PROPAGATION OF SINGULARITIES FOR THE WAVE EQUATION ON CONIC MANIFOLDS

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Abstract. For the wave equation associated to the Laplacian on a compact manifold with boundary with a conic metric (with respect to which the boundary is metrically a point) the propagation of singularities through the boundary is analyzed. Under appropriate regularity assumptions the diffracted, non-direct, wave produced by the boundary is shown to have Sobolev regularity greater than the incoming wave.

Introduction

Solutions to the wave equation (for the Friedrichs extension of the Laplacian) associated to a conic metric on a compact manifold with boundary exhibit a diffractive, or "ringing," effect when singularities strike the boundary. The main results of this paper describe the relationship between the strength of the singularities incident on the boundary and the strength of the diffracted singularities. We first show that if *no* singularities arrive at the boundary at a time $\bar{t}$ then the solution is smooth near the boundary at that time, in the sense that it is locally in the intersection of the domains of all powers of the Laplacian. We then show that if there are singularities incident on the boundary at time $\bar{t}$ and in addition the solution satisfies an appropriate *nonfocusing* condition with respect to the boundary, then the strongest singularities leaving the boundary at that time are on the geometric continuations of those incoming bicharacteristics which carry singularities, whereas on the diffracted, i.e. not geometrically continued, rays the singularities are weaker. If the incident wave satisfies a conormality condition, then the singularity on the diffracted front is shown to be conormal. Applying this analysis to the forward fundamental solution gives an extension of results of Cheeger and Taylor ([2, 3]) from the product-conic to the general conic case. The results contained in this paper represent a refinement of those previously announced in [15].

The problem of diffraction is an old one, with rigorous treatment stretching back to the work of Sommerfeld, who, in 1896, discussed diffraction around edges in the plane [22]; this includes the case of Dirichlet boundary conditions at the slit $[0,\infty) \subset \mathbb{C}$. Using the method of images, this problem may be reduced to the study of the wave equation on the cone over the circle of circumference 4π. An overview of this and many other problems of diffraction around obstacles in $\mathbb{R}^n$ is given in Friedlander [5].

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Diffractive effects were extensively studied by Cheeger and Taylor [2, 3] in the special case of *product-conic metrics*, i.e. $\mathbb{R}_+$-homogeneous metrics on $(0, \infty) \times Y$ of the form

$$g = dx^2 + x^2 h(y, dy).$$

Separation of variables can be used to give an explicit description of the fundamental solution in terms of functions of the tangential Laplacian. See also the discussion by Kalka and Menikoff [11]. Rouleux [19] obtained a version of the results of Cheeger-Taylor in the analytic category. Lebeau [12, 13] has also obtained a diffractive theorem in the setting of manifolds with corners in the analytic category, and Gérard-Lebeau [6] have explicitly analyzed the problem of an analytic conormal wave incident on an analytic corner in $\mathbb{R}^2$. Estimates in L^p spaces have been obtained for product-conic metrics by Müller-Seeger [18].

Let X be an n-dimensional conic manifold, that is to say, a compact manifold with boundary, with a Riemannian metric g on the interior which, near the boundary, takes the degenerate form

$$g = dx^2 + x^2 h, \tag{I.1}$$

where x is a boundary defining function and $h \in \mathcal{C}^\infty(X; \mathrm{Sym}^2(T^*X))$ restricts to be a metric on ∂X. Each boundary component of X is thus a "cone point" in the metric sense. A trivial example of a conic metric is obtained by blowing up a point p in a Riemannian manifold; near p we can take x to be the distance function to p, and (I.1) is simply the expression of the metric in Riemannian polar coordinates. Such conic metrics exist on any manifold with boundary.

The subject of this paper is the wave equation on a conic manifold, and in particular the propagation of singularities for its solutions. Let Δ be the Friedrichs extension of the (non-negative) Laplace-Beltrami operator on X. We consider solutions to

$$(D_t^2 - \Delta)u = 0 \tag{I.2}$$

on $\mathbb{R} \times X^\circ$ which are *admissible* in the sense that

$$u \in \mathcal{C}^{-\infty}(\mathbb{R}; \mathrm{Dom}(\Delta^{s/2})) \text{ for some } s \in \mathbb{R}$$

and the equation holds in $\mathcal{C}^{-\infty}(\mathbb{R}; \mathrm{Dom}(\Delta^{s/2-1}))$.

Hörmander's theorem [9] on the propagation of singularities for operators of real principal type yields rather complete information about the location of the singularities of any solution of (I.2) away from ∂X. Namely, $\mathrm{WF}^r(u)$ (the wavefront set computed with respect to the scale of Sobolev spaces) is contained in the characteristic variety and is a union of maximally extended null bicharacteristics. Since the null bicharacteristics are essentially time-parametrized geodesics over X°, this means, somewhat loosely speaking, that the singularities travel with unit speed along geodesics.

Hörmander's theorem does not address the question of what happens when singularities reach ∂X. That is, it leaves open the question of how singularity-carrying geodesics in ∂X *terminating* at ∂X are connected with those *emanating* from ∂X. The answer involves both the geometric propagation of singularities that one would expect from the limiting behavior of geodesics that come close to the tip of the cone without striking it, and an additional, diffractive, effect.

By the finiteness of the propagation speed for the wave equation it suffices to consider a single component Y of ∂X (i.e. a single metric cone point). We first note that for every $y \in Y$, there is a unique (maximal, unparametrized) geodesic in $X^\circ = X \setminus \partial X$ with y in its closure; conversely there is a neighborhood of Y in which each point is the end point of a unique, short, geodesic segment terminating at Y. This corresponds to a product decomposition $[0,\epsilon)_x \times Y$ of a neighborhood of Y in which projection to Y is the map to the end-point and x is the length of this segment. We henceforth take x to be the boundary defining function; the metric still has the form (I.1) but now with $h(\partial_x, \cdot) = 0$, so h is a family of metrics on Y parameterized by x.

It follows that the surfaces $t \pm x = c$, for any constant c, which are well-defined near each boundary component, are characteristic for the wave equation. These radial surfaces carry the extra, diffractive, singularities, the existence and regularity of which is our main object of study.

Our results are best illustrated by the fundamental solution itself. Consider an initial point $\bar{m} \in X^\circ$, which we shall take to be close to some boundary component Y. There is then a unique short geodesic interval from $\bar{m}$ to the boundary; its length is $d(\bar{m}, Y)$. The wave cone emanating from $\bar{m}$ is smooth for small positive times but for longer times generally becomes singular. However it is the projection of a smooth Lagrangian submanifold $I_{\bar{m}} \subset T^*(\mathbb{R} \times X^\circ)$ consisting of the union of those maximally extended null bicharacteristics of the wave operator which pass above $\bar{m}$ at $t = 0$. In addition we consider the radial surface mentioned above, $D_{\bar{m}} = \{t = x + d(\bar{m}, Y)\}$, which is well defined if $d(\bar{m}, Y)$ is small enough and which emanates from the boundary at the time of arrival of the geodesic from $\bar{m}$.

Theorem I.1 (Fundamental solution)**.** *Let $E_{\bar{m}}$ be the fundamental solution of the wave equation for the Friedrichs extension of the Laplacian of a conic metric on a compact manifold with boundary X, with pole at $\bar{m} \in X^\circ$. If $d(\bar{m}, Y)$ is sufficiently small then, regarding $E_{\bar{m}}$ as a distribution on $\mathbb{R} \times X^\circ$,*

$$\mathrm{WF}(E_{\bar{m}}) \subset I_{\bar{m}} \cup N^* D_{\bar{m}} \text{ in } |t| < 2d(\bar{m}, Y) \tag{I.3}$$

and $E_{\bar{m}}$ is conormal with respect to $D_{\bar{m}}$ away from the projection of the closure of $I_{\bar{m}}$ and is of Sobolev order $\frac{1}{2} - \delta$ there for any $\delta > 0$.

A comparison with the explicit results of Cheeger-Taylor [2, 3] in the special case of product cones shows that this result is optimal as far as Sobolev regularity is concerned.

The first step in proving this is to show that no singularities arise from the boundary spontaneously.

Theorem I.2 (Diffractive Theorem)**.** *If an admissible solution to the wave equation on a conic manifold has singularities, of Sobolev order r, on at least one null bicharacteristic hitting a boundary component at time $\bar{t}$ then it has singularities, of Sobolev order r, on at least one null bicharacteristic leaving that boundary component at time $\bar{t}$.*

This diffractive theorem does not give any localization of singularities in the boundary component Y. It in no way distinguishes between the different "outgoing" null bicharacteristics leaving the boundary component at $t = \bar{t}$ although in fact it is easily strengthened to show that singularities in the two components of the characteristic set, corresponding to the sign of the dual variable to t, do not interact at all.

In order to state a more refined theorem, which distinguishes between different points in Y, we need first to consider the possible *geometric continuations* of a geodesic terminating at $y \in Y$. Geometric continuation of geodesics corresponds to the relation

(I.4)
$$G(y) = \{y' \in Y; y \text{ and } y' \text{ are endpoints of a geodesic segment of length } \pi \text{ in } Y\},$$

where Y is endowed with the metric $h\restriction_{x=0}$. The geodesic segments of length π in Y arise naturally as limits of geodesics in X° which narrowly miss Y.

Our second main result shows that under certain circumstances, a singularity arriving at a point $y \in Y$ at $t = \bar{t}$ will produce only *weaker* singularities along rays emanating from points in $Y \setminus G(y)$ than on the geometrically continued rays, which are those emanating from $G(y)$. In other words, we show that diffracted singularities are weaker than geometrically propagated ones.

The further hypothesis that we make is one of nonfocusing of singularities. Suppose that the solution u is microlocally of Sobolev order r on all incoming null bicharacteristics arriving at Y at $t = \bar{t}$. We require that *tangential smoothing* improves this regularity; for our purposes the amount of tangential smoothing required to gain regularity is irrelevant. Let Δ_Y denote the Laplace-Beltrami operator on Y with respect to the metric $h\restriction_{x=0}$, extended to operate on a collar neighborhood of Y by the metric product decomposition described above. We assume that for N sufficiently large $(1 + \Delta_Y)^{-N} u$ is more regular (microlocally) than u in the sense that

$$\text{(I.5)} \quad p \notin \mathrm{WF}^{r'}\left((1 + \Delta_Y)^{-N} u\right) \text{ for } p \text{ on}$$
$$\text{the incoming null bicharacteristic reaching } Y \text{ at } y \in Y,\ t = \bar{t}.$$

Theorem I.3 (Geometric Propagation)**.** *If u is an admissible solution of the wave equation, then the outgoing null bicharacteristic emanating from a point $y \in Y$ at $t = \bar{t}$ is disjoint from $\mathrm{WF}^R(u)$ provided all the incoming null bicharacteristics reaching $G(y)$ at $t = \bar{t}$ are outside $\mathrm{WF}^R(u)$ and* (I.5) *holds with $r' > R$ microlocally for all other incoming null bicharacteristics meeting the boundary at time $\bar{t}$.*

This theorem may not be strengthened by dropping the nonfocusing assumption (I.5) as is shown by a counterexample in §16. On the other hand the two components of the characteristic variety are again completely independent.

At a time $\bar{t}$ the *background regularity,* r, for an admissible solution near a boundary component is the Sobolev regularity that holds on all incoming null bicharacteristics arriving at that boundary component at $t = \bar{t}$. Theorem I.2 shows that this background regularity propagates to all outgoing null bicharacteristics leaving the boundary at time $\bar{t}$. Theorem I.3 above shows that there is additionally a gain of any number, l, of Sobolev derivatives on any one of these outgoing bicharacteristics, say emanating from $\bar{y} \in Y$, provided that u has Sobolev regularity $r + l$ on all incoming bicharacteristics which are geometrically related to this one and that u has the property that sufficient tangential smoothing increases its regularity microlocally near all other (and hence all) incoming null bicharacteristics, reaching the boundary at time $\bar{t}$, to greater than $r + l$.

Although there is in principle no upper limit to the gain of regularity compared to background available from Theorem I.3, there are practical limits. The most important case of nontrivial tangential smoothing is when the solution is, for $t < \bar{t}$

and away from the boundary, a conormal distribution associated to a hypersurface that is simply tangent to the incoming radial surface $x = \bar{t} - t$. For such a distribution, the lemma of stationary phase shows that tangential smoothing gains almost $(n-1)/2$ derivatives, hence Theorem I.3 guarantees that the diffracted wave is almost $(n-1)/2$ derivatives smoother than the main singularity. In addition, the diffracted wave for such a solution will itself be conormal.

More precise statements of the theorems above as well as further results are given in Section 4. We now briefly describe the ingredients in the proofs.

We make extensive use of the calculus of edge pseudodifferential operators as developed by Mazzeo [14]. A related calculus of pseudodifferential operators adapted to edge structures has also been constructed by Schulze [20] but the lack of "completeness" in this calculus makes it much less applicable; in particular it does not seem to have associated with it a useful notion of wavefront set. We develop and use just such a notion for Mazzeo's calculus measuring microlocal regularity with respect to the intrinsic weighted Sobolev spaces. We use this to obtain a result on the propagation of singularities at ∂X analogous to Hörmander's interior result. The technique of the proof is a positive commutator construction similar to that used in [9], but with the crucial distinction, familiar from scattering theory, that the bicharacteristic flow now has radial points, thus necessitating a more subtle construction. At the radial surfaces the propagation arguments are only valid for a limited range of weighted (edge) Sobolev spaces, corresponding to a linear estimate on the weight in terms of the regularity. Such an estimate amounts to a divisibility property for the solution in terms of powers of the boundary defining function.

The proof of the diffractive theorem, Theorem I.2, relies on the extraction of a leading part, the non-commutative normal operator in the edge calculus, via an appropriately rescaled FBI (Fourier-Bros-Iagolnitzer) transform S near Y. The model operator is $\Delta_0 - 1$, where Δ_0 is the Laplacian on the tangent cone $\mathbb{R}_+ \times Y$ for the product-conic metric $dx^2 + x^2(h\restriction_{x=0})$. Iterative application of the outgoing resolvent for this model operator, corresponding to the scattering structure at the infinite volume end of the cone, combined with the microlocal propagation results discussed above, yields the regularity of u. We use estimates from [16], although in this product case we could instead rely on direct methods, as used for instance in [2].

To prove the geometric propagation theorem, we begin by establishing a *division theorem* to the effect that the nonfocusing assumption (I.5) implies that the solution has better decay in x than would be predicted merely by energy conservation. This allows us to apply the propagation results to prove a special case of Theorem I.3.

We also show that conormality on a radial surfaces persists for solutions. That is, if $(u(0), D_t u(0))$ are conormal distributions at a surface $x = \bar{x}$, supported away from a boundary component Y, and if $\bar{x}$ is sufficiently small then by standard interior regularity results $(u(t), D_t u(t))$, is conormal with respect to $\{x = \bar{x} - t\} \cup \{x = \bar{x} + t\}$ for small positive $t < \bar{x}$; we show that it continues to be conormal with respect to $\{x = |\bar{x} - t|\} \cup \{x = \bar{x} + t\}$ for small $t > \bar{x}$, i.e. after the wave leaves the boundary. Such conormal solutions provide the counterexamples mentioned following Theorem I.3.

For solutions, such as the fundamental solution, that are initially conormal at a surface meeting the radial surfaces $x = \bar{x} - t$ with at most simple tangency, we are additionally able to prove that the diffracted front is conormal; this is in effect a microlocal version of the radial conormality argument discussed above. The conormality of the diffracted front and the special case of Theorem I.3 mentioned above suffice to establish conormality and (sharp) regularity for the diffracted wave of the fundamental solution, which is then used to prove Theorem I.3 in its full generality.

The outline of the paper is as follows. In §1 we prove the existence of a product decomposition for a conic metric in a neighborhood of ∂X; this is equivalent to reducing the metric to the normal form

$$g = dx^2 + x^2 h(x, y, dy).$$

Then in §2 and §3, we discuss the mapping properties and domains of the Laplacian and its powers on a conic manifold. This enables us to give precise statements of the main theorems in §4. Sections 5–8 build up the machinery of the edge calculus and culminate in the proof of the propagation theorem for the edge wavefront set. Then in §9 and §10 we discuss the rescaled FBI transform and the normal operator in the edge calculus, which we use to prove Theorem I.2 in §11. In §12 we demonstrate the conservation of tangential regularity, i.e. iterated regularity under powers of $\varDelta_Y$. This is then used in two ways. In §13 we use conservation of tangential regularity together with conservation of iterated regularity under vector fields of the form (xD_x+tD_t) to prove conservation of radial conormality. In §14, we use the conservation of tangential regularity to prove the division theorem; the proof of the sharper division result for conormal initial data is closely related to that of radial conormality. In §15 we prove a preliminary version of Theorem I.3 and establish a theorem on conormality of the diffracted front. In §16 we discuss a class of examples including the fundamental solution and prove Theorem I.1; as a consequence we then prove Theorem I.3 in full generality, and discuss a counterexample to that theorem when the nonfocusing condition is omitted.

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1. Conic metrics; geodesics and normal form

Let X be an n-dimensional manifold with compact boundary, ∂X its boundary and $X^\circ = X \setminus \partial X$ its interior.

Definition 1.1. A *conic metric* on X is a Riemannian metric g on X° such that in a neighborhood of any boundary component Y of X, there exists a boundary defining function x ($x \geq 0$, $\{x = 0\} = \partial X$, $dx\restriction_{\partial X} \neq 0$) in terms of which

$$g = dx^2 + x^2 h \tag{1.1}$$

where $h \in \mathcal{C}^\infty(\operatorname{Sym}^2 T^*X)$ and $h\restriction_Y$ is a metric. A *conic manifold* is a compact manifold with boundary endowed with a conic metric.

Let Y be a compact manifold without boundary. The *product-conic* metrics on $[0, \infty)_x \times Y$ are the conic metrics of the form

$$g = dx^2 + x^2 h_0 \tag{1.2}$$

where h_0 is a metric on Y. These are the model cases which form the basis of our analysis below. They also motivate the basic normal form for conic metrics which we discuss next.

Theorem 1.2. *Let g be a conic metric on X. There exists a collar neighborhood $\mathcal{O}$ of ∂X and an isomorphism $(x, \Pi) : \mathcal{O} \longrightarrow [0, \epsilon) \times \partial X$, $\epsilon > 0$, such that in terms of this product decomposition*

$$g = dx^2 + x^2 \Pi^* h(x) \text{ near } \partial X, \tag{1.3}$$

where $h(x) \in \mathcal{C}^\infty([0, \epsilon); \text{metrics on } \partial X)$.

Proof. The existence of such a normal form distinguishes a vector field V by

$$Vx = 1 \text{ and } \iota_V h(x) = 0 \text{ near } \partial X. \tag{1.4}$$

The integral curves of this vector field are geodesics. Our main task is thus to show that through any point in a neighborhood of ∂X, there is a unique short geodesic reaching ∂X; the length of this short geodesic segment will then furnish the desired defining function for ∂X.

As our construction is local near a boundary component, we assume without loss of generality that ∂X is connected. We begin with any collar neighborhood and associated projection $\pi_{\partial X} : \mathcal{O} \to \partial X$, and choose a boundary defining function ρ for ∂X and coordinates $\upsilon = \pi^*_{\partial X} \bar{y}$ with $\bar{y}$ any coordinates on ∂X. We may assume ρ to have been chosen in accordance with our definition of a conic metric, so that

$$g = d\rho^2 + \rho^2 h$$

where h is a smooth, symmetric two-tensor. Let

$$h = \bullet d\rho^2 + \bullet d\rho \, d\upsilon + k_{ij}(\upsilon) d\upsilon^i d\upsilon^j + \rho \bullet d\upsilon^i d\upsilon^j;$$

then

$$\langle \cdot, \cdot \rangle_g^* = (1 + \mathcal{O}(\rho^2)) \partial_\rho^2 + \mathcal{O}(1) \partial_\rho \partial_\upsilon + (\rho^{-2} k^{ij}(\upsilon) + \mathcal{O}(\rho^{-1})) \partial_{\upsilon_i} \partial_{\upsilon_j}, \tag{1.5}$$

where, here and henceforth, $\mathcal{O}(\rho^k)$ means ρ^k times a smooth function of (ρ, υ).

We now regard the dual metric as a function on a rescaled version of the cotangent bundle. Let ${}^{\mathrm{b}}TX$ denote the *b-tangent bundle* whose sections are the vector fields tangent to ∂X. Let ${}^{\mathrm{b}}T^*X$ be its dual. Then sections of ${}^{\mathrm{b}}T^*X$ are $\mathcal{C}^\infty$-linear combinations of $d\rho/\rho$ and $d\upsilon_i$'s. Writing the canonical one-form on ${}^{\mathrm{b}}T^*X$ as

$$\xi \frac{d\rho}{\rho} + \eta \cdot d\upsilon, \tag{1.6}$$

we can now write the dual metric as

$$\langle (\xi, \eta), (\xi, \eta) \rangle_g^* = \frac{\xi^2 + k^{ij} \eta_i \eta_j}{\rho^2} + \mathcal{O}(1) \xi^2 + \mathcal{O}(\rho^{-1}) \xi \eta + \mathcal{O}(\rho^{-1}) \eta^2. \tag{1.7}$$

Over the interior ${}^{\mathrm{b}}T^*X^\circ \equiv T^*X^\circ$ and the canonical symplectic form on T^*X lifts to a form on ${}^{b}T^*X$, singular at the boundary, given in canonical coordinates by $d(\xi d\rho/\rho + \eta \cdot d\upsilon)$. Associated with this symplectic form and the energy function $\langle \cdot, \cdot \rangle_g^*$ is the Hamilton vector field

$$H_g = \frac{2}{\rho^2} (\xi \rho \partial_\rho + (\xi^2 + k^{ij} \eta_i \eta_j) \partial_\xi + H_{\partial X}) + P,$$

where

$$P = \frac{1}{\rho^2} ((\mathcal{O}(\rho^3) \xi + \mathcal{O}(\rho^2) \eta) \partial_\rho + (\mathcal{O}(\rho^3) \xi^2 + \mathcal{O}(\rho) \xi \eta + \mathcal{O}(\rho) \eta^2) \partial_\xi$$
$$+ (\mathcal{O}(\rho) \xi + \mathcal{O}(\rho) \eta) \partial_\upsilon + (\mathcal{O}(\rho^2) \xi^2 + \mathcal{O}(\rho) \xi \eta + \mathcal{O}(\rho) \eta^2) \partial_\eta,$$

and $H_{\partial X}$ is geodesic spray in the (υ, η) variables with respect to the metric k. The projections of the integral curves of H_g to X are geodesics.

Let $H_0 = H_g - P$ be the main term in the Hamilton vector field. The vector field H_0 is thus tangent to the submanifold $N = \{\eta = 0\}$ of "normal" directions to ∂X. Since on N, the flowout of $(\rho^2/2)H_0$ is $\rho = \rho_0/(1-\xi_0 s)$ and $\xi = \xi_0/(1-\xi_0 s)$, the map

$$p \mapsto \lim_{s\to+\infty} \exp\left[-s \operatorname{sgn}\xi \frac{\rho^2}{2} H_0\right]$$

maps N to ∂X smoothly in a neighborhood of the boundary, taking $p \mapsto \upsilon(p) \in \partial X$. The projections of integral curves of H_0 on N would be geodesics reaching the cone point were the metric of the desired form (1.3).

We now show that an analogous *normal manifold,* given by a perturbation of N, exists for the full vector field H_g. The flow along this manifold to the boundary will provide the desired geodesics to the cone point.

To simplify the discussion of the vector field, H_g, we scale away its homogeneity in (ξ, η) as follows. Consider the smooth function

$$\sigma = \frac{1}{\rho(\langle\cdot,\cdot\rangle_g^*)^{1/2}}$$

on the complement of the zero-section in ${}^{\mathrm{b}}T^*X$. Note that σ is approximately equal to $(\xi^2 + k^{ij}\eta_i\eta_j)^{-1/2}$, and is homogeneous of degree -1 in the fibers of ${}^{\mathrm{b}}T^*X$. We further set

$$\bar\xi = \sigma\xi,\ \ \bar\eta = \sigma\eta. \tag{1.8}$$

Since $\langle\cdot,\cdot\rangle_g^*$ is preserved under the flow H_g, $(x^2/2)H_g\sigma = -\bar\xi + \mathcal{O}(\rho^2)\bar\xi + \mathcal{O}(\rho)\bar\eta$. Note that the error terms in this expression are at least quadratic in $(\rho, \bar\eta)$. Thus we can write

$$\frac{1}{2}\rho^2\sigma H_g = \bar\xi\sigma\partial_\sigma + \bar\xi\rho\partial_\rho + k^{ij}\bar\eta_j\partial_{\upsilon_i} + k^{ij}\bar\eta_i\bar\eta_j\partial_{\bar\xi} - \left(\bar\xi_k + \frac{1}{2}\frac{\partial k^{ij}}{\partial\upsilon_k}\bar\eta_i\bar\eta_j\right)\partial_{\bar\eta_k} + P' \tag{1.9}$$

where the perturbation term has the form

$$P' = \mathcal{O}(\rho)\bar\xi\partial_\upsilon + \mathcal{O}(\rho^2+\bar\eta^2)\,(\mathcal{C}^\infty \text{ vector field})\,;$$

the single non-quadratic error term comes directly from the corresponding term in P. By homogeneity, the error terms above are independent of σ, i.e. $(1/2)\rho^2\sigma H_g$ pushes forward to a vector field on the unit sphere bundle ${}^{\mathrm{b}}S^*X \equiv {}^{\mathrm{b}}T^*X/\mathbb{R}_+$. On ${}^{\mathrm{b}}S^*X$, $(\rho, \upsilon, \bar\eta)$ are coordinates near $\bar\eta = 0$, since $\big\langle(\bar\xi,\bar\eta),(\bar\xi,\bar\eta)\big\rangle_g^* = 1$ at $\bar\eta = 0$, $\bar\xi = \pm 1$. Moreover we easily see that $\bar\xi \mp 1$ vanishes to second order at $\rho = \bar\eta = 0$; of crucial importance is the fact that the power of ρ multiplying the ξ^2 error term in (1.7) is larger than those multiplying the $\xi\eta$ and η^2 terms.

