$

*where $E_{2n+1}$ denotes a tangent number.*

*Proof.* Let $A_e(x,t)$ and $A_o(x,t)$ be the even and odd parts of $A(x,t)$ as in equation (10). By the definition of $A_e(x)$ we have

$$A_e(x/\sqrt{1+t}, t) = \sum_{n\geq 0} \frac{T_{2n}(t)}{(1+t)^n} \frac{x^{2n}}{(2n)!}.$$

With the help of the computer we compute that

$$\begin{aligned}
\lim_{t\to -1} A_e(x/\sqrt{1+t}, t) &= \mathrm{sech}^2 \frac{x}{\sqrt{2}}\\
&= \sum_{n\geq 0} \frac{(-1)^n E_{2n+1}}{2^n} \frac{x^{2n}}{(2n)!}.
\end{aligned}$$

Hence the desired result is true for $T_{2n}(t)$. Similarly,

$$\begin{aligned}\lim_{t\to -1} \sqrt{1+t}\, A_o(x/\sqrt{1+t}, t) &= -\sqrt{2}\tanh \frac{x}{\sqrt{2}} \\ &= -\sum_{n\geq 0} \frac{(-1)^n E_{2n+1}}{2^n} \frac{x^{2n+1}}{(2n+1)!},\end{aligned}$$

proving the result for $T_{2n+1}(t)$. □

By Corollary 3.2 we have $T_n(-1) = 0$ for $n \geq 2$. In other words, for $n \geq 2$ we have

$$\#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}_n(w) \text{ even}\} = \#\{w \in \mathfrak{S}_n \,:\, \mathrm{as}_n(w) \text{ odd}\} = \frac{n!}{2}.$$

A simple combinatorial proof of this fact follows from switching the last two elements of $w$; it is easy to see that this operation either increases or decreases $as_n(w)$ by 1, as first pointed out by M. Bóna and P. Pylyavskyy. More generally, a combinatorial proof of Corollary (3.2) is a consequence of equation (27) below and an argument of Bóna [6, Lemma 1.40].

The formulas (22) and (23) for the mean and variance of $\mathrm{as}_n$ suggest in analogy with (2) that $\mathrm{as}_n$ will have a limiting distribution $K(t)$ defined by

$$K(t) = \lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{as}_n(w) - 2n/3}{\sqrt{n}} \leq t\right),$$

for all $t \in \mathbb{R}$, where $w$ is chosen uniformly from $\mathfrak{S}_n$. Indeed, we have that $K(t)$ is a Gaussian distribution with variance 8/45:

$$K(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2} ds. \tag{25}$$

It was pointed out by Pemantle (private communication) that equation (25) is a consequence of the result [13, Thms. 3.1, 3.3, or 3.5] and possibly also [5]. An independent proof was also given by Widom [19], and in the next section we explain an additional method of proof.

# 4 Relationship to alternating runs.

A *run* of a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ is a maximal factor (subsequence of consecutive elements) which is increasing. An *alternating run* is a maximal factor that is increasing or decreasing. (Perhaps "birun" would be a better term.) For instance, the permutation 64283157 has four alternating runs, viz., 642, 28, 831, and 157. Let $g_k(n)$ be the number of permutations $w \in \mathfrak{S}_n$ with $k$ alternating runs. It is easy to see, as pointed out by Bóna [7], that

$$a_k(n) = \frac{1}{2}(g_{k-1}(n) + g_k(n)), \quad n \geq 2. \tag{26}$$

If we define $G_n(t) = \sum_k g_k(n)t^k$, then equation (26) is equivalent to the formula

$$T_n(t) = \frac{1}{2}(1+t)G_n(t), \tag{27}$$

where $T_n(t)$ is defined by (24).

Research on the numbers $g_k(n)$ go back to the nineteenth century; for references see Bona [6, §1.2] and Knuth [10, Exer. 5.1.3.15–16]. In particular, let $A_n(t)$ denote the $n$th *Eulerian polynomial*, i.e.,

$$A_n(t) = \sum_{w \in \mathfrak{S}_n} t^{1+\mathrm{des}(w)},$$

where $\mathrm{des}(w)$ denotes the number of descents of $w$ (the size of the descent set defined in equation (28)). It was shown by David and Barton [8, pp. 157–162] and stated more concisely by Knuth [10, p. 605] that

$$G_n(t) = \left(\frac{1+t}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right), \quad n \geq 2,$$

where $w = \sqrt{\frac{1-t}{1+t}}$. Theorem 2.3 is then a straightforward consequence of the well-known generating function (e.g., [6, Thm. 1.7])

$$\sum_{n \geq 0} A_n(t)\frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}.$$

It is also well-known (e.g., [6, Thm. 1.10]) that the Eulerian polynomial $A_n(t)$ has only real zeros, and that the zeros of $A_n(t)$ and $A_{n+1}(t)$ interlace. From this fact Wilf [20] showed that the polynomials $G_n(t)$ have (interlacing) real zeros, and hence by (27) the polynomials $T_n(t)$ also have real zeros. It is then a consequence of standard results (e.g., [4, Thm. 2]) that the numbers $a_k(n)$ for fixed $n$ are asymptotically normal as $n \to \infty$, yielding another proof of (25).

