Solutions to Exercises on Catalan and Related Numbers

19. It would require a treatise in itself to discuss thoroughly all the known interconnections among these problems. We will content ourselves with some brief hints and references that should serve as a means of further exposure to "Catalan disease" or "Catalania" (= Catalan mania). One interesting item omitted from the list because of its complicated description is the number of flexagons of order $n+1$; see C. O. Oakley and R. J. Wisner, *Amer. Math. Monthly* **64** (1957), 143–154 (esp. p. 152) for more information.

(a,b,c,h,i,r) These are covered by Corollary 2.3.

(d) This is covered by Example 3.12. Alternatively, do a depth-first search, ignoring the root edge, recording 1 when a left edge is first encountered, and recording -1 when a right edge is first encountered. This gives a bijection with (r). Note also that when all endpoints are removed (together with the incident edges), we obtain the trees of (c).

(e) This is covered by Example 2.8. For a bijection with (r), do a depth-first (preorder) search through the tree. When going "down" an edge (away from the root) record a 1, and when going up an edge record a -1. For further information and references, see D. A. Klarner, *J. Combinatorial Theory* **9** (1970), 401–411.

(f) When the root is removed we obtain the trees of (d). See also Klarner, *op. cit.*

(g) The bijection between parts (i) and (iv) of Proposition 2.1 gives a bijection between the present problem and (j). An elegant bijection with (e) was given by F. Bernhart (private communication, 1996).

(j) Let $A(x) = x + x^3 + 2x^4 + 6x^5 + \cdots$ (respectively, $B(x) = x^2 + x^3 + 3x^4 + 8x^5 + \cdots$) be the generating function for Dyck paths from $(0,0)$ to $(2n,0)$ $(n > 0)$ such that the path only touches the x-axis at the beginning and end, and the number of steps $(1,-1)$ at the end of the path is odd (respectively, even). Let $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ be the generating function for all Dyck paths from $(0,0)$ to $(2n,0)$, so the coefficients are Catalan numbers by (i). It is easy to see that $A = x(1 + CB)$ and $B = xCA$. (Also $C = 1/(1 - A - B)$, though we don't need this fact here.) Solving for A gives $A = x/(1 - x^2C^2)$. The generating function we want is $1/(1-A)$, which simplifies (using $1 + xC^2 = C$) to $1 + xC$, and the proof follows. This result is due to E. Deutsch (private communication, 1996).

(k) This result is due to P. Peart and W. Woan, Dyck paths with no peaks at height 2, preprint. The authors give a generating function proof and a simple bijection with (i).

(l) The region bounded by the two paths is called a *parallelogram polyomino*. It is an array of unit squares, say with k columns $C_1, \ldots, C_k$. Let a_i be the number of squares in column C_i, for $1 \le i \le k$, and let b_i be the number of rows in common to C_i and C_{i+1}, for $1 \le i \le k-1$. Define a sequence σ of 1's and -1's as follows (where exponentiation denotes repetition):

$$\sigma = 1^{a_1}(-1)^{a_1-b_1+1}1^{a_2-b_1+1}(-1)^{a_2-b_2+1}1^{a_3-b_2+1}\cdots 1^{a_k-b_{k-1}+1}(-1)^{a_k}.$$

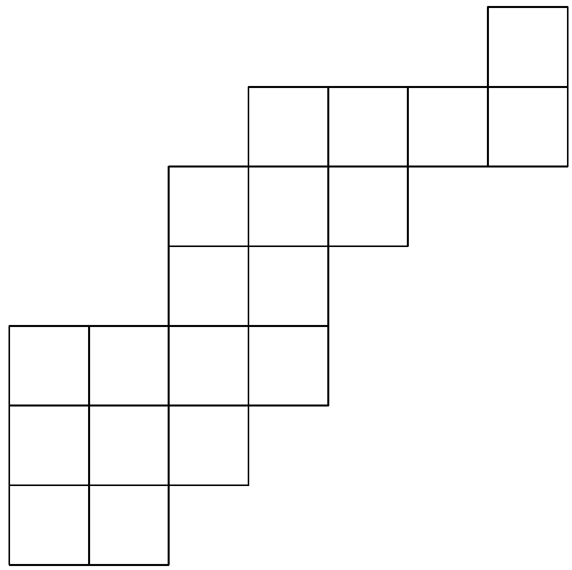

Figure 12: A parallelogram polyomino

This sets up a bijection with (r). For the parallelogram polyomino of Figure 12 we have $(a_1, \ldots, a_7) = (3,3,4,4,2,1,2)$ and $(b_1, \ldots, b_6) = (3,2,3,2,1,1)$. Hence (writing $-$ for -1)

$$\sigma = 111 - 1 - -111 - -11 - - - 1 - -1 - 11 - -.$$

The enumeration of parallelogram polyominoes is due to J. Levine, *Scripta Math.* **24** (1959), 335–338, and later G. Pólya, *J. Combinatorial Theory* **6** (1969) 102–105. See also L. W. Shapiro, *Discrete Math.* **14** (1976), 83–90; W.-J. Woan, L. W. Shapiro, and D. G. Rogers, *Amer. Math. Monthly* **104** (1997), 926–931; G. Louchard, *Random Structures and Algorithms* **11** (1997), 151–178; and R. A. Sulanke, *J. Difference Equations and Applications*, to appear. For more information on the fascinating topic of polyomino enumeration, see M.-P. Delest and G. Viennot, *Theoretical Computer Science* **34** (1984), 169–206, and X. G. Viennot, in *Séries formelles et combinatoire algébrique* (P. Leroux and C. Reutenauer, eds.), Publications de Laboratoire de Combinatoire er d'Informatique Mathématique, vol. 11, Université du Québec à Montréal, 1992, pp. 399–420.

(m) Regarding a path as a sequence of steps, remove the first and last steps from the two paths in (l). This variation of (l) was suggested by L. W. Shapiro (private communication, 1998).

(n) Fix a vertex v. Starting clockwise from v, at each vertex write 1 if encountering a chord for the first time and -1 otherwise. This gives a bijection with (r). This result is apparently due to A. Errera, *Mém. Acad. Roy. Belgique Coll. 8º (2)* **11** (1931), 26 pp. See also J. Riordan, *Math. Comp.* **29** (1975), 215–222, and S. Dulucq and J.-G. Penaud, *Discrete Math.* **17** (1993), 89–105.

(o) Cut the circle in (n) between two fixed vertices and "straighten out."

(p,q) These results are due to I. M. Gelfand, M. I. Graev and A. Postnikov, in *The Arnold-Gelfand Mathematical Seminars*, Birkhäuser, Boston, 1997, pp. 205–221 (§6). For (p), note that there is always an arc from the leftmost to the rightmost vertex. When this arc is removed, we obtain two smaller trees satisfying the conditions of the problem. This leads to an easy bijection with (c). The trees of (p) are called *noncrossing alternating trees*.

An equivalent way of stating the above bijection is as follows. Let T be a noncrossing alternating tree on the vertex set $1, 2, \ldots, n+1$ (in that order from left to right). Suppose that vertex i has r_i neighbors that are larger than i. Let u_i be the word in the alphabet $\{1, -1\}$ consisting of r_i 1's followed by a -1. Let $u(T) = u_1 u_2 \cdots u_{n+1}$. Then u is a bijection between the objects counted by (p) and (r). It was shown by M. Schlosser that exactly the same definition of u gives a bijection between (q) and (r)! The proof, however, is considerably more difficult than in the case of (p). (A more complicated bijection was given earlier by C. Krattenthaler.)

For further information on trees satisfying conditions (α), (β), and (δ) (called *alternating trees*), see Exercise 5.41.

(s) Consider a lattice path P of the type (h). Let a_{i-1} be the area above the x-axis between $x = i-1$ and $x = i$, and below P. This sets up a bijection as desired.

(t) Subtract $i-1$ from a_i and append a one at the beginning to get (s). This result is closely related to Exercise 25(c). If we replace the alphabet $1, 2, \ldots, 2(n-1)$ with the alphabet $\overline{n-1}$, $n-1$, $\overline{n-2}$, $n-2, \ldots, \overline{1}, 1$ (in that order) and write the new sequence $b_1, b_2, \ldots, b_{n-1}$ in reverse order in a column, then we obtain the arrays of R. King, in *Lecture Notes in Physics*, vol. 50, Springer-Verlag, Berlin/Heidelberg/New York, 1975, pp. 490-499 (see also S. Sundaram, *J. Combinatorial Theory (A)* **53** (1990), 209–238 (Def. 1.1)) that index the weights of the $(n-1)$-st fundamental representation of $\mathrm{Sp}(2(n-1), \mathbb{C})$.

(u) Let $b_i = a_i - a_{i+1} + 1$. Replace a_i with one 1 followed by b_i -1's for $1 \leq i \leq n$ (with $a_{n+1} = 0$) to get (r).

(v) Take the first differences of the sequences in (u).

(w) Do a depth-first search through a plane tree with $n+1$ vertices as in (e). When a vertex is encountered for the first time, record one less than its number of successors, except that the last vertex is ignored. This gives a bijection with (e).

(x) These sequences are just the inversion tables (as defined in Section 1.3) of the 321-avoiding permutations of (ee). For a proof see S. C. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (Thm. 2.1). (The previous reference deals with the *code* $c(w)$ of a permutation w rather than the inversion table $I(w)$. They are related by $c(w) = I(w^{-1})$. Since w is 321-avoiding if and only if w^{-1} is 321-avoiding, it makes no difference whether we work with the code or with the inversion table.)

