GENERALIZED BLOW-UP OF CORNERS AND FIBER PRODUCTS

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ABSTRACT. Radial blow-up, including inhomogeneous versions, of boundary faces of a manifold (always with corners) is an important tool for resolving singularities, degeneracies and competing notions of homogeneity. These constructions are shown to be particular cases of *generalized boundary blow-up* in which a new manifold and blow-down map are constructed from, and conversely determine, combinatorial data at the boundary faces in the form of a refinement of the *basic monoidal complex* of the manifold. This data specifies which notion of homogeneity is realized at each of the boundary hypersurfaces in the blown-up space.

As an application of this theory, the existence of fiber products is examined for the natural smooth maps in this context, the b-maps. Transversality of the b-differentials is shown to ensure that the set-theoretic fiber product of two maps is a *binomial variety*. Properties of these (extrinsically defined) spaces, which generalize manifolds but have mild singularities at the boundary, are investigated and a condition on the basic monoidal complex is found under which the variety has a smooth structure. Applied to b-maps this additional condition with transversality leads to a universal fiber product in the context of manifolds with corners. Under the transversality condition alone the fiber product is resolvable to a smooth manifold by generalized blow-up and then has a weaker form of the universal mapping property requiring blow-up of the domain.

INTRODUCTION

Real blow-up of a submanifold introduces a new manifold (always here implicitly meaning 'with corners') in which the submanifold in question is replaced by one or more boundary hypersurfaces in a prescribed manner. Here we analyze the notion of the *generalized boundary blow-up* of a manifold with corners, Y, which is a new manifold X along with a smooth and proper 'blow-down' map $\beta : X \longrightarrow Y$. The latter is by definition a b-map, restricting to a diffeomorphism on the interiors, with bijective b-differential. It includes standard radial blow-up of boundary faces, iterated boundary blow-up, and inhomogeneous blow-up as special cases.

It is shown here that such blow-down maps are, up to diffeomorphism, characterized in an essentially algebraic manner by a 'monoidal complex', a consistent choice of some combinatorial data at each face of Y. One application of the constructive part of this result is to fiber products of maps. Working in the category of smooth manifolds and b-maps, it is shown that under the natural condition of

b-transversality the fiber product of two maps can be decomposed and smoothed by generalized blow-up and this resolved fiber product has a weakened form of the universal factoring property for fiber products in which generalized blow-up of the domain may be required.

Although this paper is concerned with geometric and algebraic questions, one of the contexts in which blow-up appears is analytic. Most analytic problems on a manifold with corners do not have solutions 'within the smooth category'. For instance, harmonic forms with absolute or relative boundary conditions are not typically smooth up to corners. Rather they lie in conormal spaces which themselves are only 'resolved' to polyhomogeneity by blow-up. Here polyhomogeneity is to be thought of as the natural extension of smoothness, in which $C^\infty(Y)$ is extended to include functions which have non-integral (including possibly negative) powers in their 'Taylor series' at the boundary. The selection of the *correct* resolution, in the sense of blow-up, of spaces and their (fiber) products, on which the kernels of operators are defined, is one of the prime motivations for the discussion here.

A manifold (meaning from now on a manifold with corners) Y is a topological manifold with boundary which is locally diffeomorphic to the model space $[0,\infty)^k \times \mathbb{R}^{n-k}$ with its sheaf of smooth functions. The set of (connected) boundary faces of Y is denoted $\mathcal{M}(Y)$, with $\mathcal{M}_l(Y)$ denoting those faces of codimension $l \in \{0, \ldots, k\}$. We require that boundary hypersurfaces of a manifold be embedded, which means that the functions vanishing on each such hypersurface form a principal ideal $\mathcal{I}_H \subset C^\infty(Y)$, $H \in \mathcal{M}_1(Y)$. A smooth map $f : X \longrightarrow Y$ is an (interior) boundary map or 'b-map' as defined in [Mel, Mel92] provided it pulls back each of these principal ideals to a product

$$(I.1) \qquad f^* \mathcal{I}_H = \prod_{G \in \mathcal{M}_1(Y)} \mathcal{I}_G^{\alpha(H,G)}, \quad \alpha(\cdot, \cdot) \in \mathbb{Z}_+$$

of similar ideals in $C^\infty(X)$

The boundary faces of a manifold have natural b-normal bundles,

$$^{\mathrm{b}}NF \subset {}^{\mathrm{b}}T_F X \longrightarrow F, \quad F \in \mathcal{M}_*(X),$$

with inclusions $^{\mathrm{b}}N_p G \subset {}^{\mathrm{b}}N_p F$ whenever $p \in F \subset G$. At each point these are spanned by the 'radial' vector fields with respect to the boundary face in question. As a result such a bundle has a canonical global frame, realized by $\{\rho_i \partial_{\rho_i}\}$ in terms of defining functions for the boundary face, which identifies the fibers $^{\mathrm{b}}N_p F$ with the fixed vector space denoted $^{\mathrm{b}}NF$. This carries a well-defined lattice structure, $\mathrm{span}_{\mathbb{Z}} \{\rho_i \partial_{\rho_i}\}$, and taking the inward-pointing lattice points defines a 'smooth', which is to say freely generated, monoid

$$\sigma_F = \mathrm{span}_{\mathbb{Z}_+} \{\rho_i \partial_{\rho_i}\}.$$

The collection of these along with the inclusions $i_{GF} : \sigma_G \hookrightarrow \sigma_F$ for $F \subseteq G$ constitutes what we call the 'basic monoidal complex' of X:

$$\mathcal{P}_X = \{(\sigma_F, \ i_{GF}) \ ; \ F \subseteq G \in \mathcal{M}(X)\}.$$

A b-map $f : X \longrightarrow Y$ has a tangent (or dually 'logarithmic') differential which acts on the b-tangent bundles: $^{\mathrm{b}}f_* : {}^{\mathrm{b}}T_p X \longrightarrow {}^{\mathrm{b}}T_{f(p)} Y$, and at a boundary face $F \in \mathcal{M}(X)$ restricts to a well-defined homomorphism (i.e. additive map) of the monoids

$$f_\natural : \sigma_F \longrightarrow \sigma_{f_\#(F)}$$

where $f_{\#} : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ maps each boundary face of X to the boundary face of largest codimension in Y which contains its image. Viewed as a matrix, the coefficients of $f_{\natural}$ are the relevant exponents $\alpha(\cdot, \cdot) \in \mathbb{Z}_{+}$ occuring in (I.1). The collection of these monoid homomorphisms patch together to form a morphism of the complexes

$$f_{\natural} : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$$

which is fundamental to our discussion.

In general, the monoidal complexes and their morphisms capture only the combinatorial relationships between boundary faces of X, those of Y, and the order of vanishing of boundary defining functions with respect to these faces. However, in the case of boundary blow-up, where X is the blow-up of a boundary face F in Y, this is enough to completely specify, up to diffeomorphism, the domain $X = [Y; F]$ and the map $\beta : X \longrightarrow Y$ in terms of the range space Y.

Indeed, in this case β satisfies additional properties, namely

(I.2) $\beta : X \setminus \partial X \longrightarrow Y \setminus \partial Y$ is a diffeomorphism, and

(I.3) $^{\mathrm{b}}\beta_{*} : {}^{\mathrm{b}}T_p X \longrightarrow {}^{\mathrm{b}}T_{\beta(p)} Y$ is an isomorphism for all $p \in X$,

and the morphism $\beta_{\natural} : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$ forms what is called below a 'smooth refinement' of $\mathcal{P}_Y$. Abstracting this, we call a smooth proper map satisfying (I.1), (I.2) and (I.3) a 'generalized blow-down map.'

Theorem A. *A generalized blow-down map $f : X \longrightarrow Y$ determines a smooth refinement $\mathcal{P}_X \longrightarrow \mathcal{P}_Y$ of the monoidal complex on Y, and conversely to any smooth refinement $\mathcal{R} \longrightarrow \mathcal{P}_Y$ there corresponds a manifold denoted $X = [Y; \mathcal{R}]$, unique up to diffeomorphism, with $\mathcal{P}_X = \mathcal{R}$ and a generalized blow-down map $f : X = [Y; \mathcal{R}] \longrightarrow Y$.*

The manifold $[Y; \mathcal{R}]$ is referred to as the 'generalized blow-up' of Y by the refinement $\mathcal{R}$, and the important question of the lifting of b-maps under generalized blow-ups of the domain and/or range is resolved by examining the corresponding lifting of the monoidal complex morphisms.

In particular this theory is applied to the problem of the smoothness (or lack thereof) of fiber products. Recall that the fiber product of two maps $f_i : X_i \longrightarrow Y$, $i = 1, 2$ is an object (in the appropriate category) X with maps $h_i : X \longrightarrow X_i$ such that $f_1 \circ h_1 = f_2 \circ h_2$, and which has the *universal property* that for any other maps $g_i : Z \longrightarrow X_i$ such that $g_2 \circ f_2 = g_1 \circ f_1$ there is a unique map $g : Z \longrightarrow X$ through which they factor:

In the category of sets there is a unique fiber product

(I.4)
$$X_1 \times_Y X_2 = \{(p_1, p_2) \; ; \; f_1(p_1) = f_2(p_2)\} \subset X_1 \times X_2,$$
$$h_i(p_1, p_2) = p_i,$$

however, in the setting of manifolds, (I.4) is not generally smooth and fiber products do not necessarily exist. In the category of manifolds without boundary, there is a well-known sufficient condition for existence, namely that f_1 and f_2 be *transversal*, meaning that if $f_1(p_1) = f_2(p_2) = q \in Y$ then

$$(I.5) \qquad (f_1)_*(T_{p_1}X_1) + (f_2)_*(T_{p_2}X_2) = T_qY.$$

In this case the set-theoretic fiber product (I.4) is a smooth manifold and the h_i are smooth maps.

The natural analog of (I.5) in the setting of manifolds (now with corners) is 'b-transversality,' namely the requirement that

$$(I.6) \qquad {}^{\mathrm{b}}(f_1)_*({}^{\mathrm{b}}T_{p_1}X_1) + {}^{\mathrm{b}}(f_2)_*({}^{\mathrm{b}}T_{p_2}X_2) = {}^{\mathrm{b}}T_qY.$$

Under this condition, (I.4) is not necessarily a manifold, but can be decomposed as a union of what are here called 'interior binomial subvarieties'. These are objects generalizing manifolds with corners, with smooth interiors but mild singularities at the boundary and boundary faces which are of the same type. They can be resolved, by generalized boundary blow-up, to manifolds with corners.

As for a manifold, there is a natural monoidal complex $\mathcal{P}_D$ defined over the boundary faces of an interior binomial subvariety $D \subset X$, the difference being that the monoids may not be freely generated, i.e. may not be smooth. If the monoidal complex is smooth, then D has a natural structure of a smooth manifold although this may not be induced from X, in that D may not be embedded. Even if the complex is not smooth, there is a smooth manifold $[D; \mathcal{R}] \longrightarrow D$ corresponding to every smooth refinement $\mathcal{R} \longrightarrow \mathcal{P}_D$.

In the case of fiber products, the monoids in the monoidal complex over $X_1 \times_Y X_2$ are of the form

$$(I.7) \qquad \sigma_{F_1} \times_{\sigma_G} \sigma_{F_2}, \quad F_i \in \mathcal{M}(X_i),$$

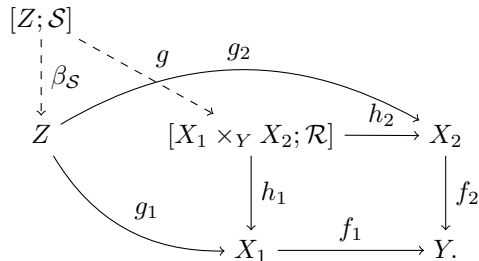where $G = (f_1)_\#(F_1) \cap (f_2)_\#(F_2) \in \mathcal{M}(Y)$.

Theorem B. *If $f_i : X_i \longrightarrow Y$ are b-maps which satisfy (I.6), and if each of the monoids (I.7) is freely generated, then there exists a universal fiber product of f_1 and f_2 in the context of manifolds with corners, which is given by a decomposition of (I.4) into a finite union of sets with natural smooth structures.*

This result extends a theorem of Joyce [Joy09] on fiber products of more restricted maps.

In the general case, when the monoids are not necessarily smooth, there is no such universal object and it is in general necessary to blow up to get smoothness and also to factor maps.

Theorem C. *For every smooth refinement $\mathcal{R}$ of the complex $\mathcal{P}_{X_1 \times_Y X_2}$, there is a manifold $[X_1 \times_Y X_2 ; \mathcal{R}]$ with b-maps to X_i commuting with the $f_i : X_i \longrightarrow Y$ and such that if $g_i : Z \longrightarrow X_i$, $i = 1, 2$ are b-maps commuting with the f_i then there exists a generalized blow-up $[Z; \mathcal{S}] \longrightarrow Z$ and a unique map $g : [Z; \mathcal{S}] \longrightarrow$*

$[X_1 \times_Y X_2 \; ; \; \mathcal{R}]$ *giving a commutative diagram*

$$
\begin{array}{ccccc}
[Z;\mathcal{S}] & & & & \\
\downarrow \beta_{\mathcal{S}} & \overset{g}{} \quad \overset{g_2}{} & & & \\
Z & & [X_1 \times_Y X_2; \mathcal{R}] \overset{h_2}{\longrightarrow} & X_2 & \\
& & \downarrow h_1 & \downarrow f_2 & \\
& \overset{g_1}{} & X_1 \overset{f_1}{\longrightarrow} & Y. &
\end{array}
$$

There is a clear relationship between the material discussed here and the theory of logarithmic structures in algebraic geometry. The monoidal complexes we describe are related to 'fans' as defined by Kato [Kat94], which essentially are to (toric) monoids what schemes are to rings. Our generalized blow-up is then related to a result of Kato, which produces a resolution of a logarithmic scheme with 'toric singularities' (also called a 'log-smooth scheme') from a subdivision of the fan associated to the scheme. Our manifolds with corners and binomial subvarieties can be viewed as log smooth spaces, albeit with a stronger differentiable structure than is usually considered in algebraic geometry.

In Section 1 we briefly recall the theory of manifolds with corners and b-maps, and establish some notation. Section 2 contains a discussion of monoids (here always meaning what are often called 'toric monoids') and their refinements, which is enough to establish the local version of generalized blow-up in Section 3. In Section 4 we discuss the theory of monoidal complexes and refinements and in Section 5 extend this to fiber products at this level. In Section 6 we complete one half of Theorem A, showing the existence of a generalized blow-up of X given a smooth refinement of $\mathcal{P}_X$, along with the lifting results for b-maps, before exhibiting ordinary boundary blow-up and its iterated version as special cases in Section 7. Section 8 completes Theorem A, giving the characterization of generalized blow-down maps. In Sections 9 and 10 we discuss the theory of interior binomial varieties and their resolution, finally applying this to fiber products in Section 11.

1. Manifolds with corners

In this section, we fix notation used for manifolds with corners and b-maps. For background, see [Mel] and [Mel92]. Set

$$
\mathbb{R}_+ = [0, \infty) \text{ and } \mathbb{Z}_+ = \{0, 1, 2, \ldots\}.
$$

The model manifold with corners is a product

$$
\mathbb{R}^{n,k} = \mathbb{R}_+^k \times \mathbb{R}^{n-k},
$$

for $k \in \{0, \ldots, n\}$, on which the smooth functions, forming the ring $C^\infty(\mathbb{R}^{n,k}) = C^\infty(\mathbb{R}^n)\big|_{\mathbb{R}^{n,k}}$, are taken to be those obtained by restriction from the smooth functions on $\mathbb{R}^n$.