In these coordinates, the linearization of the vector field $(1/2)\rho^2\sigma H_g$ in $(\rho, \bar\eta)$ near $\rho = \bar\eta = 0$ and $\bar\xi = \pm 1$ is simply

$$L = \pm\rho\partial_\rho + (k^{ij}\bar\eta_j + \mathcal{O}(\rho))\partial_{\upsilon_i} \mp \bar\eta_i\partial_{\bar\eta_i}$$

Since $(1/2)\rho^2\sigma H_g$ vanishes identically at $\rho = \bar\eta = 0$, and its linearization has eigenvalues ± 1 in the normal directions, $(1/2)\rho^2\sigma H_g$ is r-normally hyperbolic near $\rho = \bar\eta = 0$ for all r, in the notation of [7]. Hence by the Stable/Unstable Manifold Theorem as stated in Theorem 4.1 of [7], near $\{\rho = \bar\eta = 0,\ \bar\xi = 1\}$ there exists a stable invariant manifold N for the flow of $(1/2)\rho^2\sigma H_g$, with N tangent to $\{\bar\eta = 0\}$ at $\{\rho = \bar\eta = 0\}$. Because N is tangent to $\{\bar\eta = 0\}$, the projection map

$$\pi : {}^{\mathrm{b}}S^*X \to X$$

restricts to give a diffeomorphism $N \cong X$ (locally, for ρ sufficiently small).

On N, $\lim_{s\to-\infty} \rho = 0$, where s parametrizes the flow along $(1/2)\rho^2\sigma H_g$; using ρ as a parameter along the flow on N yields, by (1.8) and (1.9),

$$\frac{d\upsilon}{d\rho} = \mathcal{O}(1) + \mathcal{O}(\bar\eta/\rho);$$

the latter term is in fact $\mathcal{O}(1)$ since N is tangent to $\{\bar\eta = 0\}$ at $\rho = 0$. Thus, $\lim_{s\to-\infty} \upsilon$ is in fact a smooth map. Hence

$$\Pi : p \mapsto \lim_{s\to-\infty} \exp_{\pi^{-1}(p)} \frac{1}{2}\rho^2\sigma H_g \tag{1.10}$$

is a smooth map from a neighborhood $\mathcal{O}'$ of ∂X in X to ∂X. Indeed Π is a fibration, with fibers $\mathbb{R}_+$ given by the projections of integral curves of $(1/2)\rho^2\sigma H_g$, i.e. by geodesics hitting the “cone point” ∂X.

On $\mathcal{O}'$, set $y_i(p) = \upsilon_i(\Pi(p))$ and $x(p) = d_g(p, \Pi(p))$. To first order at ∂X, $x = \rho$ and $y = \upsilon$. Hence $(x, y_1, \dots, y_{n-1})$ form a coordinate system on a neighborhood $\mathcal{O}''$. In these coordinates, $\langle \partial_x, \partial_x\rangle_g = 1$ since $\exp t\partial_x$ is unit speed flow along geodesics reaching ∂X. Furthermore, $\langle \partial_x, \partial_{y_i}\rangle_g = 0$ for all i by Gauss's Lemma[1] and $\langle \partial_{y_i}, \partial_{y_j}\rangle_g = \mathcal{O}(x^2)$. Hence in the coordinates $(x,\ y)$, g takes the form (1.3). $\square$

Since x is the distance along the normal geodesics it is uniquely determined by (1.3); the vector field V determining the product decomposition is also fixed geometrically by (1.4). The choice of a conic metric additionally induces a metric on ∂X, namely

$$h_0 = h(0).$$

Henceforth, x will always denote this distance function for the given conic metric.

The proof of Theorem 1.2 used in a crucial way the existence of a unique normal geodesic starting at each point of the boundary. These geodesics foliate a neighborhood of the boundary, and indeed, the existence of this foliation characterizes conic metrics among more general nondegenerate forms in dx and xdy_i. On $\mathbb{R} \times X^\circ$, we thus obtain a foliation of a neighborhood of $\mathbb{R} \times \partial X$ by projections of null bicharacteristics for the symbol of the d'Alembertian, $D_t^2 - \Delta$.

We shall fix notation for various sets corresponding to the geodesic segments that hit the boundary and are within small distance $\epsilon > 0$ of it.

Definition 1.3. Let $N^\epsilon = \{\alpha dt + \beta dx : |\alpha| = |\beta|,\ x < \epsilon\} \subset T^*(\mathbb{R} \times X^\circ)$. For $\bar t \in \mathbb{R}$, $\bar y \in Y$, let

$$R^\epsilon_{\pm,I}(\bar t, \bar y) = N^\epsilon \cap \{\operatorname{sgn} \alpha\beta = 1,\ \operatorname{sgn} \alpha = \pm,\ t = \bar t - x,\ y = \bar y\},$$
$$R^\epsilon_{\pm,O}(\bar t, \bar y) = N^\epsilon \cap \{\operatorname{sgn} \alpha\beta = 1,\ \operatorname{sgn} \alpha = \pm,\ t = \bar t + x,\ y = \bar y\}.$$

[1]The usual proof of Gauss's Lemma using the first variation formula works even at a cone point, i.e. geodesics with one endpoint on ∂X are orthogonal to the hypersurfaces $d_g(p, \partial X) = \epsilon$. We simply let $y(u)$ be a curve in ∂X, and apply the first variation formula to the family of geodesics connecting $x = x_0$, $y = y(u)$ to $x = 0$, $y = y(u)$.

If any of the parameters $\bullet = I$ or O, $\pm$, $\bar{t}$, or $\bar{y}$ is omitted, the resulting set is defined as the union over all possible values of that parameter. If Y is a single component of ∂X, let

$$R^{\epsilon}_{\pm,\bullet}(\bar{t}, Y) = \bigcup_{y \in Y} R^{\epsilon}_{\pm,\bullet}(\bar{t}, y).$$

The set N^ϵ is the *normal set* near the boundary of X of points in the cotangent bundle which lie along geodesics (projections of bicharacteristics) entering and leaving ∂X. Here "I" and "O" stand for the "incoming" and "outgoing" components of N, on which dx/dt is respectively negative and positive. The additional sign $\pm$ microlocalizes in the sign of the dual variable to t. The sets $R^{\epsilon}_{\pm,\bullet}(\bar{t}, \bar{y})$ are the points through which the short geodesic to the boundary arrives at or departs from ∂X at time $t = \bar{t}$ and at the point $\bar{y}$.

If X is a compact conic manifold then every geodesic starting at an interior point can be extended maximally in both directions until and unless it terminates at the boundary. This naturally suggests the question of how such geodesics can, or should, be further extended.

Definition 1.4. By a *limiting geodesic* in a conic manifold we mean a continuous piecewise smooth curve $c : I \longrightarrow X$, where $I = \cup_j I_j$ is decomposed as a locally finite union of relatively closed subintervals on each of which c restricts to be a smooth curve c_j and such that

(1) Each c_j is either a geodesic in X or a geodesic (for h_0) in ∂X and such segments alternate.
(2) Boundary segments are of length at most π and if such a boundary segment is not the first or last segment then its length is exactly π.

Lemma 1.5. *If ξ_i is a sequence of geodesics in X° which converges uniformly as curves in X then its limit is a limiting geodesic and conversely an open neighborhood of each boundary segment of any limiting geodesic arises as such a limit.*

This result will not be used except as motivation, hence we omit its proof.

In view of this behavior of the geodesics we define a singular relation, interpreted as a set-valued map

$$\Gamma^\epsilon \subset R^\epsilon_O \times R^\epsilon_I$$

such that

$$\text{for } p \in R^\epsilon_{\pm,O}(\bar{t}),\ \Gamma^\epsilon(p) = \{q \in R^\epsilon_{\pm,I}(\bar{t}) : \Pi(q) \in G(\Pi(p))\}, \tag{1.11}$$

where Π is the projection map from a collar neighborhood of ∂X to ∂X given by (1.10), and for $y \in \partial X$, $G(y)$ is the geodesic relation defined in (I.4). Clearly,

$$R^\epsilon_{\pm,I}(\bar{t}) = \bigcup_{p \in R_{\pm,O}(\bar{t})} \Gamma^\epsilon(p). \tag{1.12}$$

In fact $\Gamma^\epsilon(p)$ is generically of codimension one in $R^\epsilon_{\pm,I}(\bar{t})$.

To the given conic metric and a boundary component Y we associated the limiting product metric

$$g_0 = dx^2 + x^2 h_0 \tag{1.13}$$

on the normal bundle to Y, which we may identify with $[0, \infty) \times Y$ using the decomposition associated with (1.3). Let Δ denote the (nonnegative) Laplace-Beltrami operator with respect to the metric g and Δ_0 that with respect to g_0.

Then, letting $(w_0,\dots,w_{n-1})=(x,y_1,\dots,y_{n-1})$,

$$\Delta=\sum_{j,k=0}^{n}\frac{1}{\sqrt{g}}D_{w_j}g^{jk}\sqrt{g}D_{w_k}$$

becomes, near Y,

$$\Delta=D_x^2-\frac{i[(n-1)+xe]}{x}D_x-\frac{i}{2}D_x+\frac{\Delta_h}{x^2},\quad \text{where } e=\frac{1}{2}\frac{\partial\log\det h(x)}{\partial x}=\frac{1}{2}\operatorname{tr}\left(h^{-1}(x)\frac{\partial h(x)}{\partial x}\right). \tag{1.14}$$

Here Δ_h is the Laplacian on ∂X with respect to the (x-dependent) metric $h(x)$ on ∂X. Similarly

$$\Delta_0=D_x^2-\frac{i(n-1)}{x}D_x+\frac{\Delta_{h_0}}{x^2}, \tag{1.15}$$

hence if we identify $[0,\infty)\times Y$ with the metric product decomposition near Y,

$$\Delta-\Delta_0\in x^{-1}\left(\mathcal{C}^\infty(x,y)(xD_x)+\sum\mathcal{C}^\infty(x,y)D_{y_i}D_{y_j}+\sum\mathcal{C}^\infty(x,y)D_{y_i}\right) \tag{1.16}$$

We will also use the notation Δ_Y for Δ_{h_0}, when restricting our attention to a single boundary component.

If g is a product-conic metric, it is easy to check that

$$[\Box,\Delta_Y]=0,$$
$$[\Box,(xD_x+(t-\bar t)D_t)]=-2i\Box.$$

In the general conic case, these "symmetries" are broken. It is crucial for our purposes, though, that perturbed versions of the above identities still hold. In particular, if for brevity we set

$$R=xD_x+(t-\bar t)D_t, \tag{1.17}$$

then

$$[\Box,\Delta_Y]=QD_x+x^{-1}P,\ \text{with}\quad Q\in\mathcal{C}^\infty([0,\epsilon);\mathrm{Diff}^1(Y)),\ \text{and}\ P\in\mathcal{C}^\infty([0,\epsilon);\mathrm{Diff}^3(Y)) \tag{1.18}$$

and

$$[\Box,R]=-2i\Box+aD_x+x^{-1}P,\text{with}\quad a\in\mathcal{C}^\infty([0,\epsilon)\times Y)\ \text{and}\ P\in\mathcal{C}^\infty([0,\epsilon);\mathrm{Diff}^2(Y)). \tag{1.19}$$

More generally,

Lemma 1.6. *For any $q,k\in\mathbb{N}$,*

$$\Box R^k=\sum_{j=0}^{k}c_{k,j}R^j\Box+\sum_{j=0}^{k-1}\left(a_{k,j}D_x+\frac{1}{x}P_{k,j}\right)R^j,\ \textit{where}\quad c_{k,j}\in\mathbb{C},\ a_{k,j}\in\mathcal{C}^\infty([0,\epsilon)\times Y)\ \textit{and}\ P_{k,j}\in\mathcal{C}^\infty([0,\epsilon);\mathrm{Diff}^2(Y)), \tag{1.20}$$

$$\Box \Delta_Y^q = \Delta_Y^q \Box + \sum_{r=0}^{q-1} (Q_{q,r} D_x + \frac{1}{x} P_{q,r}) \Delta_Y^r \tag{1.21}$$
$$\text{with } Q_{q,r} \in \mathcal{C}^\infty([0,\epsilon); \mathrm{Diff}^{q-r}(Y)),\ P_{q,r} \in \mathcal{C}^\infty([0,\epsilon); \mathrm{Diff}^{q-r+2}(Y)),$$

and

$$\Box \Delta_Y^q R^k = \sum_{j=0}^{k} c_{k,j} \Delta_Y^q R^j \Box + \sum_{\substack{0 \le r \le q,\ 0 \le j \le k \\ r+j < k+q}} (Q_{k,l,q,r} D_x + \frac{1}{x} P_{k,l,q,r}) \Delta_Y^r R^j \tag{1.22}$$
$$\text{where } Q_{k,l,q,r} \in \mathcal{C}^\infty([0,\epsilon); \mathrm{Diff}^{q-r}(Y)),\ P_{k,l,q,r} \in \mathcal{C}^\infty([0,\epsilon); \mathrm{Diff}^{q-r+2}(Y)).$$

Furthermore, all the differential operators $P_{k,j}$, $P_{q,r}$, $P_{k,l,q,r}$ have vanishing constant terms.

Proof. For $k = 1$ (1.20) follows directly from (1.19). For general k we may use this together with (1.20), as an inductive hypothesis, to see that

$$\Box R^{k+1} = R \Box R^k - [R, \Box] R^k$$

is of the stated form. Equation (1.21) follows similarly, and (1.22) from combining (1.20) and (1.21). □

It is also convenient to record a version of (1.21) which holds for real powers of Δ_Y in the tangential pseudodifferential calculus. Let $Y_s = (1 + \Delta_Y)^{s/2}$.

Lemma 1.7. *For all $s \in \mathbb{R}$,*

$$[\Box, Y_s] Y_{-s} = P D_x + \frac{1}{x} Q,$$
$$Y_{-s} [\Box, Y_s] = P' D_x + \frac{1}{x} Q',$$

with $P, P' \in \mathcal{C}^\infty([0,\epsilon); \Psi^{-1}(Y))$ and $Q, Q' \in \mathcal{C}^\infty([0,\epsilon); \Psi^1(Y))$ and where Q, Q' annihilate constants at $x = 0$.

2. Mapping properties of the Laplacian

By definition, Δ is symmetric as an operator on $\mathcal{C}_c^\infty(X)$ with respect to the Riemannian volume form

$$dg = x^{n-1} dx\, dh(x). \tag{2.1}$$

To keep track of the weighted L^2 and Sobolev spaces which necessarily appear here, we shall refer all weights to the intrinsic boundary weights. These correspond to a non-vanishing positive smooth density on X° with is of "logarithmic" form near the boundary

$$\nu_b = a \frac{dx}{x} dh_0,\ 0 < a \in \mathcal{C}^\infty(X) \text{ near } \partial X. \tag{2.2}$$

Thus we set

$$L_{\mathrm{b}}^2(X) = \left\{ u \in L_{\mathrm{loc}}^2(X^\circ); \int_X |u|^2 \nu_b < \infty \right\}. \tag{2.3}$$

Since the conic metric volume form is $dg = x^n a' \nu_b$ with $0 < a' \in \mathcal{C}^\infty(X)$, the metric L^2 space is

$$L^2_g(X) = \left\{ u \in L^2_{\mathrm{loc}}(X); \int_X |u|^2 dg < \infty \right\} = x^{-\frac{n}{2}} L^2_{\mathrm{b}}(X). \tag{2.4}$$

From (1.14) it follows that

$$\Delta = x^{-2} \left[(xD_x)^2 - \frac{i(n-1)+xe}{x} D_x - i(n-2)xD_x + \Delta_h \right] \in x^{-2}\,\mathrm{Diff}^2_{\mathrm{b}}(X). \tag{2.5}$$

Here, $\mathrm{Diff}^*_{\mathrm{b}}(X)$ is the filtered algebra of differential operators on X which is the enveloping algebra of the Lie algebra $\mathcal{V}_{\mathrm{b}}(X)$ of all smooth vector fields on X which are tangent to the boundary.

The weighted b-Sobolev spaces $x^p H^l_{\mathrm{b}}(X)$ are essentially defined by the mapping properties of these b-differential, and the corresponding b-pseudodifferential, operators. They may also be defined directly using the Mellin transform. Any weighted b-differential operator defines a continuous linear map:

$$P \in x^r\,\mathrm{Diff}^m_{\mathrm{b}}(X) \Longrightarrow P : x^p H^l_{\mathrm{b}}(X) \longrightarrow x^{p+r} H^{l-m}_{\mathrm{b}}(X)\ \forall\ p, l \in \mathbb{R}. \tag{2.6}$$

An *elliptic* b-differential operator, i.e. one for which the characteristic polynomial in tangential vector fields is invertible off the zero section of ${}^{\mathrm{b}}T^*X$, has the inverse property with respect to regularity

$$\begin{aligned} P \in x^r\,\mathrm{Diff}^m_{\mathrm{b}}(X),\ x^{-r}P \text{ elliptic },\ u \in x^p H^{-\infty}_{\mathrm{b}}(X),\ Pu \in x^{p+r} H^{l-m}_{\mathrm{b}}(X) \\ \Longrightarrow u \in x^p H^l_{\mathrm{b}}(X). \end{aligned} \tag{2.7}$$

(See [17] for a detailed discussion of b-differential and -pseudodifferential operators.)

Such an elliptic operator is Fredholm as an operator (2.6) for all but a discrete set of $p \in \mathbb{R}$. These correspond to the indicial roots, those values of the complex parameter for which the indicial operator is not invertible. The indicial operator is defined in general by

$$P \in x^r\,\mathrm{Diff}^m_{\mathrm{b}}(X) \Longrightarrow P(x^{is}v) = x^{is+r}(I(P,s)v + \mathcal{O}(x)). \tag{2.8}$$

This definition depends on the differential, at the boundary, of the defining function chosen. Rather than carry the normal bundle information to make this invariant we shall simply choose x to be the defining function in (1.3).

From (1.14), the indicial family of the Laplacian at a boundary component Y is

$$I(\Delta, s) = \Delta_Y - i(n-2)s + s^2. \tag{2.9}$$

If $0 = \lambda_0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ is the sequence of eigenvalues of Δ_Y, repeated with multiplicity, then the indicial roots of the Laplacian are

$$s^\pm_j = i\frac{(n-2)}{2} \pm \frac{i}{2}\sqrt{(n-2)^2 + \lambda^2_j}. \tag{2.10}$$

In general, assuming $P \in x^r\,\mathrm{Diff}^m_{\mathrm{b}}(X)$ to be elliptic,

(2.11) P is Fredholm as an operator (2.6)

$\Longleftrightarrow -p$ is not the imaginary part of an indicial root.

Thus, 0 and $-n+2$ are always singular values of p for the Laplacian. For $n > 2$ there is a gap in between:

(2.12) $(-n+2, 0)$ is free of singular values.

Such a gap corresponds to boundary regularity for solutions. Thus

$$\begin{aligned}(2.13)\quad & P\in x^r\operatorname{Diff}^m_{\mathrm{b}}(X)\text{ elliptic, } u\in x^pH^l_{\mathrm{b}}(X),\ Pu\in x^{q+r}H^{l-m}_{\mathrm{b}}(X)\\ &\Longrightarrow u\in x^qH^l_{\mathrm{b}}(X)\text{ provided } (p,q]\text{ is free of singular values.}\end{aligned}$$

The conclusion of (2.13) (where we assume that $p<q$ to avoid triviality) does not follow if $(p,q]$ contains a singular value. For our purposes it is enough to consider the special case that

$$(2.14)\quad p^*\in(p,q)\text{ is the unique singular value in }(p,q)\text{ and }p^*<p+1,\ q<p^*+1.$$

In fact let us further suppose that there is only one singular value $s\in\mathbb{C}$ such that $p^*=-\operatorname{Im}s$, and $I(P,s)$ is not invertible. Then

$$\begin{aligned}(2.15)\quad & u\in x^pH^l_{\mathrm{b}}(X),\ Pu\in x^{r+q}H^{l-m}_{\mathrm{b}}\Longrightarrow u=u'+u'',\\ & Pu'\in\dot{\mathcal{C}}^\infty(X),\ u''\in x^qH^l_{\mathrm{b}}(X),\ u'=\sum_{0\le j\le k-1}x^{is}(\log x)^jv_j\chi+\tilde u'\text{ and}\\ & \tilde u'\in x^qH^l_{\mathrm{b}}(X)\text{ where } I(P,xD_x)\left(\sum_{0\le j\le k-1}x^{is}(\log x)^jv_j\right)=0.\end{aligned}$$

Here, k is the order of s as a pole of $I(P,s)^{-1}$ and $\chi=\chi(x)$ is a cutoff which is identically 1 near the boundary. If there are several values of s with $\operatorname{Im}s=-p$ at which $I(P,s)^{-1}$ is singular then it is only necessary to add corresponding sums to (2.15).

3. Domains and powers

Applying the general results above for b-differential operators to the Laplacian we find

Proposition 3.1. *If $n\ge 4$ then*

$$(3.1)\qquad \operatorname{Dom}(\Delta)=\left\{u\in x^wL^2_{\mathrm{b}}(X);\Delta u\in L^2_g(X)\right\}$$

is independent of w in the range

$$(3.2)\qquad -n+2<w<-\frac{n}{2}+2.$$

If $n=3$ the same is true for w in the range $-1<w<0$ and for $n=2$

$$(3.3)\quad \operatorname{Dom}(\Delta)=\left\{u\in L^2_g(X);u=c+u',\ c\in\mathbb{C},\ u'\in x^wL^2_{\mathrm{b}}(X),\ \Delta u'\in L^2_g(X)\right\}$$

is independent of w for $w>0$ sufficiently small. In all cases, Δ is an unbounded self-adjoint operator

$$(3.4)\qquad \Delta:\operatorname{Dom}(\Delta)\longrightarrow L^2_g(X)$$

and $\operatorname{Dom}(\Delta)$ coincides with the domain of the Friedrichs extension; if $n\ge 3$, then Δ is essentially self-adjoint.

Proof. For $n\ge 3$ the constancy of $\operatorname{Dom}(\Delta)$ in terms of w follows from (2.12) and (2.13). Thus, by hypothesis, $u\in x^wL^2_{\mathrm{b}}(X)$, with w in the gap $(-n+2,0)$ and $Pu=x^2\Delta u\in x^{-\frac{n}{2}+2}L^2_b$. If $n>4$ then $-\frac{n}{2}+2$ also lies in the gap, (2.12), so by (2.7) and (2.13)

$$n>4,\ u\in\operatorname{Dom}(\Delta)\Longrightarrow u\in x^{-\frac{n}{2}+2}H^2_{\mathrm{b}}(X)$$

so then

$$\mathrm{Dom}(\Delta) = x^{2-\frac{n}{2}} H^2_{\mathrm{b}}(X),\ n > 4. \tag{3.5}$$

For $n = 4$, $-\frac{n}{2} + 2 = 0$ is the top of the gap whereas for $n = 3$ the gap is $(-1, 0)$. In these cases we deduce only that

$$\mathrm{Dom}(\Delta) = \left\{ u \in \bigcap_{w<0} x^w H^2_{\mathrm{b}}(X) :\ \Delta u \in L^2_g(X) \right\},\ n = 3, 4. \tag{3.6}$$

On the other hand, for $n = 2$ there is no gap. The hypothesis $\Delta u \in L^2_g(X)$ is equivalent to $Pu = x^2 \Delta u \in x L^2_g(X)$. The collapsed gap, at $p = 0$, corresponds to a double root of $I(P, s) = \Delta_Y + s^2$. Thus, (2.15) becomes

$$\begin{gathered} u \in x^{-\epsilon} H^2_{\mathrm{b}}(X),\ \Delta u \in L^2_g(X) \Longrightarrow \\ u = c + c' \log x + u'',\ u'' \in x^{\epsilon} H^2_{\mathrm{b}}(X),\ \epsilon > 0 \text{ sufficiently small.} \end{gathered} \tag{3.7}$$

In particular ϵ must be smaller than the smallest non-zero eigenvalue of Δ_Y, $\lambda_1(\Delta_Y)$. The hypothesis (3.3) on $u \in \mathrm{Dom}(\Delta)$ is therefore just the vanishing of the coefficient c' in (3.7). It follows that (3.3) is also independent of w provided $w < \lambda_1(\Delta_Y)$.

We now demonstrate selfadjointness by showing that the unbounded operators with the domains described above coincide with the Friedrichs extension of Δ. By definition, Δ is associated to the Dirichlet form

$$F(u, v) = \int_X \langle du, dv \rangle_g \, dg\,,\ u, v \in \mathcal{C}^\infty_c(X^\circ) \tag{3.8}$$

where dg is the metric volume form (2.1). The inner product in (3.8) is that induced, by duality, by the metric on T^*X°. Following Friedrichs we define

$$\mathcal{D} = \mathrm{Dom}(\Delta^{\frac12}) = \mathrm{cl}\left\{ \mathcal{C}^\infty_c(X^\circ) \text{ w.r.t. } F(u, u) + \|u\|^2_{L^2_g} \right\}, \tag{3.9}$$

whenever X is a compact conic manifold with boundary of dimension $n \geq 2$. Then the Friedrichs extension of Δ is the unbounded operator with domain

$$\mathrm{Dom}(\Delta_{\mathrm{Fr}}) = \left\{ u \in \mathcal{D}; \Delta u \in L^2_g(X) \right\}. \tag{3.10}$$

where Δ is the bounded operator $\mathcal{D} \longrightarrow \mathcal{D}'$ and $L^2_g(X) \subset \mathcal{D}'$ is a well-defined subspace since $\mathcal{D} \subset L^2_g(X)$ is dense.