(y) If we replace a_i by $n - a_i$, then the resulting sequences are just the inversion tables of 213-avoiding permutations w (i.e., there does not exist $i < j < k$ such that

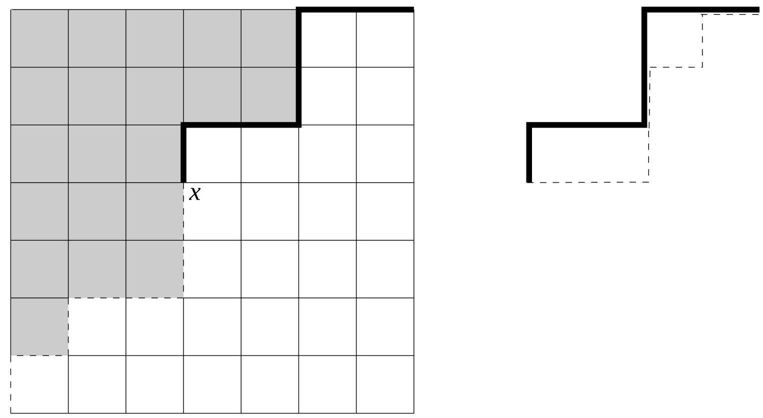

Figure 13: A bijection between (ccc) and (bb)

$w_j < w_i < w_k$). Such permutations are in obvious bijection with the 312-avoiding permutations of (ff). For further aspects of this exercise, see Exercise 32.

(z) Given a sequence $a_1, \ldots, a_n$ of the type being counted, define recursively a binary tree $T(a_1, \ldots, a_n)$ as follows. Set $T(\emptyset) = \emptyset$. If $n > 0$, then let the left subtree of the root of $T(a_1, \ldots, a_n)$ be $T(a_1, a_2, \ldots, a_{n-a_n})$ and the right subtree of the root be $T(a_{n-a_n+1}, a_{n-a_n+2}, \ldots, a_{n-1})$. This sets up a bijection with (c). Alternatively, the sequences $a_n - 1, a_{n-1} - 1, \ldots, a_1 - 1$ are just the inversion tables of the 312-avoiding permutations of (ff). Let us also note that the sequences $a_1, a_2, \ldots, a_n$ are precisely the sequences $\tau(u)$, $u \in \mathfrak{S}_n$, of Exercise 5.49(d).

(aa) If $a = a_1 a_2 \cdots a_k$ is a word in the alphabet $[n-1]$, then let $w(a) = s_{a_1} s_{a_2} \cdots s_{a_k} \in \mathfrak{S}_n$, where s_i denotes the adjacent transposition $(i, i+1)$. Then $w(a) = w(b)$ if $a \sim b$; and the permutations $w(a)$, as a ranges over a set of representatives for the classes B being counted, are just those enumerated by (ee). This statement follows from S. C. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (Theorem 2.1).

(bb) Regard a partition whose diagram fits in an $(n-1) \times (n-1)$ square as an order ideal of the poset $(\mathbf{n-1}) \times (\mathbf{n-1})$ in an obvious way. Then the partitions being counted correspond to the order ideals of (ccc). Bijections with other Catalan families were given by D. E. Knuth and A. Postnikov. Postnikov's bijection is the following. Let λ be a partition whose diagram is contained in an $(n-1) \times (n-1)$ square S. Let x be the lower right corner of the Durfee square of λ. Let L_1 be the lattice path from the upper-right corner of S to x that follows the boundary of λ. Similarly let L_2 be the lattice path from the lower left corner of S to x that follows the boundary of λ. Reflect L_2 about the main diagonal of S. The paths L_1 and the reflection of L_2 form a pair of paths as in (m). Figure 13 illustrates this bijection for $n = 8$ and $\lambda = (5, 5, 3, 3, 3, 1)$. The path L_1 is shown in dark lines and L_2 and its reflection in dashed lines.

(cc) Remove the first occurrence of each number. What remains is a permutation w of $[n]$ that uniquely determines the original sequence. These permutations are precisely the ones in (ff). There is also an obvious bijection between the sequences being counted and the nonintersecting arcs of (o).

(dd) Replace each odd number by 1 and even number by -1 to get a bijection with (r).

(ee) Corollary 7.23.11 shows that the RSK algorithm (Section 7.11) establishes a bijection with (xx). See also D. E. Knuth, *The Art of Computer Programming*, vol. 3, *Sorting and Searching*, Addison-Wesley, Reading, Massachusetts, 1973 (p. 64).

The earliest explicit enumeration of 321-avoiding permutations seems to be due to J. M. Hammersley, in *Proc. Sixth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, University of California Press, Berkeley/Los Angeles, 1972, pp. 345–394. In equation (15.9) he states the result, saying "and this can be proved in general." The first published proof is a combinatorial proof due to D. G. Rogers, *Discrete Math.* **22** (1978), 35–40. Another direct combinatorial proof, based on an idea of Goodman, de la Harpe, and Jones, appears in S. C. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (after the proof of Theorem 2.1). A sketch of this proof goes as follows. Given the 321-avoiding permutation $w = a_1 a_2 \cdots a_n$, define $c_i = \#\{j : j > i, w_j < w_i\}$. Let $\{j_1, \dots, j_\ell\}_< = \{j : c_j > 0\}$. Define a lattice path from $(0,0)$ to (n,n) as follows. Walk horizontally from $(0,0)$ to $(c_{j_1} + j_1 - 1, 0)$, then vertically to $(c_{j_1} + j_1 - 1, j_1)$, then horizontally to $(c_{j_2} + j_2 - 1, j_1)$, then vertically to $(c_{j_2} + j_2 - 1, j_2)$, etc. The last part of the path is a vertical line from $(c_{j_\ell} + j_\ell - 1, j_{\ell-1})$ to $(c_{j_\ell} + j_\ell - 1, j_\ell)$, then (if needed) a horizontal line to $(c_{j_\ell} + j_\ell - 1, n)$, and finally a vertical line to (n,n). This establishes a bijection with (h).

For an elegant bijection with (ff), see R. Simion and F. W. Schmidt, *Europ. J. Combinatorics* **6** (1985), 383–406 (Prop. 19). Two other bijections with (ff) appear in D. Richards, *Ars Combinatoria* **25** (1988), 83–86, and J. West, *Discrete Math.* **146** (1995) 247–262 (Thm. 2.8).

(ff) There is an obvious bijection between 312-avoiding and 231-avoiding permutations, viz., $a_1 a_2 \dots a_n \mapsto n+1-a_n, \dots, n+1-a_2, n+1-a_1$. It is easily seen that the 231-avoiding permutations are the same as those of (ii), as first observed by D. E. Knuth [41, Exer. 2.2.1.5]$_5$. The enumeration *via* Catalan numbers appears in *ibid.*, Exer. 2.2.1.4. References to bijections with (ee) are given in the solution to (ee). For the problem of counting permutations in $\mathfrak{S}_n$ according to the number of subsequences with the pattern 132 (equivalently, 213, 231, or 312), see M. Bóna, in *Conference Proceedings*, vol. 1, *Formal Power Series and Algebraic Combinatorics, July 14 – July 18, 1997, Universität Wien*, pp. 107–118.

(gg) This result appears on p. 796 of D. M. Jackson, *Trans. Amer. Math. Soc.* **299** (1987), 785–801, but probably goes back much earlier. For a direct bijective proof, it is not hard to show that the involutions counted here are the same as those in (kk).

(hh) A coding of planar maps due to R. Cori, *Astérisque* **27** (1975), 169 pp., when restricted to plane trees, sets up a bijection with (e).

(ii) When an element a_i is put on the stack, record a 1. When it is taken off, record a -1. This sets up a bijection with (r). This result is due to D. E. Knuth [41, Exer. 2.2.1.4]$_5$. The permutations being counted are just the 231-avoiding permutations, which are in obvious bijection with the 312-avoiding permutations of (ff) (see Knuth, *ibid.*, Exercise 2.2.1.5).

(jj) Same set as (ee), as first observed by R. Tarjan, *J. Assoc. Comput. Mach.* **19** (1972), 341–346 (the case $m = 2$ of Lemma 2). The concept of queue sorting is due to Knuth [41, Ch. 2.2.1]$_5$.

(kk) Obvious bijection with (o).

(ll) See I. M. Gessel and C. Reutenauer, *J. Combinatorial Theory (A)* **64** (1993), 189–215 (Thm. 9.4 and discussion following).

(mm) This result is due to O. Guibert and S. Linusson, in *Conference Proceedings*, vol. 2, *Formal Power Series and Algebraic Combinatorics, July 14 – July 18, 1997, Universität Wien*, pp. 243–252. Is there a nice formula for the number of alternating Baxter permutations of $[m]$?

(nn) This is the same set as (ee). See Theorem 2.1 of the reference given in (ee) to S. C. Billey *et al.* For a generalization to other Coxeter groups, see J. R. Stembridge, *J. Alg. Combinatorics* **5** (1996), 353–385.