In general, a *manifold with corners* X is a (paracompact, Hausdorff) topological manifold with boundary, X, with a ring of smooth functions, $C^\infty(X)$ with respect to which it is everywhere locally diffeomorphic to one of these model spaces. Thus, X has a covering by coordinate patches with homeomorphisms to open subsets of the local model spaces such that $u \in C^\infty(X)$ if and only if its image in each coordinate system is smooth on the model. The pull-backs of the functions on the model spaces give local coordinate systems.

Each point $p \in X$ necessarily has a well-defined (boundary) *codimension* given by the number of independent non-negative coordinate functions vanishing at p in such a coordinate system. A boundary face of codimension k is the closure of one of the components of the set of points of codimension k; the set of such boundary faces is denoted $\mathcal{M}_k(X)$. In particular $\mathcal{M}_1(X)$ consists of *boundary hypersurfaces* and $\mathcal{M}_0(X) = X$ (or the set of components of X if it is not connected). We set $\mathcal{M}(X) = \bigcup_k \mathcal{M}_k(X)$, this is an ordered set under inclusion.

As part of the definition of a manifold with corners, we require that all boundary hypersurfaces $H \in \mathcal{M}_1(X)$ be embedded; this is equivalent to insisting that the ideal of smooth functions vanishing on H, $\mathcal{I}_H \subset C^\infty(X)$, is principal. A non-negative generator $\rho_H \in C^\infty(X)$ of this ideal is a *defining function,* so $\mathcal{I}_H = \rho_H \cdot C^\infty(X)$. It follows that each boundary face of X is itself a manifold with corners.

The diffeomorphisms of X, homeomorphisms which map $C^\infty(X)$ to itself, must preserve the stratification by boundary codimension, and the infinitesimal diffeomorphisms correspond to the Lie subalgebra

$$(1.1) \qquad\qquad\qquad \mathcal{V}_{\mathrm{b}}(X) \subset \mathcal{V}(X)$$

of the usual algebra of real smooth vector fields, consisting of those vector fields which are tangent to each of the boundary faces of X. This subalgebra is the space of global smooth sections of the *b-tangent bundle,* ${}^{\mathrm{b}}TX \longrightarrow X$. Locally, whenever $(x, y) : U \longrightarrow \mathbb{R}_+^k \times \mathbb{R}^{n-k}$ are coordinates centered at p,

$$ {}^{\mathrm{b}}T_p X = \mathrm{span}_\mathbb{R} \left\{ x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k}, \partial_{y_{k+1}}, \ldots, \partial_{y_n} \right\}. $$

From (1.1), there is a natural evaluation map

$$(1.2) \qquad\qquad\qquad {}^{\mathrm{b}}TX \longrightarrow TX$$

which is an isomorphism over the interior of X but not over the boundary. Over points $p \in F \in \mathcal{M}_k(X)$, the kernel of (1.2) is the *b-normal bundle:*

$$ {}^{\mathrm{b}}NF \longrightarrow F, \quad {}^{\mathrm{b}}N_p F = \mathrm{span}_\mathbb{R} \left\{ x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k} \right\}, \text{ locally.} $$

In fact, up to reordering, the sections $x_i \partial_{x_i}$, $i = 1, \ldots, k$ of ${}^{\mathrm{b}}NF$ are well-defined independent of coordinates since any other set $(x_1', \ldots, x_k')$ must have the form (reordering if necessary) $x_i' = a_i(x, y) x_i$ with $a_i > 0$, so

$$ x_i' \partial_{x_i'} = x_i \partial_{x_i} + \mathcal{O}(x) \mathcal{V}_{\mathrm{b}}(X), $$

and hence $x_i' \partial_{x_i'} \equiv x_i \partial_{x_i}$ at F.

Thus, ${}^{\mathrm{b}}NF \longrightarrow F$ is canonically trivial, with a well-defined lattice

$$(1.3) \qquad\qquad ({}^{\mathrm{b}}NF)_\mathbb{Z} = \mathrm{span}_\mathbb{Z} \left\{ x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k} \right\}$$

and cone of *inward-pointing* vectors

$$^{\mathrm{b}}N_+F = \mathrm{span}_{\mathbb{R}_+}\left\{x_1\partial_{x_1},\ldots,x_k\partial_{x_k}\right\}.$$

These play a fundamental role in the discussion here.

A *b-map* $f : X \longrightarrow Y$ between manifolds with corners is a map which is smooth, meaning $f^*C^\infty(Y) \subset C^\infty(X)$, and is such that f pulls back each (principal) ideal $\mathcal{I}_H$ to either a product of powers of such ideals on X,

$$(1.4) \qquad f^*\mathcal{I}_H = \prod_{G\in\mathcal{M}_1(X)} \mathcal{I}_G^{\alpha(G,H)}, \ \alpha(G,H) \in \mathbb{Z}_+$$

to the zero ideal, $f^*\mathcal{I}_H = 0$ or to $C^\infty(X)$. If the second case does not occur we say f is an *interior* b-map. Otherwise $f : X \longrightarrow F$ is an interior b-map for some $F \in \mathcal{M}(Y)$. We shall mostly be concerned with interior b-maps in this paper.

For an interior b-map the differential $f_* : TX \longrightarrow TY$ extends by continuity from the interior to the *b-differential*

$$^{\mathrm{b}}f_* : {}^{\mathrm{b}}T_pX \longrightarrow {}^{\mathrm{b}}T_{f(p)}Y, \ \forall \ p \in X.$$

We denote by $f_\# : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ the map which assigns to each boundary face of X the boundary face of Y of largest codimension which contains it. Then the differential restricts to a map

$$(1.5) \qquad\qquad {}^{\mathrm{b}}f_* : {}^{\mathrm{b}}NF \longrightarrow {}^{\mathrm{b}}Nf_\#(F)$$

which is integral with respect to the lattices (1.3). Indeed, (1.5) is given by a matrix with integer entries coming from the exponents $\alpha(G,H)$ in (1.4), and $^{\mathrm{b}}f_*$ maps inward-pointing vectors to inward-pointing vectors since these entries are non-negative.

It is convenient to use multi-index notation for certain maps in coordinates. If $t = (t_1,\ldots,t_n)$ and $x = (x_1,\ldots,x_k)$ are local coordinates on spaces U and V respectively, and $\mu \in \mathrm{Mat}(n\times k,\mathbb{R})$ is a matrix, the map $(t_1,\ldots,t_n) \longmapsto (x_1,\ldots,x_n)$ where

$$x_i = \prod_j t_j^{\mu_{ji}} \text{ is denoted } t \longmapsto t^\mu = x.$$

With this convention on the order of the indices, $(t^\mu)^\nu = t^{\mu\nu}$.

2. Monoids

In general terms a monoid is a set which is closed under an associative binary operation, so in essence a group without inverses. Here we restrict attention to monoids of the special type usually known as *toric monoids*; but for brevity we shall simply call them monoids. These are distinguished among monoids in general by being commutative, torsion free, integral and saturated, though we rely on the following more explicit definition.

Definition 2.1. A monoid is a set σ, closed under a binary operation of addition, which can be expressed as the intersection

$$\sigma = N_\sigma \cap C$$

of an integral lattice N_σ and a proper convex polyhedral cone C in the vector space $N_\sigma^{\mathbb{R}} = N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$ which is integral with respect to N_σ. The cone, which is determined

from the monoid by $C = \mathrm{span}_{\mathbb{R}_+} \sigma \subset N_\sigma^{\mathbb{R}}$, will be called the *support* of σ and denoted by
$$\mathrm{supp}(\sigma) = C \subset N_\sigma^{\mathbb{R}}.$$
The minimal generators $\{v_1, \ldots, v_k\}$ of the cone $\mathrm{supp}(\sigma)$ (which do not necessarily generate σ as a monoid) with respect to N_σ are called the *extremals* of σ; by assumption no v_i is a combination of the others with non-negative coefficients but every point of $\mathrm{supp}(\sigma)$ is a non-negative linear combination of the v_i.

It follows from the definition that σ is *sharp*, meaning it has no invertible elements besides 0; *saturated*, meaning that if $nv \in \sigma$ for some $n \in \mathbb{N}$ and $v \in N_\sigma$, then $v \in \sigma$; *torsion free*, meaning that $nv = 0 \in \sigma$ if and only if $v = 0$; and *integral*, meaning that cancellation holds: if $x + y = z + y \in \sigma$ then $x = z \in \sigma$. In fact these properties characterize the monoids of Definition 2.1 (see [Ogu06]).

The *dimension* $\dim(\sigma)$ is the dimension of $N_\sigma^{\mathbb{R}}$; equivalently, $\dim(\sigma)$ is the maximum number of linearly independent extremals. A monoid σ is said to be *simplicial* if the extremals $\{v_1, \ldots, v_n\}$ are independent, in which case $\dim(\sigma) = n$ and $\mathrm{supp}(\sigma)$ is a cone over the $n-1$ simplex defined by $\{v_1, \ldots, v_n\}$. A simplicial monoid is further said to be *smooth* if it is generated as a monoid by its extremals. A smooth monoid is therefore freely generated and isomorphic to $\mathbb{Z}_+^n$. We will use the notation $\sigma = \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle$ to denote a smooth monoid freely generated by independent vectors $v_1, \ldots, v_n$.

One monoid σ' in N is a *submonoid* of another σ if $\sigma' \subset \sigma$ so the generators of σ' are non-negative integral combinations of the generators of σ.

A monoid τ is a *face* of σ, written $\tau \leq \sigma$, if τ is a submonoid such that whenever $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$. Equivalently, τ is the largest submonoid contained in a face (in the sense of cones) of $\mathrm{supp}(\sigma)$, which is in turn equivalent to the existence of a functional $u \in N_\sigma^*$ such that
$$\langle u, v \rangle = 0 \quad \text{for } v \in \tau,$$
$$\langle u, v \rangle > 0 \quad \text{for } v \in \sigma \setminus \tau.$$

In particular, the trivial monoid $\{0\}$ with no generators is a face of every monoid.

Example 2.2 (A non-simplicial monoid). In $\mathbb{R}^4$ with its standard basis, consider the monoid σ generated by the vectors (which are also its extremals)
$$v_1 = (1, 0, 1, 0) \qquad\qquad v_2 = (1, 0, 0, 1)$$
$$v_3 = (0, 1, 1, 0) \qquad\qquad v_4 = (0, 1, 0, 1).$$

It is 3-dimensional (since any three of the generators can be seen to be independent), with four 2-dimensional faces generated by the pairs $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, and $\{v_1, v_4\}$, four 1-dimensional faces generated by each of the v_i, and the 0-dimensional face $\{0\}$. It is not simplicial.

Example 2.3 (A non-smooth simplicial monoid). In $\mathbb{R}^2$, let σ be the monoid generated by $(2, 0)$, $(1, 1)$ and $(0, 2)$. (See Figure 1.) It is simplicial, with the support equal to the positive quadrant, and its extremals are $(2, 0)$ and $(0, 2)$. It is not smooth since it contains elements $((1, 1)$, for example) which are not positive integral combinations of its extremals.

A monoid homomorphism $\phi : \sigma' \longrightarrow \sigma$ is just an addition-preserving map. We will also denote by ϕ the induced linear maps $\phi : N_{\sigma'} \longrightarrow N_\sigma$, $\phi : N_{\sigma'}^{\mathbb{R}} \longrightarrow N_\sigma^{\mathbb{R}}$
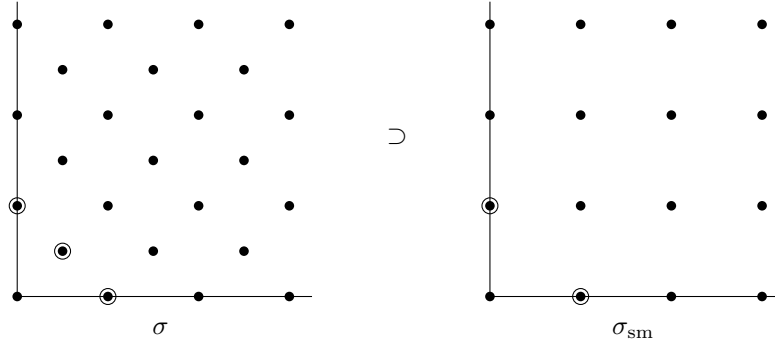
FIGURE 1. A non-smooth, simplicial monoid σ, along with its smoothing σ_{sm}. Generators of each monoid are highlighted.

and $\phi : \mathrm{supp}(\sigma') \longrightarrow \mathrm{supp}(\sigma)$. Note that unless ϕ is injective, the image of σ' in σ need not be a monoid in the sense we have defined, since it may not be saturated (consider the map $\mathbb{Z}_+^2 \longrightarrow \mathbb{Z}_+$ sending $(1,0)$ to 2 and $(0,1)$ to 3, for instance).

Definition 2.4. Given a monoid σ, a *refinement* of σ is a collection R of submonoids of σ, such that

(i) if $\sigma' \in R$ and $\tau' \leq \sigma'$, then $\tau' \in R$,
(ii) for any two monoids σ_1', $\sigma_2' \in R$, $\sigma_1' \cap \sigma_2'$ must be a face of both σ_1' and σ_2', and
(iii) $\bigcup_i \mathrm{supp}(\sigma_i) = \mathrm{supp}(\sigma)$

We say R is a *simplicial* (resp. *smooth*) refinement if each $\sigma_i \in R$ is simplicial (resp. smooth).

These conditions imply that the cones $\mathrm{supp}(\sigma_i)$ form a *fan*, in the classical sense used in the theory of toric varieties [Ful93], whose support is equal to that of σ.

The set of faces of σ (including σ itself) form the *trivial refinement* of σ.

Example 2.5. A simplicial monoid, σ, with extremals $\{v_1, \ldots, v_n\}$, has a canonical smooth refinement consisting of the faces of the smooth monoid

$$\sigma_{\mathrm{sm}} := \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle.$$

These are of the form $\mathbb{Z}_+ \langle w_1, \ldots, w_k \rangle$ for all subsets $\{w_1, \ldots, w_k\} \subseteq \{v_1, \ldots, v_n\}$; this refinement will be called the *smoothing* of σ. See Figure 1.

This example illustrates the point that while $\mathrm{supp}(\sigma)$ is covered by supports of the $\tau \in R$, σ need not be covered as a monoid by the $\tau \in R$. A simpler example of this phenomenon is the refinement of $\mathbb{Z}_+$ given by the submonoid $\mathbb{Z}_+(2) = \{2, 4, \ldots\}$.

A refinement of σ is a special case of a *monoidal complex*, discussed in Section 4.

Lemma 2.6. *If R is a refinement of σ and $\tau \leq \sigma$ is a face then*

$$(2.1) \qquad R(\tau) = \{\sigma' \in R \ ; \ \sigma' \subset \tau\}$$

is a refinement of τ.

Proof. Clearly $R(\tau)$ satisfies the first two properties of a refinement, since if $\sigma' \in R(\tau)$ and $\tau' \leq \sigma'$, then $\tau' \subset \tau$ and hence $\tau' \in R(\tau)$. For the third property, choose any $v \in \mathrm{supp}(\tau)$. Since also $v \in \mathrm{supp}(\sigma)$, there must be a $\sigma' \in R$ for which $v \in \mathrm{supp}(\sigma')$. By the support condition, $\mathrm{supp}(\sigma') \cap \mathrm{supp}(\tau)$ is a face of $\mathrm{supp}(\sigma')$, which corresponds to a monoid $\tau' \leq \sigma'$ such that $\mathrm{supp}(\tau') \subset \mathrm{supp}(\tau)$. □

A particularly important example of a refinement is the operation of *star subdivision*, which is well-known in toric geometry [Ful93] and its generalizations.