The space $\mathrm{Dom}(\Delta^{\frac12})$ is independent of which conic metric on X is used to define it, since different conic metrics give equivalent norms in (3.8). Moreover it can be localized by use of a partition of unity. Thus it is the same, locally, as the same space on a compact manifold blown up at a single point. This is well-known and easily leads to the characterization

$$\mathrm{Dom}(\Delta^{\frac12}) = \begin{cases} x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X) + \rho(x)\tilde{\mathcal{D}} & n = 2, \\ x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X), & n \geq 3, \end{cases} \tag{3.11}$$

where $\rho(x)$ is a cutoff near the boundary in terms of a boundary defining function and $\tilde{\mathcal{D}}$ is the analogous domain of the 1-dimensional operator D_x, acting on $x^{-1}L^2_{\mathrm{b}}([0,\infty))$. Using Paley's theorem it is straightforward to characterize $\tilde{\mathcal{D}} \subset x^{-1}L^2_{\mathrm{b}}([0,\infty))$ in terms of the Mellin transform:

$$
\begin{aligned}
(3.12)\quad & u \in \tilde{\mathcal{D}} \Longleftrightarrow u \in x^{-\delta}L^2_{\mathrm{b}}([0,\infty)),\ \delta > 0,\ \text{and} \\
& u_M(x) = \int x^{is}u(x)\frac{dx}{x} \text{ is holomorphic in } -\infty < \operatorname{Im} s < 0 \text{ with} \\
& \int_{\operatorname{Im} s = r} |su_M(s)|^2 d\operatorname{Re} s \text{ uniformly bounded in } r \in (-\infty, 0).
\end{aligned}
$$

In the definition of the Friedrichs domain, (3.10), the action of Δ is distributional. Thus from (3.1) and (3.2) it follows that

$$
(3.13)\quad \operatorname{Dom}(\Delta_{\mathrm{Fr}}) = \left\{u \in x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X); \Delta u \in x^{-\frac{n}{2}}L^2_{\mathrm{b}}(X)\right\} = \operatorname{Dom}(\Delta),\ n > 2.
$$

For $n = 2$ the argument only needs to be modified slightly. It follows directly from (3.3) that $\operatorname{Dom}(\Delta) \subset \operatorname{Dom}(\Delta_{\mathrm{Fr}})$. Moreover (2.15) shows that $u \in \operatorname{Dom}(\Delta_{\mathrm{Fr}})$ has an expansion as in (3.3) except for the possibility of a logarithmic term. This however is excluded by (3.12) since it would correspond to a double pole of the Mellin transform at $s = 0$.

Thus in all case we have shown that the Friedrichs extension has domain $\operatorname{Dom}(\Delta)$ as given in Proposition 3.1. □

We also need to describe the domains of the complex powers of Δ. For integral powers it is straightforward to do so. Since

$$
(3.14)\quad I(\Delta^k, s) = I(\Delta, s)I(\Delta, s + 2i)\dots I(\Delta, s + 2(k-1)i),
$$

it follows that the singular values of Δ^k are just the unions of the shifts of those of Δ, with the possibility of accidental multiplicity to be borne in mind. We are particularly interested in the domains of the small powers with real part up to $n/4$. Note that it follows from arguments directly analogous to those above that

$$
(3.15)\quad \operatorname{Dom}(\Delta^p) = x^{-\frac{n}{2}+2p}H^{2p}_{\mathrm{b}}(X) \text{ for } p < \frac{n}{4},\ 2p \in \mathbb{N}.
$$

For later applications we need to find the largest real p for which this remains true. To do so it is convenient to use complex interpolation.

Lemma 3.2. *For real $0 \le p < n/4$ the identification* (3.15) *remains true. The domain of $\Delta^{n/4}$ is independent of the conic metric defining the Laplacian and is given explicitly by*

$$
(3.16)\quad \operatorname{Dom}(\Delta^{\frac{n}{4}}) = H^{\frac{n}{2}}_{\mathrm{b}}(X) + \rho(x)\tilde{\mathcal{D}}_n
$$

where $\tilde{\mathcal{D}}_n$ reduces to (3.12) *in case $n = 2$ and in general is defined by*

$$
\begin{aligned}
(3.17)\quad & u \in \tilde{\mathcal{D}}_n \Longleftrightarrow u \in x^{-\delta}L^2_{\mathrm{b}}([0,\infty)),\ \delta > 0 \text{ and} \\
& u_M(x) = \int x^{is}u(x)\frac{dx}{x} \text{ is holomorphic in } -\infty < \operatorname{Im} s < 0 \text{ with} \\
& \int_{\operatorname{Im} s = r} (|s|^2 + 1)^{\frac{n}{2}-1}|su_M(s)|^2 d\operatorname{Re} s \text{ uniformly bounded in } r \in (-\infty, 0).
\end{aligned}
$$

There exists $\delta_0 > 0$, depending on the metric and manifold, such that for all $\delta \in (0, \delta_0)$,

$$
(3.18)\quad \operatorname{Dom}(\Delta^{\frac{n}{4}+\delta}) = x^{2\delta}H^{\frac{n}{2}+2\delta}_{\mathrm{b}}(X) + \mathbb{C}
$$

and

$$\mathrm{Dom}(\varDelta^p) \supset x^{-\frac{n}{2}+2p} H_b^{2p}(X),\ \forall\ p \geq n/4; \tag{3.19}$$

by duality there is a restriction map

$$\mathrm{Dom}(\varDelta^{-p}) \longrightarrow x^{-\frac{n}{2}-2p} H_b^{-2p}(X). \tag{3.20}$$

The map (3.20) is *not* injective for $p \geq \frac{n}{4}$ since the inclusion (3.19) does not then have dense range, as follows from the computation of the domain of $\varDelta^{n/4}$.

Proof. We first use complex interpolation from the characterization of the domain of $\varDelta^{\frac{1}{4}}$ above. For two Banach spaces X and Y, let $[X, Y]_\theta$ denote the complex interpolation space at parameter θ as discussed, for example, in §1.4 of [23]. For any positive self-adjoint operator the complex powers satisfy complex interpolation in the sense that

$$\mathrm{Dom}(A^\theta) = [\mathrm{Dom}(A), L^2_g]_\theta. \tag{3.21}$$

Furthermore the usual arguments with Sobolev spaces show that the weighted b-Sobolev spaces exhibit the same interpolation property

$$[x^t H_{\mathrm{b}}^k(X), L^2_{\mathrm{b}}(X)]_\theta = x^{t\theta} H_{\mathrm{b}}^{k\theta}(X),\ 0 \leq \theta \leq 1.$$

Applying this to (3.15) we conclude that it remains true for p smaller than the greatest half-integer smaller than $n/4$. For $n > 2$ we may apply the same argument again by noting that

$$\mathrm{Dom}(\varDelta^q) = \left\{u \in \mathrm{Dom}(\varDelta); \varDelta u \in \mathrm{Dom}(\varDelta^{q-1})\right\},\ 1 \leq q < n/4.$$

This proves (3.15) for all real $p < n/4$.

In fact essentially the same method applies to $\mathrm{Dom}(\varDelta^{\frac{n}{4}})$ since we have computed $\mathrm{Dom}(\varDelta^{\frac{n}{4}-1})$. The condition on the Mellin transform in (3.17) just represents decay at infinity like $|s|^{-n/2}$ in a uniform L^2 sense except for the single factor of s which allows more general behavior at $s = 0$.

The final characterization of the domains in (3.16) now follows by use of the Mellin transform near the boundary and reduces to the same argument as in the two-dimensional case. Thus

$$\mathrm{Dom}(\varDelta^{\frac{n}{4}}) = \left\{u \in x^{-n/2} H_{\mathrm{b}}^{\frac{n}{2}}(X),\ \varDelta u \in H_{\mathrm{b}}^{\frac{n}{2}-2}(X)\right\} \tag{3.22}$$

from which (3.16) follows as before.

For $p \geq n/4$, the fact that $\mathrm{Dom}(\varDelta^p) \supset x^{-\frac{n}{2}+2p} H_b^{2p}(X)$ follows from (2.15) and (3.18) follows by similar arguments as those above, with the upper bound on δ arising from the first positive eigenvalue of the boundary Laplacian. □

The identification, (3.18), of the domain of the powers just larger than $\frac{n}{4}$ gives a convenient mapping property independent of dimension and of the conic metric involved

$$\varDelta : x^{2\delta} H_{\mathrm{b}}^{\frac{n}{2}+2\delta}(X) + \mathbb{C} \longrightarrow x^{-2+2\delta} H_{\mathrm{b}}^{\frac{n}{2}-2+2\delta}(X),\ \delta > 0 \text{ small} \tag{3.23}$$

which has null space $\mathbb{C}$ and closed range which is a complement to $\mathbb{C}$. Indeed $\varDelta - \lambda$ also defines such a map for any $\lambda \in \mathbb{C}$ and this map is an isomorphism if λ is not in the spectrum.

4. Statement of the theorems

Let $\mathcal{E}_s$ denote the energy space, of order $s \in \mathbb{R}$, of Cauchy data for the wave equation:

$$\mathcal{E}_s = \mathcal{D}_s \oplus \mathcal{D}_{s-1} \text{ where } \mathcal{D}_s = \operatorname{Dom}(\Delta^{s/2})$$

as discussed above. If $\mathcal{O}$ is an open set containing a component of ∂X, we will denote by $\mathcal{D}_s(\mathcal{O})$ the corresponding local space which is well defined since $\mathcal{D}_s$ reduces to $H^s(X)$ locally away from the boundary. We define $\mathcal{E}_s(\mathcal{O})$ similarly.

The Cauchy problem for the wave equation

$$(D_t^2 - \Delta)u(t) = 0, \tag{4.1}$$

$$u(0) = u_0,\ D_t u(0) = u_1 \tag{4.2}$$

has a unique solution

$$u \in \mathcal{C}^0(\mathbb{R}; \mathcal{D}_s) \cap \mathcal{C}^1(\mathbb{R}; \mathcal{D}_{s-1}) \tag{4.3}$$

for all $(u_0, u_1) \in \mathcal{E}_s$. Similarly the inhomogeneous forcing problem

$$(D_t^2 - \Delta)v(t) = f,\ f \in \mathcal{C}^{-\infty}(\mathbb{R}; \mathcal{D}_{s-1}),\ f = 0 \text{ in } t < 0 \tag{4.4}$$

has a unique forward solution $v \in \mathcal{C}^{-\infty}(\mathbb{R}; \mathcal{D}_s)$ with $v = 0$ in $t < 0$.

If the conic manifold is complete rather than compact, away from the conic ends, then a similar result holds. In the general case of a manifold with conic ends the wave equation (4.1) has a unique solution with compact support in a finite interval $[-T, T]$ in place of $\mathbb{R}$ in (4.3) provided the initial data has compact support. Since the results below are all local near the boundary and the general case can be reduced to this one, for simplicity of presentation we consider only the case of a compact conic manifold.

Definition 4.1. An *admissible* solution to (4.1) is one of the form (4.3) for some $s \in \mathbb{R}$ with the equation holding in $\mathcal{C}^{-\infty}(\mathbb{R}; \mathcal{D}_t)$ for some $t \in \mathbb{R}$.

We deal here with admissible solutions to the wave equation, i.e. solutions corresponding to the Friedrichs realization of the Laplacian.

The time-translation invariance of the wave equation means that if u is an admissible solution and $e \in \mathcal{C}_c^{-\infty}(\mathbb{R})$ then $e(t) * u$ is also an admissible solution. We shall use this below to decompose solutions into positive and negative parts by choosing a decomposition

$$\begin{aligned}\delta(t) = e_+(t) + e_-(t) + e_\infty(t),\ e_\pm \in \mathcal{S}'(\mathbb{R}),& \\ e_\infty \in \mathcal{S}(\mathbb{R}),\ \operatorname{WF}(e_\pm) = \{(0, \pm\infty)\} \subset S^*\mathbb{R}.&\end{aligned} \tag{4.5}$$

The corresponding decomposition of an admissible solution u is then

$$u = u_+ + u_- + u_\infty,\ u_\infty \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}_\infty),\ u_\pm = e_\pm(t) * u, \tag{4.6}$$

where all three terms are admissible solutions. Typically we choose e_+ and e_- to have Fourier transforms supported in $[1, \infty)$ and $(-\infty, -1]$.

Global regularity theory for the wave equation shows that the strongly continuous group of bounded operators defined by (4.1) – (4.3)

$$U(t) : \mathcal{D}_1 \oplus L^2_g \longrightarrow \mathcal{D}_1 \oplus L^2_g \tag{4.7}$$

satisfies

$$U(t) : \mathcal{E}_s \longrightarrow \mathcal{E}_s\ \forall\ s \in \mathbb{R}, \tag{4.8}$$

by continuous extension for $s < 0$. The solution to (4.4) is then given by Duhamel's principle.

It is useful at various point in the discussion below to change the degree of regularity of an admissible solution; this can always be accomplished by convolution in t.

Definition 4.2. Let Θ_s be the operator on $\mathbb{R}$ given by

$$\kappa(\Theta_s)(t,t') = \psi(t-t')\kappa(|D_t|^s)(t,t') \tag{4.9}$$

where $\psi(t)$ is a smooth function of compact support, equal to one near $t = 0$ and κ denotes Schwartz kernel.

Lemma 4.3. *Let $u \in \mathcal{C}(\mathbb{R};\mathcal{E}_r)$ be a solution to the wave equation. Then $\Theta_s u \in \mathcal{C}(\mathbb{R};\mathcal{E}_{r-s})$ is also a solution, and for all $s \in \mathbb{R}$,*

$$\Theta_s\Theta_{-s}u = u \bmod \mathcal{C}^\infty(\mathbb{R};\mathcal{E}_\infty).$$

Away from the boundary the wave operator $D_t^2 - \Delta$ is smooth with principal symbol $\tau^2 - |\zeta|_z^2$ at the point $(t,z;\tau,\zeta)$ in terms of the canonical coordinates associated to the coordinates (t,z). Hörmander's theorem on the propagation of wavefront set for operators of real principal type therefore applies and shows that microlocal regularity, in terms of the wavefront set relative to Sobolev spaces, is contained in the characteristic variety and is constant along the null bicharacteristic which foliate it. Thus the regularity of a solution at any point of $T^*(\mathbb{R}\times X^\circ)\backslash 0$ is readily describable in terms of the regularity of the initial data unless the null bicharacteristic through the point hits the boundary at some intervening time.

We first state a diffractive theorem, which simply says that if there are no incoming singularities at the boundary component Y at some time $\bar{t}$ then there are no outgoing singularities arising at that time. Moreover this regularity can be microlocalized in τ, the dual variable to t.

Theorem 4.4 (Diffractive regularity)**.** *If u is an admissible solution to the conic wave equation and for some small $\epsilon > 0$, $s \in \mathbb{R}$, $R^\epsilon_{\pm,I}(\bar{t},Y) \cap \mathrm{WF}^s(u) = \emptyset$, then $R^\epsilon_{\pm,O}(\bar{t},Y) \cap \mathrm{WF}^s(u) = \emptyset$. If $R^\epsilon_I(\bar{t},Y) \cap \mathrm{WF}^s(u) = \emptyset$, then for some open sets $I \ni \bar{t}$ and $\mathcal{O} \supset Y$, $u \in \mathcal{C}(I;\mathcal{D}_s(\mathcal{O}))$.*

By time-reversibility of the equation, this theorem implies Theorem I.2. It is proved in Section 11.

In order to state the geometric theorem, we need to introduce a second-microlocal condition on the incoming singularities at time $t = \bar{t}$.

Definition 4.5 (Nonfocusing conditions). We say that an admissible solution u to the conic wave equation satisfies the **nonfocusing condition** at time $\bar{t}$ and at boundary component Y, with background regularity r and relative regularity l, if for some positive integer N and small $\epsilon > 0$

$$\mathrm{WF}^r(u) \cap R^\epsilon_{\pm,I}(\bar{t},Y) = \emptyset \text{ and} \tag{4.10}$$

$$\mathrm{WF}^{r+l}\left((1+\Delta_Y)^{-N}u\right) \cap R^\epsilon_{\pm,I}(\bar{t},Y) = \emptyset. \tag{4.11}$$

If, for some $N = N(k) \in \mathbb{N}$,

$$\mathrm{WF}^r\left((xD_x + (t-\bar{t})D_t)^p(1+\Delta_Y)^{-N}u\right) \cap R^\epsilon_{\pm,I}(\bar{t},Y) = \emptyset,\ p \le k, \tag{4.12}$$

we say that u satisfies the **radial regularity condition** to order k (and with regularity r). The combination of (4.10) with (4.12), but for regularity $r+l$:

$$\mathrm{WF}^{r+l}\left((xD_x+(t-\bar{t})D_t)^p(1+\Delta_Y)^{-N}u\right)\cap R^{\epsilon}_{\pm,I}(\bar{t},Y)=\emptyset,\ p\le k, \tag{4.13}$$

will be called the **conormal nonfocusing condition** (to order k with background regularity r and relative regularity l).

These conditions measure the extent to which the solution contains a wave collapsing radially onto the boundary, up to relative regularity order l. The strongest, conormal, version asserts that after smoothing u in the tangential variables a conormal estimate at the surface $x=\bar{t}-t$ holds to order k. As an important example, the fundamental solution $\sin t\sqrt{\Delta}/\sqrt{\Delta}$ with pole close to the boundary satisfies the conormal nonfocusing condition with $r<-n/2+1$ and $l<(n-1)/2$ for any p (see §14).

Theorem 4.6 (Geometric propagation)**.** *Let u be an admissible solution to the conic wave equation. If $p\in R^{\epsilon}_{\pm,O}(\bar{t},Y)$, $\Gamma^{\epsilon}(p)\cap\mathrm{WF}^{r'}(u)=\emptyset$ for small $\epsilon>0$ and u satisfies the nonfocusing condition of Definition 4.5 with regularity $r+l>r'$, then $p\notin\mathrm{WF}^{r'}(u)$.*

This theorem represents a sharpening of Theorem I.3. We prove a weaker version of it in Section 15 and then obtain the full theorem in Section 16.

The "edge structure" on the product $\mathbb{R}_t\times X$ is discussed in the next section and in particular the related scale of weighted edge Sobolev spaces is defined there. These spaces are appropriate for the description of the boundary regularity of admissible solutions to the wave equation. A crucial role in our proof of the geometric propagation theorem above is played by the following result on decay relative to weighted edge Sobolev spaces.

Theorem 4.7 (Division theorem)**.** *If u is an admissible solution to the conic wave equation satisfying the nonfocusing condition in Definition 4.5 with $r+l<n/2$ then there are open sets $I\ni\bar{t}$ and $\mathcal{O}\supset Y$ in X such that*

$$u\in x^{r+l-n/2}H^{r+l-k}_{\mathrm{e}}(I\times\mathcal{O}). \tag{4.14}$$

If u satisfies the conormal nonfocusing condition to order $k=1$ and if $r+l\le 1$ then in addition

$$D_t u\in x^{r+l-n/2-1/2-\epsilon}H^{r+l-k-1}_{\mathrm{e}}(I\times\mathcal{O})\ \forall\ \epsilon>0.$$

We prove this theorem in Section 14.

One can use energy conservation and the diffractive theorem to show that any solution to (4.1)–(4.3) is in $x^{s-n/2}H^s_{\mathrm{e}}$ as long as $-n/2<s<n/2$ and that any solution satisfying $\mathrm{WF}^r(u)\cap R^{\epsilon}_{\pm,I}(\bar{t})=\emptyset$ is in $x^{r-n/2}H^r_e(\mathbb{R}\times X)$, locally in time near $t=\bar{t}$, as long as $-n/2<r<n/2$. The nonfocusing condition thus leads to a stronger growth estimate than is given by energy estimates alone.

The fact that the division theorem yields a stronger result in the presence of a (first-order) conormality assumption is closely related to the fact that if a solution to the wave equation is conormal with respect to the incoming surface $x-t=\bar{x}$ in $t<0$ it is conormal with respect to the corresponding outgoing surface $x=\bar{x}+t$ in $t>0$. This follows in turn from the corresponding result on the Cauchy problem.

Theorem 4.8 (Conservation of conormality)**.** *Let* $(u_0, u_1) \in \mathcal{E}_s$ *be conormal with respect to the hypersurface* $\{x = \bar{x}\}$ *for some* $\bar{x} > 0$ *sufficiently small, then in* $x > 0$, *the solution to* (4.1)–(4.2) *is conormal with respect to*

$$\{x = |\bar{x} - t|\} \cup \{x = \bar{x} + t\}$$

for $0 < t < 2\bar{x}$.

The proof of this theorem appears in Section 13.

A microlocalized version of Theorem 4.8 allows us to show the conormality of the diffracted front, subject to radial regularity of the incident wave.

Theorem 4.9 (Conormality of the diffracted front)**.** *Let* u *be an admissible solution to the conic wave equation satisfying the radial regularity condition in Definition 4.5 at* $\bar{t}, Y$ *to every order* $k \in \mathbb{N}_0$ *and suppose that for some small* $\epsilon > 0$ *and some* $p \in R^{\epsilon}_{O}(\bar{t}, Y)$, $\Gamma^{\epsilon}(p) \cap \mathrm{WF}(u) = \emptyset$ *then, microlocally near the outgoing bicharacteristic through* p, u *is conormal with respect to* $x = t - \bar{t}$.

The proof of this theorem is in Section 15. As a corollary of this result we may refine Theorem 4.6, concluding that if in addition the full conormal nonfocusing condition, (4.13), holds for all k, then the solution is conormal to the surface $x = t - \bar{t}$ near p.

Although it follows from the results above, we nonetheless restate part of the result on the fundamental solution discussed in the introduction; in fact, the proof of this result, which occurs Section 16, is crucial in the proof of the full version of Theorem 4.6.

Theorem 4.10 (Fundamental solution)**.** *Let* $E_{\bar{m}}$ *be the fundamental solution to the conic wave equation with pole at* $\bar{m} = (\bar{x}, \bar{y}) \in X^{\circ}$. *If* $\bar{x}$ *is sufficiently small then* $E_{\bar{m}}$ *is conormal with respect to* $\{x + \bar{x} = t\}$ *away from the wave cone emanating from* $\bar{m}$, *and is of Sobolev order* $\frac{1}{2} - \delta$ *there for any* $\delta > 0$.

This is equivalent to Theorem I.1 of the Introduction.

5. Edge pseudodifferential calculus

The edge calculus of pseudodifferential operators was introduced by Mazzeo [14] as a class of operators on any compact manifold M with boundary having a fibration ∂M, $\pi : \partial M \to N$ with fiber F. In this paper, $M = \mathbb{R} \times X$, $N = \mathbb{R}$, and π is the product fibration

$$\pi : \mathbb{R} \times \partial X \to \mathbb{R},\ \pi(t, p) = t.$$

The noncompactness of M in the situation at hand necessitates only minor changes to the calculus, namely keeping supports proper. Our operators generally have compactly supported kernels. In this section, we discuss the edge calculus in the general setting, as this involves no increase in complication over the special case of $\mathbb{R} \times X$. Although there is no treatment in [14] of edge microsupport or wavefront set, the properties of these objects follow easily from the properties of the calculus discussed in [14] much as the properties of the conventional wavefront set and microsupport follow from the properties of the pseudodifferential calculus on closed manifolds. First we give a brief synopsis of the edge calculus and its properties.

A Lie algebra of $\mathcal{C}^{\infty}$ vector fields $\mathcal{V}_{\mathrm{e}}(M)$, associated to π, is given by

$$V \in \mathcal{V}_{\mathrm{e}} \Longleftrightarrow V \text{ is tangent to the fibers of } \pi \text{ at } \partial M.$$

If (x, y, z) are local coordinates with x a defining function for ∂M, y coordinates on N lifted and extended to functions on M, and z restricting to coordinates on the fibers, then $\mathcal{V}_{\mathrm{e}}(M)$ is locally spanned over $\mathcal{C}^\infty(M)$ by

$$xD_x,\ xD_y \text{ and } D_z.$$

Thus, there exists a vector bundle ${}^{\mathrm{e}}TM$ (the *edge tangent bundle*) such that

$$\mathcal{V}_{\mathrm{e}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{e}}TM).$$

Let ${}^{\mathrm{e}}T^*M$ (the *edge cotangent bundle*) denote the dual of ${}^{\mathrm{e}}TM$; sections of ${}^{\mathrm{e}}T^*M$ are then locally spanned over $\mathcal{C}^\infty(M)$ by dx/x, dy/x and dz. By the *edge cosphere bundle* we mean the quotient

$${}^{\mathrm{e}}S^*M = \left({}^{\mathrm{e}}T^*M \setminus 0\right)/\mathbb{R}_+.$$

There is a canonical bundle map ${}^{\mathrm{e}}T^*M \to T^*M$ since edge vector fields are smooth up to ∂M. This map is an isomorphism over M° so over the interior ${}^{\mathrm{e}}T^*M$ is a symplectic manifold, with symplectic form given as usual by the exterior derivative of its canonical one-form; this form becomes singular at the boundary.

For $k \in \mathbb{N}$, let $\mathrm{Diff}^k_{\mathrm{e}}(M)$ be the space of differential operators spanned over $\mathcal{C}^\infty(M)$ by operators $V_1 \dots V_m$, $V_i \in \mathcal{V}_{\mathrm{e}}(M)$, $m \leq k$. There exists a canonical (principal) symbol map, ${}^{\mathrm{e}}\sigma_m$, associating to $P \in \mathrm{Diff}^m_{\mathrm{e}}(M)$ the polynomial function on the fibers of ${}^{\mathrm{e}}T^*M$ extending the usual symbol map over the interior; it gives a surjective map

$${}^{\mathrm{e}}\sigma_m : \mathrm{Diff}^m_{\mathrm{e}}(M) \longrightarrow \text{homogeneous polynomials of degree } m \text{ on } {}^{\mathrm{e}}T^*M$$

with null space precisely $\mathrm{Diff}^{m-1}_{\mathrm{e}}(M)$.

In the particular case of interest for this paper, $\mathcal{V}_{\mathrm{e}}(\mathbb{R} \times X)$ is locally spanned by the vector fields xD_x, xD_t, D_y over $\mathcal{C}^\infty(\mathbb{R} \times X)$. If we let

$$\xi \frac{dx}{x} + \lambda \frac{dt}{x} + \eta \cdot dy$$

be the canonical one-form on ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$, then naturally

$${}^{\mathrm{e}}\sigma_1(xD_x) = \xi,\ {}^{\mathrm{e}}\sigma_1(xD_t) = \lambda,\ {}^{\mathrm{e}}\sigma_1(D_{y_i}) = \eta_i$$

are dual coordinates on the fibers of ${}^{\mathrm{e}}T^*M$.