(oo) These are just the 132-avoiding permutations $w_1 \cdots w_n$ of $[n]$ (i.e., there does not exist $i < j < k$ such that $w_i < w_k < w_j$), which are in obvious bijection with the 312-avoiding permutations of (ff). This result is an immediate consequence of the following results: (i) I. G. Macdonald, *Notes on Schubert Polynomials*, Publications du LACIM, vol. 6, Univ. du Québec à Montréal, 1991, (4.7) and its converse stated on p. 46 (due to A. Lascoux and M. P. Schützenberger), (ii) *ibid.*, eqn. (6.11) (due to Macdonald), (iii) part (ff) of this exercise, and (iv) the easy characterization of dominant permutations (as defined in Macdonald, *ibid.*) as 132-avoiding permutations. For a simpler proof of the crucial (6.11) of Macdonald, see S. Fomin and R. Stanley, *Advances in Math.* **103** (1994), 196–207 (Lemma 2.3).

(pp) See Exercise 3.68(b). Noncrossing partitions first arose in the work of H. W. Becker, *Math. Mag.* **22** (1948–49), 23–26, in the form of *planar rhyme schemes*, i.e., rhyme schemes with no crossings in the *Puttenham diagram*, defined by G. Puttenham, *The Arte of English Poesie*, London, 1589 (pp. 86–88). Further results on noncrossing partitions are given by H. Prodinger, *Discrete Math.* **46** (1983), 205–206; N. Dershowitz and S. Zaks, *Discrete Math.* **62** (1986), 215–218; R. Simion and D. Ullman, *Discrete Math.* **98** (1991), 193–206; P. H. Edelman and R. Simion, *Discrete Math.* **126** (1994), 107–119; R. Simion, *J. Combinatorial Theory (A)* **65** (1994); R. Speicher, *Math. Ann.* **298** (1994), 611–628; A. Nica and R. Speicher, *J. Algebraic Combinatorics*, **6** (1997), 141–160; R. Stanley, *Electron. J. Combinatorics* **4**, R20 (1997), 14 pp. See also Exercise 5.35.

(qq) These partitions are clearly the same as the noncrossing partitions of (pp). This description of noncrossing partitions is due to R. Steinberg (private communication).

(rr) Obvious bijection with (pp). (Vertical lines are in the same block if they are connected by a horizontal line.) As mentioned in the Notes to Chapter 1, Murasaki diagrams were used in *The Tale of Genji* to represent the 52 partitions of a five-element set. The noncrossing Murasaki diagrams correspond exactly to the noncrossing partitions. The statement that noncrossing Murasaki diagrams are enumerated by Catalan numbers seems first to have been observed by H. W. Gould, who pointed it out to M. Gardner, leading to its mention in [27]. Murasaki diagrams were not actually used by Lady Murasaki herself. It wasn't until the Wasan period of old Japanese mathematics, from the late 1600s well into the 1700s, that the Wasanists started attaching the Murasaki diagrams (which were actually incense diagrams) to illustrated editions of *The Tale of Genji*.

(ss) This result was proved by M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 56), using generating function techniques.

(tt) See R. C. Mullin and R. G. Stanton, *Pacific J. Math.* **40** (1972), 167-172 (p. 168). They set up a bijection with (e). They also show that $2n+1$ is the largest possible value of k for which there exists a noncrossing partition of $[k]$ with $n+1$ blocks such that no block contains two consecutive integers. A simple bijection with (a) was given by D. P. Roselle, *Utilitas Math.* **6** (1974), 91–93. The following bijection with (e) is due to A. Vetta (1997). Label the vertices $1, 2, \ldots, 2n+1$ of a tree in (e) in preorder. Define i and j to be in the same block of $\pi \in \Pi_{2n+1}$ if j is a right child of i.

(uu) Let P_n denote the poset of intervals with at least two elements of the chain $\mathbf{n}$, ordered by inclusion. Let $\mathcal{A}_n$ denote the set of antichains of P_n. By the last paragraph of Section 3.1, $\#\mathcal{A}_n$ is equal to the number of order ideals of P_n. But P_n is isomorphic to the poset $\mathrm{Int}(\mathbf{n-1})$ of *all* (nonempty) intervals of $\mathbf{n-1}$, so by (bbb) we have $\#\mathcal{A}_n = C_n$. Given an antichain $A \in \mathcal{A}_n$, define a partition π of $[n]$ by the condition that i and j (with $i < j$) belong to the same block of π if $[i, j] \in A$ (and no other conditions not implied by these). This establishes a bijection between $\mathcal{A}_n$ and the nonnesting partitions of $[n]$. For a further result on nonnesting partitions, see the solution to Exercise 5.44. The present exercise was obtained in collaboration with A. Postnikov. The concept of nonnesting partitions for any reflection group (with the present case corresponding to the symmetric group $\mathfrak{S}_n$) is due to Postnikov and is further developed in C. A. Athanasiadis, On noncrossing and nonnesting partitions for classical reflection groups, preprint, 1998.

(vv) If $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \subseteq (n-1, n-2, \ldots, 1)$, then the sequences $(1, \lambda_{n-1}+1, \ldots, \lambda_1+1)$ are in bijection with (s). Note also that the set of Young diagrams contained in $(n-1, n-2, \ldots, 1)$, ordered by inclusion (i.e., the interval $[\emptyset, (n-1, n-2, \ldots, 0)]$ in Young's lattice, as defined in Exercise 3.63), is isomorphic to $J(\mathrm{Int}(\mathbf{n-1}))^*$, thereby setting up a bijection with (bbb).

(ww) Given a standard Young tableau T of shape (n, n), define $a_1 a_2 \cdots a_{2n}$ by $a_i = 1$ if i appears in row 1 of T, while $a_i = -1$ if i appears in row 2. This sets up a bijection with (r). See also $[72, \text{p. } 63]_7$ and our Proposition 7.10.3.

(xx) See the solution to (ee) (first paragraph) for a bijection with 321-avoiding permutations. An elegant bijection with (ww) appears in [15, vol. 1, p. 131]$_2$ (repeated in [72, p. 63]$_7$). Namely, given a standard Young tableau T of shape (n, n), let P consist of the part of T containing the entries $1, 2, \ldots, n$; while Q consists of the complement in T of P, rotated 180°, with the entry i replaced by $2n+1-i$. See also Corollary 7.23.12.

(yy) Let b_i be the number of entries in row i that are equal to $n-i+1$ (so $b_n = 0$). The sequences $b_n+1, b_{n-1}+1, \ldots, b_1+1$ obtained in this way are in bijection with (s).

(zz) This result is equivalent to Prop. 2.1 of S. C. Billey, W. Jockusch, and R. Stanley, *J. Algebraic Combinatorics* **2** (1993), 345–374. See also the last paragraph on p. 363 of this reference.

(aaa) Obvious bijection with (ww). This interpretation of Catalan numbers appears in [34, p. 222]$_3$. Note also that if we label the elements of $\mathbf{2} \times \mathbf{n}$ analogously to what was illustrated for $n = 3$, then the linear extensions coincide with the permutations of (dd).

(bbb) There is an obvious bijection with order ideals I of Int($\mathbf{n}$) that contain every one-element interval of $\mathbf{n}$. But the "upper boundary" of the Hasse diagram of I "looks like" the Dyck paths of (i). See [34, bottom of p. 222]$_3$.

(ccc) This result is equivalent to the $q = 1$ case of G. E. Andrews, *J. Stat. Plan. Inf.* **34** (1993), 19–22 (Corollary 1). For a more explicit statement and some generalizations, see R. G. Donnelly, Ph.D. thesis, University of North Carolina, 1997, and Symplectic and odd orthogonal analogues of $L(m, n)$, preprint. For a bijective proof, see the solution to (bb). A sequence of posets interpolating between the poset Int($\boldsymbol{n-1}$) of (bbb) and A_{n-1}, and all having C_n order ideals, was given by D. E. Knuth (private communication, 9 December 1997).

(ddd) Given a sequence $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$, define a poset P on the set $\{x_1, \ldots, x_n\}$ by the condition that $x_i < x_j$ if and only if $j + a_{n+1-i} \geq n+1$. (Equivalently, if Z is the matrix of the zeta function of P, then the 1's in $Z - I$ form the shape of the Young diagram of a partition, rotated 90° clockwise and justified into the upper right-hand corner.) This yields a bijection with (s). This result is due to R. L. Wine and J. E. Freund, *Ann. Math. Statis.* **28** (1957), 256–259. See also R. A. Dean and G. Keller, *Canad. J. Math.* **20** (1968), 535–554. Such posets are now called *semiorders.* For further information, see P. C. Fishburn, *Interval Orders and Interval Graphs*, Wiley-Interscience, New York, 1985, and W. T. Trotter, *Combinatorics and Partially Ordered Sets*, Johns Hopkins University Press, Baltimore/London, 1992 (Ch. 8). For the labelled version of this exercise, see Exercise 30.

(eee) The lattice $J(P)$ of order ideals of the poset P has a natural planar Hasse diagram. There will be two elements covering $\hat{0}$, corresponding to the two minimal elements of P. Draw the Hasse diagram of $J(P)$ so that the rooted minimal element of P goes to the left of $\hat{0}$ (so the other minimal element goes to the right). The "outside boundary" of the Hasse diagram then "looks like" the pair of paths in (l) (rotated 45° counterclockwise).

(fff) These relations are called *similarity relations.* See L. W. Shapiro, *Discrete Math.* **14** (1976), 83–90; V. Strehl, *Discrete Mathematics* **19** (1977), 99–101; D. G. Rogers, *J. Combinatorial Theory (A)* **23** (1977), 88–98; J. W. Moon, *Discrete Math.* **26** (1979), 251–260. Moon gives a bijection with (r). E. Deutsch (private communication) has pointed out an elegant bijection with (h), viz., the set enclosed by a path and its reflection in the diagonal *is* a similarity relation (as a subset of $[n]\times[n]$). The connectedness of the columns ensures the last requirement in the definition of a similarity relation.