Proposition 2.7. *If σ is a monoid and $0 \neq v \in \sigma$ the collection, $S(\sigma, v)$, of monoids consisting of*

(i) *all faces $\tau \leq \sigma$ such that $v \notin \tau$, and*
(ii) *monoids $\tau + \mathbb{Z}_+ v := \mathrm{span}_{\mathbb{Z}_+} \tau \cup \{v\}$ where $\tau \leq \sigma$ and $v \notin \tau$*

is a refinement of σ. If all proper faces of σ are smooth, then $S(\sigma, v)$ is a smooth refinement.

Proof. For two monoids τ_1, τ_2 of the first type, the intersection $\tau_1 \cap \tau_2$ is a face of σ not containing v, hence a monoid of the first type. For two monoids $\tau_1 + \mathbb{Z}_+ v$ and $\tau_2 + \mathbb{Z}_+ v$ of the second type, the intersection is a monoid $(\tau_1 \cap \tau_2) + \mathbb{Z}_+ v$ of the second type, and for one of each type, the intersection $(\tau_1 + \mathbb{Z}_+ v) \cap \tau_2 = \tau_1 \cap \tau_2$ is of the first type. The support condition follows from the fact that any $w \in \mathrm{supp}(\sigma)$ lies in some cone of the form $\mathrm{supp}(\tau + \mathbb{Z}_+ v)$, with $\tau \leq \sigma$.

If $\tau = \mathbb{Z}_+ \langle v_1, \ldots, v_k \rangle$ is smooth and $v \notin \tau$, then $\tau + \mathbb{Z}_+ v$ has independent generators $\{v_1, \ldots, v_k, v\}$, so $\tau + \mathbb{Z}_+ v = \mathbb{Z}_+ \langle v_1, \ldots, v_k, v \rangle$ is smooth. Since σ itself is not in $S(\sigma, v)$ smoothness follows from the smoothness of all the proper faces of σ. □

In the context of generalized blow-up, star subdivision is the operation which realizes the ordinary blow-up of a boundary face of a manifold. There is a similar construction, "planar refinement," which we discuss next although it is not used until Section 10, in which v is replaced by the intersection of σ with a subspace.

First, as a matter of notation, suppose $\tau_i \subset \sigma$, $i = 1, 2$ are *full submonoids* in the sense that each

$$(2.2) \qquad \tau_i = \sigma \cap \mathrm{supp}(\tau_i).$$

Then we define the *join* of τ_1 and τ_2 to be the full submonoid

$$(2.3) \qquad \tau_1 * \tau_2 := \sigma \cap \big(\mathrm{supp}(\tau_1) + \mathrm{supp}(\tau_2)\big)$$

where $\mathrm{supp}(\tau_1) + \mathrm{supp}(\tau_2) = \mathrm{span}_{\mathbb{R}_+} (\tau_1 \cup \tau_2)$ is the convex hull of the supports of the τ_i.

Let σ be a smooth monoid, and $\mu \subset \sigma$ be a submonoid which is obtained by intersection with an integral subspace of $N_\sigma^{\mathbb{R}}$:

$$(2.4) \qquad \mu = \sigma \cap M, \quad M \subset N_\sigma^{\mathbb{R}} \text{ integral.}$$

Note that μ need not be simplicial, though it is a full submonoid in the sense of (2.2). Let Λ consist of the maximal faces of σ meeting μ trivially:

$$(2.5) \qquad \Lambda = \{\tau \leq \sigma \; ; \; \tau \cap \mu = \{0\}, \text{ but for all } \tau' > \tau, \; \tau' \cap \mu \neq \{0\}\}.$$

Lemma 2.8. *If σ is a smooth monoid, M is an integral subspace and μ and Λ are defined by (2.4) and (2.5) then for any $\tau_1 \neq \tau_2 \in \Lambda$, there exists a functional*

$u \in N_\sigma^*$ such that $\langle u, \tau_1 \rangle \geq 0$, $\langle u, \tau_2 \rangle \leq 0$, $\langle u, \mu \rangle = 0$ and if v is a generator of either τ_1 or τ_2 then $v \in \tau_1 \cap \tau_2$ if and only if $\langle u, v \rangle = 0$.

Proof. If M has full dimension the result is trivial, so we may assume that M is a proper subspace. If M is a hyperplane, then u is a defining functional, $u^\perp = M$, and is determined up to non-vanishing constant. Since $\tau_i \cap M = \{0\}$, u is definite on each and cannot have the same sign on both or they would be equal by maximality. Thus, either $\langle u, \tau_1 \rangle > 0$, $\langle u, \tau_2 \rangle < 0$ or with signs reversed. In this case their intersection is trivial and the result clearly holds. Conversely if $\tau_1 \cap \tau_2 \neq \{0\}$, then $\mathrm{codim}(M) > 1$.

In the general case we proceed by induction on $\mathrm{codim}(M)$. Choose a generator $w \in \tau_1 \cap \tau_2$, and consider the smooth monoid $\sigma' < \sigma$ generated by all the generators of σ except w, with $\tau_i' := \tau_i \cap \sigma'$. Consider the subspace $M' = (M + \mathbb{R} w) \cap N_{\sigma'} \subset N_{\sigma'}$ and set $\mu' = M' \cap \sigma'$. Since $w \notin M$, M' has the same dimension as M, hence the codimension of M' in $N_{\sigma'}$ is one less than the codimension of M in N_σ.

Now, the τ_i' are maximal faces of σ' not meeting μ'. To see this first note that $\tau_i' \cap M' = \{0\}$ since the generators of μ can be written $e_l = e_l' + c_l w$ where the e_l' are generators for μ' and $c_l \geq 0$. Thus, if there was a point $p \in \tau_i' \cap \mu'$, necessarily of the form $p = \sum_l d_l e_l'$ with the $d_l \geq 0$, then there would be a point $p + \sum_l d_l c_l w \in \tau_i \cap \mu$.

Similarly, the τ_i' are maximal since if $\tau' > \tau_i'$ is a face of σ' which meets μ' trivially then $\tau' + \mathbb{Z}_+ w > \tau_i$ meets μ trivially, contradicting the maximality of τ_i.

Thus by induction on the codimension of M' in $N_{\sigma'}$ there exists a functional $u' \in N_{\sigma'}^*$ with the desired separating property for the τ_i' and M'. Extending u' to u by requiring $u(w) = 0$ gives a functional in N_σ^* with the desired properties and completes the inductive step. $\qquad\square$

Now set $\mu * \Lambda = \{\mu * \tau \ ; \ \tau \in \Lambda\}$, defined by (2.3), and consider

$$(2.6) \qquad\qquad S(\sigma, \mu) = \{\gamma \leq \sigma' \ ; \ \sigma' \in \mu * \Lambda\}.$$

Proposition 2.9 (Planar refinement)**.** *If σ is smooth monoid and $\mu = \sigma \cap M$ is the intersection with an integral subspace $M \subset N_\sigma^{\mathbb{R}}$ then (2.6) gives a refinement of σ containing μ; this operation commutes with faces in the sense that for any face $\tau \leq \sigma$, $S(\tau, \tau \cap \mu) = \big(S(\sigma, \mu)\big)(\tau)$ with notation as in (2.1).*

Proof. If $M = N_\sigma^{\mathbb{R}}$, then $\mu = \sigma$, $\Lambda = \{0\}$ and $\mathcal{S}(\sigma, \mu) = \mu * \{0\}$ consists of σ and its faces, which is the trivial refinement of σ.

First we show that the supports of $S(\sigma, \mu)$ cover $\mathrm{supp}(\sigma)$, in fact

$$(2.7) \qquad\qquad \sigma = \bigcup_{\tau \in \Lambda} \mu * \tau,$$

since all the monoids are full as above. Certainly $\mu \subset \tau * \mu$, so consider $v \in \sigma \setminus \mu$ and choose any $m_1 \in \mu$. The ray from m_1 through v meets some face of σ, call it τ_1, so v is a positive combination of $m_1 \in \mu$ and $t_1 \in \tau_1$. If $\tau_1 \cap \mu = \{0\}$ then $\tau_1 \subset \tau \in \Lambda$ and we are finished. On the other hand, if τ_1 meets μ, then t_1 is a positive combination of some $m_2 \in \mu$ and $t_2 \in \tau_2$ with τ_2 a proper face of τ_1. Continuing this way eventually shows that $v \in \mu * \tau_k$ for some τ_k which does not meet μ, since the dimension decreases and $v \notin \mu$. Thus $v \in \mu * \tau$ for some $\tau \in \Lambda$ with $\tau \geq \tau_k$ and (2.7) follows.

Next consider the intersection of two elements in the union (2.7). We will show that

$$(2.8) \qquad (\mu * \tau_1) \cap (\mu * \tau_2) = \mu * (\tau_1 \cap \tau_2), \ \tau_i \in \Lambda.$$

Certainly the right side is contained in the left. Lemma 2.8 applies and gives a functional u. If v is in the intersection it follows that $u(v) \geq 0$ and $u(v) \leq 0$ so $u(v) = 0$ and from the last property of u this implies $v \in \mu * (\tau_1 \cap \tau_2)$. Since u is a supporting hyperplane for both $\mu * \tau_i$, it follows that $\mu * (\tau_1 \cap \tau_2)$ is a boundary face of each of these full submonoids. This in turn implies that the intersection of any two elements of $S(\sigma, \mu)$ is a common boundary face of both and hence an element of $S(\sigma, \mu)$. Indeed, such an intersection must be contained in two $\mu * \tau_i$ with $\tau_i \in \Lambda$. If they are the same, the conclusion is immediate, and if not then they are both contained in the common boundary face (2.8) and again the result follows.

That μ is a boundary face of each $\mu * \tau$, $\tau \in \Lambda$ follows from existence of a functional $u \in N_\sigma^*$ such that $u > 0$ on the generators of τ and $u(M) = 0$. $\qquad \square$

For any $v \in \sigma$, if $\mathbb{Z}_+ v$ is a full submonoid, then $S(\sigma, \mathbb{Z}_+ v)$ is just the ordinary star subdivision of σ along v. Note however, that in the case of star subdivision we can relax the condition that σ be smooth, as well as the condition that $\mathbb{Z}_+ v$ be full.

3. Generalized blow-up of $\mathbb{R}_+^n$

In preparation for the global case of generalized boundary blow-up of a manifold with corners treated in Section 6, we first discuss the 'local case' of generalized blow-up of the model space $\mathbb{R}_+^n$. Associated with this basic space is the 'basic monoid' consisting of the integral points with non-negative entries in ${}^b N \{0\}$, the b-normal to the maximum codimension boundary face $\{0\}$. We show how to construct a blown-up space mapping surjectively onto $\mathbb{R}_+^n$ corresponding to any smooth refinement of this basic monoid. The functoriality of this operation with respect to diffeomorphisms and b-maps is then considered.

Consider $\mathbb{R}_+^n$ with coordinates $(x_1, \ldots, x_n)$ and the vector space

$$ {}^b N \{0\} = \mathrm{span}_{\mathbb{R}} \{x_1 \partial_{x_1}, \ldots, x_n \partial_{x_n}\}. $$

As pointed out in Section 1, the vectors $(x_1 \partial_{x_1}, \ldots, x_n \partial_{x_n})$ are invariantly defined up to reordering with respect to diffeomorphisms (indeed, any diffeomorphism of $\mathbb{R}_+^n$ to itself must take 0 to 0). The *basic monoid of* $\mathbb{R}_+^n$ consists of the inward pointing lattice points

$$ \sigma_{\mathbb{R}_+^n} = \mathbb{Z}_+ \langle x_1 \partial_{x_1}, \ldots, x_n \partial_{x_n} \rangle \ \text{in} \ {}^b N \{0\}; $$

it is clearly smooth. To any smooth refinement R of $\sigma_{\mathbb{R}_+^n}$ we proceed to associate a *generalized blow-up* of $\mathbb{R}_+^n$, which will be denoted, with its blow-down map

$$ [\mathbb{R}_+^n; R] \xrightarrow{\ \beta\ } \mathbb{R}_+^n. $$

The blow-down map $\beta : [\mathbb{R}_+^n; R] \longrightarrow \mathbb{R}_+^n$ is an interior b-map from this new manifold with corners, which is proper, surjective and restricts to a diffeomorphism on the interior of its domain to the interior of $\mathbb{R}_+^n$. We construct $[\mathbb{R}_+^n; R]$ and β from explicit coordinate patches and transition diffeomorphisms.

For brevity we will say that a monoid $\sigma \in R$ of maximal dimension n is *maximal*. To any such maximal monoid $\sigma = \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle$, we associate a copy of the model space $U_\sigma = \mathbb{R}_+^n$ with coordinates $t = (t_1, \ldots, t_n)$, where the order of the coordinates

is associated to the order of the vectors. Let $\nu \in \mathrm{GL}(n, \mathbb{Q})$ be the matrix whose rows are the coordinates of the vectors v_i, so that ν has entries ν_{ij}, where

$$v_i = \sum_{j=1}^{n} \nu_{ij} x_j \partial_{x_j}.$$

Since R is a refinement of $\sigma_{\mathbb{R}^n_+}$, ν has non-negative integral entries:

$$\nu \in \mathrm{Mat}(n \times n, \mathbb{Z}_+) \cap \mathrm{GL}(n, \mathbb{Q}).$$

Then consider the smooth map

$$\beta_\sigma : U_\sigma \ni t \longmapsto t^\nu = x \in \mathbb{R}^n_+$$

where the image space is the fixed model manifold, and we use the notational convention established in Section 1. This is an interior b-map (which will be the local coordinate version of the blow-down map) under which

$${}^{\mathrm{b}}(\beta_\sigma)_*(t_i \partial_{t_i}) = \sum_j \nu_{ij} x_j \partial_{x_j} = v_i.$$

To each face $\tau \leq \sigma$ given by a collection of generators, $\tau = \mathbb{Z}_+ \langle v_i \rangle_{i \in I}$, $I \subset \{1, \ldots, n\}$ we associate the (relatively) open set

(3.1) $$U_{\sigma, \tau} = \{(t_1, \ldots, t_n) \in U_\sigma \; ; \; t_j \neq 0 \text{ if } j \notin I\} \subset U_\sigma.$$

Thus, $U_{\sigma, \tau} \cong \mathbb{R}^k_+ \times (0, \infty)^{n-k} \subset \mathbb{R}^n_+$, where the coordinates allowed to take the value 0 are the $t_{i_1}, \ldots, t_{i_k}$, corresponding to those generators of σ which are also generators of τ.

Proposition 3.1 (Generalized blow-up of $\mathbb{R}^n_+$). *For any two maximal monoids σ_1 and σ_2 in a smooth refinement R of $\sigma_{\mathbb{R}^n_+}$, with common face $\tau = \sigma_1 \cap \sigma_2$, the diffeomorphism of the interiors of the U_{σ_i} given by $\beta_{\sigma_2}^{-1} \beta_{\sigma_1}$ extends by continuity to a diffeomorphism $\chi_{12} : U_{\sigma_1, \tau} \longrightarrow U_{\sigma_2, \tau}$ and the quotient space*

(3.2) $$[\mathbb{R}^n_+; R] = \left(\bigsqcup U_\sigma \right) / \{\chi_*\}$$

is a manifold with corners, with coordinate charts the U_σ; it is equipped with a well-defined blow-down map $\beta : [\mathbb{R}^n_+; R] \longrightarrow \mathbb{R}^n_+$ given by β_σ on each U_σ.