The edge calculus of pseudodifferential operators, defined in [14], arises as a microlocalization of $\mathrm{Diff}^*_{\mathrm{e}}(M)$. Let $\dot{\mathcal{C}}^\infty(M)$ denote the space of smooth functions on M vanishing, with all derivatives, to infinite order at ∂M, and let $\mathcal{C}^{-\infty}(M)$ be the dual to the corresponding space of densities, $\dot{\mathcal{C}}^\infty(M; \Omega)$. The space $\Psi^*_{\mathrm{e}}(M)$ is a graded algebra of operators on $\mathcal{C}^{-\infty}(M)$; as we will frequently use *weighted* edge operators, we will depart from the notation of [14] in carrying along the weight as an index in the calculus. Thus the bigraded space of operators $\Psi^{m,l}_{\mathrm{e}}(M)$ enjoys the following properties:

- $\Psi^{m,l}_{\mathrm{e}}(M)$ is a graded $*$-algebra.
- $\Psi^{m,l}_{\mathrm{e}}(M) = x^l \Psi^{m,0}_{\mathrm{e}}(M)$ (and the latter space is the space denoted $\Psi^m_{\mathrm{e}}(M)$ in [14]).
- $x^l \,\mathrm{Diff}^m_{\mathrm{e}}(M) \subset \Psi^{m,l}_{\mathrm{e}}(M)$ for all $m \in \mathbb{N}$, $l \in \mathbb{Z}$.
- The maps ${}^{\mathrm{e}}\sigma_m$ extend to
$${}^{\mathrm{e}}\sigma_{m,l} : \Psi^{m,l}_{\mathrm{e}}(M) \longrightarrow x^l \left[S^m_{\mathrm{phg}}({}^{\mathrm{e}}T^*M)/S^{m-1}_{\mathrm{phg}}({}^{\mathrm{e}}T^*M) \right];$$
the range space for ${}^{\mathrm{e}}\sigma$ can be conveniently identified with $\mathcal{C}^\infty({}^{\mathrm{e}}S^*M)$.

- The symbol map is a homomorphism of $*$-algebras.
- The sequence

$$0 \longrightarrow \Psi_{\mathrm{e}}^{m-1,l}(M) \longrightarrow \Psi_{\mathrm{e}}^{m,l}(M) \longrightarrow x^l \left[S_{\mathrm{phg}}^{m}({}^{\mathrm{e}}T^*M)/S_{\mathrm{phg}}^{m-1}({}^{\mathrm{e}}T^*M)\right] \longrightarrow 0$$

  is exact and multiplicative.
- If $A \in \Psi_{\mathrm{e}}^{m,l}(M)$ and $B \in \Psi_{\mathrm{e}}^{m',l'}(M)$ then

$${}^{\mathrm{e}}\sigma_{m+m'-1,l+l'}([A,B]) = \frac{1}{i}\{{}^{\mathrm{e}}\sigma_{m,l}(A), {}^{\mathrm{e}}\sigma_{m',l'}(B)\},$$

  where the Poisson bracket is computed with respect to the singular symplectic structure on ${}^{\mathrm{e}}T^*M$ described above.
- ${}^{\mathrm{e}}\sigma(x^{-l}Ax^l) = {}^{\mathrm{e}}\sigma(A)$ for all $A \in \Psi_{\mathrm{e}}(M)$, $l \in \mathbb{R}$.

In the case $M = \mathbb{R} \times X$, the elements of $\Psi_{\mathrm{e}}^{m,l}(\mathbb{R} \times X)$ may be represented locally in the form

$$\text{(5.1)} \quad Au(t,x,y) = (2\pi)^{-n-2}x^l \int e^{i(s-1)\xi+(y-y')\eta+iT\lambda} \\ b(t,x,y,s,y',T,\xi,\eta,\lambda)u(xs,y',t-xT)x^{2+l}ds\,dy'\,dT,$$

where b is a classical (polyhomogeneous) symbol of order m with ξ, η, λ as fiber variables; ${}^{\mathrm{e}}\sigma_m(A)$ is then the equivalence class of b.

An operator $A \in \Psi_{\mathrm{e}}^{m,l}(M)$ is said to be *elliptic* at $p \in {}^{e}S^*M$ if $\sigma(A)$ has an inverse in $x^{-l}[S_{\mathrm{phg}}^{-m}({}^{\mathrm{e}}T^*M)/S_{\mathrm{phg}}^{-m-1}({}^{\mathrm{e}}T^*M)]$, locally near p. Assuming that the orders are clear we may suppress indices and so $p \in \mathrm{Ell}_{\mathrm{e}}(A)$. There is a related notion of microsupport for edge pseudodifferential operators, corresponding to the fact that the composition of operators gives an asymptotically local formula for the amplitude, b in (5.1). If $A \in \Psi_{\mathrm{e}}^{m,l}(M)$, the *microsupport* of A, $\mathrm{WF}_{\mathrm{e}}'(A)$, is the closed subset of ${}^{e}S^*M$ given locally by $\operatorname{ess\,supp} b$, the conic support of b; it has the following properties:

- $\mathrm{WF}_{\mathrm{e}}'(AB) \subset \mathrm{WF}_{\mathrm{e}}'(A) \cap \mathrm{WF}_{\mathrm{e}}'(B)$.
- $\mathrm{WF}_{\mathrm{e}}'(x^{-s}Ax^s) = \mathrm{WF}_{\mathrm{e}}'(A)$ for all $s \in \mathbb{R}$.
- If $A \in \Psi_{\mathrm{e}}^{m,l}(M)$ and $p \in \mathrm{Ell}_{\mathrm{e}}(A)$ there exists $Q \in \Psi_{\mathrm{e}}^{-m,-l}(M)$ such that

$$p \notin \mathrm{WF}_{\mathrm{e}}'(QA - I) \cup \mathrm{WF}_{\mathrm{e}}'(AQ - I).$$

- If $A \in \Psi_{\mathrm{e}}^{m,l}(M)$ and $\mathrm{WF}_{\mathrm{e}}'(A) = \emptyset$ then $A \in \Psi_{\mathrm{e}}^{-\infty,l}(M)$—note that this is not a totally residual operator.

There is a continuous quantization map (by no means unique)

$$\mathrm{Op}_{\mathrm{e}} : x^l S_{\mathrm{phg}}^{m}({}^{\mathrm{e}}T^*M) \to \Psi_{\mathrm{e}}^{m,l}(M)$$

which satisfies

$${}^{\mathrm{e}}\sigma_{m,l}(\mathrm{Op}_{\mathrm{e}}(a)) = [a] \in x^l S_{\mathrm{phg}}^{m}({}^{\mathrm{e}}T^*M)/S_{\mathrm{phg}}^{m-1}({}^{\mathrm{e}}T^*M)\ \forall\ a \in x^l S_{\mathrm{phg}}^{m}({}^{\mathrm{e}}T^*M) \text{ and} \\ \mathrm{WF}_{\mathrm{e}}'\,\mathrm{Op}_{\mathrm{e}}(a) \subset \operatorname{ess\,supp}(a).$$

Associated with the edge calculus there is a scale of Sobolev spaces. For integral order these may be defined directly. Thus for $k \in \mathbb{N}$ and any $s \in \mathbb{R}$ we set

$$\text{(5.2)} \quad H_{\mathrm{e}}^{k,s}(\mathbb{R} \times X) = \{u \in x^s L_{\mathrm{b,loc}}^2(\mathbb{R} \times X); \\ (xD_t)^r P_{k-r}u \in x^s L_{\mathrm{b,loc}}^2(\mathbb{R} \times X)\ \forall\ P_{k-r} \in \mathrm{Diff}_{\mathrm{b}}^{k-r}(X) \text{ and } 0 \le r \le k\},\ k \in \mathbb{N}.$$

For negative integral orders we can similarly define

$$\begin{aligned}(5.3)\qquad &\text{for } k\in -\mathbb{N},\ H^{k,s}_{\mathrm{e}}(\mathbb{R}\times X)\ni u \Longleftrightarrow u\in \mathcal{C}^{-\infty}(\mathbb{R}\times X)\\ &\text{and } \exists\ u_{i,r}\in x^s L^2_{\mathrm{b,loc}}(\mathbb{R}\times X),\ P_{k-r,i}\in \mathrm{Diff}^{k-r}_{\mathrm{b}}(X),\ i=1,\dots,N,\\ &\qquad\text{with } u=\sum_{i=1}^{N}\sum_{r=0}^{k}(xD_t)^r P_{k-r,i}u_{i,r}.\end{aligned}$$

For general orders, the edge Sobolev spaces can be defined using the calculus.

Definition 5.1. $u\in H^{m,l}_e(M)\Longleftrightarrow \Psi^{m,-l}_{\mathrm{e}}(M)\cdot u\subset L^2_{\mathrm{b}}(M)$.

Note that we have chosen to weight these Sobolev spaces with respect to the b-weight, not the metric weight. Note also the change of sign on l. Since $H^{m,l}_e(M)=x^l H^{m,0}_e(X)$, we will often use the notation $H^m_e(M)=H^{m,0}_e(M)$ and write the x-weight explicitly. In the case of interest in this paper, when $M=\mathbb{R}\times X$ is noncompact, we will consider only edge Sobolev spaces *local in* t, without writing this explicitly. The corresponding L^2-based edge wavefront set plays a fundamental result below.

Definition 5.2. If $u\in H^{-\infty,l}_e(M)$ then $\mathrm{WF}^{m,l}_{\mathrm{e}}(u)\subset {}^eS^*M$ is defined by the condition that $p\notin \mathrm{WF}^{m,l}_{\mathrm{e}}(u)$ iff there exists $A\in\Psi^{m,-l}_{\mathrm{e}}(M)$, elliptic at p, such that $Au\in L^2_{\mathrm{b}}(M)$.

The usual properties carry over to these spaces:

- $\mathrm{WF}^{m,l}_{\mathrm{e}}(M)$ is closed.
- For all $l\in\mathbb{R}$, $\mathrm{WF}^{m,l}_{\mathrm{e}}(u)\cap {}^eS^*M^\circ=\mathrm{WF}^m(u)$ (recall that ${}^eS^*M^\circ$ and S^*M° are canonically isomorphic).
- $\bigcap_{m,l}H^{m,l}_e(M)=\dot{\mathcal{C}}^\infty(M),\quad \bigcup_{m,l}H^{m,l}_e(M)=\mathcal{C}^{-\infty}(M)$.
- Complex interpolation holds:
$$[H^{m,l}_e(M),H^{m',l'}_e(M)]_\theta=H^{\theta m+(1-\theta)m',\ \theta l+(1-\theta)l'}_e(M).$$
- If $A\in\Psi^{m,l}_{\mathrm{e}}(M)$ then $A:H^{m',l'}_e(M)\longrightarrow H^{m'-m,l'+l}_e(M)$.
- For $m\le m'$,
$$\mathrm{WF}^{m,l}_{\mathrm{e}}(u)\subset \mathrm{WF}^{m',l}_{\mathrm{e}}(u).$$
- If $u\in H^{-\infty,l'}_e(M)$ and $A\in\Psi^{k,l}_{\mathrm{e}}(M)$ then
$$\mathrm{WF}^{m,l+l'}_{\mathrm{e}}(Au)\subset \mathrm{WF}'_{\mathrm{e}}(A)\cap \mathrm{WF}^{m+k,l'}_{\mathrm{e}}(u).$$
- If $u\in H^{-\infty,l'}_e(M)$ and $A\in\Psi^{k,l}_{\mathrm{e}}(M)$ then
$$\mathrm{WF}^{m+k,l'}_{\mathrm{e}}(u)\setminus \mathrm{WF}^{m,l+l'}_{\mathrm{e}}(Au)\subset (\mathrm{Ell}_{\mathrm{e}}A)^{\complement}.$$

We now prove a less standard result.

Proposition 5.3. *If $u\in H^{-\infty,l}_e(M)$ and $l'\le l$ then for all $\theta\in[0,1]$ and $m,m'\in\mathbb{R}$,*
$$\mathrm{WF}^{\theta m+(1-\theta)m',\theta l+(1-\theta)l'}_{\mathrm{e}}(u)\subset \mathrm{WF}^{m',l'}_{\mathrm{e}}(u)\cap \mathrm{WF}^{m,l}_{\mathrm{e}}(u).$$

Proof. Suppose $p\notin \mathrm{WF}^{m',l'}_{\mathrm{e}}(u)\cap \mathrm{WF}^{m,l}_{\mathrm{e}}(u)$. Then there exists $A'\in\Psi^{m',-l'}_{\mathrm{e}}(M)$ and $A\in\Psi^{m,-l}_{\mathrm{e}}(M)$ both elliptic at p, such that Au, $A'u\in L^2_{\mathrm{b}}$. From elliptic regularity it follows that if $B\in\Psi^{0,0}_{\mathrm{e}}(M)$ has $\mathrm{WF}'_{\mathrm{e}}$ concentrated near p then $Bu\in$

$H_e^{m,l}(M) \cap H_e^{m',l'}(M)$. By interpolation of weighted edge Sobolev spaces it follows that $Bu \in H_e^{\theta(m-m'),\theta(l-l')}(M)$ and the result follows. □

We will require some results about edge-regularity of solutions to the wave equation.

Proposition 5.4. *For all $p \in \mathbb{R}_+$,*

$$x^p H_e^p(\mathbb{R} \times X) = H^p_{loc}(\mathbb{R}; L^2_{\mathrm{b}}(X)) \cap x^p L^2_{loc}(\mathbb{R}; H^p_b(X)) \text{ and}$$
$$x^{-p} H_e^{-p}(\mathbb{R} \times X) = H^{-p}_{loc}(\mathbb{R}; L^2_{\mathrm{b}}(X)) + x^{-p} L^2_{loc}(\mathbb{R}; H^{-p}_b(X)).$$

Proof. All spaces are local in t (by definition); multiplying by any $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ it suffices to assume that supports are compact. Fourier transformation in t coupled with an interpolation argument shows that for $p \in \mathbb{N}$

$$H^p_{\mathrm{loc}}(\mathbb{R}; L^2_{\mathrm{b}}(X)) \cap x^p L^2_{\mathrm{loc}}(\mathbb{R}; H^p_b(X)) = \bigcap_{j=0}^{p} x^j H^{p-j}_{\mathrm{loc}}(\mathbb{R}; H^j_b(X)).$$

The latter space is equal to $x^p H_e^p(\mathbb{R} \times X)$ as defined in (5.2), proving the first result for $p \in \mathbb{N}$; the second follows by duality. The results for general $p \in \mathbb{R}$ follow by interpolation. □

Proposition 5.5. *If u is a solution to the wave equation in $L^2_{loc}(\mathbb{R}; \mathcal{D}_s)$, with $|s| < n/2$, then $u \in x^s H_e^{s-n/2}(\mathbb{R} \times X)$.*

Proof. Certainly $u \in H^{2r}_{\mathrm{loc}}(\mathbb{R}; \mathcal{D}_{s-2r})$ for all $r \in \mathbb{Z}$. So, by interpolation, this holds for all $r \in \mathbb{R}$. By Lemma 3.2, $u \in L^2_{\mathrm{loc}}(\mathbb{R}; x^{s-n/2} H^s_b(X)) \cap H^s_{\mathrm{loc}}(\mathbb{R}; L^2_{\mathrm{b}}(X))$, hence the result follows from Proposition 5.4. □

6. Bicharacteristic flow

The canonical one-form on ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$ is

$$\lambda \frac{dt}{x} + \xi \frac{dx}{x} + \eta \cdot dy$$

hence the symplectic form is

$$\omega = \frac{d\lambda \wedge dt}{x} + \frac{d\xi \wedge dx}{x} - \frac{\lambda dx \wedge dt}{x^2} + d\eta \wedge dy.$$

We can now write the symbol of the d'Alembertian

$$p = \sigma(\Box) = \lambda^2/x^2 - g(x, y, \xi dx/x + \eta \cdot dy) = \frac{\lambda^2 - \xi^2 - h(x, y, \eta)}{x^2}$$

and

$$p_0 = \sigma(D_t^2 - \Delta_0) = \frac{\lambda^2 - \xi^2 - h_0(y, \eta)}{x^2}$$

where Δ_0 is defined by (1.15). Let $\Sigma = \{p = 0\} \subset {}^{\mathrm{e}}T^*(\mathbb{R} \times X)$ denote the characteristic variety of the d'Alembertian.

Let H_g and H_{g_0} denote the respective Hamilton vector fields of p and p_0 on ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$, near a boundary component Y of X. Thus

$$\frac{x^2}{2} H_g = \frac{x^2}{2} H_{g_0} + W = H_Y + (\xi^2 + h_0(y, \eta))\partial_\xi + \lambda\xi\partial_\lambda + \xi x \partial_x - \lambda x \partial_t + W, \tag{6.1}$$

where W is the Hamilton vector field of $p - p_0$, hence

$$\frac{x^2}{2}W = \frac{x}{2}\frac{\partial h(\eta)}{\partial x}\partial_\xi + (h - h_0)\partial_\xi - \frac{1}{2}\frac{\partial(h(\eta) - h_0(\eta))}{\partial y}\cdot\partial_\eta + (h^{ij} - (h_0)^{ij})\eta_i\partial_{y_j}, \tag{6.2}$$

and where H_Y is the Hamilton vector field in (y, η) for $(1/2)h_0(y, \eta)$, i.e. is the geodesic spray in Y.

Note that H_g, H_{g_0}, and W are all homogeneous of degree 1 in ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$, and that $(x^2/2)W$ is less singular than $(x^2/2)H_g$ at $x = 0$, and vanishes at $\eta = 0$. Thus if ${}^{\mathrm{e}}\overline{T}^*(\mathbb{R} \times X)$ denotes the fiberwise radial compactification of ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$,

$$(x^2/2)H_g,\ (x^2/2)H_{g_0} \in h(\eta)^{\frac{1}{2}}\mathcal{V}_b({}^{\mathrm{e}}\overline{T}^*(\mathbb{R} \times X)\backslash 0), \tag{6.3}$$

$$(x^2/2)W \in xh(\eta)^{\frac{1}{2}}\mathcal{V}_b({}^{\mathrm{e}}\overline{T}^*(\mathbb{R} \times X)\backslash 0). \tag{6.4}$$

Note also that the vector field H_g is tangent to the incoming and outgoing sets $R_{\pm,I}(Y)$, $R_{\pm,O}(Y)$, which we now regard (by homogeneity) as subsets of ${}^{e}S^*(\mathbb{R} \times X)$. These incoming and outgoing manifolds are the interiors of smooth manifolds with boundary in ${}^{e}S^*(\mathbb{R} \times X)$, and we define their boundaries as follows, with IC and OG standing for incoming and outgoing manifolds respectively.

Definition 6.1. Let

$$\mathsf{IC}_\pm(\bar{t}, \bar{y}) = \overline{R^{\epsilon}_{\pm,I}(\bar{t}, \bar{y})} \cap {}^{e}S^*_{\mathbb{R}\times\partial X}(\mathbb{R} \times X)$$
$$\mathsf{OG}_\pm(\bar{t}, \bar{y}) = \overline{R^{\epsilon}_{\pm,O}(\bar{t}, \bar{y})} \cap {}^{e}S^*_{\mathbb{R}\times\partial X}(\mathbb{R} \times X)$$

with the same convention for omitted indices as was used for $R^{\epsilon}_{\pm,\bullet}(\bar{t}, \bar{y})$.

For Y a boundary component of X, we define a map

$$\Upsilon : {}^{\mathrm{e}}T^*(\mathbb{R} \times X)\backslash 0 \supset U \to Y$$

constant in the fibers, which is approximately invariant under the flow of H_g and which will serve as a useful localizer. The subset U on which Υ is defined is a conic neighborhood of IC. The map Υ is constructed as follows: consider the data (x, Π) of Theorem 1.2 as identifying a neighborhood of Y in X with a neighborhood of $x = 0$ in the *model cone* $\tilde{X} = \mathbb{R}_+ \times Y$, which we now equip with the *model metric* g_0. Then for any point q near IC in ${}^{\mathrm{e}}T^*(\mathbb{R} \times \tilde{X})$, set

$$\Upsilon(q) = \lim_{s\to s_\infty} \pi_Y \exp_q(s(x^2/2)H_{g_0}), \tag{6.5}$$

where

$$s_\infty = h_0(\eta(q))^{-\frac{1}{2}}\left((\operatorname{sgn}\theta)\frac{\pi}{2} - \arctan\theta\right),\ \text{with } \theta = \frac{\xi(q)}{h_0(\eta(q))^{\frac{1}{2}}},$$

and where π_Y is the projection onto the factor Y of $\tilde{X}$. As will be shown below, the signs are chosen so that $\Upsilon(q)$ is the limit of the projection on Y of the unique geodesic through q, as it heads toward the *large end* of the model cone $\tilde{X}$, i.e. the point at infinity from which the geodesic emanated. To see that Υ is well-defined and smooth, note that under the flow along $(x^2/2)H_{g_0}$, $\xi'' = 2\xi\xi'$, hence

$$\begin{aligned}
\xi(s) &= C\tan(Cs + \theta) \qquad \lambda(s) = D\sec(Cs + \theta)\\
x(s) &= E\sec(Cs + \theta) \qquad t(s) = -E\tan(Cs + \theta) + F\\
h_0(\eta) &= G.
\end{aligned} \tag{6.6}$$

Since $\xi' = \xi^2 + h_0(\eta)$, we compute $C = h_0(\eta(q))^{\frac{1}{2}}$ and $\theta = \arctan \xi(q)/h_0(\eta(q))^{\frac{1}{2}}$, so that

$$s_\infty = C^{-1}\left((\operatorname{sgn}\theta)\frac{\pi}{2} - \theta\right)$$

depends smoothly on q if U is chosen small enough ($\operatorname{sgn}\theta$ is constant on components of U), and as $s \to s_\infty$, $x \to +\infty$ and is strictly increasing on $s \in [0, \infty)$; simultaneously, $t \to \pm\infty$. Thus, since (y, η) are undergoing geodesic flow,

$$\Upsilon(q) = y(\exp_{(y(q),\eta(q))} s_\infty H_Y)$$

(where H_Y is geodesic flow on Y with metric h_0) is manifestly smooth in q. By the definition as a limit along flow-lines of H_{g_0}, we also have

$$\Upsilon_* \frac{x^2}{2} H_{g_0} = 0.$$

Now we turn to the perturbed flow H_g: Let V denote the rescaled vector field $(x^2/2)(\lambda^2 + \xi^2 + h(\eta))^{-1/2} H_g$ on ${}^{\mathrm{e}}S^*(\mathbb{R} \times X)$, and let the flow along V be parametrized by s. Note that under the bicharacteristic flow of $(x^2/2)H_g$ on ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$, as $|t| \to \infty$, $s \to C^{-1}(\pm\pi/2 - \theta)$, hence $\xi/\lambda \to \pm C/D$; moreover, $|\xi|$, $|\lambda| \to \infty$ in this limit, while η remains bounded. Thus we have established:

Lemma 6.2. *Every maximally extended integral curve of V over ∂X contains in its closure exactly one point in IC and one in OG; the former lies over the point $\Upsilon(p) \in Y$, for any p along the integral curve.*

7. Construction of symbols of test operators

We now write down the symbols of the operators to be used in the commutator estimates in §8. As usual, we work in a product neighborhood of a boundary component Y as described in Theorem 1.2. For points y_1, $y_2 \in Y$, we let $d(y_1, y_2)$ denote distance with respect to the metric h_0.

Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ vanish for $x < 0$, be equal to 1 for $x > 1$, and be nondecreasing, with smooth square root, such that χ' also has smooth square root. Choose $\psi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ to be equal to 1 at $x = 0$, be supported in $(-1, 1)$, with derivative supported in $(-1, -1/2) \cup (1/2, 1)$, and to be the square of a smooth function; let $(\operatorname{sgn} x)\psi'(x)$ also be the square of a smooth function. For positive constants ϵ_i let $\psi_i(x) = \psi(x/\epsilon_i)$ and $\chi_i(x) = \chi(x/\epsilon_i)$.

First we consider test symbols at incoming radial points.

Given $m, l \in \mathbb{R}$, $\bar{y} \in Y$, and $\bar{x} \in \mathbb{R}_+$, define a nonnegative symbol on ${}^{\mathrm{e}}T^*X$ by setting

$$\begin{aligned}(a^{\mathrm{in}}_{m,l,\pm})^2 &= \chi(\pm\lambda)\chi(\pm\xi)\chi_1(x - x_0 + vt)\chi_1(-x + x_1 - v't)\\ &\cdot \psi_2((d(\Upsilon, \bar{y})^2 - \delta x)_+)\psi_3(h(\eta)^{\frac{1}{2}}/|\lambda|)\chi_4(t + \epsilon_4)\psi_5(p(t, x, y, \lambda, \xi, \eta)/\lambda^2)(\pm\lambda)^m x^l.\end{aligned}$$

where $v < 1 < v'$, $x_0 < \bar{x} < x_1$, $\delta > 0$, and we have written $\Upsilon = \Upsilon(t, x, y, \lambda, \xi, \eta)$ for the map defined by (6.5). We will assume that x_1 is sufficiently small that the perturbation term $W = H_g - H_{g_0}$ has the property

$$(x^2/2)W = h_0(\eta)^{\frac{1}{2}}(\mathcal{O}(x)\partial_\xi + \mathcal{O}(x)\partial_\eta + \mathcal{O}(x)\partial_y)$$

(from (6.2)) where

(7.1) all $\mathcal{O}(x)$ terms are bounded by 10^{-2} when $x < x_1$

and furthermore that

$$(x/2)\frac{\partial \log h}{\partial x} < 10^{-2} \text{ for } x < x_1. \tag{7.2}$$

Since $(x^2/2)H_{g_0}d(\Upsilon,\bar{y})^2$ vanishes, (6.4) yields

$$\left|(x^2/2)H_g d(\Upsilon,\bar{y})^2\right| \leq Axh(\eta)^{\frac{1}{2}} \text{ when } x < x_1 \tag{7.3}$$

for some constant A.

Observe that on $\operatorname{supp} a^{\mathrm{in}}_{m,l,\pm}$, both

$$\frac{\left|\lambda^2 - \xi^2 - h(\eta)\right|}{\lambda^2} < \epsilon_5$$

and

$$\frac{h(\eta)^{\frac{1}{2}}}{|\lambda|} < \epsilon_3.$$

Hence

$$\sqrt{1-\epsilon_5-\epsilon_3^2} < |\xi|/|\lambda| < \sqrt{1+\epsilon_5} \tag{7.4}$$

so that $\operatorname{supp} a^{\mathrm{in}}_{m,l,\pm}$ is localized arbitrarily near $R_{\pm,I}$.