(ggg) A simple combinatorial proof was given by L. W. Shapiro, *J. Combinatorial Theory* **20** (1976), 375–376. Shapiro observes that this result is a combinatorial manifestation of the identity

$$\sum_{k\geq 0}\binom{n}{2k}2^{n-2k}C_k = C_{n+1},$$

due to J. Touchard, in *Proc. Int. Math. Congress, Toronto (1924)*, vol. 1, 1928 (p. 465).

(hhh) Obvious bijection with (bbb). This interpretation in terms of stacking coins is due to J. Propp. See A. M. Odlyzko and H. S. Wilf, *Amer. Math. Monthly* **95** (1988), 840–843 (Rmk. 1).

(iii) See J. H. van Lint, *Combinatorial Theory Seminar, Eindhoven University of Technology*, Lecture Notes in Mathematics, no. 382, Springer-Verlag, Berlin/Heidelberg/ New York, 1974 (pp. 22 and 26–27).

(jjj) The total number of n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ is $\binom{2n}{n}$ (see Section 1.2). Call two such multisets M and N *equivalent* if for some $k\in\mathbb{Z}/(n+1)\mathbb{Z}$ we have $M=\{a_1,\ldots,a_n\}$ and $N=\{a_1+k,\ldots,a_n+k\}$. This defines an equivalence relation in which each equivalence class contains $n+1$ elements, exactly one of which has its elements summing to 0. Hence the number of multisets with elements summing to 0 (or to any other fixed element of $\mathbb{Z}/(n+1)\mathbb{Z}$) is $\frac{1}{n+1}\binom{2n}{n}$. This result appears in R. K. Guy, *Amer. Math. Monthly* **100** (1993), 287–289 (with a more complicated proof due to I. Gessel).

(kkk) This result is implicit in the paper G. X. Viennot, *Astérisque* **121–122** (1985), 225–246. Specifically, the bijection used to prove (12), when restricted to Dyck words, gives the desired bijection. A simpler bijection follows from the work of J.-G. Penaud, in *Séminaire Lotharingien de Combinatoire*, 22^{e} Session, Université Louis Pasteur, Strasbourg, 1990, pp. 93–130 (Cor. IV-2-8). Yet another proof follows from more general results of J. Bétréma and J.-G. Penaud, *Theoret. Comput. Sci.* **117** (1993), 67–88. For some related problems, see Exercise 46.

(lll) Let I be an order ideal of the poset $\mathrm{Int}(\mathbf{n-1})$ defined in (bbb). Associate with I the set R_I of all points $(x_1,\ldots,x_n)\in\mathbb{R}^n$ satisfying $x_1>\cdots>x_n$ and $x_i-x_j<1$ if $[i,j-1]\in I$. This sets up a bijection between (bbb) and the regions R_I being counted. This result is implicit in R. Stanley, *Proc. Natl. Acad. Sci. U.S.A.* **93** (1996), 2620–2625 (§2), and also appears (as part of more general results) in C. A. Athanasiadis, Ph.D. thesis, Massachusetts Institute of Technology, 1996 (Cor. 7.1.3) and A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint, (Prop. 7.2), available at http://front.math.ucdavis.edu/math.CO/9712213.

(mmm) Let P be a convex $(n+2)$-gon with vertices $v_1, v_2, \ldots, v_{n+2}$ in clockwise order. Let T be a triangulation of T as in (a), and let a_i be the number of triangles incident to v_i. Then the map $T \mapsto (a_1, \ldots, a_{n+2})$ establishes a bijection with (a). This remarkable result is due to J. H. Conway and H. S. M. Coxeter [20, problems (28) and (29)]. The arrays (54) are called *frieze patterns*.

(nnn) See F. T. Leighton and M. Newman, *Proc. Amer. Math. Soc.* **79** (1980), 177–180, and L. W. Shapiro, *Proc. Amer. Math. Soc.* **90** (1984), 488–496.

20. (a) Given a path P of the first type, let (i, i) be the first point on P that intersects $y = x$. Replace the portion of P from $(1, 0)$ to (i, i) by its reflection about $y = x$. This yields the desired bijection.

This argument is the famous "reflection principle" of D. André, *C. R. Acad. Sci. Paris* **105** (1887), 436–437. The application (b) below is also due to André. The importance of the reflection principle in combinatorics and probability theory was realized by W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, John Wiley and Sons, New York, 1950 (3rd edition, 1968). For a vast number of extensions and ramifications, see L. Takács, *Combinatorial Methods in the Theory of Stochastic Processes*, John Wiley and Sons, New York, 1967; T. V. Narayana, *Lattice Path Combinatorics with Statistical Applications*, Mathematical Expositions no. 23, University of Toronto Press, Toronto, 1979; and S. G. Mohanty, *Lattice Path Counting and Applications*, Academic Press, New York, 1979. For a profound generalization of the reflection principle based on the theory of Coxeter groups, as well as some additional references, see I. Gessel and D. Zeilberger, *Proc. Amer. Math. Soc.* **115** (1992), 27–31.

(b) The first step in such a lattice path must be from $(0, 0)$ to $(1, 0)$. Hence we must subtract from the total number of paths from $(1, 0)$ to (m, n) the number that intersect $y = x$, so by (a) we get $\binom{m+n-1}{n} - \binom{m+n-1}{m} = \frac{m-n}{m+n}\binom{m+n}{n}$.

(c) Move the path one unit to the right to obtain the case $m = n + 1$ of (b).

21. (a) Given a path $P \in X_n$, define $c(P) = (c_0, c_1, \ldots, c_n)$, where c_i is the number of horizontal steps of P at height $y = i$. It is not difficult to verify that the cyclic permutations $C_j = (c_j, c_{j+1}, \ldots, c_n, c_1, \ldots, c_{j-1})$ of $c(P)$ are all distinct, and for each such there is a unique $P_j \in X_n$ with $c(P_j) = C_j$. Moreover, the number of excedances of the paths $P = P_0, P_1, \ldots, P_n$ are just the numbers $0, 1, \ldots, n$ in some order. From these observations the proof is immediate.

This result, known as the *Chung-Feller theorem*, is due to K. L. Chung and W. Feller, *Proc. Nat. Acad. Sci. U.S.A.* **35** (1949), 605–608. A refinement was given by T. V. Narayana, *Skand. Aktuarietidskr.* (1967), 23–30. For further information, see the books by Narayana (§ I.2) and Mohanty (§ 3.3) mentioned in the solution to Exercise 20(a).

(b) Immediate from (a).

22. This result was given by L. W. Shapiro, problem E2903, *Amer. Math. Monthly* **88** (1981), 619. An incorrect solution appeared in **90** (1983), 483–484. A correct but nonbijective solution was given by D. M. Bloom, **92** (1985), 430. The editors asked for a bijective proof in problem E3096, **92** (1985), 428, and such a proof was given by W. Nichols, **94** (1987), 465–466. Nichols's bijection is the following. Regard a lattice path as a sequence of E's (for the step $(1,0)$) and N's (for the step $(0,1)$). Given a path P of the type we are enumerating, define recursively a new path $\psi(P)$ as follows:

$$\psi(\emptyset) = \emptyset, \quad \psi(DX) = D\psi(X), \quad \psi(D'X) = E\psi(X)ND^*,$$

where (a) D is a path of positive length, with endpoints on the diagonal $x = y$ and all other points below the diagonal, (b) D' denotes the path obtained from D by interchanging E's and N's, and (c) $D = ED^*N$. Then ψ establishes a bijection between the paths we are enumerating and the paths of Exercise 19(h) with n replaced by $2n$. For an explicit description of ψ^{-1} and a proof that ψ is indeed a bijection, see the solution of Nichols cited above.

23. The Black pawn on a6 must promote to a knight and then move (in a unique way) to h7 in five additional moves. The Black pawn on a7 must also promote to a knight and then move (in a unique way) to f8 in four additional moves. White then plays Pf7 mate. The first move must be Pa5, after which the number of solutions is the same as if the pawn on a7 were on a6. Each pawn then makes nine moves (including moves after promotion). After the first move Pa5, denote a move by the pawn on a5 by $+1$ and a move by the pawn on a7 by -1. Since the pawn on a7 can never overtake the pawn on a5 (even after promotion), it follows that the number of solutions is just the number of sequences of nine 1's and nine -1's with all partial sums nonnegative. By Exercise 19(r), the number of solutions is therefore the Catalan number $C_9 = 4862$.

This problem is due to Kauko Väisänen, and appears in A. Puusa, *Queue Problems*, Finnish Chess Problem Society, Helsinki, 1992 (Problem 2). This booklet contains fifteen problems of a similar nature. See also Exercise 7.18. For more information on serieshelpmates in general, see A. Dickins, *A Guide to Fairy Chess*, Dover, New York, 1971, p. 10, and J. M. Rice and A. Dickins, *The Serieshelpmate*, second edition, Q Press, Kew Gardens, 1978.

24. These are just Catalan numbers! See for instance J. Gili, *Catalan Grammar*, Dolphin, Oxford, 1993, p. 39. A related question appears in *Amer. Math. Monthly* **103** (1996), 538 and 577.

25. (a) Follows from Exercise 3.29(b) and 19(bbb). See L. W. Shapiro, *American Math. Monthly* **82** (1975), 634–637.