Proof. The space U_σ, with its local blow-down map β_σ, really depends on the choice of the order of the generators implicit in the definition of the map. However, the coordinates t_j are each naturally associated to the corresponding generator and change of order of the generators corresponds to the same reordering of the coordinates in U_σ. Thus we can freely reorder the generators.

For two maximal monoids, σ_1 and σ_2, $\tau = \sigma_1 \cap \sigma_2$ is generated by the common generators of σ_1 and σ_2. For convenience of notation we can assume that these are the first k generators:

$$\tau = \mathbb{Z}_+ \langle v_1, \ldots, v_k \rangle = \sigma_1 \cap \sigma_2,$$

$$\sigma_1 = \mathbb{Z}_+ \langle v_1, \ldots, v_k, v_{k+1}, \ldots, v_n \rangle, \; \sigma_2 = \mathbb{Z}_+ \langle v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_n \rangle.$$

Since the σ_i are smooth, their generators form bases so for $i > k$,

$$v_i = \sum_{j=1}^{k} b_{ij} v_j + \sum_{j=k+1}^{n} c_{ij} v'_j, \; b_{ij}, \; c_{ij} \in \mathbb{Q},$$

where the coefficients are rational since both bases are integral with respect to $\{x_1 \partial_{x_1}, \ldots, x_n \partial_{x_n}\}$.

Now, the map $\chi_{12} = \beta_{\sigma_2}^{-1} \beta_{\sigma_1}$ between the interiors of the two cones is given by $t \longmapsto t^\mu = t'$, where

$$\mu = \begin{pmatrix} \mathrm{Id} & 0 \\ b & c \end{pmatrix} \quad [b]_{ij} = b_{ij}, \ [c]_{ij} = c_{ij}.$$

In other words,

$$\chi_{12}(t_1, \ldots, t_n) = (t'_1, \ldots, t'_n) \in \mathbb{R}^n_+ = U_{\sigma_2}, \text{ where}$$
$$t'_i = t_i \prod_{j>k} t_j^{b_{ji}}, \ i \le k, \quad t'_i = \prod_{j>k} t_j^{c_{ji}}, \ i > k.$$

Since, in $U_{\sigma_1,\tau}$, $t_j > 0$ for $j > k$, χ_{12} extends smoothly from the interior to $U_{\sigma_1,\tau}$. Since the inverse, $\chi_{21}(t') = (t')^{\nu^{-1}}$, has a similar form, χ_{12} is a diffeomorphism onto $U_{\sigma_2,\tau}$.

Next we show that the quotient (3.2) is Hausdorff. Consider points $p_i \in U_{\sigma_i}$, $i = 1, 2$ which are not identified by χ_{12}. If $p_1 \in U_{\sigma_1,\tau}$ then its image lies in $U_{\sigma_2,\tau}$ and a sufficiently small neighborhood of p_2 does not contain $\chi_{12}(p_1)$ in its closure, so the points have neighborhoods in the U_{σ_i} with no χ_{12} related points. Using the inverse the same is true if $p_2 \in U_{\sigma_2,\tau}$ so we may assume both are in the complements of the respective $U_{\sigma_i,\tau}$.

Since $\sigma_1 \cap \sigma_2 = \tau$, there is a separating hyperplane, $u^\perp$ for $u \in {}^{\mathrm{b}}N^* \{0\}$ so that

$$\langle u, v_i \rangle = 0, \ i = 1, \ldots, k$$
$$\langle u, v_i \rangle > 0, \ \langle u, v'_i \rangle < 0, \ i = k+1, \ldots, n.$$

Let w and w' be the coordinates of u with respect to the bases dual to the generators of σ_1 and σ_2, respectively. Thus $w_i = w'_i = \langle u, v_i \rangle = 0$ for $i = 1, \ldots, k$ and $w_i = \langle u, v_i \rangle > 0$, $w'_i = \langle u, v'_i \rangle < 0$ for $i > k$.

Then $\chi_{12}^*(t')^{w'} = t^w$, and p_1 and p_2 can be separated by the explicit open sets $\{t^w < \epsilon\} \subset U_{\sigma_1}$ and $\{(t')^{-w'}\} \subset U_{\sigma_2}$ since

$$\chi_{12}\left(\{t^w < \epsilon\} \cap U_{\sigma_1,\tau}\right) = \left\{(t')^{w'} < \epsilon\right\} \cap U_{\sigma_2,\tau} = \left\{(t')^{-w'} > 1/\epsilon\right\} \cap U_{\sigma_2,\tau},$$

so these sets do not meet in the quotient by χ_{12}.

Thus $[\mathbb{R}^n_+; R]$ defined by the quotient (3.2) has the structure of a Hausdorff, paracompact smooth manifold with corners arising from the covering by the coordinate charts which are the images in $[\mathbb{R}^n_+, R]$ of the U_σ. Smooth functions on $[\mathbb{R}^n_+; R]$ are those which are smooth in each of the coordinate patches and the blow-down map

$$\beta : [\mathbb{R}^n_+; R] \longrightarrow \mathbb{R}^n_+$$

is a well defined smooth b-map since it is such on each coordinate patch and these maps form, by construction, a commutative diagram with the transition maps. $\square$

The coordinate chart U_σ is related to the affine variety $\mathrm{Spec}(\mathbb{R}[\hat{\sigma}])$, where $\hat{\sigma}$ is the dual monoid of σ, which is realized as a monoid in the dual lattice to N_σ.

In fact, U_σ consists precisely of the "$\mathbb{R}_+$-valued" points in $\mathrm{Spec}(\mathbb{R}[\hat{\sigma}])$, which are given by monoid homomorphisms $\mathrm{Hom}_{\mathrm{mon}}(\hat{\sigma}, \mathbb{R}_+)$, with $\mathbb{R}_+$ considered as a multiplicative monoid. This is the "singular manifold with corners" [Ful93] in the case of a toric variety, though our setup is more closely related to toroidal embeddings [KKMSD73] and logarithmic geometry [Kat94], [Ogu06]. In any case,

our explicit construction gives immediately a topology and smooth structure to this set.

The boundary faces of $[\mathbb{R}^n_+; R]$ correspond precisely to the monoids in the refinement R. If $\tau \in R$ and $\sigma \geq \tau$ is a maximal monoid of which it is a face then τ defines the boundary face $F_{\sigma,\tau} \subset U_\sigma = \mathbb{R}^n_+$ which is the closure of $U_{\sigma,\tau} \setminus (0, \infty)^n$, the part of the boundary where the coordinates corresponding to the generators of τ vanish (and the other coordinates may or may not vanish).

Proposition 3.2. *The manifold $[\mathbb{R}^n_+; R]$, fixed by a smooth resolution of $\sigma_{\mathbb{R}^n_+}$, has interior diffeomorphic to $(0, \infty)^n$ and its boundary faces are in 1-1 correspondence with the monoids $\tau \in R$ where the face F_τ corresponding to τ is the quotient in (3.2) of the union of the $F_{\sigma,\tau} \subset U_\sigma$ for the maximal monoids σ with $\tau \leq \sigma$; this correspondence satisfies $\mathrm{codim}(F_\tau) = \dim(\tau)$ and is inclusion-reversing:*

$$F_{\tau'} \subseteq F_\tau \iff \tau \leq \tau'.$$

Proof. The boundary faces of each $U_\sigma = \mathbb{R}^n_+$ are, as follows from (3.1), precisely the $F_{\sigma,\tau}$ for all faces $\tau \leq \sigma$, since these correspond to all subsets of the coordinate functions. Under the transition maps χ_{12} defined above and corresponding to two monoids, σ_1 and σ_2, of maximal dimension, the interior of each $F_{\sigma_1,\tau}$ is mapped onto the interior of $F_{\sigma_2,\tau}$, if $\tau \leq \sigma_1 \cap \sigma_2$. Otherwise if τ is not contained in either σ_1 or σ_2 then $F_{\sigma_1,\tau}$ does not meet the domain of definition of χ_{12} or does not meet the range.

Thus the interiors of the $F_{\sigma,\tau}$ are globally well-defined subsets of $[\mathbb{R}^n_+; R]$. For each $\tau \in R$, the closure in $[\mathbb{R}^n_+; R]$ of the interior of $F_{\sigma,\tau}$ is the image of the union of the closures in the U_σ that it meets. It is therefore everywhere locally a manifold with corners, and hence is globally a manifold with corners. In particular each boundary face is embedded in $[\mathbb{R}^n_+; R]$. Finally, the fact that $\tau \leq \tau' \iff F_{\tau'} \subseteq F_\tau$ is evident in U_σ for a maximal monoid $\sigma \geq \tau' \geq \tau$. $\qquad\square$

Next consider the relationship between the blow-up, $[\mathbb{R}^n_+; R]$, of $\mathbb{R}^n_+$ defined by a refinement R of $\sigma_{\mathbb{R}^n_+}$, and the blow-up $[\mathbb{R}^k_+; R(\tau)]$ where $R(\tau)$ is the induced refinement of $\tau \leq \sigma_{\mathbb{R}^n_+}$ as in Lemma 2.6 and $k = \dim(\tau)$.

Lemma 3.3. *If $R(\tau)$ is the refinement of $\tau = \mathbb{Z}_+ \langle x_i \partial_{x_i} \rangle_{i \in I} \leq \sigma_{\mathbb{R}^n_+}$ obtained from a smooth refinement R of $\sigma_{\mathbb{R}^n_+}$, where $I \subset \{1, \ldots, n\}$, and $\dim(\tau) = \#I = k$, there is an injection*

$$(3.3) \qquad [\mathbb{R}^k_+; R(\tau)] \times (0, \infty)^{n-k} \longleftrightarrow [\mathbb{R}^n_+; R]$$

giving a commutative diagram

$$
\begin{array}{ccc}
[\mathbb{R}^k_+; R(\tau)] \times (0, \infty)^{n-k} & \longleftrightarrow & [\mathbb{R}^n_+; R] \\
\downarrow & \beta & \downarrow \\
\mathbb{R}^k_+ \times (0, \infty)^{n-k} & \longleftrightarrow & \mathbb{R}^n_+
\end{array}
$$

where the bottom map is the inclusion

$$(3.4) \qquad \mathbb{R}^k_+ \times (0, \infty)^{n-k} \cong \{x_i \neq 0 \; ; \; i \notin I\} \subset \mathbb{R}^n_+.$$

Proof. Consider a maximal monoid $\sigma' \in R(\tau)$ (so $\dim(\sigma') = k$). As a monoid in $R(\tau)$, σ' gives rise to a coordinate chart

$$\mathbb{R}_+^k \cong U'_{\sigma'} \subset [\mathbb{R}_+^k; R(\tau)],$$

while considered as a monoid in R, σ' corresponds to the open set

$$\mathbb{R}_+^k \times (0, \infty)^{n-k} \cong U_{\sigma, \sigma'} \subset U_\sigma \cong \mathbb{R}_+^n,$$

for a maximal monoid $\sigma \in R$ such that $\sigma' \leq \sigma$. This σ is unique since $\dim(\sigma') = \dim(\tau)$ and τ is a face of $\sigma_{\mathbb{R}_+^n}$.

Thus there are natural identifications $U'_{\sigma'} \times (0, \infty)^{n-k} \subset U_{\sigma'}$, which are compatible with the face relations, and so patch together to form the map

$$[\mathbb{R}_+^k; R(\tau)] \times (0, \infty)^{n-k} \hookrightarrow [\mathbb{R}_+^n; R]$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a crucial step in the passage from the local construction of generalized blow-up above to the global setting of general manifolds with corners, we consider the lifting of smooth diffeomorphisms to the generalized blow-up. As for standard radial blow-up although the local construction is algebraic, it turns out to be compatible with diffeomorphisms.

Proposition 3.4. *Any diffeomorphism of open submanifolds of $\mathbb{R}_+^n$, $f : O_1 \subset \mathbb{R}_+^n \xrightarrow{\cong} O_2 \subset \mathbb{R}_+^n$, which maps boundary hypersurfaces onto themselves lifts to a unique diffeomorphism,*

$$f' : O'_1 \subset [\mathbb{R}_+^n; R] \longrightarrow O'_2 \subset [\mathbb{R}_+^n; R]$$

where $O'_i = \beta^{-1}(O_i)$, such that the diagram

$$
\begin{array}{ccc}
O'_1 & \longrightarrow & O'_2 \\
\downarrow & \;\;\;\not\!\beta & \downarrow \\
O_1 & \longrightarrow & O_2
\end{array}
$$

commutes and then f' also maps boundary hypersurfaces onto themselves.

Proof. Fix an n-dimensional monoid $\sigma = \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle \in R$. Let $\nu \in \mathrm{Mat}(n \times n, \mathbb{Z}_+) \cap \mathrm{GL}(n, \mathbb{Q})$ be the matrix for which β acts by $\beta : t \longmapsto t^\nu$ on U_σ (recall that the rows of ν are the coordinates of the v_i).

By the assumption that f maps boundary faces onto themselves, $f : O_1 \subset \mathbb{R}_+^n \longrightarrow O_2 \subset \mathbb{R}_+^n$ has an expression in coordinates of the form

$$f : (x_1, \ldots, x_n) \longmapsto \big(f_1(x), \ldots, f_n(x)\big) = \big(a_1(x)x_1, \ldots, a_n(x)x_n\big),$$

where $a_i > 0$, $i = 1, \ldots, n$. Indeed, if O_1 meets part of the hypersurface $x_i = 0$, then $f_i(x)$ must obviously have this form, and otherwise (if $O_1 \cap \{x_i = 0\} = \emptyset$), we can set $a_i = f_i/x_i$. We use the notational shorthand

$$a(x)\, x = \big(a_1(x)x_1, \ldots, a_n(x)x_n\big),$$

where $a(x) = \big(a_1(x), \ldots, a_n(x)\big)$.

Define f'_σ on U_σ by

$$f'_\sigma : t \longmapsto a\, (t^\nu)^{\nu^{-1}}\, t,$$

that is,

$$f'_\sigma{}^*(t_i) = \left(\prod_j a_j (t^\nu)^{\nu_{ij}^{-1}}\right) t_i.$$

The coefficients are smooth functions since $t \longmapsto t^\nu$ is smooth, the coefficients of ν^{-1} are in $\mathbb{Q}$, and the a_i are strictly positive.