We choose δ small enough that $\delta x_1 < \epsilon_2/2$, hence $\operatorname{supp}\psi_2(\cdot)\cap\{x=x_1\}\neq\emptyset$. We now choose the other ϵ_i sufficiently small and v, v' sufficiently close to 1, so that $a^{\mathrm{in}}_{m,l,\pm}$ has support in an arbitrarily small neighborhood of the closure of a bicharacteristic segment

$$\{t=s,\ x=\bar{x}-s,\ y=\bar{y},\ \lambda=\pm1,\ \xi=\pm1,\ \eta=0; s\in[0,\bar{x}]\}$$

passing through the point q with coordinates $x=\bar{x}$, $y=\bar{y}$, $t=0$, $\xi=\pm1$, $\eta=0$ and hitting IC at time $\bar{x}$. Note that we may translate the t variable freely without changing any of the properties of a^{in}.

We now evaluate, term by term,

$$\frac{x^2}{2}H_g(a^{\mathrm{in}}_{m,l,\pm})^2.$$

The term containing

$$\frac{x^2}{2}H_g\chi_1(x-x_0+vt) = \chi_1'(x-x_0+vt)(\xi - v\lambda + \mathcal{O}(x)h_0(\eta)^{\frac{1}{2}})x$$

has sign $\pm$ on $\operatorname{supp} a^{\mathrm{in}}_{m,l,\pm}$, since (7.4) implies $|\xi|\geq|\lambda|(1-\epsilon_5-\epsilon_3)$, hence by (7.1), $\operatorname{sgn}(\xi-v\lambda)=\operatorname{sgn}\xi$ on $\operatorname{supp} a^{\mathrm{in}}$ provided $v<1-\epsilon_5-\epsilon_3-10^{-2}\epsilon_3$.

Similarly, the term involving $(x^2/2)H_g\chi_1(-x+x_1-v't)$ can also be made to have a positive derivative if v' is sufficiently greater than 1. Shrinking ϵ_3 and ϵ_5 as necessary, these conditions can always be achieved.

The term containing $\frac{x^2}{2}H_g\psi_2((d(\Upsilon,\bar{y})^2-\delta x)_+)$ gives

$$\psi_2'((d(\Upsilon,\bar{y})^2-\delta x)_+)\cdot(-\delta\xi x + \mathcal{O}(x)h(\eta)^{\frac{1}{2}})$$

where, by (7.3), the $\mathcal{O}(x)$ stands for something bounded by Ax. Hence if

$$\delta|\xi| > Ah(\eta)^{\frac{1}{2}}, \tag{7.5}$$

this term has sign $\pm$; by (7.4) and the support property of $\psi_3(h(\eta)^{\frac{1}{2}}/|\lambda|)$, the condition

$$\delta > \frac{A\epsilon_3}{\sqrt{1-\epsilon_5-\epsilon_3^2}}$$

suffices to ensure (7.5). This can be achieved by further shrinking ϵ_3 as necessary.

The term involving $(1/2)x^2 H_g\psi_3(h(\eta)^{\frac{1}{2}}/|\lambda|)$ can be evaluated using (6.1)–(6.2), which show that $H_g h(\eta) = \xi x\partial h/\partial x$ and hence

$$\frac{x^2}{2}H_g(h^{\frac{1}{2}}/\lambda) = \frac{\xi h^{\frac{1}{2}}}{\lambda}\left(\frac{\partial h}{\partial x}\frac{x}{2h} - 1\right),$$

so by (7.2) this term has sign $\pm$.

The term involving $(1/2)x^2 H_g\chi_4(t+\epsilon_4)$, unlike those discussed previously, has sign $\mp$. This term is supported in a region in which we will assume microlocal regularity: its support is in

$$\{|t| < \epsilon_4,\ x \in (x_0, x_1),\ d(\Upsilon, \bar{y})^2 < \delta x_1 + \epsilon_2,\ h(\eta)^{1/2}/|\lambda| < \epsilon_3\};$$

this lies away from ∂X but inside an arbitrarily small neighborhood in ${}^eS^*(\mathbb{R}\times X)$ of an arbitrarily specified point in $R^{\epsilon}_{\pm,I}$. Call this term e, for "error."

The term involving $\frac{x^2}{2}H_g\psi_5$ is supported in $p/\lambda^2 > \epsilon_5/2$, hence vanishes identically on the characteristic variety Σ. Denote this term k. The terms arising from $(1/2)x^2 H_g\chi(\pm\lambda)$ and $(1/2)x^2 H_g\chi(\pm\xi)$ are supported in a compact subset of ${}^eT^*(\mathbb{R}\times X)$; let c be their sum.

Finally, the factor $(\pm\lambda)^m x^l$ has derivative $(m+l)\xi(\pm\lambda)^m$, which has sign $\pm\operatorname{sgn} m$, i.e. $\pm$ as long as $m+l>0$; this term, of course, has the same support as $a^{\mathrm{in}}_{m,l,\pm}$ itself; denote it $\pm(a')^2$.

All the nonnegative resp. nonpositive terms described above can be arranged to be squares of smooth functions resp. minus squares of smooth functions. We organize the information gleaned above as follows. Let

$$q = (\bar{t},\ \bar{x},\ \bar{y},\ \lambda = \pm 1,\ \xi = \pm 1,\ \eta = 0)$$

be a given point in $R_{\pm,I}$ with $\bar{x}$ sufficiently small; let Ω be the closure of the bicharacteristic connecting q to the boundary:

$$\Omega = \{(\bar{t}+s,\ \bar{x}-s,\ \bar{y},\ \lambda=\pm 1,\ \xi = \pm 1,\ \eta = 0) : s \in [0,\bar{x}]\}.$$

Thus we have shown:

Lemma 7.1. *Provided $m+l>0$ there exists a symbol $a^{in}_{m,l,\pm}$ of order m and weight l in ${}^eT^*(\mathbb{R}\times X)$ such that*

$$\frac{x^2}{2}H_{\bar{g}}(a^{in}_{m,l,\pm})^2 = \pm(a')^2 \pm \sum_j b_j^2 + e + c + k \tag{7.6}$$

where $a' = a\cdot(\pm(m+l)\xi)^{\frac{1}{2}}$, $\operatorname{supp} a^{in}_{m,l,\pm}$ is an arbitrarily small neighborhood of Ω, $\operatorname{supp} e$ is an arbitrarily small neighborhood of q, $\operatorname{supp} c$ is compact in ${}^eT^*(\mathbb{R}\times X)$ and $\Sigma\cap\operatorname{supp}(k) = \emptyset$.

Next we consider test symbols at outgoing radial points. Let χ, ψ be as above. Given $\bar{x}$, $\bar{y}$, set

$$\begin{aligned}(a^{\mathrm{out}}_{m,l,\pm})^2 &= \chi(\pm\lambda)\chi(\mp\xi)\chi_1(-x+x_1+vt)\chi_1(x-x_0-v't)\\ &\cdot\psi_2((d(\Upsilon,\bar{y})^2+\delta x)_+)\psi_3(h(\eta)^{\frac{1}{2}}/|\lambda|)\chi_4(\epsilon_4-t)\psi_5(p(t,x,y,\lambda,\xi,\eta)/\lambda^2)(\pm\lambda)^m x^l\end{aligned}$$

where $v<1<v'$ and $x_0<\bar{x}<x_1$. As with a^{in}, we can choose constants ϵ_i small enough that $\operatorname{supp} a^{\mathrm{out}}_{m,l,\pm}$ lies in a small neighborhood of the closure of a

bicharacteristic segment

$$\Omega = \{(t = -s,\ x = \bar{x} - s,\ y = \bar{y},\ \lambda = \pm 1,\ \xi = \mp 1,\ \eta = 0); s \in [0, \bar{x}]\}$$

passing through the point q with coordinates $x = \bar{x}$, $y = \bar{y}$, $t = 0$, $\xi = \pm 1$, $\eta = 0$ and emanating from $\partial(\mathbb{R} \times X)$ at time $-\bar{x}$. Moreover if $m + l < 0$, all terms in $(x^2/2)H_g a^{\mathrm{out}}_{m,l,\pm}$ can be arranged to be $\pm$ squares of smooth functions, with the exception of a compactly supported term, a term supported away from Σ, and, most importantly, the term involving

$$\frac{x^2}{2} H_g(\psi_3(h(\eta)^{\frac{1}{2}}/|\lambda|)).$$

Let e denote this error term. Then

$$\mathrm{supp}(e) \subset \mathrm{supp}(a^{\mathrm{out}}_{m,l,\pm}) \cap \{h(\eta)^{\frac{1}{2}}/|\lambda| \in [\epsilon_3/2, \epsilon_3]\}.$$

This is a subset of the complement of $R_{\pm,O}$ inside any given positive conic neighborhood of Ω.

More generally, let

$$q = (\bar{t},\ x = 0,\ \bar{y},\ \lambda = \pm 1,\ \xi = \mp 1,\ \eta = 0)$$

be any point in OG; let Ω be the closure of the short bicharacteristic extending from q to $(t = \bar{t} + \bar{x},\ x = \bar{x},\ y = \bar{y},\ \lambda = \pm 1,\ \xi = \mp 1,\ \eta = 0)$, so

$$\Omega = \{(t = \bar{t} + s,\ x = s,\ \bar{y},\ \lambda = \pm 1,\ \xi = \pm 1,\ \eta = 0) : s \in [0, \bar{x}]\}.$$

Lemma 7.2. *Provided $m + l < 0$ there is a symbol $a^{out}_{m,l,\pm}$ of order m in ${}^{\mathrm{e}}T^*(\mathbb{R} \times X)$ such that*

$$\frac{x^2}{2} H_{\bar{g}}(a^{out}_{m,l,\pm})^2 = \pm(a')^2 \pm \sum_j b_j^2 + e + c + k \tag{7.7}$$

where $\mathrm{supp}(a^{out}_{m,l,\pm})$ *is an arbitrarily small neighborhood of* Ω, $a' = a(\mp(m + l)\xi)^{\frac{1}{2}}$, $\mathrm{supp}(e)$ *is contained in the complement of* $R_{\pm,O}$ *in an arbitrarily small neighborhood of* Ω, $\mathrm{supp}(c)$ *is compact and* $\Sigma \cap \mathrm{supp}(k) = \emptyset$.

8. Propagation of edge wavefront set

In this section, we prove a theorem on propagation of singularities for the edge wavefront set which is a central ingredient in the diffractive theorem (Theorem 4.4), the geometric propagation theorem (Theorem 4.6) and the proof of the conormal regularity of the diffracted front in Section 15.

Theorem 8.1. *For $u \in H_e^{-\infty,l}(I \times [0, \epsilon)_x \times Y)$, a distributional solution to the wave equation $\Box u = 0$, with $\bar{t} \in I \subset \mathbb{R}$ open, the following four propagation results hold.*

(i) *If $p \in \mathsf{IC}(\bar{t}, Y)$, $m > l + (n-1)/2$ and $\mathrm{WF}^m(u) \cap R^{\epsilon}_{\pm,I}(\bar{t}, y(p)) = \emptyset$ then $p \notin \mathrm{WF}_e^{m,l'}(u)$ for all $l' < l$.*

(ii) *For any $m \in \mathbb{R}$, $\mathrm{WF}_e^{m,l}(u) \cap {}^e S^*_{\mathbb{R}\times\partial X}(\mathbb{R} \times X) \backslash (\mathsf{IC}(\bar{t}) \cup \mathsf{OG}(\bar{t}))$ is a union of maximally extended integral curves of $V = (x^2/2)(\lambda^2 + \xi^2 + h_0(\eta))^{-1/2} H_g$.*

(iii) *If $U \subset {}^e S^*_{\mathbb{R}\times\partial X}(\mathbb{R} \times X)$ is a neighborhood of $p \in \mathsf{OG}(\bar{t}, Y)$ and $\mathrm{WF}_e^{m,l}(u) \cap U \subset \mathsf{OG}$ then $p \notin \mathrm{WF}_e^{M,l}(\Delta_Y^k u)$ for $k \in \mathbb{N}_0$ provided $M \leq m - 2k$ and $M < l + (n-1)/2$.*

(iv) *If* $R^j u \in H_e^{-\infty,l}(\mathbb{R}\times X)$, *with* R *given by* (1.17), *and* $p \in \mathsf{OG}(\bar t)$ *has a neighborhood* $U \subset {}^{e}S^*_{\mathbb{R}\times Y}(\mathbb{R}\times X)$ *such that* $\mathrm{WF}_e^{m,l}(R^{j'}u)\cap U \subset \mathsf{OG}$, *for* $0 \le j' \le j$, *then* $p \notin \mathrm{WF}_e^{M,l}(R^j u)$ *for* $j \in \mathbb{N}_0$ *provided* $M \le m - j$ *and* $M < l + (n-1)/2$.

Remark 8.2. This theorem correspond to propagation into, within and out of the boundary. The first two parts, together with the fact that $\mathrm{WF}_e^{m,l}(u)$ is closed, can be combined with Lemma 6.2 to conclude that for $p \in {}^{e}S^*_{\mathbb{R}\times\partial X}(\mathbb{R}\times X)\backslash\mathsf{OG}$, if $\mathrm{WF}^m(u)\cap R^{\epsilon}_{\pm,I}(t(p),\Upsilon(p)) = \emptyset$ then $p \notin \mathrm{WF}_e^{m,l'}(u)$, for $l' < l$.

The conclusion of (iii) implies, by closedness of the edge wavefront set and Hörmander's Theorem, that in fact all of $R^{\epsilon}_{\pm,O}(\bar t, y(p))$ is absent from $\mathrm{WF}^m(u)$. This third part of the theorem is trivial, however, when it is applied with $l = s - n/2$ to a solution in $\mathcal{D}_s$. Therefore this part of the theorem (and the fourth part likewise) is useless in the absence of a "division theorem" yielding better x decay of u than is given by energy estimates.

Proof. First consider (i). By assumption, $u \in H_e^{q,l}(\mathbb{R}\times X)$, locally near $\{\bar t\}\times Y$ for some $q \in \mathbb{R}$ and $p \in \mathsf{IC}(\bar t, Y)$. We shall prove the following statement:

$$\begin{gathered}\text{If } m' > l' + n/2 - 1,\ u \in H_e^{-\infty,l'}(I\times[0,\epsilon)\times Y),\ p \notin \mathrm{WF}_e^{m',l'}(u),\\ \text{and } R_{\pm,I}(\bar t, y(p)) \cap \mathrm{WF}^{m'+1/2}(u) = \emptyset \text{ then } p \notin \mathrm{WF}_e^{m'+1/2,l'}(u).\end{gathered} \tag{8.1}$$

To do so, choose

$$A_\delta \in \Psi_e^{m',-l'-n/2+1}(\mathbb{R}\times X) = \mathrm{Op_e}\left[\psi_\delta(\lambda,\xi,\eta)a^{\mathrm{in}}_{m',-l'-n/2+1,\pm}\right], \tag{8.2}$$

where a^{in} is constructed in Lemma 7.1 and

$$\psi_\delta(\lambda,\xi,\eta) = \psi((\lambda^2+\xi^2+h(\eta))\delta)$$

with $\psi(x)$ smooth, equal to 1 for $x < 1/2$ and 0 for $x > 1$. Choosing the supports sufficiently small, we see that the error term e in (7.6) will have support in the complement of $\mathrm{WF}^{m'+1/2}(u)$. Note that $H_g\psi_\delta$ is supported in $\{1/(2\delta) \le \lambda^2+\xi^2+h(\eta) \le 1/\delta\}$.

Thus if A'_δ has symbol $\psi_\delta a'$ with a' as in (7.6),

$$[\Box, A^*_\delta A_\delta] = \pm(A'_\delta)^*(A'_\delta) \pm \sum_j B^*_{\delta,j}B_{\delta,j} + E_\delta + K_\delta + R_\delta + S_\delta \tag{8.3}$$

where

- $\mathrm{WF}_e'(K_\delta)\cap\Sigma = \emptyset$
- $A'_\delta \in \Psi_e^{m'+\frac12,-l'-\frac n2}(\mathbb{R}\times X)$ with $A'_\delta \to A' \equiv A'_0$ in $\Psi_e^{m'+\frac12+\epsilon,-l'-\frac n2}(\mathbb{R}\times X)$ for all $\epsilon > 0$ as $\delta \to 0$.
- $E_\delta \in \Psi_e^{2m'+1,-2l'-n}(\mathbb{R}\times X)$ is bounded in δ with $\mathrm{WF}_e'(E_\delta)$ uniformly bounded away from ∂X and contained in $(\mathrm{WF}^{m'+1/2}\,u)^{\complement}$,
- R_δ bounded in $\Psi_e^{2m',-2l'-n}(\mathbb{R}\times X)$
- S_δ is bounded in $\Psi_e^{2m'+1,-2l'-n}(\mathbb{R}\times X)$ and, as $\delta\to 0$, converges to 0 in $\Psi_e^{2m'+1+\epsilon,-2l'-n}(\mathbb{R}\times X)$ for all $\epsilon > 0$ (this is the term whose symbol involves $H_g\psi_\delta$).

Applying (8.3) to u and pairing with u *with respect to the inner product on* L^2_g yields

$$\|A'_\delta u\|_g^2 - \langle S_\delta u, u\rangle_g \leq \left|\langle E_\delta u, u\rangle_g\right| + \left|\langle K_\delta u, u\rangle_g\right| + \left|\langle R_\delta u, u\rangle_g\right|; \tag{8.4}$$

the integration by parts is justified since $u \in H_e^{-\infty,l'}(\mathbb{R}\times X)$ and $\Box A_\delta^* A_\delta u \in H_e^{\infty,-l'-n}(\mathbb{R}\times X)$. All terms on the right-hand side are bounded uniformly as $\delta \to 0$. A weak convergence argument now shows that $\|A'_0 u\|_g < \infty$, hence $\mathrm{WF}_{\mathrm{e}}^{m'+\frac{1}{2},l'}(u) = \emptyset$ is disjoint from the elliptic set of A'_0; the shift by $n/2$ in the x weight here comes from the difference between L^2_g and L^2_{b}. This proves (8.1).

If $q > l + (n-2)/2$, then iterative application of (8.1) proves that $p \notin \mathrm{WF}_{\mathrm{e}}^{m,l}(u)$ directly. If $q \leq l + (n-2)/2$, however, a further argument is needed.

Supposing $l_0 = \sup\{r; p \notin \mathrm{WF}_{\mathrm{e}}^{m,r}(u)\} < l$, we wish to arrive at a contradiction. We will employ an interpolation argument illustrated in Figure 1. We have already shown that $l_0 \geq q - (n-2)/2$ and by hypothesis, $u \in H_e^{q,l}(\mathbb{R}\times X)$ (at least locally), so Proposition 5.3 shows that

$$p \notin \mathrm{WF}_{\mathrm{e}}^{\theta q+(1-\theta)m,\theta l+(1-\theta)l_0}(u)\ \forall\ \theta \in [0,1]. \tag{8.5}$$

In particular, since $l < m - (n-1)/2$ (by the hypothesis of the theorem) and $q \leq l + n/2 - 1$,

$$\theta' = (m - l_0 - n/2 + 1)/((m - l_0 - n/2 + 1) + (l - q + n/2 - 1)) \in [0,1].$$

If $m' = \theta' q + (1-\theta')m$ and $l' = \theta' l + (1-\theta')l_0$, then $m' = l' + n/2 - 1$, and by (8.5), $p \notin \mathrm{WF}_{\mathrm{e}}^{m',l'-\epsilon}(u)$ for all $\epsilon > 0$. Hence applying (8.1) iteratively shows that $p \notin \mathrm{WF}_{\mathrm{e}}^{m,l'-\epsilon}(u)$, with $l' - \epsilon > l_0$, which is the desired contradiction. Thus (i) is proved.

l''

(q,l)

$(m', l'-\epsilon)$ (m,l)

(m, l_0)

$l'' = m'' - n/2 + 1$

m''

FIGURE 1. The interpolation argument in part one of Theorem 8.1: we begin with global regularity of order (q,l) and microlocal regularity of order (m,l_0). Interpolation gives microlocal regularity of order (m',l') and iterative application of (8.1) is used to move along the horizontal line and obtain microlocal regularity of order $(m, l'-\epsilon)$ with $l'-\epsilon > l_0$.

To prove (ii), i.e. to show that regularity propagates across ${}^{e}S^*_{\partial X}(\mathbb{R}\times X)$ up to (but not including) points in OG, we appeal to the standard proof of Hörmander's theorem on propagation of singularities for operators of real principal type by use of positive commutator estimates. This applies microlocally near all characteristic points where the rescaled Hamilton vector field $V=(x^2/2)(\lambda^2+\xi^2+|\eta|^2)^{-1/2}H_g$ is non-zero, hence away from $\mathsf{IC}\cup\mathsf{OG}$. See [16] for an analogous discussion in the context of the scattering calculus; note that in that case the propagation result may be reduced to Hörmander's theorem whereas in this case it is an analogue of it.

Now consider (iii), first for $k=0$. In this case we may simply suppose that $M=m<l+(n-1)/2$. We proceed much as in the proof of (i) above. The result follows by iterative application of the following assertion:

(8.6) If $m'<l+n/2-1,\ u\in H_e^{-\infty,l}(I\times X)$ and $p\notin \mathrm{WF}_e^{m',l}(u)$ then
$$p\notin \overline{{}^{e}S^*_{\mathbb{R}\times\partial X}(\mathbb{R}\times X)\cap \mathrm{WF}_e^{m'+1/2,l}(u)\setminus \mathsf{OG}}\Longrightarrow p\notin \mathrm{WF}_e^{m'+1/2,l}(u).$$

Note that the final hypothesis on p here is equivalent to the existence of a neighborhood U of p such that $U\cap \mathrm{WF}_e^{m'+1/2,l}(u)\subset \mathsf{OG}$.

To prove (8.6), choose $A_\delta\in\Psi_e^{m',-l-n/2+1}(\mathbb{R}\times X)$ as in (8.2) with $a^{\mathrm{in}}_{m',-l-n/2+1,\pm}$ replaced by $a^{\mathrm{out}}_{m',-l-n/2+1,\pm}$, where a^{out} is constructed in Lemma 7.2 and supports are kept small, corresponding to the implicit neighborhood, U, in (8.6). If A'_δ has symbol $\psi_\delta a'$ with a' as in (7.7), then
$$[\Box,A_\delta^*A_\delta]=\pm(A'_\delta)^*(A'_\delta)\pm\sum_j B^2_{\delta,j}+E_\delta+K_\delta+R_\delta+S_\delta$$
where

- $\mathrm{WF}'_e(K_\delta)\cap\Sigma=\emptyset$
- $A'_\delta\in\Psi_e^{m'+\frac12,-2l-n}(\mathbb{R}\times X)$ and $A'_\delta\to A'\equiv A'_0$ in $\Psi_e^{m'+\frac12+\epsilon,-l-\frac n2}(\mathbb{R}\times X)$ for all $\epsilon>0$ as $\delta\to0$.
- $E_\delta\in\Psi_e^{2m'+1,-2l-n}(\mathbb{R}\times X)$ and $\mathrm{WF}'_e(E_\delta)\subset U\backslash\mathsf{OG}(\bar t)$, uniformly in δ.
- R_δ is uniformly bounded in $\Psi_e^{2m',-2l'-n}(\mathbb{R}\times X)$
- S_δ is bounded in $\Psi_e^{2m'+1,-2l-n}(\mathbb{R}\times X)$ and, as $\delta\to0$, converges to 0 in $\Psi_e^{2m'+1+\epsilon,-2l-n}(\mathbb{R}\times X)$ for all $\epsilon>0$.

Hence

(8.7) $$\|A'_\delta u\|_g^2-\langle S_\delta u,u\rangle_g\le\Big|\langle E_\delta u,u\rangle_g\Big|+\Big|\langle K_\delta u,u\rangle_g\Big|+\Big|\langle R_\delta u,u\rangle_g\Big|,$$

so $\mathrm{WF}_e^{m'+\frac12,l}(u)$ is disjoint from the elliptic set of A'. This proves (8.6).

Finally consider (iii). We work by induction, assuming that the result is known for all nonnegative integers smaller than a given k. Note that Δ_Y is an edge (pseudo)differential operator, so the conclusion of (iii) for positive k is only nontrivial if $m\ge l+(n-1)/2$. In particular $p\notin \mathrm{WF}_e^{m',l}(\Delta_Y^k)$ for m' sufficiently negative. We therefore prove the following analogue of (8.6) with u replaced by $\Delta_Y^k u$:

(8.8) If $m'<l+n/2-1,\ u\in H_e^{-\infty,l}(I\times X)$ and $p\notin \mathrm{WF}_e^{m,l}(u)$ then
$$p\notin \overline{{}^{e}S^*_{\mathbb{R}\times\partial X}(\mathbb{R}\times X)\cap \mathrm{WF}_e^{m'+1/2,l}(\Delta_Y^k u)\setminus \mathsf{OG}}\Longrightarrow p\notin \mathrm{WF}_e^{m'+1/2,l}(\Delta_Y^k u),$$

proceeding much as before.