(b) We assume knowledge of Chapter 7. It follows from the results of Appendix 2 of Chapter 7 that we want the coefficient of the trivial Schur function $s_\emptyset$ in the Schur function expansion of $(x_1+x_2)^{2n}$ in the ring $\Xi_2 = \Lambda_2/(x_1x_2-1)$. Since $s_\emptyset = s_{(n,n)}$ in Ξ_2, the number we want is just $\langle s_1^{2n}, s_{(n,n)}\rangle = f^{(n,n)}$ (using Corollary 7.12.5), and the result follows from Exercise 19(ww).

(c) See R. Stanley, *Ann. New York Acad. Sci.*, vol. 576, 1989, pp. 500–535 (Example 4 on p. 523).

(d) See R. Stanley, in *Advanced Studies in Pure Math.*, vol. 11, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, 1987, pp. 187–213 (bottom of p. 194). A simpler proof follows from R. Stanley, *J. Amer. Math. Soc.* **5** (1992), 805–851 (Prop. 8.6). For a related result, see C. Chan, *SIAM J. Disc. Math.* **4** (1991), 568–574.

(e) See L. R. Goldberg, *Adv. Math.* **85** (1991), 129–144 (Thm. 1.7).

(f) See D. Tischler, *J. Complexity* **5** (1989), 438–456.

(g) This algebra is the *Temperley-Lieb algebra* $A_{\beta,n}$ (over K), with many interesting combinatorial properties. For its basic structure see F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer-Verlag, New York, 1989, p. 33 and §2.8. For a direct connection with 321-avoiding permutations (defined in Exercise 19(ee)), see S. C. Billey, W. Jockusch, and R. Stanley, *J. Algebraic Combinatorics* **2** (1993), 345–374 (pp. 360–361).

(h) See J.-Y. Shi, *Quart. J. Math.* **48** (1997), 93–105 (Thm. 3.2(a)).

(i) This remarkable conjecture is a small part of a vast conjectured edifice due to M. Haiman, *J. Algebraic Combinatorics* **3** (1994), 17–76. See also A. M. Garsia and M. Haiman, *J. Algebraic Combinatorics* **5** (1996), 191–244; A. M. Garsia and M. Haiman, *Electron. J. Combinatorics* **3** (1996), no. 2, Paper 24; and M. Haiman, (t,q)-Catalan numbers and the Hilbert scheme, *Discrete Math.*, to appear.

26. (a) This curious result can be proved by induction using suitable row and column operations. It arose from a problem posed by E. Berlekamp and was solved by L. Carlitz, D. P. Roselle, and R. A. Scoville, *J. Combinatorial Theory* **11** (1971), 258–271. A slightly different way of stating the result appears in [34, p. 223]$_3$.

(b) *Answer:* $a_n = C_n$, the nth Catalan number. One way (of many) to prove this result is to apply part (a) to the cases $\lambda = (2n+1, 2n, \dots, 2, 1)$ and $\lambda = (2n, 2n-1, \dots, 2, 1)$, and to use the interpretation of Catalan numbers given by Corollary 2.3(v). Related work appears in A. Kellogg (proposer), Problem 10585, *Amer. Math. Monthly* **104** (1997), 361, and C. Radoux, *Bull. Belgian Math. Soc. (Simon Stevin)* **4** (1997), 289–292.

27. (a) The unique such basis $y_0, y_1, \dots, y_n$, up to sign and order, is given by

$$y_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{i+j}{2i} x_i.$$

(b) Now

$$y_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{i+j+1}{2i+1} x_i.$$

28. (a) The problem of computing the probability of convexity was raised by J. van de Lune and solved by R. B. Eggleton and R. K. Guy, *Math. Mag.* **61** (1988), 211–219, by a clever integration argument. The proof of Eggleton and Guy can be "combinatorialized" so that integration is avoided. The decomposition of $\mathcal{C}_d$ given below in the solution to (c) also yields a proof. For a more general result, see P. Valtr, in *Intuitive Geometry (Budapest, 1995)*, Bolyai Soc. Math. Stud. **6**, János Bolyai Math. Soc., Budapest, 1997, pp. 441–443.

(b) Suppose that $x = (x_1, x_2, \ldots, x_d) \in \mathcal{C}_d$. We say that an index i is *slack* if $2 \leq i \leq d-1$ and $x_{i-1} + x_{i+1} > 2x_i$. If no index is slack, then either $x = (0, 0, \ldots, 0)$, $x = (1, 1, \ldots, 1)$, or $x = \lambda(1, 1, \ldots, 1) + (1-\lambda)y$ for $y \in \mathcal{C}_d$ and sufficiently small $\lambda > 0$. Hence in this last case x is not a vertex. So suppose that x has a slack index. If for all slack indices i we have $x_i = 0$, then x is of the stated form (55). Otherwise, let i be a slack index such that $x_i > 0$. Let $j = i - p$ be the largest index such that $j < i$ and j is not slack. Similarly, let $k = i + q$ be the smallest index such that $k > i$ and k is not slack. Let

$$A(\epsilon) =$$

$$\left(x_1, \ldots, x_j, x_{j+1} + \tfrac{\epsilon}{p}, x_{j+2} + \tfrac{2\epsilon}{p}, \ldots, x_i + \epsilon, \ldots, x_{k-2} + \tfrac{2\epsilon}{q}, x_{k-1} + \tfrac{\epsilon}{q}, x_k, \ldots, x_n\right).$$

For small $\epsilon > 0$, both $A(\epsilon)$ and $A(-\epsilon)$ are in $\mathcal{C}_d$. Since $x = \frac{1}{2}\left(A(\epsilon) + \frac{1}{2}A(-\epsilon)\right)$, it follows that x is not a vertex. The main idea of this argument is due to A. Postnikov.

(c) For $1 \leq r \leq s \leq d$, let

$$\begin{aligned}
F_{rs} &= \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_r = x_{r+1} = \cdots = x_s = 0\} \\
F_r^- &= \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_r = x_{r+1} = \cdots = x_d = 0\} \\
F_s^+ &= \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_1 = x_2 = \cdots = x_s = 0\}.
\end{aligned}$$

Now F_r^- is a simplex with vertices $(1, \frac{k-1}{k}, \frac{k-2}{k}, \ldots, \frac{1}{k}, 0, 0, \ldots, 0)$ for $1 \leq k \leq r-1$, together with $(0, 0, \ldots, 0)$. These vertices have denominators (i.e., the smallest positive integer whose product with the vertex has integer coordinates) $1, 2, 3, \ldots, r-1, 1$, respectively. Hence

$$\sum_{n \geq 0} i(F_r^-, n)x^n = \frac{1}{[1][r-1]!}.$$

Similarly

$$\sum_{n \geq 0} i(F_s^+, n)x^n = \frac{1}{[1][d-s]!}.$$

Since $F_{rs} \cong F_r^- \times F_s^+$, we have $i(F_{rs}, n) = i(F_r^-, n)i(F_s^+, n)$ and

$$\sum_{n \geq 0} i(F_{rs}, n)x^n = \frac{1}{[1][r]!} * \frac{1}{[1][d-s]!}.$$

Let P be the poset of all F_{rs}'s, ordered by inclusion, and let μ denote the Möbius function of $P \cup \{\hat{1}\}$. Let $G = \bigcup_{r=1}^{d} F_{rr}$, a polyhedral complex in $\mathbb{R}^d$. By Möbius inversion we have

$$i(G,n) = -\sum_{F_{st}\in P} \mu(F_{st},\hat{1})\, i(F_{st},n).$$

But $F_{tu} \subseteq F_{rs}$ if and only if $t \leq r \leq s \leq u$, from which it is immediate that

$$-\mu(F_{st},\hat{1}) = \begin{cases} 1, & s=t \\ -1, & s=t-1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n\geq 0} i(G,n)x^n = \sum_{r=1}^{d} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-r]!} - \sum_{r=1}^{d-1} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-1-r]!}.$$

Now the entire polytope $\mathcal{C}_d$ is just a cone over G with apex $(1,1,\ldots,1)$. From this it is not hard to deduce that

$$\sum_{n\geq 0} i(\mathcal{C}_d,n)x^n = \frac{1}{1-x}\sum_{n\geq 0} i(G,n)x^n,$$

and the proof follows.

29. See P. Valtr, *Discrete Comput. Geom.* **13** (1995), 637–643. Valtr also shows in *Combinatorica* **16** (1996), 567–573, that if n points are chosen uniformly and independently from inside a triangle, then the probability that the points are in convex position is $\frac{2^n}{(2n)!}\binom{3n-3}{n-1,n-1,n-1}$.

30. Equation (57) is equivalent to

$$\sum_{n\geq 0} f_n \frac{1}{n!}\left(\log(1-x)^{-1}\right)^n x^n = C(x).$$

Hence by (5.25) we need to show that

$$n!\, C_n = \sum_{k=1}^{n} c(n,k) f_k,$$

where $c(n,k)$ is the number of permutations $w \in \mathfrak{S}_n$ with k cycles. Choose a permutation $w \in \mathfrak{S}_n$ with k cycles in $c(n,k)$ ways. Let the cycles of w be the elements of a semiorder P in f_k ways. For each cycle $(a_1,\ldots,a_i)$ of w, replace this element of P with an antichain whose elements are labelled $a_1,\ldots,a_i$. If $a = (a_1,\ldots,a_i)$ and $b = (b_1,\ldots,b_j)$ are two cycles of w, then define $a_r < b_s$ if and only if $a < b$ in P. In this way we get a poset $\rho(P,w)$ with vertices $1,2,\ldots,n$. It is not hard to see that $\rho(P,w)$ is a semiorder, and that every isomorphism class of n-element semiorders occurs exactly $n!$ times among the posets $\rho(P,w)$. Since by Exercise 19(ddd) there are C_n nonisomorphic n-element semiorders, the proof follows.