The composition $f \circ \beta : O'_1 \cap U_\sigma \longrightarrow O_1 \subset \mathbb{R}^n_+$ has the form

$$f \circ \beta : t \xrightarrow{\beta} t^\nu = x \xrightarrow{f} a(x)x = a\left(t^\nu\right) t^\nu$$

while $\beta \circ f'_\sigma$ has the form

$$\beta \circ f'_\sigma : t \xrightarrow{f'} a\left(t^\nu\right)^{\nu^{-1}} t \xrightarrow{\beta} \left(a\left(t^\nu\right)^{\nu^{-1}} t\right)^\nu = a\left(t^\nu\right) t^\nu$$

and so they agree. It is easy to see that the f'_σ defined on each coordinate chart U_σ are compatible on overlaps, i.e. that f'_{σ_i} commute with the transition maps $\chi_{12} : U_{\sigma_1,\tau} \longrightarrow U_{\sigma_2,\tau}$, and so these maps patch together to define a map

$$f' : \beta^{-1}(O_1) \subset [\mathbb{R}^n_+; R] \longrightarrow \beta^{-1}(O_2) \subset [\mathbb{R}^n_+; R].$$

We next verify that this lifting is functorial; that is, $(g \circ f)' = g' \circ f'$, which will also prove that f' is a diffeomorphism with inverse $\left(f^{-1}\right)'$, since clearly $\mathrm{Id}' = \mathrm{Id}$. Suppose then that $f : O_1 \xrightarrow{\cong} O_2$ has the form

$$f : x \longmapsto a(x)\, x$$

while $g : O_2 \xrightarrow{\cong} O_3$ has the form

$$f : x \longmapsto b(x)\, x.$$

Thus $g \circ f$ has the form

$$g \circ f : x \longmapsto c(x)\, x \qquad c(x) = b\big(a(x)x\big)a(x),$$

and therefore on a coordinate chart U_σ as above, $(g \circ f)'$ has the form

$$(g \circ f)' : t \longmapsto c\left(t^\nu\right)^{\nu^{-1}} t = b\big(a\left(t^\nu\right)t^\nu\big)^{\nu^{-1}} a\left(t^\nu\right)^{\nu^{-1}} t.$$

On the other hand, $g' \circ f'$ acts by

$$g' \circ f' : t \longmapsto b\left(\left(a\left(t^\nu\right)^{\nu^{-1}} t\right)^\nu\right)^{\nu^{-1}} a\left(t^\nu\right)^{\nu^{-1}} t = b\big(a\left(t^\nu\right)t^\nu\big)^{\nu^{-1}} a\left(t^\nu\right)^{\nu^{-1}} t,$$

which is the same. $\qquad\qquad\qquad\square$

In fact we will need a slightly more technical version, involving passage to the induced refinements $R(\tau)$ as in Lemma 3.3.

Corollary 3.5. *Let $R(\tau_I)$ and $R(\tau_J)$ be the refinements of $\tau_I, \tau_J \leq \sigma_{\mathbb{R}^n_+}$ induced by R for $I, J \subset \{1, \ldots, n\}$, where $\dim(\tau_I) = k$ and $\dim(\tau_J) = l$. Suppose*

$$f : O_1 \subset \mathbb{R}^k_+ \times (0,\infty)^{n-k} \xrightarrow{\cong} O_2 \subset \mathbb{R}^l_+ \times (0,\infty)^{n-l}$$

is a diffeomorphism of open submanifolds, which maps boundary faces to themselves when both sides are included in $\mathbb{R}^n_+$ as in (3.4). Then f lifts to a unique diffeomorphism

$$f' : O'_1 \subset [\mathbb{R}^k_+; R(\tau_I)] \times (0,\infty)^{n-k} \longrightarrow O'_2 \subset [\mathbb{R}^l_+; R(\tau_J)] \times (0,\infty)^{n-l}$$

where $O_i' = (\beta \times \mathrm{Id})^{-1}(O_i)$, *which commutes with blow-down maps as above, and also maps boundary hypersurfaces to themselves when both sides are considered as subsets of $[\mathbb{R}_+^n; R]$ as in (3.3).*

Proof. Using the inclusions $\mathbb{R}_+^k \times (0,\infty)^{n-k} \subset \mathbb{R}_+^n$ and $\mathbb{R}_+^l \times (0,\infty)^{k-l} \subset \mathbb{R}_+^n$ corresponding to sets $\{x_i \neq 0\}_{i \notin I}$ and $\{x_j \neq 0\}_{j \notin J}$, respectively, we consider O_1 and O_2 to be open sets in $\mathbb{R}_+^n$, and then the result follows immediately from Lemma 3.3 and Proposition 3.4. $\qquad\square$

Finally, we establish a local version of a result we shall prove in more generality in Section 6 about lifting b-maps to a generalized blow-up. Suppose

$$f : \mathbb{R}_+^m \longrightarrow \mathbb{R}_+^n$$

is an interior b-map for which $f(0) = 0$. Then f has the explicit coordinate expression

$$f : x \longmapsto x' = a(x)\, x^\delta = \left(a_1(x) \prod_{j=1}^m x_j^{\delta_{j1}}, \ldots, a_n(x) \prod_{j=1}^m x_j^{\delta_{jn}} \right)$$

where $\delta \in \mathrm{Mat}(m \times n, \mathbb{Z}_+)$ and $0 < a_i(x) \in C^\infty(\mathbb{R}_+^m)$. Because of the non-negativity and integrality of the δ_{ij}, the b-differential ${}^{\mathrm{b}}f_* : {}^{\mathrm{b}}N_0\{0\} \longrightarrow {}^{\mathrm{b}}N_0\{0\}$ (which is represented by the matrix δ^{T} with respect to the bases $\{x_i \partial_{x_i}\}$ and $\{x_i' \partial_{x_i'}\}$) restricts to a monoid homomorphism

$$^{\mathrm{b}}f_* = \delta^{\mathrm{T}} : \sigma_{\mathbb{R}_+^m} \longrightarrow \sigma_{\mathbb{R}_+^n}.$$

Proposition 3.6. *If $f : \mathbb{R}_+^m \longrightarrow \mathbb{R}_+^n$ is an interior b-map such that ${}^{\mathrm{b}}f_* : \sigma_{\mathbb{R}_+^m} \longrightarrow \sigma_{\mathbb{R}_+^n}$ factors through a monoid homomorphism $\phi : \sigma_{\mathbb{R}_+^m} \longrightarrow \tau$ for some $\tau \in R$ where R is a smooth refinement of $\sigma_{\mathbb{R}_+^n}$ then there exists a unique b-map $f' : \mathbb{R}_+^m \longrightarrow [\mathbb{R}_+^n; R]$ such that*

$$
\begin{array}{ccc}
 & & [\mathbb{R}_+^n; R] \\
 & {\nearrow}^{f'} & \downarrow {\scriptstyle \beta} \\
\mathbb{R}_+^m & \xrightarrow{\ f\ } & \mathbb{R}_+^n
\end{array}
$$

commutes and the range of f' is contained in the coordinate chart $U_\tau \subset [\mathbb{R}_+^n; R]$. Moreover this construction is functorial in that for a b-map $g : \mathbb{R}_+^l \longrightarrow \mathbb{R}_+^m$, ${}^{\mathrm{b}}(f \circ g)_*$ factors through $\tau \in R$ if and only if ${}^{\mathrm{b}}f_*$ does, and then $(f \circ g)' = f' \circ g$.

Proof. Taking a maximal dimension monoid containing τ if necessary, it suffices to assume that τ is maximal, $\dim(\tau) = n$. Then let $(U_\tau, t = (t_1, \ldots, t_n))$ be the coordinate chart associated to $\tau = \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle \in R$, and let $\nu \in \mathrm{GL}(n, \mathbb{Z}_+)$ denote the matrix such that $v_i = \sum_j \nu_{ij} x_j' \partial_{x_j'}$ as in the construction of generalized blow-up, so that the blow-down acts by $\beta : t \longmapsto x = t^\nu$ on U_τ. Alternatively, one can view ν^{T} as the matrix defining the monoid inclusion $\nu^{\mathrm{T}} : \tau \hookrightarrow \sigma_{\mathbb{R}_+^n}$ with respect to the bases $\{v_i\}$ and $\{x_i' \partial_{x_i'}\}$.

In a similar manner, let $\mu \in \mathrm{Mat}(m \times n, \mathbb{Z}_+)$ denote the matrix whose transpose represents the given homomorphism

$$\phi = \mu^{\mathrm{T}} : \sigma_{\mathbb{R}_+^m} \longrightarrow \tau = \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle$$

with respect to the bases $\{x_i \partial_{x_i}\}$ and $\{v_i\}$. From the assumption that $^\mathrm{b} f_*$ factors through ϕ it follows that
$$\delta^\mathrm{T} = \nu^\mathrm{T} \mu^\mathrm{T} = (\mu \nu)^\mathrm{T}.$$

Define $f' : \mathbb{R}_+^m \longrightarrow U_\tau$ in coordinates by
$$f' : x \longmapsto a(x)^{\nu^{-1}} x^\mu = t.$$

Then observe that $\beta \circ f'$ has the form
$$\beta \circ f' : x \longmapsto \left(a(x)^{\nu^{-1}} x^\mu\right)^\nu = a(x) x^{\mu \nu} = a(x) x^\delta,$$

and is therefore equal to f. The form of any $f' : \mathbb{R}_+^m \longrightarrow U_\tau$ such that $\beta \circ f' = f$ is determined in these local coordinates, giving uniqueness.

To show functoriality, suppose that $g : \mathbb{R}_+^l \longrightarrow \mathbb{R}_+^m$ acts by $z \longmapsto b(z) z^\kappa = x$. We have the commutative diagram

$$
\begin{array}{ccccc}
 & & & \nearrow^{\mu^\mathrm{T}} & \raisebox{1em}{τ} \\
 & & & & \big\downarrow \nu^\mathrm{T} \\
\sigma_{\mathbb{R}_+^l} & \xrightarrow{\kappa^\mathrm{T}} & \sigma_{\mathbb{R}_+^m} & \xrightarrow{\delta^\mathrm{T}} & \sigma_{\mathbb{R}_+^n}.
\end{array}
$$

The composition $f \circ g$ acts by
$$f \circ g : z \longmapsto a\left(b(z) z^\kappa\right) b^\delta z^{\kappa \delta} =: c(z) z^{\kappa \delta}.$$

Then on one hand $(f \circ g)'$ has the form
$$(f \circ g)' : z \longmapsto c(z)^{\nu^{-1}} z^{\kappa \mu} = a\left(b(z) z^\kappa\right)^{\nu^{-1}} b^{\delta \nu^{-1}} z^{\kappa \mu},$$

while on the other hand $f' \circ g$ acts by
$$f' \circ g : z \longmapsto a\left(b(z) z^\kappa\right)^{\nu^{-1}} (b z^\kappa)^\mu,$$

which is the same since $\delta \nu^{-1} = \mu$. $\qquad\square$

Modifying the proof slightly, we obtain a version of the above which includes "interior" coordinates.

Corollary 3.7. *If $f : \mathbb{R}_+^l \times \mathbb{R}^{m-l} \longrightarrow \mathbb{R}_+^k \times \mathbb{R}^{n-k}$ is an interior b-map such that $f(0) = 0$ and $^\mathrm{b} f_* : \sigma_{\mathbb{R}_+^l} \longrightarrow \sigma_{\mathbb{R}_+^k}$ factors through a monoid homomorphism $\sigma_{\mathbb{R}_+^l} \longrightarrow \tau \in R$, then f lifts to a unique b-map $f' : \mathbb{R}_+^l \times \mathbb{R}^{m-l} \longrightarrow [\mathbb{R}_+^k; R] \times \mathbb{R}^{n-k}$ such that $f = (\beta \times \mathrm{Id}) \circ f'$, and this is functorial with respect to composition.*

Proof. Such a map f has the coordinate representation
$$f : (x, y) \longmapsto \left(a(x, y) x^\delta, b(x, y)\right)$$

with $a, b > 0$. With μ, ν as in the proof above, $f' : \mathbb{R}_+^l \times \mathbb{R}^{m-l} \longrightarrow U_\tau \times \mathbb{R}^{n-k}$ defined by
$$f' : (x, y) \longmapsto \left(a(x, y)^{\nu^{-1}} x^\mu, b(x, y)\right)$$

has the desired properties. The proof of functoriality is similar. $\qquad\square$

4. Monoidal complexes

In Section 6 we show how to associate a natural monoid to each boundary face of a manifold with corners, with the compatibility condition that whenever $F \subseteq G$, the monoid associated to G is a face of the one associated to F, i.e. that face relations persist in an inclusion-reversing sense. We have already seen such an example, which is the association between monoids in R and faces in $[\mathbb{R}^n_+; R]$, where R is a refinement of $\sigma_{\mathbb{R}^n_+}$. The notion of a refinement (where all the monoids reside in the same vector space) is not sufficiently flexible to capture the ways in which boundary faces may be related for general X so we introduce the concept of a *monoidal complex*; this should be thought of by analogy to a simplicial or CW complex. It is a generalization of the structure of a refinement in which the monoids are still "attached together nicely along faces", but in which the monoids reside in separate vector spaces and there is no 'base' monoid (as when R is a refinement "of σ"). The contents of this section are all that is needed for the discussion of the generalized blow-up of a manifold in Section 6, apart from Theorem 6.5 which uses the fiber product of monoidal complexes discussed in the next section.

Consider a finite partially ordered set by $A = (A, \leq)$. It is convenient to think of A as a *category*, with objects the elements $a \in A$ and morphisms given by the order relation: $a \to b \in \mathrm{Hom}(a, b) \iff a \leq b$.

Definition 4.1. Let $(A, \leq)$ be a finite ordered set. A *monoidal complex* $\mathcal{Q}$ over A is a covariant functor from A to the category of monoids, where all morphisms are isomorphisms onto faces; subsequently these are required to be *complete and reduced*.

That is, $\mathcal{Q}$ consists of a monoid σ_a for each $a \in A$, along with isomorphisms called *face maps*

$$(4.1) \qquad\qquad i_{ab} : \sigma_a \xrightarrow{\cong} \tau \leq \sigma_b \text{ whenever } a \leq b,$$

for some $\tau \leq \sigma_b$. This relation will be denoted

$$(4.2) \qquad\qquad\qquad \sigma_a \leq \sigma_b, \text{ for } a \leq b.$$

We say $\mathcal{Q}$ is *complete*, respectively *reduced*, if for every $b \in A$ and every face $\tau \leq \sigma_b$, there exists at least one (resp. at most one) $a \in A$ such that $a \leq b$ and $i_{ab} : \sigma_a \cong \tau$.

Lemma 4.2. *If $\mathcal{Q}$ is complete and reduced, then the directed sets $(A, \leq)$ and $\Gamma_{\mathcal{Q}} = \{\tau \leq \sigma \; ; \; \sigma \in \mathcal{Q}\} / i_*$ with the order (4.2) are isomorphic.*

Proof. Consider the composition of the map $A \ni a \longmapsto \sigma_a \in \mathcal{Q}$ with the quotient by the face maps (4.1). This map $A \longrightarrow \Gamma_{\mathcal{Q}}$ intertwines the orders by (4.2); it must be injective by the assumption that $\mathcal{Q}$ is reduced, and surjective by the assumption that it is complete. $\square$

From this it follows that for a complete, reduced monoidal complex, A is entirely determined by the set of $\sigma \in \mathcal{Q}$ and the face relations (4.2). Nevertheless, it is often convenient to be able to refer explicitly to an indexing set A, as in the case $(A, \leq) = (\mathcal{M}(X), \leq)$ below, which we will most often encounter.

There are evidently some obstructions for a general ordered set to index a complete, reduced monoidal complex, but we shall not concern ourselves with them here; all complete reduced monoidal complexes will arise naturally. From now on, all monoidal complexes will be complete and reduced.
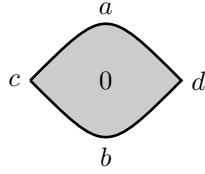
FIGURE 2. A manifold whose boundary face relations do not index a monoid refinement.

Example 4.3. The faces of a single monoid σ form a monoidal complex over the ordered set $\{\tau \; ; \; \tau \leq \sigma\}$.