To prove (8.8), note that Lemma 1.6 gives a distributional equation for $u_k = \Delta_Y^k u$ of the form

$$\Box u_k + \left(Q_k D_x + \frac{1}{x} P_k\right) u = 0, \text{ where } P_k \in \mathcal{C}^\infty([0,\epsilon); \mathrm{Diff}^{2k+1}(Y)),\\ Q_k \in \mathcal{C}^\infty([0,\epsilon); \mathrm{Diff}^{2k-1}(Y)). \tag{8.9}$$

Now applying the test operator A_δ and pairing with u_k gives an estimate similar to (8.7) with an extra term:

$$\|A'_\delta u_k\|_g^2 - \langle S_\delta u_k, u_k\rangle_g \le \left|\langle E_\delta u_k, u_k\rangle_g\right| + \left|\langle K_\delta u_k, u_k\rangle_g\right|\\ + \left|\langle R_\delta u_k, u_k\rangle_g\right| + \left|\left\langle A_\delta \Delta_Y^k u, A_\delta B_k u\right\rangle_g\right|. \tag{8.10}$$

The first term on the left can be reorganized, modulo terms uniformly controlled by the inductive hypothesis, to bound a positive multiple of $\|A'_\delta u\|_{H^{2k}(Y)}$, the tangential Sobolev norm of $A'_\delta u$. We now show that the last term on the right can be estimated by a small multiple of this norm, modulo the inductive bounds. Note that $A_\delta = G_\delta A'_\delta + C_\delta + D_\delta$ where $G_\delta \in \Psi_{\mathrm{e}}^{-1/2,1}(\mathbb{R}\times X)$ is uniformly bounded as $\delta \downarrow 0$, C_δ is lower order and D_δ is supported in the region of known regularity. Modulo terms bounded by the inductive hypothesis or by the hypothesis of (8.8), we can thus write the last term as

$$\left\langle \Delta_Y^k A'_\delta u, B_k G_\delta^* G_\delta A'_\delta u\right\rangle_g.$$

We may rewrite B_k as a sum of terms $x^{-1} C_k S_i$ with $C_k \in \mathrm{Diff}^{2k}(Y)$ and where S_i are smooth b-vector fields. Hence modulo controllable terms the inner product above is estimated by a sum of terms of the form

$$\left\|\Delta_Y^k A'_\delta u\right\| \left\|x^{-1} C_k S_i G_\delta^* G_\delta A'_\delta u\right\|_g.$$

The latter norm is (again modulo known terms) controlled by $\epsilon\|A'_\delta u\|_{H^{2k}(Y)}$, with the ϵ coming from the small support in x. Thus the last term in (8.10) can be absorbed in the left, giving the inductive estimate and proving (8.8).

To prove (iv), we appeal once again to Lemma 1.6 and proceed as with (iii). $\Box$

Theorem 8.3. *Let $u(t)$ be the solution of the Cauchy problem* (4.1)–(4.3) *and suppose that $u \in H_e^{-\infty,l}(I\times X)$ for some open $I \ni \bar t$ and some $l \in \mathbb{R}$ then*

$$p \in R^\epsilon_{\pm,O}(\bar t),\ \Gamma^\epsilon(p) \cap \mathrm{WF}^m(u) = \emptyset \text{ for } m < l + (n-1)/2 \text{ and } \epsilon > 0\\ \Longrightarrow p \notin \mathrm{WF}^{m-\delta}(u)\ \forall\ \delta > 0.$$

Proof. By (i) and (ii) of Theorem 8.1, there is no edge wavefront set of order $(m, m-(n-1)/2-\delta)$ along all bicharacteristics in ${}^eS^*_{\mathbb{R}\times\partial X}(\mathbb{R}\times X)$ terminating at $t=\bar t$, $y = y(p)$. Hence by (iii) of Theorem 8.1, closedness of edge wavefront set, and Hörmander's theorem, $p \notin \mathrm{WF}^{m-2\delta}(u)$. $\Box$

9. The rescaled FBI transformation

To analyze the boundary behavior of solutions to the wave equation, we employ a rescaled version of the one-dimensional FBI ("Fourier-Bros-Iagolnitzer") transformation in the time variable. The FBI transform is a variant of the Bargmann transform which was employed by Bros and Iagolnitzer [10] to study microlocal regularity in the analytic setting. Its properties were further elaborated by Sjöstrand

[21]. For our purposes, the FBI transform could be dispensed with in favor of a composition of localization in time and Fourier transform. Use of the FBI transform, however, seems more likely to admit generalization. The $\mathcal{C}^\infty$ approach to the FBI used here follows the spirit of [24], which deals with compact manifolds. A partial FBI transform is also used in a related analysis of the wave equation by Gérard and Lebeau [6].

Consider the complex phase function $\phi(t,\tau,t') = i(t-t')^2\langle\tau\rangle/2 + (t-t')\tau$. The associated FBI transform applied just to the time variable is

$$Tu(t,\tau,x,y) = \int_X e^{i\phi(t,\tau,t')} a(t,t',\tau) u(t',x,y)\,dt'$$

where a is a polyhomogeneous symbol of order 1/4 with proper support in t,t'. It is an elementary consequence of the stationary phase lemma, demonstrated in [24], that T^*T is a pseudodifferential operator on $\mathbb{R}$ of order 0 and hence is L^2-bounded. It is elliptic at $(\bar{t},\pm\infty) \in S^*\mathbb{R}$ if a is elliptic at $t = t' = \bar{t}$, $\tau = \pm\infty$. Thus the ellipticity of a at $(\bar{t},\infty)$ implies that there exists $G \in \Psi^0(\mathbb{R})$ with $(\bar{t},\infty) \notin \mathrm{WF}'(G)$ such that

$$T^*T = \mathrm{Id} + G, \text{ so } Tu = 0 \text{ in } \tau > 1 \text{ near } \{t = \bar{t}\} \Longrightarrow u = -Gu \text{ near } \{t = \bar{t}\}. \tag{9.1}$$

The basic intertwining property of T corresponds to the boundedness of

$$TD_t - \tau T : L^2(\mathbb{R}) \longrightarrow \langle\tau\rangle^{\frac{1}{2}} L^2(\mathbb{R}^2). \tag{9.2}$$

We shall choose a to have support in $\tau > 1$, the region $\tau < -1$ being handled by reflection in t. Fixing a product decomposition near the boundary and inserting a cutoff $\chi(x)$ with $\chi(x) = 1$ for x sufficiently small and with support in the product neighborhood of a boundary component Y, we scale the x-variable and define

$$Su(t,\tau,\tilde{x},y) = \int \chi(\frac{\tilde{x}}{\tau}) e^{i\phi(t,\tau,t')} a(t,t',\tau) u(t',\frac{\tilde{x}}{\tau},y)\,dt'. \tag{9.3}$$

If we let $\tilde{X}$ denote the model cone $[0,\infty) \times Y$ then

$$S : \mathcal{C}^{-\infty}(\mathbb{R} \times X) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R} \times [1,\infty) \times \tilde{X}). \tag{9.4}$$

In fact the assumed properness of the support in t,t' ensures that Su is $\mathcal{C}^\infty$ in τ and t. However, it is the growth in τ, uniformly in $\tilde{x}$, that will interest us here. In view of the assumption on the support of a is we will use a decomposition of the type described in (4.6) of $u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$ under convolution in t and examine u_+.

Taking into account the scaling, the boundedness of T implies that S is bounded on the scale-invariant spaces in the x variable

$$S : L^2_{\mathrm{b}}(\mathbb{R} \times X) \longrightarrow L^2_{\mathrm{b}}(\mathbb{R} \times [1,\infty) \times \tilde{X}) \tag{9.5}$$

where the measure on the left is $dx\,dt\,dy/x$ and on the right $dt\,d\tau\,d\tilde{x}\,dy/\tilde{x}$ and we assume that a is a function of $t - t'$ only to get the global estimate. The scale-invariance of the measue shows that

$$S^*S = \chi(x)G \tag{9.6}$$

where G is the pseudodifferential operator on $\mathbb{R}$ in (9.1).

The simplest intertwining properties of S follow directly from (9.3):

$$\begin{gathered} S \circ Qu = Q \circ Su,\ Q \in \mathrm{Diff}^*(Y) \text{ and} \\ S(xu) = \frac{\tilde{x}}{\tau} Su. \end{gathered} \tag{9.7}$$

It follows from the second of these that if $f \in \dot{\mathcal{C}}^\infty(X)$ then composition with f as a multiplication operator gives

$$S \circ f : L^2_{\mathrm{b}}(\mathbb{R} \times X) \longrightarrow \bigcap_k \left(\frac{\tilde{x}}{\tau}\right)^k L^2_{\mathrm{b}}(\mathbb{R} \times [1,\infty) \times \tilde{X}). \tag{9.8}$$

Note that $\tilde{x}/\tau$ is bounded on the support of Su and $\tau > 1$ by assumption. Since we will only use the FBI transform to examine behavior in Taylor series at the boundary, we shall denote the space of operators with the property (9.8) as E^∞ and consider these as error terms. The presence of the localizing function χ in the definition of S means for instance that

$$S \circ xD_x = \tilde{x}D_{\tilde{x}} \circ S + S' \circ f',\ f' \in \dot{\mathcal{C}}^\infty(X),\ S' \circ f' \in E^\infty. \tag{9.9}$$

Here, S' is an operator of the same form as S with a different cutoff χ.

Scaling (9.2) and using (9.7) we conclude that

$$S \circ (xD_t) - \tilde{x}S : L^2_{\mathrm{b}}(\mathbb{R} \times X) \longrightarrow \tau^{-\frac{1}{2}} \langle \tilde{x} \rangle L^2_{\mathrm{b}}(\mathbb{R} \times [1,\infty) \times \tilde{X}). \tag{9.10}$$

The scaling of x makes $\tilde{x}$ a global variable on $\mathbb{R}_+$, and we are primarily concerned with behavior as $\tilde{x} \to \infty$. The structure which arises at infinity here corresponds to the $\mathbb{R}_+$-homogeneous metric in (1.13). This is a conic metric on $\tilde{X} = [0,\infty) \times Y$, but its uniform behavior near infinity is rather different from its behavior near $\tilde{x} = 0$. Consider the inversion

$$\tilde{X}^\circ = (0,\infty) \times Y \ni (\tilde{x}, y) \longmapsto (1/\tilde{x}, y) \in (0,\infty) \times Y = \tilde{X}^\circ \tag{9.11}$$

which induces an isomorphism of $\mathcal{C}^{-\infty}(\tilde{X})$. Under this transformation, the metric becomes a scattering metric in the sense of [16] at the boundary $w = 1/\tilde{x} = 0$, which is to say it is a particular type of asymptotically locally Euclidean metric.

In [16] the associated compactly-supported Sobolev spaces $H^m_{\mathrm{sc,c}}(\tilde{X})$ are defined for any manifold with boundary. It is natural then to introduce Sobolev spaces which are of "b-type" near $\tilde{x} = 0$ and of "scattering type" near $\tilde{x} = \infty$, based however on the $\mathbb{R}_+$-invariant measure $d\tilde{x}\, dy/\tilde{x}$ (note that this convention differs from the weight used in [16]). Thus if we choose $\phi \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty))$, with $\phi(\tilde{x}) = 1$ near $\tilde{x} = 0$ then we may define

$$\begin{aligned} H^m_{\text{b-sc}}(\tilde{X}) = \{u \in \mathcal{C}^{-\infty}(\tilde{X}); \phi(\tilde{x})u \in H^m_{\mathrm{b,c}}(\tilde{X}),& \\ (1-\phi)(1/w)u(1/w,y) \in H^m_{\mathrm{sc,c}}(\tilde{X})\},\ m \in \mathbb{R}.& \end{aligned} \tag{9.12}$$

Since both the "b" and the "sc" Sobolev space reduce to the standard Sobolev spaces in the interior, this is independent of the choice of ϕ. We shall also employ weighted versions of these spaces which we will generally write in terms of the weights $\tilde{x}$ and $\langle \tilde{x} \rangle$. The former is a boundary defining function near 0 but is also, near infinity, of the form of $1/w$ where $w = 0$ is a defining function for inverted infinity. On the other hand $1/\langle \tilde{x} \rangle$ is just a defining function for inverted infinity. Thus we consider the weighted spaces

$$\tilde{x}^l \langle \tilde{x} \rangle^k H^m_{\text{b-sc}}(\tilde{X}),\ l, k, m \in \mathbb{R}. \tag{9.13}$$

Directly from the definition, these scales of Hilbertable spaces satisfy complex interpolation.

The spaces which arise here correspond to functions of τ and t with values in these weighted Sobolev spaces. We are primarily interested in global behavior in τ but local behavior in t. We therefore define

$$L^2_{\mathrm{loc}}(\mathbb{R}_t \times [1,\infty]; H^m_{\text{b-sc}}(\tilde{X})) = \big\{u \in \mathcal{C}^{-\infty}(\mathbb{R}\times\mathbb{R}\times\tilde{X}); u = 0 \text{ in } \tau < 1 \text{ and } \phi(t)u \in L^2(\mathbb{R}\times\mathbb{R}; H^m_{\text{b-sc}}(\tilde{X}))\ \forall\ \phi \in \mathcal{C}^\infty_c(\mathbb{R})\big\}. \tag{9.14}$$

More generally weights will be written out as in (9.13). Note that the closed bracket at $\tau = \infty$ in (9.14) is intended to indicate that these spaces are indeed global in τ.

Lemma 9.1. *For any m, $l \in \mathbb{R}$*

$$S : x^l H^m_e(\mathbb{R}\times X) \longrightarrow \bigcap_{\alpha\in[0,m]} \tau^{-l}\tilde{x}^l\langle\tilde{x}\rangle^{-\alpha} L^2_{loc}(\mathbb{R}_t\times[1,\infty]; H^{m-\alpha}_{b\text{-}sc}(\tilde{X})). \tag{9.15}$$

Conversely provided $\chi' \in \mathcal{C}^\infty(X)$, $\chi'\chi = \chi'$ and $m \geq 0$,

$$\text{if } u \in L^2_{\mathrm{b}}(\mathbb{R}\times X) \text{ then } Su \in \bigcap_{\alpha\in[0,m]} \tau^{-l}\tilde{x}^l\langle\tilde{x}\rangle^{-\alpha} L^2_{loc}(\mathbb{R}_t\times[1,\infty]; H^{m-\alpha}_{b\text{-}sc}(\tilde{X})) \text{ near } \{t=\bar{t}\} \Longrightarrow \chi' u_+ = u_1 + u_2,\ u_1 \in x^l H^m_e(\mathbb{R}\times X),\ u_2 \in \mathcal{C}^\infty(\mathbb{R}; L^2_{\mathrm{b}}(X)) \text{ near } \{t=\bar{t}\}. \tag{9.16}$$

Proof. For positive integral m, $u \in x^l H^m_e(\mathbb{R}\times X)$ with support near the boundary if $(xD_x)^k(xD_t)^p Qu \in x^l L^2_{\mathrm{b}}(\mathbb{R}\times X)$ for all $Q \in \mathrm{Diff}^*(Y)$ and all $k+p+\mathrm{ord}(Q) \leq m$. The continuity estimates in (9.15) then follow from (9.5), (9.7), (9.8), (9.9) and (9.10). Similarly, for negative integral m, $u \in H^m_e(\mathbb{R}\times X)$ may be written as a finite superposition of edge operators of order $-m$ applied to elements of $L^2_{\mathrm{b}}(\mathbb{R}\times X)$ and continuity follows similarly. Complex interpolation on both sides then gives the general case of (9.15).

To see the partial converse, (9.16), first replace u by u_+ as in (4.6). Since the amplitude of S is assumed to be supported in $\tau > 1$ this simply changes S by a similar operator with rapidly decreasing amplitude. Thus we may suppose that the condition holds for u_+. The invertibility of S in (9.6) shows that $S^*S \equiv \chi$ modulo a term arising from G; applying the arguments above to S^* therefore gives (9.16). □

It follows from (9.15) that if $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ is 1 near 0

$$(1-\phi(\tilde{x}))S\circ B : x^l H^m_{\mathrm{e}}(\mathbb{R}\times X) \longrightarrow \tau^{-l}\langle\tilde{x}\rangle^{-M} H^M_{\text{b-sc}}(\mathbb{R}\times[1,\infty)\times\tilde{X})\ \forall\ M,\ B \in \Psi^{-\infty}_{\mathrm{e},c}(\mathbb{R}\times X). \tag{9.17}$$

Thus Su is rapidly decreasing in the sense of Schwartz near infinity. Thus the boundary part of the scattering wavefront set, in the sense of [16], of Su is determined by the edge wavefront set, over the boundary, of u. In fact we need to use a notion of scattering wavefront set in a uniform sense in t and τ. Such uniform versions of the scattering wavefront set are also discussed in [16]. As there, we are only really interested in the scattering wavefront set near $\tilde{x} = \infty$.

On a compact manifold with boundary, X, the scattering wavefront set is a subset of the boundary of the radial compactification of the intrinsic scattering cotangent bundle. This bundle reduces to the usual cotangent bundle in the interior but has basis at any boundary point p the differentials dx/x^2 and dy_j/x, where where x is a local boundary defining function and $y = 0$ at p, are boundary coordinates. The part of the scattering wavefront set over the boundary can be defined explicitly as follows. If $(\bar{\zeta}, \bar{\eta}) \in {}^{\mathrm{sc}}T^*_p X$ is the coordinate representation of a general point in the fiber over p, $\bar{\zeta}\frac{dx}{x^2} + \bar{\eta} \cdot \frac{dy}{x} \in {}^{\mathrm{sc}}T^*_p X$ then $(\bar{\zeta}, \bar{\eta}) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ if for some $\psi \in \mathcal{C}^\infty(X)$ supported in the coordinate patch and with $\psi(p) \neq 0$,

$$\int e^{-i\frac{\zeta}{x} + i\frac{\eta \cdot y}{x}} \psi(x, y) u(x, y) \frac{dx}{x} dy \text{ is } \mathcal{C}^\infty \text{ near } (\bar{\zeta}, \bar{\eta}). \tag{9.18}$$

Note that changing to the singular variables $1/x, y/x$ identifies the complement of the boundary in a neighborhood of p with an open set in $\mathbb{R}^n$. Then the formal integral in (9.18) may be identified with the Fourier transform. Since u is an extendible (hence Schwartz) distribution, (9.18) is well defined; this condition is also independent of the choice of coordinates.

The *uniform* version of the scattering wavefront we need is obtained by strengthening (9.18).

Definition 9.2. If $v \in \tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty]; \langle\tilde{x}\rangle^p H^m_{\text{b-sc}}(\tilde{X}))$ for some m, l and p then

$$(\bar{t}, \infty; \bar{y}, \bar{\zeta}, \bar{\eta}) \notin \mathrm{WF}_{\mathrm{sc},\partial}(v; \tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty]; \langle\tilde{x}\rangle^q H^*_{\text{b-sc}}(\tilde{X}))) \tag{9.19}$$

if $v = v_1 + v_2$ where, for some m, $v_1 \in \tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty]; \langle\tilde{x}\rangle^q H^m_{\text{b-sc}}(\tilde{X}))$ and

$$\int e^{-i\zeta\tilde{x} + i\eta\cdot y\tilde{x}} \psi(\frac{1}{\tilde{x}}, y) u(\tilde{x}, y) \frac{d\tilde{x}}{\tilde{x}} dy \in \tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty]; \mathcal{C}^\infty(\mathbb{R}^n)) \text{ near } (\bar{t}, \bar{\zeta}, \bar{\eta}), \tag{9.20}$$

where $\psi(x, y) \in \mathcal{C}^\infty_c([0, \infty) \times Y)$ is equal to 1 near $(0, \bar{y})$.

Note that the regularity, m, here is irrelevant since it is the boundary part of the scattering wavefront set we are examining.

We will show that the wavefront relation of S is related to

$$\mathcal{R} = \{(\xi', y, \mu, t; \xi, y', \eta, t', \lambda); \lambda > 0,\ t = t',\ y = y',\ (\xi', \mu) = \lambda^{-1}(\xi, -\eta)\}. \tag{9.21}$$

Proposition 9.3. *If* $u \in x^l H^{-\infty}_{\mathrm{e},c}(\mathbb{R} \times X)$ *has* $\mathrm{WF}^{\infty,l}_{\mathrm{e}}(u) \subset \{\lambda >\}$ *then*

$$\begin{aligned} \mathrm{WF}_{\mathrm{sc},\partial}(Su; \tau^{-l} L^2_{loc}(\mathbb{R} \times [1, \infty]; \langle\tilde{x}\rangle^{l-m} H^*_{b\text{-}sc}(\tilde{X}))) \\ \subset \mathcal{R} \circ \left(\mathrm{WF}^{m,l}_{\mathrm{e}}(u) \cap {}^e S^*_{\mathbb{R}\times\partial X}(\mathbb{R} \times X)\right), \end{aligned} \tag{9.22}$$

in the sense of Definition 9.2.

Proof. If $u \in x^l H^m_{\mathrm{e}}(\mathbb{R} \times X)$ then, by Lemma 9.1 the scattering wavefront set of Su relative to $\tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty]; \langle\tilde{x}\rangle^{l-m} H^*_{\text{b-sc}}(\tilde{X}))$ is empty, since Su lies in the space $\tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty]; \langle\tilde{x}\rangle^{l-m} H^0_{\text{b-sc}}(\tilde{X}))$. Thus it suffices to prove (9.22) for $m = \infty$.

It is therefore enough to consider $u \in x^l H^m_{\mathrm{e},c}(\mathbb{R} \times X)$, for some m, having support in the product neighborhood of the boundary and having scattering wavefront set contained in $\{\lambda > 0\}$. Then, using a partition of unity in the edge calculus, we may decompose

$$u = u' + \sum_j B_j u$$

where $u' \in x^l H^\infty_{\mathrm{e},c}$ and the B_j have small wavefront sets, in $\lambda > 0$. Again by Lemma 9.1 the term u' produces a term which is Schwartz near $\tilde{x} = \infty$ as a function with values in the weighted L^2 space $\tau^{-l} L^2_{\mathrm{loc}}(\mathbb{R} \times [1, \infty))$. This corresponds to the absence of any scattering wavefront set. So we may replace u by Bu, with an edge pseudodifferential operator, $B \in \Psi^0_{\mathrm{e}}(\mathbb{R} \times X)$, with essential support concentrated near some boundary point $(\bar{t}, 0, \bar{y}, \bar{\lambda}, \bar{\eta}, \bar{\xi})$. We may also suppose that B has its support near the boundary component Y. Since the edge wavefront set is a conic notion this means that we may take $\bar{\lambda} = 1$. In local coordinates

$$Bu(t,x,y) = (2\pi)^{-1} \int e^{i((s-1)\xi + (y-y')\eta + T\lambda)} b(t,x,y,s,y',T,\xi,\eta,\lambda) u(xs, y', t - xT) x^2 \, ds \, dy' \, dT \, d\xi \, d\eta \, d\lambda, \tag{9.23}$$

where, at the expense of another error of order $-\infty$ in the edge calculus, we may assume that

$$\text{the amplitude } b \text{ has small conic support (as } \lambda \to \infty\text{) around } t = \bar{t},\ x = 0,\ y = \bar{y},\ s = 1,\ y' = \bar{y},\ T = 0,\ \xi = \bar{\xi}\lambda \text{ and } \eta = \bar{\eta}\lambda. \tag{9.24}$$

Since we are interested in the scattering wavefront set of the image we can localize near $\tilde{x} = \infty$; the assumption on b means the image is already effectively localized in y near $\bar{y}$ and in t near $\bar{t}$. Thus, in local coordinates based at $\bar{y} = 0$, we take the (normalized) Fourier transform of $v = (1 - \phi(\tilde{x}))SBu$,

$$\mathcal{F}v(\xi', \mu; \tau) = \int e^{i\tilde{x}\xi' + \tilde{x}y\cdot\mu} v(\tilde{x}, y; \tau) \tilde{x}^{n-1} d\tilde{x} dy. \tag{9.25}$$

We then localize near some point using a cutoff $\tilde{\phi}(\tilde{x}, y, \xi', \mu)$ to examine the regularity; such regularity allows us to conclude absence of scattering wavefront set at points in the support of $\tilde{\phi}$, where ξ' and μ are coordinates in the fibers of the scattering cotangent bundle determined by the canonical one-form

$$\xi' d\tilde{x} + \mu \cdot d(\tilde{x}y).$$

Finally, then, we arrive at the composite operator

$$\begin{gathered} Gu = \mathcal{F}(1 - \phi(\tilde{x}))SBu, \\ Gu = \int e^{i\psi} cu(\frac{\tilde{x}s}{\tau}, y', t' - \frac{\tilde{x}T}{\tau}) \tau^{-2} \tilde{x}^{n+1} ds \, dy' \, dT \, dt' \, d\tilde{x} \, dy \, d\xi \, d\eta \, d\lambda, \text{ where} \\ c = c(\tilde{x}, y, \xi', \mu, t, t', \tau, \frac{\tilde{x}}{\tau}, s, T, \xi, \eta, \lambda) \\ = \tilde{\phi}(\tilde{x}, y, \xi', \mu) a(t, t', \tau) b(\frac{\tilde{x}}{\tau}, y, t', s, T, \xi, \eta, \lambda) \text{ and} \\ \psi = \tilde{x}\xi' + \tilde{x}y\cdot\mu + (t - t')\tau + \frac{i}{2}\langle\tau\rangle(t - t')^2 + (s-1)\xi + (y - y')\cdot\eta + T\lambda. \end{gathered} \tag{9.26}$$

First suppose that $e(\tilde{x}, \lambda)$ is a conic cutoff keeping $|\tilde{x} - \lambda| \le \delta\langle\tilde{x}, \lambda\rangle$ asymptotically for some small $\delta > 0$ and equal to one on a smaller region of this type. Inserting $1 - e(\tilde{x}, \lambda)$ into the integral we may integrate by parts using the vector field

$$V_1 = \partial_T + \frac{\tilde{x}}{\tau}\partial_{t'} \text{ satisfying } V_1\psi = \lambda - \tilde{x} + \frac{i\tilde{x}\langle\tau\rangle}{\tau}(t' - t). \tag{9.27}$$

Since $t - t'$ is small, for large τ this is elliptic in λ and $\tilde{x}$ on the support of the amplitude. This gives an operator of the form (9.26) but with amplitude arbitrarily decreasing in $\tilde{x}, \lambda$. Rapid decrease in λ implies rapid decrease in η and ξ and hence again effectively replaces B by an operator of arbitrarily low order. Hence this term also has no uniform scattering wavefront set.

Thus, by inserting the cutoff e we may assume that $\tilde{x} \sim \lambda$ on the support of the amplitude. Now, consider the vector fields

$$V_2 = \partial_s - \frac{\tilde{x}}{s}\partial_{\tilde{x}} - \frac{\tilde{x}}{\tau s}T\partial_{t'} \text{ and } W = \partial_y \tag{9.28}$$

which annihilate $u(\frac{\tilde{x}s}{\tau}, y', t' - \frac{\tilde{x}T}{\tau})$ and satisfy

$$V_2\psi = \xi - \frac{\tilde{x}\xi'}{s} - \frac{\tilde{x}y\cdot\mu}{s} + \frac{\tilde{x}T}{s}\left(1 - i\frac{\langle\tau\rangle}{\tau}(t'-t)\right) \text{ and } W\psi = \eta + \tilde{x}\mu. \tag{9.29}$$

These functions, with $t - \bar{t}$, and $V_1\psi$, define $\mathcal{R}$.