This result was first proved by J. L. Chandon, J. Lemaire, and J. Pouget, *Math. Sci. Hum.* **62** (1978), 61–80, 83. For a more general situation in which the number A_n of unlabelled objects is related to the number B_n of labelled objects by $\sum B_n x^n/n! = \sum A_n(1-e^{-x})^n$, see R. Stanley, *Proc. Natl. Acad. Sci. U.S.A.* **93** (1996), 2620–2625 (Thm. 2.3) and A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint (§6), available at http://front.math.ucdavis.edu/math.CO/9712213.

31. (a) (Sketch.) We will triangulate $\mathcal{P}$ into d-dimensional simplices σ, all containing 0. Thus each σ will have d vertices of the form $e_i - e_j$, where $i < j$. Given a graph G with d edges on the vertex set $[d+1]$, let σ_G be the convex hull of all vectors $e_i - e_j$ for which ij is an edge of G with $i < j$, and let $\tilde{\sigma}_G$ be the convex hull of σ_G and the origin. It is easy to see that $\tilde{\sigma}_G$ is a d-dimensional simplex if and only if G is a tree. Moreover, it can be shown that σ_G lies on the boundary of $\mathcal{P}$ (and hence can be part of a triangulation of the type we are looking for) if and only if G is an *alternating* tree, as defined in Exercise 5.41. We therefore want to choose a set $\mathcal{T}$ of alternating trees T on $[d+1]$ such that the $\tilde{\sigma}_T$'s are the facets of a triangulation of $\mathcal{P}$. One way to do this is to take $\mathcal{T}$ to consist of the *noncrossing* alternating trees on $[d+1]$, i.e., alternating trees such that if $i < j < k < l$, then not both ik and jl are edges. By Exercise 19(p) the number of such trees is C_d. (We can also take $\mathcal{T}$ to consist of alternating trees on $[d+1]$ such that if $i < j < k < l$ then not both il and jk are edges. By Exercise 19(q) the number of such trees is again C_d.) Moreover, it is easy to see that for any tree T on $[d+1]$ we have $V(\tilde{\sigma}_T) = 1/d!$, where V denotes relative volume. Hence $V(\mathcal{P}) = C_d/d!$. This result appears in I. M. Gelfand, M. I. Graev and A. Postnikov, in *The Arnold-Gelfand Mathematical Seminars*, Birkhäuser, Boston, 1997 (Theorem 2.3(2)).

    (b) Order the $\binom{d+1}{2}$ edges ij, $1 \le i < j \le d+1$, lexicographically, e.g., $12 < 13 < 14 < 23 < 24 < 34$. Order the C_d noncrossing alternating trees $T_1, T_2, \dots, T_{C_d}$ lexicographically by edge set, i.e., $T_i < T_j$ if for some k the first (in lexicographic order) k edges of T_i and T_j coincide, while the $(k+1)$st edge of T_i precedes the $(k+1)$st edge of T_j. For instance, when $d = 3$ the ordering on the noncrossing alternating trees (denoted by their set of edges) is

$$\{12, 13, 14\},\ \{12, 14, 34\},\ \{13, 14, 23\},\ \{14, 23, 24\},\ \{14, 23, 34\}.$$

One can check that $\tilde{\sigma}_{T_i}$ intersects $\tilde{\sigma}_{T_1} \cup \cdots \cup \tilde{\sigma}_{T_{i-1}}$ in a union of $j-1$ $(d-1)$-dimensional faces of $\tilde{\sigma}_{T_i}$, where j is the number of vertices of T_i that are less than all their neighbors. It is not hard to see that the number of noncrossing alternating trees on $[d+1]$ for which exactly j vertices are less that all their neighbors is just the Narayana number $N(d, j)$ of Exercise 36. It follows from the techniques of R. Stanley, *Annals of Discrete Math.* **6** (1980), 333–342 (especially Theorem 1.6), that

$$(1-x)^{d+1} \sum_{n \ge 0} i(\mathcal{P}, n) x^n = \sum_{j=1}^{d} N(d, j) x^{j-1}.$$

32. The Tamari lattice was first considered by D. Tamari, *Nieuw Arch. Wisk.* **10** (1962), 131–146, who proved it to be a lattice. A simpler proof of this result was given

by S. Huang and D. Tamari, *J. Combinatorial Theory (A)* **13** (1972), 7–13. The proof sketched here follows J. M. Pallo, *Computer J.* **29** (1986), 171–175. For further properties of Tamari lattices and their generalizations, see P. H. Edelman and V. Reiner, *Mathematika* **43** (1996), 127–154; A. Björner and M. L. Wachs, *Trans. Amer. Math. Soc.* **349** (1997), 3945–3975 (§9); and the references given there.

33. (a) Since $\mathcal{S}$ is a simplicial complex, A_n is a simplicial semilattice. It is easy to see that it is graded of rank $n-2$, the rank of an element being its cardinality (number of diagonals). To check the Eulerian property, it remains to show that $\mu(x, \hat{1}) = (-1)^{\ell(x,\hat{1})}$ for all $x \in A_n$. If $x \in A_n$ and $x \neq \hat{1}$, then x divides the polygon C into regions $C_1, \ldots, C_j$, where each C_i is a convex n_i-gon for some n_i. Let $\bar{A}_n = A_n - \{\hat{1}\}$. It follows that the interval $[x, \hat{1}]$ is isomorphic to the product $\bar{A}_{n_1} \times \cdots \times \bar{A}_{n_j}$, with a $\hat{1}$ adjoined. It follows from Exercise 61 (dualized) that it suffices to show that $\mu(\hat{0}, \hat{1}) = (-1)^{n-2}$. Equivalently (since we have shown that every proper interval is Eulerian), we need to show that A_n has as many elements of even rank as of odd rank. One way to proceed is as follows. For any subset B of A_n, let $\eta(B)$ denote the number of elements of B of even rank minus the number of odd rank. Label the vertices of C as $1, 2, \ldots, n$ in cyclic order. For $3 \leq i \leq n-1$, let $\mathcal{S}^*$ be the set of all elements of $\mathcal{S}$ for which either there is a diagonal from vertex 1 to some other vertex, or else such a diagonal can be adjoined without introducing any interior crossings. Given $S \in \mathcal{S}^*$, let i be the least vertex that is either connected to 1 by a diagonal or for which we can connect it to vertex 1 by a diagonal without introducing any interior crossing. We can pair S with the set S' obtained by deleting or adjoining the diagonal from 1 to i. This pairing (or involution) shows that $\eta(\mathcal{S}^*) = 0$. But $A_n - \mathcal{S}^*$ is just the interval $[T, \hat{1}]$, where T contains the single diagonal connecting 2 and n. By induction (as mentioned above) we have $\eta([T, \hat{1}]) = 0$, so in fact $\eta(A_n) = 0$.

(b) See C. W. Lee, *Europ. J. Combinatorics* **10** (1989), 551–560. An independent proof was given by M. Haiman (unpublished). This polytope is called the *associahedron.* For a far-reaching generalization, see [28, Ch. 7] and the survey article C. W. Lee, in *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 4, 1991, pp. 443–456.

(c) Write

$$\begin{aligned} F(x,y) &= x + \sum_{n\geq 2} \sum_{i=1}^{n-1} W_{i-1}(n+1) x^n y^i \\ &= x + x^2y + x^3(y+2y^2) + x^4(y+5y^2+5y^3) + \cdots. \end{aligned}$$

By removing a fixed exterior edge from a dissected polygon and considering the edge-disjoint union of polygons thus formed, we get the functional equation

$$F = x + y\frac{F^2}{1-F}.$$

(Compare equation (15).) Hence by Exercise 5.59 we have

$$\begin{aligned} F &= \sum_{m\geq 1} \frac{1}{m}[t^{m-1}]\left(x + y\frac{t^2}{1-t}\right)^m \\ &= \sum_{m\geq 1} \frac{1}{m}[t^{m-1}]\sum_{n=0}^{m}\binom{m}{n}x^n\left(y\frac{t^2}{1-t}\right)^{m-n}. \end{aligned}$$

From here it is a simple matter to obtain

$$F = x + \sum_{n\geq 2}\sum_{i=1}^{n-1}\frac{1}{n+i}\binom{n+i}{i}\binom{n-2}{i-1}x^n y^i,$$

whence

$$W_i(n) = \frac{1}{n+i}\binom{n+i}{i+1}\binom{n-3}{i}. \tag{74}$$

This formula goes back to T. P. Kirkman, *Phil. Trans. Royal Soc. London* **147** (1857), 217–272; E. Prouhet, *Nouvelles Annales Math.* **5** (1866), 384; and A. Cayley, *Proc. London Math. Soc. (1)* **22** (1890–1891), 237–262, who gave the first complete proof. For Cayley's proof see also [28, Ch. 7.3]. For a completely different proof, see Exercise 7.18. Another proof appears in D. Beckwith, *Amer. Math. Monthly* **105** (1998), 256–257.

(d) We have

$$h_i = \frac{1}{n-1}\binom{n-3}{i}\binom{n-1}{i+1}.$$

This result appears in C. W. Lee, *Europ. J. Combinatorics* **10** (1989), 551–560 (Thm. 3).