More generally, any refinement R of σ forms a monoidal complex over the set R with the order coming from the face relations, if we take i_{ab} in (4.1) to be the identity maps.

For an example which is not a refinement, consider the following.

Example 4.4. Let $\sigma_0 = \{0\}$, $\sigma_a = \mathbb{Z}_+(1,0) \subset \mathbb{R}^2$, $\sigma_b = \mathbb{Z}_+(0,1) \subset \mathbb{R}^2$, and let σ_c and σ_d be two distinct copies of the monoid $\mathbb{Z}_+ \langle (1,0),(0,1) \rangle \subset \mathbb{R}^2$. Let the face maps be the obvious ones in the diagram

$$
\begin{array}{ccc}
 & \sigma_a \longrightarrow \sigma_c & \\
\sigma_0 \nearrow & \times & \\
 & \searrow & \\
 & \sigma_b \longrightarrow \sigma_d.
\end{array}
$$
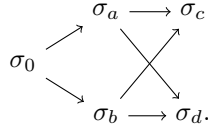
This complex cannot be realized as a refinement, since otherwise σ_c and σ_d would have to be identified; here they remain distinct. Note that the ordered set underlying this complex is the same as the one given by the boundary faces (with the order of reverse inclusion) of the 2-dimensional manifold with corners pictured in Fig. 2.

Definition 4.5. A *morphism* of monoidal complexes $\phi : \mathcal{Q}_A \longrightarrow \mathcal{Q}_B$ consists of a map of ordered sets $\phi_\# : (A, \leq) \longrightarrow (B, \leq)$ and monoid homomorphisms $\phi_{ab} : \mathcal{Q}_A \ni \sigma_a \longrightarrow \sigma_b \in \mathcal{Q}_B$ for all $a \in A$, where $b = \phi_\#(a) \in B$. These are required to commute with the homomorphisms $i_{aa'}$ for $a \leq a'$, so that

$$
\begin{array}{ccc}
\sigma_a & \longrightarrow & \sigma_b \\
\downarrow & \phi_{ab} & \downarrow \\
\sigma_{a'} & \longrightarrow & \sigma_{b'}
\end{array}
$$

commutes, where $a \leq a'$, $\phi_\#(a) = b$, $\phi_\#(a') = b'$, and thus $b \leq b'$. We say ϕ is *injective* if all the morphisms ϕ_{ab} are injective, though we do not necessarily require that $\phi_\# : A \longrightarrow B$ be injective.

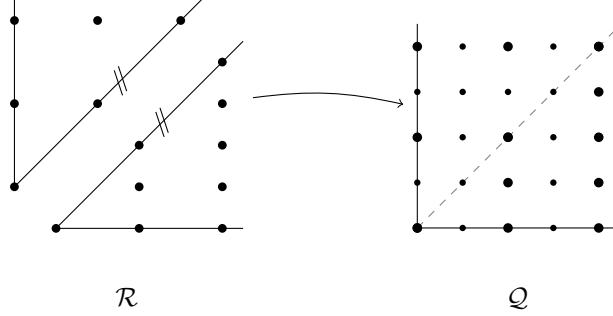An elementary example of a morphism is the inclusion of a subcomplex:

$$\mathcal{R} \qquad\qquad\qquad \mathcal{Q}$$

FIGURE 3. A refinement of monoidal complexes. Only maximal dimension monoids are pictured. In this example, $\mathcal{R} \to \mathcal{Q}$ is injective, though $\mathcal{R}$ is not smooth, and the individual monoid homomorphisms do not cover their targets.

Definition 4.6. Let $\mathcal{Q}$ be a monoidal complex over $(A, \leq)$. A *monoidal subcomplex* of $\mathcal{Q}$ is a complex $\mathcal{Q}_0$ obtained by restricting $\mathcal{Q}$ to a subset $A_0 \subset A$:

$$\mathcal{Q}_0 = \{\sigma_a \; ; \; a \in A_0\}$$

such that $\mathcal{Q}_0$ is complete and reduced. There is then an injective morphism

$$\mathcal{Q}_0 \hookrightarrow \mathcal{Q}$$

consisting of the identity homomorphisms over $A_0 \subset A$.

In Section 6, we will show that a b-map $f : X \longrightarrow Y$ gives rise to a morphism $f_\natural : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$ of the basic monoidal complexes associated to X and Y.

Another important class of morphisms consists of refinements.

Definition 4.7. Let $\mathcal{Q}$ be a monoidal complex. A *refinement* of $\mathcal{Q}$ is a morphism $\phi : \mathcal{R} \longrightarrow \mathcal{Q}$ all of whose homomorphisms are injective and for all $\sigma \in \mathcal{Q}$,

(i) $\bigcup_{\tau \in \phi_\#^{-1}(\sigma)} \phi\big(\mathrm{supp}(\tau)\big) = \mathrm{supp}(\sigma)$, and

(ii) for $\tau_1, \tau_2 \in \phi_\#^{-1}(\sigma)$, $\mathrm{relint}\big[\phi\big(\mathrm{supp}(\tau_1)\big)\big] \cap \mathrm{relint}\big[\phi\big(\mathrm{supp}(\tau_2)\big)\big] = \emptyset$ unless $\tau_1 = \tau_2$,

where the *relative interior* of a cone $C = \mathrm{span}_{\mathbb{R}_+} \{v_1, \ldots, v_k\}$ is the set $\mathrm{relint}[C] = \mathrm{span}_{(0,\infty)} \{v_1, \ldots, v_k\}$.

Condition (i) requires that the support of each monoid in $\mathcal{Q}$ is covered by the supports of monoids in $\mathcal{R}$, and condition (ii) demands that these supports only intersect along mutual faces. These, along with our assumption that $\mathcal{R}$ is complete and reduced, are analogous to the conditions in Definition 2.4 for the refinement of a monoid. Figure 3 depicts a refinement.

Proposition 4.8. *If $\phi : \mathcal{R} \longrightarrow \mathcal{Q}$ is a refinement then for each $\sigma \in \mathcal{Q}$, the collection*

$$(4.3) \qquad\qquad \mathcal{R}(\sigma) := \Big\{\phi(\sigma') \; ; \; \sigma' \in \phi_\#^{-1}(\tau), \; \tau \leq \sigma\Big\}$$

is a refinement of σ. Conversely, let $\{R(\sigma) \; ; \; \sigma \in \mathcal{Q}\}$ be a collection of refinements which are compatible in that whenever $\tau \leq \sigma$, the refinement $R(\tau)$ is identical to the induced refinement $\big(R(\sigma)\big)(\tau)$ as in Lemma 2.6. Then the quotient $\mathcal{R} = \bigcup R(\sigma)\big/i_*$

of the set of all monoids in the collection by the face maps i_* of $\mathcal{Q}$ forms a refinement $\phi : \mathcal{R} \longrightarrow \mathcal{Q}$ of monoidal complexes.

Proof. Let $\mathcal{R}(\sigma)$ be the set in (4.3). Certainly

$$\mathcal{R}(\sigma) \supseteq \left\{ \phi(\sigma') \; ; \; \sigma' \in \phi_\#^{-1}(\sigma) \right\}$$

and it follows that $\mathcal{R}(\sigma)$ satisfies property 2.4.(iii) for a refinement of σ by property (i) in Definition 4.7 above.

To see that $\mathcal{R}(\sigma)$ satisfies the other two properties of a refinement, let $\tau_i' \in \mathcal{R}$, $i = 1, 2$ with $\phi(\tau_i') \in \mathcal{R}(\sigma)$, and for convenience of notation, identify these with their images in σ. Let $\tau \leq \sigma$ be the smallest face such that $\tau_1' \cap \tau_2' \subset \tau$. Since ϕ is a morphism it follows that each $\tau_i' \cap \tau$ is a face of τ_i', hence also in $\mathcal{R}$, and it must be that $\tau_i' \cap \tau \in \phi_\#^{-1}(\tau)$ (or else there would be a smaller τ), so it follows that the face $\tau_i' \cap \tau \leq \tau$ is also in $\mathcal{R}(\sigma)$.

By property (ii) above for $\tau_i' \cap \tau \in \phi_\#^{-1}(\tau)$, it follows that $\tau_1' \cap \tau_2' = (\tau_1' \cap \tau) \cap (\tau_2' \cap \tau)$ must be a face of each. To see that $\mathcal{R}(\sigma)$ contains the faces of all its monoids, simply let τ_1' be the face in question of τ_2'; we conclude that $\tau_1' = \tau_1' \cap \tau_2' \in \mathcal{R}(\sigma)$. Thus $\mathcal{R}(\sigma)$ is a refinement of σ.

For the converse, let $\mathcal{R}$ be the set of all monoids in the refinements $R(\sigma)$, $\sigma \in \mathcal{Q}$, modulo the identification of monoids in $(R(\sigma))(\tau)$ with those in $R(\tau)$ for $\tau \leq \sigma$, as above. That $\mathcal{R}$ is a complex follows easily from the fact that each $R(\sigma)$ forms a monoidal complex.

For notational clarity, denote the ordered sets for $\mathcal{R}$ and $\mathcal{Q}$ by $(A, \leq)$ and $(B, \leq)$, respectively. We take $\phi_\# : A \longrightarrow B$ to be the map which sends a to the smallest $b \in B$ such that $\sigma_a' \in \mathcal{R}$ is in the refinement R_{σ_b}, and then $\phi_{ab} : \sigma_a' \longrightarrow \sigma_b$ is given by the inclusion $R_{\sigma_b} \ni \sigma_a' \subset \sigma_b$ which is of course injective.

Finally, it follows directly from the fact that the $R(\sigma)$ are refinements that $\phi : \mathcal{R} \longrightarrow \mathcal{Q}$ satisfies the properties in Definition 4.7. $\square$

The remainder of this section is devoted to specific algorithms for obtaining and extending various refinements of complexes.

A primary means of obtaining refinements is star subdivision; we extend this to a monoidal complex. The proof of the following follows directly from Propositions 4.8 and 2.7.

Proposition 4.9 (Star subdivision). *Let $\mathcal{Q}$ be a monoidal complex, $\sigma_a \in \mathcal{Q}$, and $v \in \sigma_a$ the monoids*

$$\mathcal{S}(\mathcal{Q}, v)(\sigma_b) = \begin{cases} S\big(\sigma_b, (i_{ab}v)\big) & \text{if } \sigma_a \leq \sigma_b, \text{ and} \\ \sigma_b & \text{otherwise} \end{cases}$$

form a monoidal complex which refines $\mathcal{Q}$; if $\mathcal{Q}$ is smooth, then $\mathcal{S}(\mathcal{Q}, v)$ is smooth.

Recall that in Section 2 an extension of star subdivision is discussed, giving a refinement $S(\sigma, \mu)$ of a smooth monoid, σ, with respect a monoid μ given by the intersection of σ with an integral subspace. We now extend this to monoidal complexes, which will be of use in Section 10.

Proposition 4.10 (Planar refinement). *Let $\mathcal{P}$ a smooth monoidal complex and suppose $i : \mathcal{Q} \longrightarrow \mathcal{P}$ is an injective morphism such that for all $\sigma \in \mathcal{P}$, $(i_\#)^{-1}(\sigma) \subset$*

$\mathcal{Q}$ contains at most one monoid μ, and $i(\mu) \subset \sigma$ is the intersection of σ with an integral subspace, then

$$\mathcal{S}(\mathcal{P}, \mathcal{Q})(\sigma) = S(\sigma, \mu), \quad \mu = (i_\#)^{-1}(\sigma)$$

form a complex, containing $\mathcal{Q}$ as a subcomplex, which refines $\mathcal{P}$.

Proof. This follows directly from Proposition 2.9, using the fact that $S(\tau, \mu \cap \tau) = S(\sigma, \mu)(\tau)$. □

The smoothing of a simplicial monoid also extends to monoidal complexes.

Proposition 4.11 (Smoothing). *Let $\mathcal{Q}$ be a simplicial complex. Then the collection of monoid refinements $\sigma_{\mathrm{sm}} \longrightarrow \sigma$, where $\sigma_{\mathrm{sm}} = \mathbb{Z}_+ \langle v_1, \ldots, v_n \rangle$ is freely generated by the extremals of σ as in Example 2.5, forms a smooth refinement $\mathcal{Q}_{\mathrm{sm}} \longrightarrow \mathcal{Q}$.*

Proof. This follows directly from the fact that the faces of σ_{sm}, namely the monoids $\mathbb{Z}_+ \langle v_i \rangle_{i \in I}$, $I \subset \{1, \ldots, n\}$ are the smoothings of the corresponding faces of σ. □

5. Fiber products of complexes

In this section we discuss the existence of fiber products for monoidal complexes and then cover some specific algorithms for producing smooth refinements and extending a refinement of a subcomplex to the whole complex. Apart from the proof of Theorem 6.5, the material here is not used until Section 10.

Given a morphism $\mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$ of complexes and a refinement $\mathcal{R}_2 \longrightarrow \mathcal{Q}_2$, one can consider the problem of finding a refinement $\mathcal{R}_1 \longrightarrow \mathcal{Q}_1$ such that

$$
\begin{array}{ccc}
\mathcal{R}_1 & \longrightarrow & \mathcal{R}_2 \\
\downarrow & & \downarrow \\
\mathcal{Q}_1 & \longrightarrow & \mathcal{Q}_2
\end{array}
$$

commutes. This leads to the consideration of the pullback, or fiber product, of two morphisms in the category of monoidal complexes.

Before discussing fiber products in this category, we examine fiber products in the categories of ordered sets and monoids. Thus, let $(A, \leq)$, $(B, \leq)$ and $(C, \leq)$ be ordered sets, with order preserving maps $(\phi_1)_\# : A \longrightarrow C$ and $(\phi_2)_\# : B \longrightarrow C$.

The product $A \times B$ is an ordered set with elements $(a, b) : a \in A, b \in B$ and order given by

$$(a_1, b_1) \leq (a_2, b_2) \iff a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

The fiber product is the subset

$$A \times_C B = \{(a, b) \in A \times B \; ; \; (\phi_1)_\#(a) = (\phi_2)_\#(b) = c\} \subset A \times B,$$

with the order induced from $A \times B$, which is well-defined since the $(\phi_i)_\#$ are order preserving.

Proposition 5.1 (Fiber products of monoids). *If $\phi_i : \sigma_i \longrightarrow \sigma$, $i = 1, 2$ are monoid homomorphisms, then there is a unique monoid $\sigma_1 \times_\sigma \sigma_2$ with homomorphisms to*

σ_i such that

$$
\begin{array}{ccc}
\sigma_1 \times_\sigma \sigma_2 & \longrightarrow & \sigma_2 \\
\downarrow & \phi_2' & \downarrow \\
\sigma_1 & \longrightarrow & \sigma
\end{array}
$$

commutes, and which has the universal property that any other monoid τ forming such a commutative square factors through a unique homomorphism to $\sigma_1 \times_\sigma \sigma_2$.

Its faces can be described as follows. Let A, B and C be the ordered sets whose elements are faces of σ_1, σ_2, and σ, respectively, and define $(\phi_i)_\# : A_i \longrightarrow A$ by taking $(\phi_i)_\#(\tau_i)$ to be the smallest $\tau \in A$ such that $\phi_i(\tau_i) \subset \tau$. Then faces of $\sigma_1 \times_\sigma \sigma_2$ are indexed by an ordered subset of $A \times_C B$.