Assuming that the support of $\tilde{\phi}$ in (9.26) is disjoint from the image of the essential support of B under $\mathcal{R}$, integration by parts now allows the amplitude, c, in (9.26) to be replaced by a symbol of arbitrarily low order in ξ, η and λ. Simpler versions of the arguments used to prove (9.15) and the boundedness of edge pseudodifferential operators show that this leads to a map

$$x^l H^m_{\mathrm{e,c}}(\mathbb{R}\times X) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}_t; \tau^{-l}L^2([1,\infty)). \tag{9.30}$$

This proves (9.22). □

We shall apply the FBI transform to solutions of $\Box u = 0$. Since S is a Fourier integral operators, with complex phase function, the transform satisfies a model equation involving the Laplacian, $\varDelta_0$, on the model cone $\tilde{X} = \mathbb{R}_+ \times Y$ with respect to the the product metric $dx^2 + x^2h_0(y, dy)$.

Lemma 9.4. *There are operators S_1 and S_2 of the same form as S with amplitudes of order $\frac{1}{4}$ and $L \in \mathrm{Diff}^2_{\mathrm{b}}(X)$ such that*

$$(\varDelta_0 - 1)S = -\tau^{-2}S\Box + \tau^{-\frac{1}{2}}S_1 + \tau^{-1}\tilde{x}^{-1}S_2L. \tag{9.31}$$

Proof. Computing directly

$$\Box\psi u = \psi\Box u - 2(D_x\psi)(D_xu) - (D_x^2\psi)u.$$

Let $\tilde{S}$ be defined as S was but with cutoff $\tilde{\psi} \in \mathcal{C}^\infty(X)$, still supported in the product neighborhood of the boundary, with $\tilde{\psi}\psi = \psi$. Then

$$\tilde{S}(\Box\psi u) = S(\Box u) + \tilde{S}(L_1u) \text{ with } L_1 \in \mathrm{Diff}^1_{\mathrm{b}}(X). \tag{9.32}$$

Now the commutation relation for D_t shows that

$$\tilde{S}D_t^2\psi u = \tau^2\tilde{S}\psi u + \tau^{\frac{3}{2}}S'u$$

where S' is of the same form as S but with a different amplitude of order $\frac{1}{4}$. Since $S\varDelta_0 = \varDelta_0 S$, $\varDelta\psi - \varDelta_0\psi \in x^{-1}\,\mathrm{Diff}^2_{\mathrm{b}}(X)$ and $\tilde{S}(\psi u) = Su$ we conclude that

$$(1-\varDelta_0)S(u) = \tau^{-2}\tilde{S}\Box\psi u - \tau^{-\frac{1}{2}}S'u + \tilde{S}(\varDelta - \varDelta_0)\psi u,$$

yielding (9.31). □

10. Reduced normal operator

The reduced normal operator of the conic wave operator, as an element of the weighted edge calculus, is $1-\Delta_0$ where Δ_0 is the Laplacian for the tangent product-conic metric on $\tilde{X} = [0,\infty)_{\tilde{x}} \times Y$. Regularity results obtained in subsequent sections depend on *global* invertibility properties for this operator. It is important to understand that its behavior at infinity in the product cone is quite different from its behavior near $\tilde{x} = 0$. In fact, near ∞, the metric (1.13) is a special case (again the model case) of a scattering metric considered in [16]. The local and microlocal estimates in [16] are combined below with the analysis of the domain for conic metrics above to get the desired invertibility estimates, which are of a standard type in scattering theory.

The analysis of the Friedrichs extension of Δ_0 on $\tilde{X}$ proceeds very much as above, using also the fact that the metric is complete at infinity. Thus the scattering Sobolev spaces are, near infinity, the natural metric Sobolev spaces and the domains of powers of Δ_0 are closely related to the "b-sc" Sobolev spaces defined in (9.12).

Proposition 10.1. *For the Friedrichs extension of Δ_0, the Laplacian for the metric* (1.13)*,*

$$\begin{aligned} \mathrm{Dom}(\Delta_0^{\frac{p}{2}}) &= \tilde{x}^{-\frac{n}{2}+p}\langle\tilde{x}\rangle^{-p} H^p_{b\text{-}sc}(\tilde{X}),\ -\frac{n}{2} < p < \frac{n}{2}, \\ \mathrm{Dom}(\Delta_0^{\frac{n}{4}+\frac{\delta}{2}}) &= \tilde{x}^{\delta}\langle\tilde{x}\rangle^{-\delta-\frac{n}{2}} H^{\frac{n}{2}+\delta}_{b\text{-}sc}(\tilde{X}) + \mathbb{C}\phi,\ 0<\delta<\delta_0 \end{aligned} \tag{10.1}$$

for $\delta_0 > 0$ sufficiently small and with $\phi \in \mathcal{C}^\infty_c(\tilde{X})$ identically equal to 1 *near $\tilde{x} = 0$.*

The spectrum of Δ_0 is the whole of $[0,\infty)$. Just as in (3.23) in the compact-conic case, Δ_0 defines a continuous map,

$$\Delta_0 : x^{\delta}\langle\tilde{x}\rangle^{-\delta-\frac{n}{2}} H^{\frac{n}{2}+\delta}_{\text{b-sc}}(\tilde{X}) + \mathbb{C}\phi \longrightarrow x^{-2+\delta}\langle\tilde{x}\rangle^{2-\delta-\frac{n}{2}} H^{\frac{n}{2}-2+\delta}_{\text{b-sc}}(\tilde{X}),\ \delta>0 \text{ small}\ , \tag{10.2}$$

however this is never an isomorphism. On the other hand, the spectral family $\Delta_0 - \lambda$, for $\lambda \in \mathbb{C}\setminus[0,\infty)$ does define a continuous map as in (10.2) which is an isomorphism.

One of the standard results of scattering theory, proved in [16] in the context of scattering metrics, is the *limiting absorption principle.* This asserts the existence of the limit of the operator $(\Delta_0 - \lambda \pm i\gamma)^{-1}$ for $\lambda \in (0,\infty)$ as $\gamma \downarrow 0$. Of course the limit cannot exist as a bounded operator inverting the resolvent, i.e. cannot be defined on the range space of (10.2). However it does exist on a somewhat smaller space.

Proposition 10.2. *For $\gamma > 0$ the resolvent*

$$(\Delta_0 - 1 \pm i\gamma)^{-1} : \tilde{x}^{-2+\delta}\langle\tilde{x}\rangle^{2-\delta-\frac{n}{2}} H^{\frac{n}{2}-2+\delta}_{b\text{-}sc}(\tilde{X}) \longrightarrow \tilde{x}^{\delta}\langle\tilde{x}\rangle^{-\delta-\frac{n}{2}} H^{\frac{n}{2}+\delta}_{b\text{-}sc}(\tilde{X}) + \mathbb{C}\phi \tag{10.3}$$

restricts to an operator

$$(\Delta_0 - 1 \pm i\gamma)^{-1} : x^{-2+\delta}\langle\tilde{x}\rangle^{2-\delta-\frac{n}{2}-s} H^{\frac{n}{2}-2+\delta}_{b\text{-}sc}(\tilde{X}) \longrightarrow x^{\delta}\langle\tilde{x}\rangle^{-\delta-\frac{n}{2}-s} H^{\frac{n}{2}+\delta}_{b\text{-}sc}(\tilde{X}) + \mathbb{C}\phi \tag{10.4}$$

for any s and as an operator

$$(\Delta_0 - 1 \pm i\gamma)^{-1} : \tilde{x}^{-2+\delta} \langle \tilde{x} \rangle^{2-\delta-\frac{n}{2}-\frac{1}{2}-\epsilon} H_{b\text{-}sc}^{\frac{n}{2}-2+\delta}(\tilde{X}) \longrightarrow \tilde{x}^{\delta} \langle \tilde{x} \rangle^{-\delta-\frac{n}{2}+\frac{1}{2}+\epsilon} H_{b\text{-}sc}^{\frac{n}{2}+\delta}(\tilde{X}) + \mathbb{C}\phi, \quad \text{for any } \epsilon > 0, \tag{10.5}$$

the strong limit exists, as $\gamma \downarrow 0$.

Proof. The only difficulty with the convergence of the resolvent is related to the large, i.e.scattering, end of the cone. Modulo compact errors the analysis in [16] therefore applies and gives the same continuity properties for the limit of the resolvent as in the scattering case itself. □

Thus we have two limiting operators $(\Delta_0 - 1 \pm i0)^{-1}$ with the mapping property (10.5). For these operators it is not possible to take $\epsilon < 0$ in the domain space, nor in the range space without further restriction. However there are "larger" spaces on which the limiting resolvent is defined; these are fixed in terms of conditions on the scattering wavefront set.

Proposition 10.3. *Let ξ' be the dual variable to $d\tilde{x}$ in the scattering cotangent bundle. Let $U_\pm$ be an open neighborhood of the part of the radial set for $\Delta_0 - 1$ on which $\pm\xi' > 0$. Then for any $m \in \mathbb{R}$ and $k' > k > -\frac{1}{2}$*

$$(\Delta_0 - 1 \pm i0)^{-1} : \left\{ f \in \tilde{x}^{-2+\delta} \langle \tilde{x} \rangle^{2-\delta-\frac{n}{2}+k} H_{b\text{-}sc}^{\frac{n}{2}-2+\delta}(\tilde{X}); \mathrm{WF}_{\mathrm{sc}}(u) \cap U_\pm = \emptyset \right\} \longrightarrow \left\{ u \in \tilde{x}^{\delta} \langle \tilde{x} \rangle^{-\delta-\frac{n}{2}+k'} H_{b\text{-}sc}^{\frac{n}{2}+\delta}(\tilde{X}) + \mathbb{C}\phi; \mathrm{WF}_{\mathrm{sc}}(u) \cap U_\pm = \emptyset \right\} \tag{10.6}$$

and this is a two sided inverse of $\Delta_0 - 1$ on the union over k and k' of these spaces.

Proof. This just follows by combining the results of [16] with the analysis above of the domain of Δ_0. All the estimates of [16] concern the behavior as $\tilde{x} \longrightarrow \infty$ and the appearance of the conic boundary at $\tilde{x} = 0$ makes essentially no difference to the argument. This result could also be proved using the analysis of the forward fundamental solution of the $\mathbb{R}_+$-invariant conic wave operator from [2]. □

11. Diffractive theorem

Proposition 11.1. *Suppose $u \in \mathcal{C}(\mathbb{R}, \mathcal{D}_{\frac{n}{2}-\delta})$, for some $\delta > 0$, is an admissible solution to the conic wave equation and*

$$\mathrm{WF}(u) \cap R_I^{\epsilon}(0) = \emptyset \tag{11.1}$$

for some $\epsilon > 0$. Then for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R} \times X)$ with support in a sufficiently small neighborhood of $\{0\} \times \partial X$

$$\phi u \in x^{-\delta'} H_{\mathrm{b}}^{\infty}(\mathbb{R} \times X), \ \forall \ \delta' > 0, \tag{11.2}$$

so u is conormal near the boundary for $|t|$ small.

Remark 11.2. Note that, since u is a solution of the wave equation, (11.2) is equivalent to the condition that for all k, $\phi(x) D_t^k u$ is an L^2 function of t near $t = 0$ with values in $\bigcap_s \mathcal{D}_s$, provided $\phi \in \mathcal{C}^\infty(X)$ is chosen to have support sufficiently near the boundary, i.e. the conclusion of Theorem 4.4 holds with $s = \infty$.

Proof. First we replace u by u_+ in terms of the decomposition (4.6), this has the same regularity properties but its Fourier transform in t has support in $\tau > 1$. The treatement of u_- is similar. Now we proceed in three steps.

First, the hypothesis (11.1) allows Theorem 8.1 to be applied directly. Since $u \in \mathcal{C}(\mathbb{R}, \mathcal{E}_{\frac{n}{2}}) \subset x^{-\delta} L^2_{\mathrm{b,loc}}(\mathbb{R} \times X)$, the first part of Theorem 8.1 shows that

$$\mathsf{IC}(0) \cap \mathrm{WF}_{\mathrm{e}}^{m,-\delta}(u) = \emptyset \ \forall\ m,\ \forall\ \delta > 0. \tag{11.3}$$

Our initial goal is to apply the identity in Lemma 9.4 to show that

$$\phi D_t^k u \in x^{-\delta} H_{\mathrm{e}}^{-\infty}(\mathbb{R} \times X) \tag{11.4}$$

for all k and a cutoff as in the statement above.

The initial hypothesis on u implies in particular that $u \in x^{-\delta} H_{\mathrm{e}}^{\frac{n}{2}-\delta}(\mathbb{R} \times X)$ near $\{0\} \times \partial X$. Applying Lemma 9.1, with the cutoff in the definition of S supported sufficiently near $x = 0$ we therefore conclude that

$$Su \in \bigcap_{\alpha \in [0, n/2-\delta]} \tilde{x}^{-\delta} \langle \tilde{x} \rangle^{-\alpha} \tau^{\delta} L^2_{\mathrm{loc}} \left([1,\infty)_\tau \times \mathbb{R}_t ; H_{\mathrm{b\text{-}sc}}^{\frac{n}{2}-\delta-\alpha}(\tilde{X}) \right) \text{ near } \bar{t} = 0. \tag{11.5}$$

Similar conclusions apply to the terms on the right in (9.31), with $S_1 u$ having the regularity property (11.5) and

$$S_2(Lu) \in \bigcap_{\alpha \in [0, n/2-\delta-2]} \tilde{x}^{-\delta} \langle \tilde{x} \rangle^{-\alpha} \langle \tilde{x} \rangle^{\frac{n}{2}-\delta} L^2_{\mathrm{loc}} \left([1,\infty)_\tau \times \mathbb{R}_t ; H_{\mathrm{b\text{-}sc}}^{\frac{n}{2}-\delta-2-\alpha}(\tilde{X}) \right) \quad \text{near } \bar{t} = 0. \tag{11.6}$$

It is also the case that near $\tilde{x} = 0$, Su is in the domain of $\Delta_0^{\frac{n}{2}+\delta}$ if $\delta > 0$ is small enough. The identity (9.31) then holds, near $\tilde{x} = 0$, in the sense of the domain of $\Delta_0^{\frac{n}{2}-2+\delta}$, which is just a weighted b-Sobolev space.

Of fundamental importance is the estimate on the scattering wavefront set which follows from (11.3), together with the initial assumption (11.1), by use of Proposition 9.3. Namely, the incoming wavefront set for all terms, Su, $S_1 u$ and $S_2(Lu)$, computed with respect to the spaces in (11.5) and (11.6) is absent—these functions are microlocally rapidly decreasing as $\tilde{x} \to \infty$ in this uniform sense. Now, Lemma 10.3 shows that solving (9.31) gives an "improvement" in the estimate (11.5) by a factor of $\tau^{-\frac{1}{2}}$ at the expense of more growth as $\tilde{x} \to \infty$. Since we may invert S, up to errors which involve a smoothing operator in t, we may iterate this argument, reapplying Theorem 8.1 to obtain absence of scattering wavefront set with the new τ weight and using the improved estimate on Su to estimate $S_1 u$ and $S_2(Lu)$. Such iteration yields an estimate

$$Su \in \bigcap_{\alpha} \tilde{x}^{-\delta} \langle \tilde{x} \rangle^{N} \tau^{-M} H_{\mathrm{b\text{-}sc}}^{\frac{n}{2}-\delta}(\mathbb{R} \times [1,\infty) \times \tilde{X}) \text{ near } \bar{t} = 0,\ \forall M \text{ with } N = N(M). \tag{11.7}$$

Using the reverse regularity estimate in Lemma 9.1 we conclude that (11.4) does indeed hold.

Finally we apply Theorem 8.1 again to deduce the claimed regularity (11.1). Indeed, we may apply the first three parts of the theorem, in succession, to $D_t^k u$ for any integer k. As before we find that over $t = 0$ and the boundary

$$\mathrm{WF}_{\mathrm{e}}^{M,-\delta}(u) \subset \mathsf{OG}(0),\ \forall\ M. \tag{11.8}$$

Now we may use the outgoing propagation result to conclude that

$$\mathsf{OG}(0) \cap \mathrm{WF}_{\mathrm{e}}^{\frac{n-1}{2}-\delta,-\delta}(D_t^k u) = \emptyset$$

for any $\delta > 0$. Since this is fixed finite regularity, certainly implying that $D_t^k u \in L_g^2(\mathbb{R} \times X)$ locally for all t derivatives, (11.2) follows. $\square$

Proof of Theorem 4.4. We use the decomposition (3.18) to assume without loss of generality that there are no singularities in $\{\tau < 0\}$. By time-translation invariance, we may assume that $\bar{t} = 0$. Choose a cutoff $\psi(t) \in \mathcal{C}^\infty(\mathbb{R})$ which is 1 in $t > -\frac{1}{2}\delta$ and 0 in $t < -\delta$ for $\delta > 0$ which will be chosen small. Then $v = \psi(t)u$ satisfies

$$\Box v = f = -2i\psi'(t)D_t u - \psi''(t)u. \tag{11.9}$$

Now, if $\delta > 0$ is small enough, the hypothesis on the wavefront set of u means that $\mathrm{WF}^{r-1}(f)$ is disjoint from all incoming rays arriving at the boundary at time 0. Thus, it may be divided into two pieces $f = f_1 + f_2$, with both f_i supported in $-\frac{1}{4}\delta > t > -2\delta$, f_1 supported away from the boundary and satisfying

$$f_1 \in \mathcal{C}^p(\mathbb{R}; \mathcal{D}_{r-1-p}),\ \forall\, p \in \mathbb{N} \tag{11.10}$$

and with $\mathrm{WF}(f_2)$ disjoint from the incoming cone at $t = 0$. It follows that the forward solution to $\Box v_1 = f_1$ is in $H^r(\mathbb{R} \times X)$ away from the boundary and is in $\mathcal{C}(\mathbb{R}; \mathcal{D}_r)$ near it. Thus it suffices to consider $\Box v_2 = f_2$ which is a solution near $t = 0$ with no incoming singularities at all at time $t = 0$; by the uniqueness of solutions it is equal to $u - v_1$ near $t = 0$. Now v_2 may be extended to a global solution u' which is equal to v_2, and hence to $u - v_1$ near $t = 0$ and this too has no incoming singularities at time 0. After a finite amount of smoothing in t using Lemma 4.3, Proposition 11.1 applies to u', showing that it has no outgoing singularities at time 0 (and indeed lies in $\mathcal{C}(\mathbb{R}; \mathcal{D}_\infty)$) locally. $\square$

12. Propagation of tangential regularity

As a prelude to the division theorem, we will prove that regularity of solutions in the tangential, i.e. boundary, directions is conserved under time-evolution. This result represents a global version, in energy spaces, of the microlocal estimates in Theorem 8.1, part iii.

Recall that for a boundary component Y of X, Δ_Y denotes the Laplace-Beltrami operator induced on Y by the metric h_0. Set

$$Y_s = (1 + \Delta_Y)^{s/2},\ s \in \mathbb{R}.$$

These tangential pseudodifferential operators act naturally on the boundary, but may be viewed as acting on the fibers of the product decomposition in Theorem 1.2. Thus they act on functions or distributions on $(a, b)_t \times \partial X \cap [0, \epsilon)_x$.

Recall that $\mathcal{E}_s$ denotes the energy space $\mathcal{D}_s \oplus \mathcal{D}_{s-1}$ where $\mathcal{D} = \mathrm{Dom}(\Delta^{s/2})$. We write $\mathcal{E} = \mathcal{E}_1$ as well as $\mathcal{D} = \mathcal{D}_1$ for convenience. $\mathcal{E}_1$ is a Hilbert space with the norm

$$\|(u, v)\|_{\mathcal{E}}^2 = \|u\|_{L_g^2}^2 + \|du\|_{L_g^2}^2 + \|v\|_{L_g^2}^2,\ (u, v) \in \mathcal{E}.$$

Note that if we let

$$M = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

denote the infinitesimal generator of $U(t)$, M is not selfadjoint with respect to the norm on $\mathcal{E}$ as defined here (owing to the $\|u\|_{L^2}^2$ term).

For use in the sequel, we note the following energy estimate: For an admissible solution $(u, D_t u)$ with $(u, D_t u) \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{E}_\infty)$, and an operator Q such that

$$Q : \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}_\infty) \to \mathcal{C}^\infty(\mathbb{R}; \mathcal{D}_{n/2}) \tag{12.1}$$

set

$$\Psi(t) = \begin{pmatrix} Qu(t) \\ D_t(Qu(t)) \end{pmatrix}.$$

We can then compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Psi(t)\|^2_{\mathcal{E}} &= \operatorname{Re}\left\langle \begin{pmatrix} iD_t(Qu) \\ iD_t^2(Qu) \end{pmatrix}, \Psi \right\rangle_{\mathcal{E}} \\ &= \operatorname{Re}\langle iM\Psi, \Psi\rangle_{\mathcal{E}} + \operatorname{Re}\left\langle \begin{pmatrix} 0 \\ i[\Box, Q]u \end{pmatrix}, \Psi \right\rangle_{\mathcal{E}} \\ &= \operatorname{Re}\langle i[\Box, Q]u, D_t(Qu)\rangle_{L^2_g} + \operatorname{Re}\langle iD_t(Qu), Qu\rangle_{L^2_g}. \end{aligned} \tag{12.2}$$

Let $\mathcal{O}$ denote a product neighborhood of the boundary and let $\mathcal{E}_c(\mathcal{O}) \subset \mathcal{E}$ and $\mathcal{D}_c(\mathcal{O}) \subset \mathcal{D}$ be the subspaces with compact supports in $\mathcal{O}$.

Lemma 12.1. *If $B(x)$ is a tangential pseudodifferential operator of order 1 depending smoothly on x and $B(0)\mathbb{C} = 0$ then $x^{-1}B(x) : \mathcal{D}_c(\mathcal{O}) \longrightarrow L^2(X)$.*

Proof. We know that operators of the form $x^{-1}V$ with V a tangential vector field are bounded in this way, as is $B(x)$ itself. The generalized inverse of the tangential Laplacian satisfies $E\Delta_Y = \operatorname{Id} - \Pi_0$ where Π_0 is orthogonal projection onto the constants. We have $B(x)\Pi_0 = x\tilde{B}(x)$, hence

$$B(x) = B(x)E\Delta_Y + B(x)\Pi_0 = \sum_j B_j(x)V_j + x\tilde{B}(x) \tag{12.3}$$

where V_j are vector fields tangent to ∂X, the B_j are of order 0 (and hence bounded on L^2) and $\tilde{B}(x)$ is again of order 1 and smooth in x. Thus

$$x^{-1}B(x) = \sum_j B_j(x)x^{-1}V_j + \tilde{B}(x) \tag{12.4}$$

is bounded as claimed by (3.9). □

Let $\mathcal{O}' \subset \mathcal{O}$ be a smaller product neighborhood of ∂X.

Proposition 12.2. *For all $|t|$ sufficiently small and $s \in \mathbb{R}$,*

$$Y_s U(t) Y_{-s} : \mathcal{E}_c(\mathcal{O}') \longrightarrow \mathcal{E}$$

is bounded.

Proof. As $(\dot{\mathcal{C}}^\infty(X) + \mathbb{C}) \oplus \dot{\mathcal{C}}^\infty(X) \subset \mathcal{E}_\infty$ is dense in $\mathcal{E}$, it suffices to prove the relevant estimate for Cauchy data in $\mathcal{E}_\infty$.

Let $\Phi(t) = (u(t), D_t u(t)) = U(t)(u_0, u_1)$ be the solution to the Cauchy problem (4.1)–(4.2) with $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subset \mathcal{O}'$. Then there is an open interval I containing 0 such that for $t \in I$, we have $\operatorname{supp} u(t)$, $\operatorname{supp} D_t u(t) \subset \mathcal{O}$. The hypothesis (12.1) is satisfied for $Q = Y_s$, hence

$$\frac{1}{2}\frac{d}{dt}\|Y_s\Phi(t)\|^2_{\mathcal{E}} = \operatorname{Re}\langle Y_s u_t, Y_s u\rangle + \operatorname{Re}\langle [\Delta, Y_s]u, Y_s u_t\rangle \tag{12.5}$$

for all $t \in I$. By Lemma 1.7 and Lemma 12.1, $\|[\Delta, Y_s]u\|$ is bounded by a multiple of $\|(u,0)\|_{\mathcal{E}}$, locally near Y. Hence

$$\frac{1}{2}\frac{d}{dt}\|Y_s\Phi(t)\|^2_{\mathcal{E}} \leq C\|Y_s\Phi(t)\|^2_{\mathcal{E}}$$

from which the boundedness follows. □

13. Global propagation of conormality

The tangential regularity discussed in the previous section is the main step to showing that incoming conormal waves with respect to the surface $R_{\pm,I}$ propagate through the boundary to be conormal on the outgoing radial surface $R_{\pm,O}$. To prove this we need further to show that regularity is preserved under the repeated action of the radial vector field

$$R = xD_x + (t-\bar{t})D_t.$$

where $t = \bar{t}$ is the time at which the cone hits the boundary. Using time-translation invariance we may always assume $\bar{t} = 0$.