34. (a)–(d) See J. Fürlinger and J. Hofbauer, *J. Combinatorial Theory (A)* **40** (1985), 248–264, and the references given there. For (b) see also G. E. Andrews, *J. Stat. Plan. Inf.* **34** (1993), 19–22 (Corollary 1). The continued fraction (59) is of a type considered by Ramanujan. It is easy to show (see for instance [1, §7.1]$_1$) that

$$F(x) = \frac{\displaystyle\sum_{n\geq 0}(-1)^n q^{n^2}\frac{x^n}{(1-q)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0}(-1)^n q^{n(n-1)}\frac{x^n}{(1-q)\cdots(1-q^n)}}.$$

(e) See R. Stanley, *Ann. New York Acad. Sci.*, vol. 574 (1989), 500–535 (Example 4 on p. 523). This result is closely related to Exercise 25(c).

35. These results are due to V. Welker, *J. Combinatorial Theory (B)* **63** (1995), 222–244 (§4).

36. (a) There are $\binom{n}{k-1}\binom{n-1}{k-1}$ pairs of compositions $A : a_1 + \cdots + a_k = n+1$ and $B : b_1 + \cdots + b_k = n$ of $n+1$ and n into k parts. Construct from these compositions a circular sequence $w = w(A,B)$ consisting of a_1 1's, then b_1 -1's, then a_2 1's, then b_2 -1's, etc. Because n and $n+1$ are relatively prime, this circular sequence w could have arisen from exactly k pairs (A_i, B_i) of compositions of $n+1$ and n into k parts, viz., $A_i : a_i + a_{i+1} + \cdots + a_k + a_1 + \cdots + a_{i-1} = n+1$ and $B_i : b_i + b_{i+1} + \cdots + b_k + b_1 + \cdots + b_{i-1} = n$, $1 \le i \le k$. By the second proof of Theorem 5.3.10 (or more specifically, the paragraph following it), there is exactly one way to break w into a linear sequence $\overline{w}$ such that $\overline{w}$ begins with a 1, and when this initial 1 is removed every partial sum is nonnegative. Clearly there are exactly k 1's in $\overline{w}$ (with or without its initial 1 removed) followed by a -1. This sets up a bijection between the set of all "circular equivalence classes" $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ and X_{nk}. Hence

$$X_{nk} = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}.$$

(b) For $k \ge 1$, let $X_k = X_{1k} \cup X_{2k} \cup \cdots$. Every $w \in X_k$ can be written uniquely in one of the forms (i) $1\ u\ {-1}\ v$, where $u \in X_j$ and $v \in X_{k-j}$ for some $1 \le j \le k-1$, (ii) $1\ {-1}\ u$, where $u \in X_{k-1}$, (iii) $1\ u\ {-1}$, where $u \in X_k$, and (iv) $1\ {-1}$ (when $k=1$). Regarding X_k as a language as in Example 6.6, and replacing for notational comprehensibility 1 by α and -1 by β, conditions (i)–(iv) are equivalent to the equation

$$X_k = \sum_{j=1}^{k-1} \alpha X_j \beta X_{k-j} + \alpha\beta X_{k-1} + \alpha X_k \beta + \delta_{1k}\alpha\beta.$$

Thus if $y_k = \sum_{n\ge1} N(n,k)x^n$, it follows that (setting $y_0 = 0$)

$$y_k = x\sum_{j=0}^{k} y_j y_{k-j} + xy_{k-1} + xy_k + \delta_{1k}x.$$

Since $F(x,t) = \sum_{k\ge1} y_k t^k$, we get (60).

Narayana numbers were introduced by T. V. Narayana, *C. R. Acad. Sci. Paris* **240** (1955), 1188–1189, and considered further by him in *Sankhya* **21** (1959), 91–98, and *Lattice Path Combinatorics with Statistical Applications*, Mathematical Expositions no. 23, University of Toronto Press, Toronto, 1979 (Section V.2). Further references include G. Kreweras and P. Moszkowski, *J. Statist. Plann. Inference* **14** (1986), 63–67; G. Kreweras and Y. Poupard, *European J. Combinatorics* **7** (1986), 141–149; R. A. Sulanke, *Bull. Inst. Combin. Anal.* **7** (1993), 60–66; and R. A. Sulanke, *J. Statist. Plann. Inference* **34** (1993), 291–303.

37. Equivalent to Exercise 1.37(c). See also the nice survey R. Donaghey and L. W. Shapiro, *J. Combinatorial Theory (A)* **23** (1977), 291–301.

38. All these results except (f), (k), (l), and (m) appear in Donaghey and Shapiro, *loc. cit.* Donaghey and Shapiro give several additional interpretations of Motzkin numbers

and state that they have found a total of about 40 interpretations. For (f), see M. S. Jiang, in *Combinatorics and Graph Theory (Hefei, 1992)*, World Scientific Publishing, River Edge, NJ, 1993, pp. 31–39. For (k), see A. Kuznetsov, I. Pak, and A. Postnikov, *J. Combinatorial Theory (A)* **76** (1996), 145–147. For (l), see M. Aigner, Motzkin numbers, preprint. Aigner calls the partitions of (l) *strongly noncrossing*. Finally, for (m) see M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (pp. 55–56) (and compare Exercise 1.29). Klazar's paper contains a number of further enumeration problems related to the present one that lead to algebraic generating functions; see Exercise 19(ss,tt) and Exercise 39(o) for three of them. The name "Motzkin number" arose from the paper T. Motzkin, *Bull. Amer. Math. Soc.* **54** (1948), 352–360.

39. (a) This was the definition of Schröder numbers given in the discussion of Schröder's second problem in Section 2.

(b,e,f,h,i) These follow from (a) using the bijections of Proposition 2.1.

(c) See D. Gouyou-Beauchamps and D. Vauquelin, *RAIRO Inform. Théor. Appl.* **22** (1988), 361–388. This paper gives some other tree representations of Schröder numbers, as well as connections with Motzkin numbers and numerous references.

(d) An easy consequence of the paper of Shapiro and Stephens cited below.

(g) Due to R. A. Sulanke, *J. Difference Equations and Applications*, to appear. The objects counted by this exercise are called *zebras*. See also E. Pergola and R. A. Sulanke, *J. Integer Sequences* (electronic) **1** (1998), Article 98.1.7, available at http://www.research.att.com/∼njas/sequences/JIS.

(j,k) See L. W. Shapiro and A. B. Stephens, *SIAM J. Discrete Math.* **4** (1991), 275–280. For (j), see also Exercise 17(b).

(l) L. W. Shapiro and S. Getu (unpublished) conjectured that the set $\mathfrak{S}_n(2413, 3142)$ and the set counted by (k) are identical (identifying a permutation matrix with the corresponding permutation). It was proved by J. West, *Discrete Math.* **146** (1995), 247–262, that $\#\mathfrak{S}_n(2413, 3142) = r_{n-1}$. Since it is easy to see that permutations counted by (k) are 2413-avoiding and 3142-avoiding, the conjecture of Shapiro and Getu follows from the fact that both sets have cardinality r_{n-1}. Presumably there is some direct proof that the set counted by (k) is identical to $\mathfrak{S}_n(2413, 3142)$.

West also showed in Theorem 5.2 of the above-mentioned paper that the sets $\mathfrak{S}_n(1342, 1324)$ and (m) are identical. The enumeration of $\mathfrak{S}_n(1342, 1432)$ was accomplished by S. Gire, Ph.D. thesis, Université Bordeaux, 1991. The remaining seven cases were enumerated by D. Kremer, Permutations with forbidden subsequences and a generalized Schröder number, preprint. Kremer also gives proofs of the three previously known cases. She proves all ten cases using the method of "generating trees" introduced by F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, *J. Combinatorial Theory (A)* **24** (1978), 382–394, and further developed by J. West, *Discrete Math.* **146** (1995), 247–262, and **157** (1996), 363–374. It has been verified by computer that there are no other pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ for which $\#\mathfrak{S}_n(u, v) = r_{n-1}$ for all n.

(m) This is a result of Knuth [41, Exercises 2.2.1.10–2.2.1.11, pp. 239 and 533–534]$_5$; these permutations are now called *deque-sortable*. A combinatorial proof appears in D. G. Rogers and L. W. Shapiro, in *Lecture Notes in Math.*, no. 884, Springer-Verlag, Berlin, 1981, pp. 293–303. Some additional combinatorial interpretations of Schröder numbers and many additional references appear in the preceding reference. For q-analogues of Schröder numbers, see J. Bonin, L. W. Shapiro, and R. Simion, *J. Stat. Planning and Inference* **34** (1993), 35–55.

(n) See D. G. Rogers and L. W. Shapiro, *Lecture Notes in Mathematics*, no. 686, Springer-Verlag, Berlin, 1978, pp. 267–276 (§5) for simple bijections with (a) and other "Schröder structures."

(o) This result is due to R. C. Mullin and R. G. Stanton, *Pacific J. Math.* **40** (1972), 167–172 (§3), using the language of "Davenport-Schinzel sequences." It is also given by M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 55).

(p) Remove a "root edge" from the polygon of (h) and "straighten out" to obtain a noncrossing graph of the type being counted.