Faces of $\sigma_1 \times_\sigma \sigma_2$ are not necessarily in 1–1 correspondence with $A \times_C B$, since there will generally be distinct pairs $(\tau_1, \tau_2) \neq (\tau_1', \tau_2') \in A \times B$ (with $\tau_i' \leq \tau_i$) such that $\tau_1 \times_\tau \tau_2 = \tau_1' \times_\tau \tau_2'$.

Proof. The set $\sigma_1 \times \sigma_2$ is a monoid in $N_{\sigma_1} \times N_{\sigma_2}$, generated by $\{(v_i, 0), (0, w_j)\}$ where $\{v_1, \ldots, v_n\}$ generate σ_1 and $\{w_1, \ldots, w_k\}$ generate σ_2. Then the set

$$\sigma_1 \times_\sigma \sigma_2 = \{(v, w) \in \sigma_1 \times \sigma_2 \; ; \; \phi_1(v) = \phi_2(w)\}$$

(which might be trivial) is evidently closed under addition. It is the intersection of $N_{\sigma_1} \times N_{\sigma_2}$ with the cone

$$\mathrm{supp}(\sigma_1) \times_{\mathrm{supp}(\sigma)} \mathrm{supp}(\sigma_2) = \mathrm{supp}(\sigma_1) \times \mathrm{supp}(\sigma_2) \cap N_{\sigma_1}^\mathbb{R} \times_{N_\sigma^\mathbb{R}} N_{\sigma_2}^\mathbb{R},$$

and hence is a monoid.

We will construct a map I from faces of $\sigma_1 \times_\sigma \sigma_2$ to $A \times_C B$. Let $\tau \leq \sigma_1 \times_\sigma \sigma_2$ be a face. Since $\sigma_1 \times_\sigma \sigma_2 \subset \sigma_1 \times \sigma_2$, there is a smallest face of $\sigma_1 \times \sigma_2$ containing τ, which must be of the form $\tau_1 \times \tau_2$ for some $\tau_i \leq \sigma_i$, $i = 1, 2$. Observe that $(\phi_1)_\#(\tau_1) = (\phi_2)_\#(\tau_2) \in C$, otherwise there would be strictly a smaller τ_1 or τ_2. Thus $(\tau_1, \tau_2) \in A \times_C B$ and we set $I(\tau) = (\tau_1, \tau_2)$. I is injective, since τ is determined by (τ_1, τ_2) by the obvious formula $\tau = \tau_1 \times_\sigma \tau_2$.

The universal property of $\sigma_1 \times_\sigma \sigma_2$ in the category of monoids follows from the analogous property in the category of sets, since the maps factoring through $\sigma_1 \times_\sigma \sigma_2$ are additive. $\qquad\square$

As a special case of the fiber product, observe that if $\phi_2 : \sigma_2 \longrightarrow \sigma$ is injective, then $\sigma_1 \times_\sigma \sigma_2 = \phi_1^{-1}(\sigma_2)$, where we identify σ_2 with its image in σ. In this case ϕ_2' is also injective since we can identify it with the inclusion $\phi_1^{-1}(\sigma_2) \subset \sigma_1$. If both ϕ_1 and ϕ_2 are injective, then we can identify $\sigma_1 \times_\sigma \sigma_2$ with $\sigma_1 \cap \sigma_2 \subset \sigma$, and both of the ϕ_i' are injective.

Note that $\sigma_1 \times \sigma_2$ is smooth if σ_1 and σ_2 are, but this is *not* true of $\sigma_1 \times_\sigma \sigma_2$, as in the following example.

Example 5.2. Let $\sigma_1 = \sigma_2 = \mathbb{Z}_+^2$ and let $\phi_1 = \phi_2 : \mathbb{Z}_+^2 \longrightarrow \mathbb{Z}_+ = \sigma$ be the map $(m, n) \longmapsto m + n$. Then it is a simple exercise to verify that $\sigma_1 \times_\sigma \sigma_2$ is the monoid of Example 2.2, which is not even simplicial.

Since the intersection of smooth monoids need not be smooth, it is similarly easy to come up with examples where the σ_i are smooth and the ϕ_i are even injective but $\sigma_1 \times_\sigma \sigma_2 = \sigma_1 \cap \sigma_2$ is not smooth.

Proposition 5.3 (Fiber product of monoidal complexes)**.** *If $\mathcal{Q}_1$, $\mathcal{Q}_2$ are monoidal complexes over A and B, respectively, and if $\phi_i : \mathcal{Q}_i \longrightarrow \mathcal{Q}$ are morphisms to a complex $\mathcal{Q}$ over C, then there exist monoidal complexes $\mathcal{Q}_1 \times \mathcal{Q}_2$ over $A \times B$ and $\mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{Q}_2$ over an ordered subset of $A \times_C B$ with morphisms ϕ_i' to $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that*

$$
\begin{array}{ccc}
\mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{Q}_2 & \longrightarrow & \mathcal{Q}_2 \\
\downarrow & \phi_2' & \downarrow \\
\mathcal{Q}_1 & \longrightarrow & \mathcal{Q}
\end{array}
$$

commutes, and this is universal with respect to complexes forming a similar commutative square with the $\mathcal{Q}_i$ and $\mathcal{Q}$.

Proof. To define $\mathcal{Q}_1 \times \mathcal{Q}_2$, let $\sigma_{(a,b)} = \sigma_a \times \sigma_b$ for each $(a,b) \in A \times B$. Likewise, to define $\mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{Q}_2$, for $(a,b) \in A \times_C B$ take $\sigma_{(a,b)} = \sigma_a \times_{\sigma_c} \sigma_b$. It suffices to verify that these collections of monoids form complexes.

Consider first $\mathcal{Q}_1 \times \mathcal{Q}_2$. The faces of $\sigma_a \times \sigma_b$ are the monoids $\sigma_{a'} \times \sigma_{b'}$ where $\sigma_{a'} \leq \sigma_a$ and $\sigma_{b'} \leq \sigma_b$. Since the $\mathcal{Q}_i$ are complete and reduced, these are in 1–1 correspondence with the elements $(a', b') \in A \times B$ such that $(a', b') \leq (a, b)$.

Next, it follows from Proposition 5.1, that the faces of $\sigma_a \times_{\sigma_c} \sigma_b$ are monoids of the form $\sigma_{a'} \times_{\sigma_c} \sigma_{b'} = \sigma_{a'} \times_{\sigma_{c'}} \sigma_{b'}$ where $(\phi_1)_{\#}(a') = (\phi_2)_{\#}(b') = c' \leq c$ and $(a', b') \leq (a, b) \in A \times_C B$.

The universal properties of $\mathcal{Q}_1 \times \mathcal{Q}_2$ and $\mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{Q}_2$ follow from the corresponding universal properties of $A \times B$ and $A \times_C B$ among ordered sets, and of $\sigma_1 \times \sigma_2$ and $\sigma_1 \times_{\sigma} \sigma_2$ among monoids. $\qquad \square$

Proposition 5.4 (Refinements pull back)**.** *If $\phi : \mathcal{R} \longrightarrow \mathcal{Q}$ is a refinement, and $\psi : \mathcal{Q}_1 \longrightarrow \mathcal{Q}$ is any morphism of complexes, then*

$$\phi' : \mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{R} \longrightarrow \mathcal{Q}_1 \text{ is a refinement.}$$

In particular, the fiber product $\mathcal{R}_1 \times_{\mathcal{Q}} \mathcal{R}_2$ of two refinements is a mutual refinement of each.

Proof. Fix $\sigma \in \mathcal{Q}$, and consider $\tau \in \mathcal{R}(\sigma)$ and $\sigma_1 \in \mathcal{Q}_1$ such that $\psi : \sigma_1 \longrightarrow \sigma$. If we identify τ with its image in σ, then as noted above,

$$\phi' : \sigma_1 \times_{\sigma} \tau \cong \psi^{-1}(\tau) \hookrightarrow \sigma_1$$

is identified with an inclusion and therefore injective. From the identity

$$\psi^{-1}(\tau_1 \cap \tau_2) = \psi^{-1}(\tau_1) \cap \psi^{-1}(\tau_2)$$

it then follows that $\sigma_1 \times_{\sigma} \mathcal{R}(\sigma) = \{\sigma_1 \times_{\sigma} \tau \; ; \; \tau \in \mathcal{R}(\sigma)\}$ is a refinement of σ_1, and by commutativity of ϕ and ψ with the face maps that $\mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{R} \longrightarrow \mathcal{Q}_1$ forms a refinement of complexes. $\qquad \square$

Note that $\mathcal{Q}_1 \times_{\mathcal{Q}} \mathcal{Q}_2$ need not be smooth even if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are smooth, and even if they are refinements as the discussion following Example 5.2 shows. For this and other reasons, it is desirable to know that smooth refinements exist.

In fact the usual algorithms for obtaining a smooth refinement of a fan in toric geometry, [Ful93], [DCP83], go through essentially unchanged in the theory of monoidal complexes. However, these algorithms are not functorial with respect to the inclusion of subcomplexes, a feature we shall later require.

As far as the authors know the following algorithm is new, though it follows such an intuitive strategy for subdivision (starting with the monoids of highest dimension and proceeding downward, in contrast to the usual algorithms which go the other way), that it would not be surprising if it has already appeared in the toric literature.

Theorem 5.5 (Natural smooth refinement). *Let $\mathcal{Q}$ be a monoidal complex. Then there exists a* natural smooth refinement $\mathrm{NS}(\mathcal{Q}) \longrightarrow \mathcal{Q}$ *with the following properties.*

(i) *If $\mathcal{Q}$ is smooth then $\mathrm{NS}(\mathcal{Q}) = \mathcal{Q}$.*
(ii) *If $\mathcal{Q}_0 \subset \mathcal{Q}$ is a subcomplex, then the corresponding subcomplex of $\mathrm{NS}(\mathcal{Q})$ is the natural smooth refinement of $\mathcal{Q}_0$, i.e.*

$$\mathrm{NS}(\mathcal{Q}_0) = \mathrm{NS}(\mathcal{Q})_0 := \{\mathrm{NS}(\mathcal{Q})(\sigma) \; ; \; \sigma \in \mathcal{Q}_0\} \, .$$

Proof. In fact we first construct a *simplicial* refinement $\mathrm{NS}'(\mathcal{Q})$ with the analogous properties, and then take $\mathrm{NS}(\mathcal{Q}) = \mathrm{NS}'(\mathcal{Q})_{\mathrm{sm}}$ to be the smoothing of $\mathrm{NS}'(\mathcal{Q})$, as in Proposition 4.11.

Consider a monoid σ with extremals $V = \{v_1, \ldots, v_n\}$. Let Λ be the set of those v_i which are linearly independent from all of the others. In other words, Λ consists of those $v \in V$ with nonzero image under the quotient $V/(V \setminus \{v\})$. Then the monoid τ which is the largest face of σ lying in the span of $V \setminus \Lambda$ is uniquely determined, and is either $\{0\}$ or non-simplicial. It represents the 'essential' non-simpliciality of σ, since σ consists of the join of τ and the smooth face generated by Λ.

We define the *non-simplicial dimension* of σ by

$$\mathrm{nsdim}(\sigma) = \dim(\tau)$$

so in particular, σ is simplicial if and only if $\mathrm{nsdim}(\sigma) = 0$. If $\mathrm{nsdim}(\sigma) = \dim(\sigma)$ (i.e. $\Lambda = \emptyset$, so $\sigma = \tau$), we call σ *fully non-simplicial.*

To construct the natural simplicial refinement, we will iteratively replace $\mathcal{Q}$ by the star subdivision $\mathcal{S}(\mathcal{Q}, v)$, where v is the sum of the extremals of a fully non-simplicial monoid σ, starting with those of largest dimension and proceeding downward. For brevity we refer to this operation as "subdividing σ" in $\mathcal{Q}$.

For $j \in \mathbb{N}$, let

$$M_j(\mathcal{Q}) = \#\{\sigma \; ; \; \mathrm{nsdim}(\sigma) = j\} \, .$$

Since $\mathcal{Q}$ has a finite number of monoids which are each finite dimensional, each $M_j(\mathcal{Q})$ is finite, and $M_j(\mathcal{Q}) = 0$ for all $j \geq N$, for some N.

Proceeding by induction, assume that $M_j(\mathcal{Q}) = 0$ for $j > k$. For the inductive step, we will produce a refinement $\mathcal{Q}' \longrightarrow \mathcal{Q}$ for which $M_j(\mathcal{Q}') = 0$ for $j \geq k$. If there are no fully non-simplicial monoids of dimension k, then $M_k(\mathcal{Q}) = 0$ (indeed, if $\mathrm{nsdim}(\sigma) = k$ then it has a fully non-simplicial face $\tau \leq \sigma$ as above of dimension k), and the induction is complete. Otherwise, let σ with extremals $\{v_1, \ldots, v_n\}$ be a fully non-simplicial monoid of dimension k, and consider the star subdivision $\mathcal{S}(\mathcal{Q}, v)$, where $v = v_1 + \cdots + v_n$. Thus we remove all monoids μ in $\mathcal{Q}$ containing v and replace them with new monoids consisting of their faces $\tau \not\ni v$ joined to v.

If v lies in a monoid μ, then necessarily $\sigma \leq \mu$, and by the induction hypothesis $k \geq \mathrm{nsdim}(\mu) \geq \mathrm{nsdim}(\sigma) = k$, so equality holds. Moreover, σ must be the *unique* fully non-simplicial face of μ with dimension k, so any face $\tau \leq \mu$ with $\mathrm{nsdim}(\tau) = k$ must have $\sigma \leq \tau$. Thus if $v \notin \tau \leq \mu$, then $\sigma \not\leq \tau$ and therefore $\mathrm{nsdim}(\tau + \mathbb{Z}_+ v) < k$ since v is independent from the generators of τ and $\mathrm{nsdim}(\tau) < k$.

Since all monoids in $\mathcal{S}(\mathcal{Q}, v)$ which are not in $\mathcal{Q}$ are of this form, we see that $M_j(\mathcal{S}(\mathcal{Q}, v)) \leq M_j(\mathcal{Q})$ for $j \geq k$. On the other hand, M_k must actually decrease by at least one, since $\mathcal{Q}$ contains a monoid of non-simplicial dimension k which is not in $\mathcal{S}(\mathcal{Q}, v)$, namely σ. Thus

$$M_k(\mathcal{S}(\mathcal{Q}, v)) < M_k(\mathcal{Q})$$

and since $M_k(\mathcal{Q}) < \infty$, after a finite number of such subdivisions, we obtain a complex $\mathcal{Q}'$ for which $M_j(\mathcal{Q}') = 0$ for $j \geq k$ and $M_j(\mathcal{Q}') < \infty$ for $j < k$, completing the inductive step.

Note that the inductive step does not depend on the order of the subdivisions. Indeed, the result of subdividing σ_1 and σ_2 of dimension k in either order is the same, unless there is a monoid μ with $\sigma_1 \leq \mu$ and $\sigma_2 \leq \mu$. But such a μ would then have $\mathrm{nsdim}(\mu) > k$, contradicting the induction hypothesis. Upon completion of the induction, we obtain a complex $\mathrm{NS}'(\mathcal{Q})$ such that $M_j\big(\mathrm{NS}'(\mathcal{Q})\big)$ vanishes for all j, thus $\mathrm{NS}'(\mathcal{Q})$ is simplicial. Since star subdivisions are refinements and refinements compose, $\mathrm{NS}'(\mathcal{Q}) \longrightarrow \mathcal{Q}$ is a simplicial refinement.