To begin, note that we can combine identities from Lemmas 1.6 and 1.7 to obtain, for any fixed s, k,

(13.1)
$$[\Box, Y_s R^k] = \sum_{j=0}^{k-1} c_j Y_{s+1} R^j \Box + \sum_{j=0}^{k-1} (a_j D_x + x^{-1}P_j) Y_{s+1} R^j + (RD_x + x^{-1}S)Y_s R^k$$

where $c_j \in \mathbb{C}$, $a_j \in \mathcal{C}^\infty([0,\epsilon) \times Y)$ and $R \in \mathcal{C}^\infty([0,\epsilon) \times Y; \Psi^{-1}(Y))$, and where S, $P_j \in \mathcal{C}^\infty([0,\epsilon) \times Y; \Psi^1(Y))$ annihilate constants at $x = 0$. Since the operator $Y_s R_k$ satisfies the hypothesis (12.1), (13.1) makes sense when applied to an element of $\mathcal{C}^\infty(\mathbb{R}; \mathcal{D}_\infty)$, with the equality holding in $\mathcal{C}^\infty(\mathbb{R}; \mathcal{D}_{n/2-2})$. Thus using Lemma 12.1, the energy identity gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big\|(Y_iR^ju, D_tY_iR^ju)\big\|^2_{\mathcal{E}} \\ \leq C\big\|(Y_iR^ju, D_tY_iR^ju)\big\|_{\mathcal{E}} \sum_{i'+j'\leq i+j} \Big\|(Y_{i'}R^{j'}u, D_tY_{i'}R^{j'}u)\Big\|_{\mathcal{E}}. \end{aligned} \tag{13.2}$$

Proposition 13.1. *If $U(t)$ is the solution operator to the Cauchy problem* (4.1)–(4.3) *then for a product boundary neighborhood $\mathcal{O}$, small time T and each $k \in \mathbb{N}$,*

(13.3)
$$\sum_{i+j\leq k} \|Y_iR^jU(t)(u_0,u_1)\|_{\mathcal{E}} \leq C \sum_{i+j\leq k} \|Y_iR^j(u_0,u_1)\|_{\mathcal{E}},\ |t| \leq T,\ (u_0,u_1) \in \mathcal{E}_c(\mathcal{O}).$$

Proof. We apply (13.2) inductively. Note that

$$\sum_{i+j\leq k} \|Y_iR^jU(t)(u(t), D_tu(t))\|_{\mathcal{E}} \sim \sum_{i+j\leq k} \|\left(Y_iR^ju(t), D_tY_iR^ju(t)\right)\|_{\mathcal{E}}$$

since

$$[D_t, Y_iR^j] = \sum_{l=0}^{j-1} c_l Y_i R^l D_t = Y_i \sum_{l=0}^{j-1} c'_l D_t R^l. \quad \Box$$

Proof of Theorem 4.8. By results going back at least to Hadamard (see also [8, 4]), under the hypotheses of the theorem, there exists $\epsilon > 0$ such that for $0 < t < \epsilon$, $u(t)$ and $D_t u(t)$ are conormal distributions in $\mathbb{R} \times X$ at $\{x = \bar{x} - t\} \cup \{x = \bar{x} + t\}$ with respect to $H^s(\mathbb{R} \times X)$ and $H^{s-1}(\mathbb{R} \times X)$ respectively. The distribution conormal to $\{x = \bar{x} + t\}$ remains conormal for small positive time, as it does not reach ∂X; we thus assume without loss of generality that this component vanishes. Hence for all s, k

$$Y_s R^k (u, D_t u) \in \mathcal{C}(I_0; \mathcal{E}_s)$$

for some interval I_0 containing 0 but not necessarily $\bar{x}$.

Note that with Θ_s from Definition 4.2, $[\Theta_s, R]$ is a properly supported pseudodifferential operator of order s on $\mathbb{R}$, hence

$$[\Theta_s, R] = Q\Theta_s + E \tag{13.4}$$

where $Q \in \Psi^0(\mathbb{R})$ and $E \in \Psi^{-\infty}(\mathbb{R})$. Since convolution with a properly supported time-translation invariant pseudodifferential operator of order 0 on $\mathbb{R}_t$ maps finite energy solutions to finite energy solutions, iterative application (13.4) shows that for all i and j

$$Y_i R^j (\Theta_{s-1} u, \Theta_{s-1} D_t u) \in \mathcal{C}(I; \mathcal{E}),$$

where $I \subset I_0$ is a time interval containing 0.

Now by Proposition 13.1, $Y_i R^j \Box^k \Theta_{s-1}(u, D_t u) \in \mathcal{E}$ for all $i, j, k \in \mathbb{N}$ and for all $t < T$. The symbols of Y_1, R, and $\Box$ are defining functions for the conormal bundle to the hypersurface $x = |t - \bar{x}|$. Hence $(\Theta_{s-1} u(t), \Theta_{s-1} D_t u(t))$ is conormal to $x = |t - \bar{x}|$ in (H^1, L^2). By Lemma 4.3, $(u(t), D_t u(t))$ is conormal to $x = |t - \bar{x}|$ in (H^s, H^{s-1}). The theorem then follows by restriction to fixed t. □

14. Proof of the division theorem

In this section, we prove Theorem 4.7. We begin with a preparation argument, allowing us to replace our hypothesis of regularity at $R^{\epsilon}_{\pm,I}$ with global regularity of the same order. As in the proof of Theorem 4.4, we may assume without loss of generality that u has edge wavefront set only in $\{\tau > 0\}$.

Let $i : 0 \times X^\circ \hookrightarrow \mathbb{R} \times X^\circ$ be the inclusion map, and i^* the induced contravariant map on cotangent bundles. Under the assumption of the nonfocusing condition (from Definition 4.5), there exists a microlocal neighborhood U of $i^* R^{\epsilon}_I(\bar{t})$ and $k \in \mathbb{N}$ such that $\mathrm{WF}^{r+l}(Y_{-k} u\restriction_{t=0}) \cap U = \emptyset$. We construct a microlocalizer in such a neighborhood which preserves tangential regularity.

Lemma 14.1. *Let $U \subset T^* X^\circ$ be an open conic neighborhood of $i^* R_I(\bar{t})$. There exist a smaller neighborhood V and an operator $H \in \Psi^0(X^\circ)$, with Schwartz kernel compactly supported in an arbitrarily small neighborhood of the diagonal, such that $\mathrm{WF}' H \subset U$, $\mathrm{WF}'(I - H) \subset V^{\complement}$, and $[Y_k, H] = 0$ for all $k \in \mathbb{Z}$.*

Proof. Let $\psi_\epsilon \in \mathcal{C}^\infty_c(\mathbb{R})$ equal 1 for $x < \epsilon/2$ and 0 for $x > \epsilon$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ vanish for $x < 0$ and equal 1 for $x > 1$. If ϵ is sufficiently small then

$$\operatorname{supp} \psi_\epsilon(|x - \bar{t}|)\psi_\epsilon(|\eta|^2/(\langle \xi \rangle^2 + |\eta|^2)) \subset U;$$

here η is dual to ∂_y and ξ to ∂_x in $T^* X^\circ$. Let

$$H = \psi_\epsilon(|x - \bar{t}|)\psi_\epsilon(\Delta_Y/(\langle D_x \rangle^2 + \Delta_Y))\psi_\epsilon(|x - \bar{t}|),$$

where the function of the operator $\Delta_Y/(\langle D_x \rangle^2 + \Delta_Y)$ may be constructed on the manifold $S^1 \times X$ and its Schwartz kernel cut off in x and glued into a product neighborhood of $\{x = \bar{t}\}$. Then H commutes with Δ_Y, and hence with Y_k for all k. $\square$

Since the kernel of H vanishes near the boundary it preserves the domains of all powers of Δ. Let $(v(t), Dv(t))$ be the solution to the Cauchy problem (4.1)–(4.3) with initial data $(Hu(0), HD_t u(0))$. By construction, $\mathrm{WF}(u-v) \cap R_I(\bar{t}) = \emptyset$, so by the diffractive theorem, $u - v \in x^{-\delta} H_b^\infty(I \times U)$ for some open interval $I \ni \bar{t}$, $U \supset \partial X$. Hence it suffices to prove the desired results for v instead of u.

Since

$$Y_{-k}(v(0), D_t v(0)) = HY_{-k}(u(0), D_t v(0)),$$

if the microlocalizer H is chosen concentrated sufficiently near the incoming set, the nonfocusing condition yields

$$Y_{-k}(v(0), D_t v(0)) \in \mathcal{E}_{r+l}.$$

Proposition 12.2 now yields

$$Y_{-k}\Theta_{s-1}U(t)(v(0), D_t v(0)) \in \mathcal{E},$$

hence

$$Y_{-k}U(t)(v(0), D_t v(0)) \in \mathcal{C}(\mathbb{R}; \mathcal{E}_s)$$

and the first part of the division theorem (that under the assumption of the nonfocusing condition) now follows by Proposition 5.5.

We now prove the second part of the theorem (under the assumption of the conormal nonfocusing condition). For simplicity, we now translate the time variable so that $\bar{t} = 0$.

We have

$$\begin{aligned} &RY_{-k}(v, D_t v)\restriction_{t=0} \\ &\qquad = (xD_x HY_{-k}u(0) + tY_{-k}D_t u(0),\ xD_x HY_{-k}D_t u(0) + t\Delta HY_{-k}u(0)). \end{aligned}$$

Since $[H, \Delta]$ and $[H, xD_x]$ are pseudodifferential operators of order 1 and 0, compactly supported in X°, this is in $\mathcal{E}_{r+l}$ provided the conormal nonfocusing condition holds.

Now we prove the second part of the division theorem for the special case $r+l = 1$. By Proposition 13.1

$$\Box RY_{-k}v = f \in \mathcal{C}(\mathbb{R}; \mathcal{D}_{-1}),$$

hence

$$(RY_{-k}v(t), D_t RY_{-k}v(t)) = U(t)[(RY_{-k}v, D_t RY_{-k}v)\restriction_{t=0}] + \int_0^t U(t-s)(0, f)ds.$$

The first term in the right is in $\mathcal{C}(\mathbb{R}; \mathcal{E}_1)$ by the conormality assumption. The second is in $\mathcal{C}^1(\mathbb{R}; \mathcal{E}_0)$, hence in $\mathcal{C}(\mathbb{R}; \mathcal{E}_1)$. Thus, $RY_{-k}v(t) \in \mathcal{C}(\mathbb{R}; \mathcal{D}_1)$, i.e.

$$D_t Y_{-k}v \in t^{-1}[\mathcal{C}(\mathbb{R}; \mathcal{D}_1) + xD_x\mathcal{C}(\mathbb{R}; \mathcal{D}_1)].$$

By Lemma 3.2, we obtain for all $\epsilon > 0$,

$$D_t Y_{-k}v \in t^{-1}\mathcal{C}(\mathbb{R}; x^{-n/2+1-\epsilon}L^2_{\mathrm{b}}(X)) \subset t^{-1/2-\epsilon}x^{-n/2+1-\epsilon}L^2_{\mathrm{b}}(I \times X)$$

for all $\epsilon > 0$. We also know a priori that

$$D_t Y_{-k}v \in \mathcal{C}(\mathbb{R}; \mathcal{D}_0) \subset t^{1/2}x^{-n/2}L^2_{\mathrm{b,loc}}(\mathbb{R} \times X).$$

Hence by interpolation,

$$D_t Y_{-k}v \in x^{-n/2+1/2-\epsilon}L^2_{\mathrm{b}}(I \times X) \text{ for all } \epsilon > 0,$$

proving the theorem in the special case $r+l=1$.

If $r+l<1$, we apply Lemma 4.3 and (13.4) to conclude

$$\Theta_{r+l-1} D_t Y_{-k} v \in x^{-n/2+1/2-\epsilon} L^2_{\mathrm{b}}(I \times X). \tag{14.1}$$

Thus

$$D_t Y_{-k} v \in x^{-n/2+1/2-\epsilon} H^{r+l-1}(I; L^2_{\mathrm{b}}(X)),$$

and since $r+l-1<0$, Proposition 5.4 yields

$$D_t v \in x^{-n/2+1/2-\epsilon} H_e^{r+l-k-1}(I \times X). \quad \square$$

15. Consequences of the division theorem

In this section, we record two consequences of the division theorem. First, we prove a slightly weakened version of Theorem 4.6, the geometric propagation theorem. Then we prove Theorem 4.9, establishing conormality of the diffracted front.

The weakened version of Theorem 4.6 we now establish is as follows

Partial Theorem 4.6. *If u is an admissible solution to the conic wave equation satisfying the nonfocusing condition in Definition 4.5 at $\{\bar{t}\} \times Y$ for some $r \in \mathbb{R}$ and $l \in (0, n/2)$ and if $p \in R^{\epsilon}_{\pm,O}(\bar{t}, Y)$, then*

$$\Gamma^{\epsilon}(p) \cap \mathrm{WF}^{r'}(u) = \emptyset \text{ for } r' \in (r, r+l-1/2) \Longrightarrow p \notin \mathrm{WF}^{r'-\delta}(u)\ \forall \delta > 0. \tag{15.1}$$

If u satisfies the conormal nonfocusing condition at $\{\bar{t}\} \times Y$, with $k=1$, then (15.1) *holds for all $r' \in (r, r+l)$.*

In other words, the theorem holds almost as originally stated, subject to limits on l and to the stronger conormal version of the nonfocusing condition, or yields one half derivative less, subject to the original hypothesis (again for a limited range of l).

Proof. Let

$$(v(t), D_t v(t)) = \Theta_{r+n/2-1}(u(t), D_t u(t)).$$

By (13.4), the new solution v satisfies the nonfocusing condition, with the same l. It also satisfies the conormal nonfocusing condition if u does, with $r+l \leq 1$. The division theorem now implies that there exists an interval $I \ni \bar{t}$, and neighborhood $\mathcal{O} \supset \partial X$ in X, such that

$$D_t v \in x^{-n+r'-r+1/2} H_{\mathrm{e}}^{-\infty}(I \times \mathcal{O}),$$

where we allow $r' \in (r, r+l-1/2)$ resp. $(r, r+l)$ in the case where u satisfies the nonfocusing condition resp. conormal nonfocusing condition. On the other hand, the incoming regularity assumption shows that $\Gamma_{\epsilon}(p) \cap \mathrm{WF}^{-n/2+r'-r}(D_t v) = \emptyset$. Theorem 8.3 now yields $p \notin \mathrm{WF}^{-n/2+r'-r-\delta}(D_t v)$, hence $p \notin \mathrm{WF}^{r'-\delta}(u)$. $\square$

We now prove Theorem 4.9 by including the effect of iterated regularity with respect to the vector field R defined in (1.17).

Proof of Theorem 4.9. We will show that away from the geometrically propagated rays, the solution maintains its regularity under application $Y_i R^j$ for all $i \in 2\mathbb{N}$, $j \in \mathbb{N}$. The proof thus amounts to a microlocalized version of the argument used previously to prove conservation of radial conormality.

Using Lemma 4.3, we may assume that $r=1$.

By Proposition 13.1, for all $i \in 2\mathbb{N}$, $j \in \mathbb{N}$, there exists $N = N(i,j) < 0$ such that

$$Y_N Y_i R^j(u, D_t u) \in \mathcal{C}(\mathbb{R}; \mathcal{E})$$

for all t small. Thus, $Y_i R^j D_t u \in x^{-\epsilon} H_e^{-\infty}(\mathbb{R} \times X)$, locally near $t = \bar{t}$.

We may apply the first two parts of Theorem 8.1 to the solution $D_t u$ and conclude that, for all m and $\epsilon > 0$, $\mathrm{WF}_{\mathrm{e}}^{m,-\epsilon}(D_t u)$ is disjoint from all incoming bicharacteristic segments, into and within the boundary, with endpoints p. Using the interpolation of wavefront sets in Proposition 5.3 we conclude that the same holds for $\mathrm{WF}_{\mathrm{e}}^{m,-\epsilon}(Y_i R^j D_t u)$ for all i, j and m. The third and fourth parts of Theorem 8.1. Then shows that $p \notin \mathrm{WF}_{\mathrm{e}}^{m,-\epsilon}(Y_i R^j u)$ provided $m < -\epsilon$. This establishes conormality. $\square$

16. Conormal Cauchy data and the proof of Theorem 4.6

The most significant class of examples to which the conormal nonfocusing condition applies is given by solutions u with Cauchy data conormal to a hypersurface W which is at most simply tangent to the radial surfaces.

Lemma 16.1. *Let W be a compact smooth hypersurface in X° such that, in a collar neighborhood of the boundary, dx restricted to W vanishes at only finitely many points $p_j = (x_j, y_j)$, $j = 1, \ldots, M$, at each of which W is simply tangent to $x = x_j$ and suppose $u_0 \in H^r(X)$ is conormal with respect to W then for any $N \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that*

$$Y_{-k} u_0 \in H^s_{loc}(X^\circ),\ V_1 \ldots V_p Y_{-k} u_0 \in H^s_{loc}(X^\circ),\ p \le N,\ s < r, \tag{16.1}$$

for all $V_i \in \mathcal{C}_c^\infty(X^\circ; TX)$ that are tangent to $\{x = x_j\}$ for all j.

Proof. Without loss of generality, we may localize and take $M = 1$. Since $Y_{-k} \in \Psi^{-k}(Y)$, for any K we may choose k so that $\kappa(Y_{-k})(y, y')$, the Schwartz kernel of Y_{-k}, is in $\mathcal{C}^K(Y \times Y)$; we choose $K > (n-1)/2 + N$.

The distribution u is of the form

$$u(x,y) = \int c(x,y,\xi) e^{i\phi(x,y)\xi}\, d\xi,$$

where $c \in S^{-r-1/2+\epsilon}(\mathbb{R} \times X^\circ)$ (for all $\epsilon > 0$) is a symbol with one phase variable and ϕ is a defining function for W. Hence

$$Y_{-k} u = \int c(x, y', \xi) \kappa(Y_{-k})(y, y') e^{i\phi(x,y')\xi}\, d\xi\, dy'.$$

If $\phi \neq 0$ or $d_y \phi \neq 0$ on $\operatorname{supp} c$, integration by parts using $\phi^{-1} D_\xi$ or $(\xi \phi'_{y'})^{-1} D_{y'}$ shows that $Y_{-k} u \in H^{K+r}(X^\circ)$. Hence may assume that, on the support of c, there is one point, which in local coordinates we may take to be $x = \bar{x}$, $y = 0$, at which $\phi = 0 = d_y \phi$. The hypothesis of simple tangency and the Morse Lemma allow us to arrange that, locally, $\phi(x,y) = (x - \bar{x}) - \sum_j \sigma_j y_j^2$ with $\sigma_j = \pm$. The method of stationary phase now yields

$$Y_{-k} u = \int \langle \xi \rangle^{-(n-1)/2} \tilde{c}(x,y,\xi) e^{i(x-\bar{x})\xi}\, d\xi$$

where $\tilde{c}$ satisfies any finite number of the symbol estimates of any order greater than $-r - 1/2$ provided that K is taken large enough. Thus $Y_{-k} u \in H_{\mathrm{loc}}^{r+(n-1)/2-\epsilon}(X^\circ)$ for any $\epsilon > 0$. Furthermore, as long as $\alpha + |\beta| \le N$, integration by parts shows that $((x - x_1) D_x)^\alpha D_y^\beta Y_{-k} u$ remains in this space. $\square$

Such conormality for the initial data leads to solutions satisfying the conormal nonfocusing condition.

Proposition 16.2. *Suppose that* $(u_0, u_1) \in \mathcal{E}_r(X)$ *vanishes near* ∂X *and* u_0 *and* u_1 *are conormal with respect to a hypersurface* $W \subset X^\circ$ *as in Lemma 16.1 then, for small* $t \in \mathbb{R}$, *the solution* u *to the Cauchy problem problem* (4.1)–(4.2) *satisfies the conormal nonfocusing condition with background regularity* r *and relative regularity* $l < (n-1)/2$ *to all orders* $k \in \mathbb{N}$.

Proof. By localization of the initial data we may again assume that there is just one point of tangency of W to one radial surface, $x = c$. Since the conormal nonfocusing condition is trivial microlocally away from the radial directions we may assume that u_0 and u_1 are supported close to the point of tangency. Initial data for the wave equation which is conormal with respect to a hypersurface, W, gives rise, by Huygen's principle to a solution which is conormal, for $t > 0$ small, to the union of the two characteristic hypersurfaces $W_\pm$ through W; this follows from the original construction of Lax or from [8]. Here $W_\pm$ are each tangent to the radial surface $\{x \pm t = c\}$. Thus W_- is outgoing: the bicharacteristics forming it do not hit the boundary small times. Now, we may apply Lemma 16.1 above, regarding t as a parameter, to conclude that the solution satisfies the conormal nonfocusing condition with background regularity r for any $l < (n-1)/2$ to all orders. □

Proof of Theorem 4.10. We can write the fundamental solution in the form

$$E_{\bar{m}} = \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}\delta_{\bar{m}} = U(t)(0, i\delta_{\bar{m}})$$

where $\bar{m} \in X^\circ$ lies sufficiently close to ∂X. Then for all $\epsilon > 0$, the initial data

$$(0, i\delta_{\bar{m}}) \in H^{-n/2+1-\epsilon}(X^\circ) \oplus H^{-n/2-\epsilon}(X^\circ)$$

hence the solution lies is $\mathcal{E}_{-n/2+1-\epsilon}$ for all $\epsilon > 0$.

For small $t > 0$ $(E_{\bar{m}}(t), D_t E_{\bar{m}}(t))$ are conormal with respect to the hypersurface W which is the geodesic sphere of radius t around $\bar{m}$. This is tangent only to the two radial surfaces $x = d(\bar{m}, \partial X) \pm t$ and the tangency is certainly simple. Hence Proposition 16.2 applies and shows that u satisfies the conormal nonfocusing condition. Now Theorem 4.4 and Partial Theorem 4.6 shows that for, $2d(\bar{m}, \partial X) > t > d(\bar{m}, \partial X)$, the inclusion (I.3) holds. On the diffracted front, but away from the direct front, application of Theorem 4.9 shows that $E_{\bar{m}}$ is conormal and of Sobolev regularity $-n/2 + 1 + (n-1)/2 - \delta = 1/2 - \delta$ for every $\delta > 0$. Iterated regularity with respect to this space then follows by interpolation. □

We now use this special case to prove Theorem 4.6.

Proof of Theorem 4.6. We begin by sharpening the regularity results of Theorem I.1 to include uniformity and regularity in the location of the pole.

Consider the Schwartz kernel of the fundamental solution, $E(t, x, y; x', y')$. For fixed $(x', y') \in (0, \epsilon) \times \partial X$ for $\epsilon > 0$ small enough and small t and x this is the distribution discussed in Theorem I.1. Moreover, by uniqueness of the solution to the Cauchy problem it depends continuously on (x', y'). It follows that the results described there hold uniformly in (x', y'). In particular near some fixed point $\bar{t} = \bar{x} + \bar{x}'$ with y, y' such that it is not on the direct front,

$$((t - x')\partial_t + x\partial_x)^k D_y^\alpha E \in L^2_{\text{loc}},\ \forall\ k, \alpha.$$

Here we have given up approximately half a derivative in the x, y variables, and continuity in x', y' to settle for iterative regularity with respect to L^2 in all variables. By the symmetry and t-translation invariance of the problem $E(t, x, y; x', y') = E(-t, x', y'; x, y)$ so it also follows that

$$((t-x)\partial_t + x'\partial_{x'})^k D_y^\alpha E \in L^2_{\mathrm{loc}}, \ \forall\ k, \alpha$$

in the same region. All these vector fields commute, so by interpolation we deduce regularity with respect to all the vector fields simultaneously. Now, together with the wave operator itself, in both sets of variables, these symbols of these operators define the (two components of) the conormal bundle to $\{t = x + x'\}$. Thus we deduce that E is, away from the direct front and for small $t, x, x' > 0$, conormal with respect to this hyperplane.

Although we have only shown iterative regularity in L^2, iterative regularity in $H^{\frac{1}{2}-\epsilon}_{\mathrm{loc}}(\mathbb{R} \times X \times X)$ in this set follows by interpolation. Hence, a fortiori, E is conormal with respect to the diffracted front with iterate regularity in $H^{\frac{1}{2}-\epsilon}(\mathbb{R} \times X \times X)$.

Now, consider an admissible solution to the wave equation satisfying the hypotheses of Theorem 4.6. Using Theorem 4.4 and a partition of unity, we may assume that the Cauchy data $(u(0), D_t u(0))$ is identically zero in a neighborhood of (the projection to X of) all points geometrically related to $p \in R^\epsilon_{\pm,O}(\bar{t})$ and that at non geometrically related points it is supported in a microlocal neighborhood of $R_{\pm,I}(\bar{t})$. The nonfocusing condition then implies that there exists $N \in \mathbb{N}$ such that locally near $x = \bar{t}$,

$$u(t, x, y) = \\ \int \frac{\partial E(t, x, y, x', y')}{\partial t} u_0(x', y') \frac{dx'}{x'}\, dy' + \int E(t, x, y, x', y') u_1(x', y') \frac{dx'}{x'}\, dy'$$

we obtain the desired boundedness, by regarding E and $\partial E/\partial t$ locally near the diffracted front as the kernels of Fourier integral operator of order ϵ resp. $1 + \epsilon$ on $\mathbb{R}_+$, smoothly parametrized by t, with values in $\Psi^{-\infty}(Y)$. $\square$

The explicit construction of the fundamental solution in the product case by Cheeger-Taylor [2, 3] can be used to show that Theorem 4.6 cannot be strengthened by omitting the hypothesis of the nonfocusing condition. In particular, in Example 4.1 of [2] the authors show that on $X = \mathbb{R}_+ \times S^1_2$, the cone over the circle of circumference 4π (with coordinate θ), the fundamental solution $E(t, x, \theta, x', \theta')$ has a jump discontinuity across the diffracted wavefront, where the value of the jump varies smoothly with θ, θ', at least in $|\theta - \theta'| < \pi$. Now let $\phi(\theta)$ be a smooth function supported in $|\theta| < \delta$. The function

$$v(t, x, \theta) = \int_{-2\pi}^{2\pi} E(t, x, \theta, x', \theta') \phi(\theta')\, d\theta'$$

is also an admissible solution to the wave equation, lying in $\mathcal{C}(\mathbb{R}; \mathcal{D}_{1/2-\epsilon})$ for all $\epsilon > 0$. For $t > x'$, it has geometrically propagated singularities contained in $(x = t - x', \theta \in (-\delta, \delta) \pm \pi)$. On the other hand, for all $\theta' \in \operatorname{supp} \phi$, $E(t, x, \theta, x', \theta')$ has a jump discontinuity along $x = t - x'$, $|\theta| < \pi - \delta$, hence v also has a jump discontinuity this surface, i.e. this diffracted singularity is in $H^{1/2-\epsilon}$ for all $\epsilon > 0$. Hence the diffracted singularity for v is no smoother than the geometrically propagated singularities.

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