(q,r) These results (which despite their similarity are not trivially equivalent) appear in D. G. Rogers, *Lecture Notes in Mathematics*, no. 622, Springer-Verlag, Berlin, 1977, pp. 175–196 (equations (38) and (39)), and are further developed in D. G. Rogers, *Quart. J. Math. (Oxford) (2)* **31** (1980), 491–506. In particular, a bijective proof that (q) and (r) are equinumerous appears in Section 3 of this latter reference. It is also easy to see that (p) and (r) are virtually identical problems. A further reference is D. G. Rogers and L. W. Shapiro, *Lecture Notes in Mathematics*, no. 686, Springer-Verlag, Berlin, 1978, pp. 267–276.

(s) See M. Ciucu, *J. Algebraic Combinatorics* **5** (1996), 87–103, Theorem 4.1.

40. Note that this exercise is the "opposite" of Exercise 39(k), i.e., here we are counting the permutation matrices P for which not even a single new 1 can be added (using the rules of Exercise 39(k)). The present exercise was solved by Shapiro and Stephens in Section 3 of the paper cited in the solution to Exercise 39(k). For a less elegant form of the answer and further references, see M. Abramson and M. O. J. Moser, *Ann. Math. Stat.* **38** (1967), 1245–1254.

41. This result was conjectured by J. West, Ph.D. thesis, M.I.T., 1990 (Conjecture 4.2.19), and first proved by D. Zeilberger, *Discrete Math.* **102** (1992), 85–93. For further proofs and related results, see M. Bóna, 2-stack sortable permutations with a given number of runs, preprint dated May 13, 1997; M. Bousquet-Mélou, *Electron. J. Combinatorics* **5** (1998), R21, 12 pp.; S. Dulucq, S. Gire, and J. West, *Discrete Math.* **153** (1996), 85–103; I. P. Goulden and J. West, *J. Combinatorial Theory (A)* **75** (1996), 220–242; J. West, *Theoret. Comput. Sci.* **117** (1993), 303–313; and M. Bousquet-Mélou, *Elec. J. Combinatorics* **5(1)** (1998), R21, 12 pp.

42. It's easy to see that $f(n)$ is the number of lattice paths with n steps $(1,0)$, $(0,1)$, and $(1,1)$, that begin at $(0,0)$ and end on the line $y = x$. Hence by equation (29) we have $F(x) = 1/\sqrt{1-2x-3x^2}$. The linear recurrence is then given by Example 4.8(b)(ii).

It's also easy to see that $f(n) = [t^0](t^{-1}+1+t)^n$. For this reason $f(n)$ is called a *middle trinomial coefficient*. Middle trinomial coefficients were first considered by L. Euler, *Opuscula analytica*, vol. 1, Petropolis, 1783, pp. 48–62, who obtained the generating function $1/\sqrt{1-2x-3x^2}$. See also Problem III.217 of $[53, \text{vol. I, pp. 147 and } 349]_5$. The interpretation of $f(n)$ in terms of chess is due to K. Fabel, problem #1413, *feenschach* **13** (October, 1974), Heft 25, p. 382; solution, **13** (May-June-July, 1975), Heft 28, p. 91. Fabel gives the solution as

$$f(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{i!^2(n-2i)!},$$

but does not consider the generating function $F(x)$.

43. (a) Define a secondary structure to be *irreducible* if either $n = 1$ or there is an edge from 1 to n. Let $t(n)$ be the number of irreducible secondary structures with n vertices, and set

$$T(x) = \sum_{n \geq 1} t(n)x^n = x + x^3 + x^4 + 2x^5 + 4x^6 + \cdots.$$

It is easy to see that $S = 1/(1-T)$ and $T = x^2S + x - x^2$. Eliminating T and solving for S yields the desired formula.

This result is due to P. R. Stein and M. S. Waterman, *Discrete Math.* **26** (1978), 261–272 (the case $m = 1$ of (10)). For further information and references, including connections with biological molecules such as RNA, see W. R. Schmitt and M. S. Waterman, *Discrete Applied Math.* **51** (1994), 317–323.

(b) See A. Nkwanta, in *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* **34**, 1997, pp. 137–147.

44. This result is due to P. H. Edelman and V. Reiner, *Graphs and Combinatorics* **13** (1997), 231–243 (Theorem 1).

45. See R. Stanley, solution to 6342, *American Math. Monthly* **90** (1983), 61–62. A labelled version of this result was given by S. Goodall, The number of labelled posets of width two, London School of Economics, Mathematics Preprint Series LSE-MPS-46, March, 1993.

46. These remarkable results are due to G. Viennot and D. Gouyou-Beauchamps, *Advances in Appl. Math.* **9** (1988), 334–357. The subsets being enumerated are called *directed animals*. For a survey of related work, see G. Viennot, *Astérisque* **121–122** (1985), 225–246. See also the two other papers cited in the solution to Exercise 19(kkk), as well as the paper M. Bousquet-Mélou, *Discrete Math.* **180** (1998), 73–106. Let us also mention that Viennot and Goyou-Beauchamps show that $f(n)$ is the number of sequences of length $n-1$ over the alphabet $\{-1, 0, 1\}$ with nonnegative partial sums. Moreover, M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 64) shows that $f(n)$ is the number of partitions of $[n+1]$ such that no block contains two consecutive integers, and such that if $a < b < c < d$, a and d belong to the same block B_1, and b and c belong to the same block B_2, then $B_1 = B_2$.

47. (a) There is a third condition equivalent to (i) and (ii) of (a) that motivated this work. Every permutation $w \in \mathfrak{S}_n$ indexes a closed Schubert cell $\overline{\Omega}_w$ in the complete flag variety $\mathrm{GL}(n,\mathbb{C})/B$. Then w is smooth if and only if the variety $\overline{\Omega}_w$ is smooth. The equivalence of this result to (i) and (ii) is implicit in K. M. Ryan, *Math. Ann.* **276** (1987), 205–244, and is based on earlier work of Lakshmibai, Seshadri, and Deodhar. An explicit statement that the smoothness of $\overline{\Omega}_w$ is equivalent to (ii) appears in V. Lakshmibai and B. Sandhya, *Proc. Indian Acad. Sci. (Math. Sci.)* **100** (1990), 45–52.

(b) This generating function is due to M. Haiman (unpublished).

NOTE. It was shown by M. Bóna, *Electron. J. Combinatorics* **5**, R31 (1998), 12 pp., that there are four other inequivalent (in the sense of Exercise 39(l)) pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ such that the number of permutations in $\mathfrak{S}_n$ that avoid them is equal to $f(n)$, viz., $(1324, 2413)$, $(1342, 2314)$, $(1342, 2431)$, and $(1342, 3241)$. (The case $(1342, 2431)$ is implicit in Z. Stankova, *Discrete Math.* **132** (1994), 291–316.)

48. This result is due to M. Bóna, *J. Combinatorial Theory (A)* **80** (1997), 257–272.

49. (a) Given a domino tiling of B_n, we will define a path P from the center of the left-hand edge of the middle row to the center of the right-hand edge of the middle row. Namely, each step of the path is from the center of a domino edge (where we regard a domino as having six edges of unit length) to the center of another edge of the same domino D, such that the step is symmetric with respect to the center of D. One can check that for each tiling there is a unique such path P. Replace a horizontal step of P by $(1, 1)$, a northeast step by $(1, 0)$, and a southeast step by $(0, 1)$ (no other steps are possible), and we obtain a lattice path from $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$, and conversely any such lattice path corresponds to a unique domino tiling of B_n. This establishes the desired bijection. For instance, Figure 14 shows a tiling of B_3 and the corresponding path P (as a dotted line). The steps in the lattice path from $(0, 0)$ to $(3, 3)$ are $(1, 0)$, $(1, 1)$, $(1, 0)$, $(0, 1)$, $(0, 1)$.

The board B_n is called the *augmented Aztec diamond*, and its number of domino tilings was computed by H. Sachs and H. Zernitz, *Discrete Appl. Math.* **51** (1994), 171–179. The proof sketched above is based on an explication of the proof of Sachs and Zernitz due to Dana Randall (unpublished).

(b) The board is called an *Aztec diamond*, and the number of tilings is now $2^{\binom{n+1}{2}}$. (Note how much larger this number is than the solution $f(n)$ to (a).) Four proofs of this result appear in N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, *J. Alg. Combinatorics* **1** (1992), 111–132, 219–234. For some related work, see M. Ciucu, *J. Alg. Combinatorics* **5** (1996), 87–103, and H. Cohn, N. Elkies, and J. Propp, *Duke Math. J.* **85** (1996), 117–166. Domino tilings of the Aztec diamond and augmented Aztec diamond had actually been considered earlier by physicists, beginning with I. Carlsen, D. Grensing, and H.-Chr. Zapp, *Philos. Mag. A* **41** (1980), 777–781.

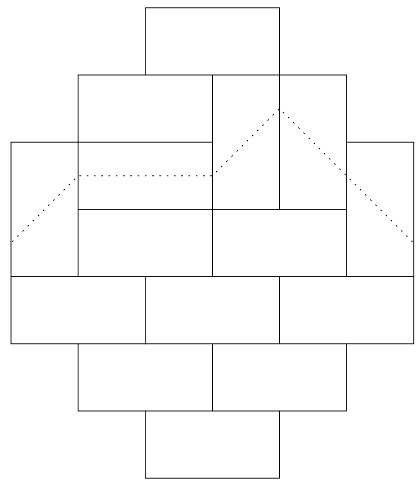

Figure 14: A path on the augmented Aztec diamond B_3

50. This result appears in F. R. K. Chung, R. L. Graham, J. Morrison, and A. M. Odlyzko, *Amer. Math. Monthly* **102** (1995), 113–123 (equation (11)). This paper contains a number of other interesting results related to pebbling.