# 5 Open problems.

In this section we mention three directions of possible generalization of our work above.

1. Let $\text{is}(m, w)$ denote the length of the longest subsequence of $w \in \mathfrak{S}_n$ that is a union of $m$ increasing subsequences, so $\text{is}(w) = \text{is}(1, w)$. The numbers $\text{is}(m, w)$ have many interesting properties, summarized in [17, §4]. Can anything be said about the analogue for alternating sequences, i.e., the length $\text{as}(m, w)$ of the longest subsequence of $w$ that is a union of $m$ alternating subsequences? This question can also be formulated in terms of the lengths of the alternating runs of $w$.

2. Can the results for increasing subsequences and alternating subsequences be generalized to other "patterns"? More specifically, let $\sigma$ be a (finite) word in the letters $U$ and $D$, e.g., $\sigma = UUDUD$. Let $\sigma^\infty$ denote the infinite word $\sigma\sigma\sigma\cdots$, e.g.,

$$(UUD)^\infty = UUDUUDUUD\cdots.$$

   For this example, we have for instance that $UUDUUDU$ is a prefix of $\sigma^\infty$ of length 7.

   Let $\tau = a_1 a_2 \cdots a_{m-1}$ be a word of length $m-1$ in the letters $U$ and $D$. A sequence $v = v_1 v_2 \cdots v_m$ of integers is said to have *descent word* $\tau$ if $v_i > v_{i+1}$ whenever $a_i = D$, and $v_i < v_{i+1}$ whenever $a_i = U$. Thus $v$ is increasing if and only if

$\tau = U^{m-1}$, and $v$ is alternating if and only if $\tau = (DU)^{j-1}$ or $\tau = (DU)^{j-1}D$ depending on whether $m = 2j-1$ or $m = 2j$.

Now let $w \in \mathfrak{S}_n$ and define $\mathrm{len}_\sigma(w)$ to be the length of longest subsequence of $w$ whose descent word is a prefix of $\sigma^\infty$. Thus $\mathrm{len}_U(w) = \mathrm{is}_n(w)$ and $\mathrm{len}_{DU}(w) = \mathrm{as}_n(w)$. What can be said in general about $\mathrm{len}_\sigma(w)$? In particular, let

$$E_\sigma(n) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \mathrm{len}_\sigma(w),$$

the expectation of $\mathrm{len}_\sigma(w)$ for $w \in \mathfrak{S}_n$. Note that $E_U(n) \sim 2\sqrt{n}$ by (1), and $E_{DU}(n) \sim 2n/3$ by (7). Is it true that for any $\sigma$ we have $E_\sigma(n) \sim \alpha n^c$ for some $\alpha, c > 0$? Or at least that for some $c > 0$ (depending on $\sigma$) we have

$$\lim_{n\to\infty} \frac{\log E_\sigma(n)}{\log n} = c,$$

in which case can we determine $c$ explicitly?

3. The *descent set* $D(w)$ of a permutation $w = w_1 \cdots w_n$ is defined by
$$D(w) = \{i : w_i > w_{i+1}\} \subseteq [n-1], \tag{28}$$
where $[n-1] = \{1, 2, \ldots, n-1\}$. Thus $w$ is alternating if and only if $D(w) = \{1, 3, 5, \ldots\} \cap [n-1]$. Let $S \subseteq [k-1]$. What can be said about the number $b_{k,S}(n)$ of permutations $w \in \mathfrak{S}_n$ that avoid all $v \in \mathfrak{S}_k$ satisfying $D(v) = S$? In particular, what is the value $L_{k,S} = \lim_{n\to\infty} b_{k,S}(n)^{1/n}$? (It follows from [2] and [12], generalized in an obvious way, that this limit exists and is finite.) For instance, if $S = \emptyset$ or $S = [k-1]$, then it follows from [14] that $L_{k,S} = (k-1)^2$. On the other hand, if $S = \{1, 3, 5, \ldots\} \cap [k-1]$ then it follows from (19) that $L_{k,S} = k-1$.

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