Finally set $\mathrm{NS}(\mathcal{Q}) = \mathrm{NS}'(\mathcal{Q})_{\mathrm{sm}}$. This involves a local operation on each monoid, and does not depend on any choice of order of the monoids. The first property of $\mathrm{NS}(\mathcal{Q})$ is clear, since if $\mathcal{Q}$ is smooth then $M_j(\mathcal{Q}) = 0$ for all j, so $\mathrm{NS}'(\mathcal{Q}) = \mathcal{Q}$ and $\mathcal{Q}_{\mathrm{sm}} = \mathcal{Q}$.

The second property follows from the fact that either $\big(\mathcal{S}(\mathcal{Q}, v)\big)_0 = \mathcal{Q}_0$ if $v \in \sigma \notin \mathcal{Q}_0$, or $\big(\mathcal{S}(\mathcal{Q}, v)\big)_0 = \mathcal{S}(\mathcal{Q}_0, v)$ otherwise, along with the fact that $\mathrm{NS}'(\mathcal{Q})$ depends only on the set of fully non-simplicial monoids, independent of any choice of order. $\square$

The following is an immediate corollary of Proposition 5.3 and Theorem 5.5.

Corollary 5.6. *Let $\mathcal{R}_i$ $i = 1, 2$ be refinements of $\mathcal{Q}$. Then a mutual smooth refinement exists, namely $\mathrm{NS}(\mathcal{R}_1 \times_\mathcal{Q} \mathcal{R}_2) \longrightarrow \mathcal{Q}$.*

Finally, we include here a lemma which will be of use in Section 9.

Lemma 5.7. *Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be a monoidal subcomplex, and $\mathcal{R}_0 \longrightarrow \mathcal{Q}_0$ a refinement. Then there exists a refinement $\mathcal{R} \longrightarrow \mathcal{Q}$ which extends $\mathcal{R}_0$, meaning that it contains $\mathcal{R}_0$ as a subcomplex. If $\mathcal{R}_0$ is smooth, then a smooth extension exists.*

Proof. Say a monoid $\tau \in \mathcal{Q}$ is "refined by $\mathcal{R}_0$" if $\tau \in \mathcal{Q}_0$ and $\mathcal{R}_0(\tau) \longrightarrow \tau$ is a nontrivial refinement (i.e. $\mathcal{R}_0(\tau) \neq \tau$). We then say that $\sigma \in \mathcal{Q}$ is "damaged by $\mathcal{R}_0$" if some face $\tau \leq \sigma$ is refined by $\mathcal{R}_0$, but σ itself is not. In particular, note that if σ is damaged, then $\sigma \notin \mathcal{Q}_0$, otherwise $\mathcal{R}_0 \longrightarrow \mathcal{Q}_0$ would fail to be a refinement.

If there are no monoids which are damaged by $\mathcal{R}_0$, then

$$\mathcal{R} := \mathcal{R}_0 \cup \mathcal{Q} \setminus \mathcal{Q}_0 \longrightarrow \mathcal{Q}$$

is an extension of $\mathcal{R}_0$.

Let $d(\mathcal{Q}, \mathcal{R}_0)$ denote the minimum dimension of $\sigma \in \mathcal{Q}$ such that σ is damaged by $\mathcal{R}_0$. We will produce a refinement $\mathcal{R}_1$ of a subcomplex $\mathcal{Q}_1 \subset \mathcal{Q}$ which extends $\mathcal{R}_0$, and for which $d(\mathcal{Q}, \mathcal{R}_1) > d(\mathcal{Q}, \mathcal{R}_0)$. Proceeding by induction, we eventually obtain a refinement $\mathcal{R}_k$ of $\mathcal{Q}_k \subset \mathcal{Q}$ extending $\mathcal{R}_0$ which damages no monoids on $\mathcal{Q}$, since $d(\mathcal{Q}, \mathcal{R}_k)$ is bounded by the maximum dimension of a monoid in $\mathcal{Q}$, and then we can take $\mathcal{R} = \mathcal{R}_k \cup \mathcal{Q} \setminus \mathcal{Q}_k$ as above.

For the induction, assume a refinement $\mathcal{R}_{j-1} \longrightarrow \mathcal{Q}_{j-1} \subset \mathcal{Q}$ is given, which extends $\mathcal{R}_0$. Let $\mathcal{Q}_j$ be the subcomplex of $\mathcal{Q}$ consisting of $\mathcal{Q}_{j-1}$ and all monoids

of dimension $d(\mathcal{Q}, \mathcal{R}_{j-1})$ which are damaged by $\mathcal{R}_{j-1}$ along with their faces. Note that none of their proper faces are damaged by definition of $d(\mathcal{Q}, \mathcal{R}_{j-1})$. Let Λ_j be the set of all monoids of the form $\tau + \mathbb{Z}_+ v$ where $v = v_1 + \cdots v_n$ is the sum over extremals of a damaged monoid σ in $\mathcal{Q}_j$, and either

(i) $\tau \leq \sigma$, such that $\tau \notin \mathcal{Q}_{j-1}$, or
(ii) $\tau \in \mathcal{R}_{j-1}(\tau')$ for some $\tau' \leq \sigma$.

Then set

$$\mathcal{R}_j = \mathcal{R}_{j-1} \cup \Lambda_j.$$

We claim $\mathcal{R}_j$ refines $\mathcal{Q}_j$, and $d(\mathcal{Q}, \mathcal{R}_j) > d(\mathcal{Q}, \mathcal{R}_{j-1})$.

Indeed, identifying monoids in $\mathcal{R}_j$ with their images in monoids in $\mathcal{Q}$, the intersection of a monoid in $\mathcal{R}_{j-1}$ and one in Λ_j must be a face of each in $\mathcal{R}_{j-1}$; the intersection of two monoids in Λ_j is a face of each in Λ_j; and it is clear that the support of any $\sigma \in \mathcal{Q}_j$ is covered by the supports of monoids in $\mathcal{R}_j$, so $\mathcal{R}_j \longrightarrow \mathcal{Q}_j$ is a refinement.

For the second claim, suppose there was a monoid $\sigma \in \mathcal{Q}$ with $\dim(\sigma) \leq d(\mathcal{Q}, \mathcal{R}_{j-1})$ which was damaged by $\mathcal{R}_j$. As noted above, $\sigma \notin \mathcal{Q}_j$, which means that σ is not damaged by $\mathcal{R}_{j-1}$ and therefore must be damaged by $\mathcal{R}_j \setminus \mathcal{R}_{j-1} = \Lambda_j$. In other words, σ has a proper face τ which is non-trivially refined by Λ_j, but then $\tau \in \mathcal{Q}_j$ and $\dim(\sigma) > \dim(\tau) = d(\mathcal{Q}, \mathcal{R}_{j-1})$, a contradiction.

Finally, if $\mathcal{R}_0$ is smooth, then we can replace $\mathcal{R}$ by $\mathrm{NS}(\mathcal{R})$ since $(\mathrm{NS}(\mathcal{R}))_0 = \mathrm{NS}(\mathcal{R}_0) = \mathcal{R}_0$ by Theorem 5.5. $\square$

6. Generalized blow-up of a manifold with corners

We first describe a functor $X \longmapsto \mathcal{P}_X$ which assigns to a manifold with corners its 'basic smooth monoidal complex' over the ordered set of the boundary faces of X; and assigns to a b-map $f : X \longrightarrow Y$ a morphism $f_\sharp : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$. Next we show that for any smooth refinement $\mathcal{R} \longrightarrow \mathcal{P}_X$, the local construction of generalized blow-up in Section 3 extends to give a new manifold with corners $[X; \mathcal{R}]$ with basic complex realizing $\mathcal{R}$.

Let X be a compact manifold with corners, and consider the set $\mathcal{M}(X) = \bigcup_{k=0}^n \mathcal{M}_k(X)$ of its boundary faces. It is partially ordered with respect to inclusion, but here we will equip it with the *reverse order* $(\mathcal{M}(X), \leq)$, in which

(6.1) $$(\mathcal{M}(X), \leq) \ni G \leq F \iff G \supseteq F.$$

If $F \in \mathcal{M}_k(X)$, recall that $^{\mathrm{b}}NF \longrightarrow F$ is canonically trivialized by the frame $(x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k})$ where the x_i are any boundary defining functions for the hypersurfaces H_i such that F is a component of $\bigcap_i H_i$. These elements are well-defined independent of the choices of the x_i, and they therefore generate a smooth monoid.

Proposition 6.1 (Basic monoidal complex). *For a manifold X, the* basic monoidal complex $\mathcal{P}_X$ *over* $(\mathcal{M}(X), \leq)$ *consisting of the monoids*

$$\sigma_F = \mathbb{Z}_+ \langle x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k} \rangle \subset {}^{\mathrm{b}}NF, \quad F \in \mathcal{M}(X)$$

is smooth, complete and reduced.

Proof. Let $F \in \mathcal{M}_k(X)$, and let $(x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k})$ be the trivializing frame for $^{\mathrm{b}}NF$. Since the $x_i \partial_{x_i}$ are independent, σ_F is smooth. If $F \subseteq G$, then $^{\mathrm{b}}NG$ is

trivialized by a frame $(x_i \partial_{x_i})_{i \in I}$, $I \subset \{1, \ldots, k\}$, and at any $p \in F$, we have a natural inclusion $^{\mathrm{b}}N_p G \subset {}^{\mathrm{b}}N_p F$ which induces over $G \leq F$ the natural homomorphism

$$i_{GF} : \sigma_G = \mathbb{Z}_+ \langle x_i \partial_{x_i} \rangle_{i \in I} \hookrightarrow \sigma_F = \mathbb{Z}_+ \langle x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k} \rangle$$

which is an isomorphism onto the corresponding face of σ_F. Moreover, every face of σ_F corresponds in this way to σ_G for a unique $G \supseteq F$, so the complex is complete and reduced. $\qquad \square$

If $f : X \longrightarrow Y$ is a b-map

(6.2) $$f_\# : (\mathcal{M}(X), \leq) \longrightarrow (\mathcal{M}(Y), \leq)$$

is order preserving.

Proposition 6.2 (Morphism of a b-map)**.** *Let $f : X \longrightarrow Y$ be an interior b-map. Then $^{\mathrm{b}}f_* : {}^{\mathrm{b}}NF \longrightarrow {}^{\mathrm{b}}Nf_\#(F)$ induces a morphism*

(6.3) $$f_\flat : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$$

of monoidal complexes over $f_\# : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$.

Proof. If $f_\#(F) = G$, then at any point $p \in F$,

$$^{\mathrm{b}}f_* : {}^{\mathrm{b}}N_p F \longrightarrow {}^{\mathrm{b}}N_{f(p)} G$$

is linear, and with respect to bases $\{x_i \partial_{x_i}\}$ for $^{\mathrm{b}}NF$ and $\{x'_j \partial_{x'_j}\}$ for $^{\mathrm{b}}NG$, $^{\mathrm{b}}f_*$ is integral with non-negative coefficients, thus

$$f_\flat := {}^{\mathrm{b}}f_* : \sigma_F \longrightarrow \sigma_G$$

is a monoid homomorphism.

If $F \subseteq G$, the face map $i_{GF} : \sigma_G \longrightarrow \sigma_F$ comes from the inclusion $^{\mathrm{b}}NG \subseteq {}^{\mathrm{b}}NF$ as a subspace and $^{\mathrm{b}}f_*$ intertwines this with the corresponding inclusion $^{\mathrm{b}}Nf_\#(G) \subseteq {}^{\mathrm{b}}Nf_\#(F)$, from which it follows that $f_\flat \circ i_{GF} = i_{f_\#(G)f_\#(F)} \circ f_\flat$. $\qquad \square$

Theorem 6.3 (Generalized blow-up of a manifold)**.** *If X is a manifold with corners then any smooth refinement, $\mathcal{R}$ of $\mathcal{P}_X$, defines a manifold $[X; \mathcal{R}]$, the generalized blow-up of X with respect to $\mathcal{R}$, equipped with a blow-down map*

$$\beta : [X; \mathcal{R}] \longrightarrow X,$$

restricting to a diffeomorphism of the interiors and such that $\beta_\flat : \mathcal{P}_{[X;\mathcal{R}]} \longrightarrow \mathcal{P}_X$ factors through an isomorphism

$$\mathcal{P}_{[X;\mathcal{R}]} \xrightarrow{\cong} \mathcal{R}$$

of monoidal complexes.

In particular $\mathcal{M}([X; \mathcal{R}])$ is determined as an ordered set by $\mathcal{R}$, so $[X; \mathcal{R}]$ has a unique boundary face F_τ for each $\tau \in \mathcal{R}$, with $\mathrm{codim}(F_\tau) = \dim(\tau)$ and

$$F_\sigma \subseteq F_\tau \iff \tau \leq \sigma.$$

We prove below in Corollary 8.8 that

$$[X; \mathcal{R}_1] \cong [X; \mathcal{R}_2] \iff \mathcal{R}_1 = \mathcal{R}_2.$$

Proof. As a manifold $X = \bigcup W_i$ has a locally finite open covering by coordinate charts $\phi_i : W_i \cong V_i \subset \mathbb{R}_+^{k(i)} \times \mathbb{R}^{n-k(i)}$. It can further be arranged that the cover is 'good' in the sense that all intersections of the coordinate sets are contractible and that the origin is in the image of each coordinate chart, so that the codimension $k(i)$ is achieved. Then the image of each coordinate chart is actually diffeomorphic to $\mathbb{R}_+^{k(i)} \times \mathbb{R}^{n-k(i)}$ so, by composing with such a diffeomorphism, it can be assumed that each coordinate chart is surjective. The manifold is then recovered, up to global diffeomorphism, by gluing

$$X \cong \bigsqcup_i V_i / \sim$$

where the equivalence relation is generated by the transition maps, $p \sim q$ if and only if $f_{ij}(p) = q$ for some i and j. Here, $f_{ij} = \phi_j \circ \phi_i^{-1} : O_{ij} \longrightarrow O_{ji}$ is a diffeomorphism on the sets $O_{ij} = \phi_i(W_i \cap W_j)$, whenever $W_i \cap W_j \neq \emptyset$.

In essence the blown-up manifold is obtained by blowing up each coordinate chart and showing that the transition maps lift to be smooth.

For each i, let $F_i \in \mathcal{M}_{k(i)}(X)$ be the unique boundary face of codimension $k(i)$ such that $F_i \cap W_i \neq \emptyset$ and let $\mathcal{R}(F_i) = \mathcal{R}(\sigma_{F_i})$ be the induced refinement of σ_{F_i}, interpreted as a refinement of the basic monoid $\sigma_{F_i} \cong \sigma_{\mathbb{R}_+^n}$ of $\mathcal{R}_+^n$. Whenever $W_i \cap W_j \neq \emptyset$, there is a unique smallest boundary face $G_{ij} \in \mathcal{M}(X)$ which meets O_{ij} and contains both F_i and F_j, so that $\mathcal{R}(F_i) = (\mathcal{R}(G_{ij}))(\sigma_{F_i})$. Then set

$$V_i' = [\mathbb{R}_+^{k(i)}; \mathcal{R}(F_i)] \times \mathbb{R}^{n-k(i)}$$

and for each pair (i, j), set

$$O_{ij}' = (\beta_i \times \mathrm{Id})^{-1}(O_{ij}) \subset [\mathbb{R}_+^{k(i)}; \mathcal{R}(F_i)] \times \mathbb{R}^{n-k(i)}.$$

In light of Corollary 3.5, there are unique diffeomorphisms

$$f_{ij}' : O_{ij}' \xrightarrow{\cong} O_{ji}'$$

lifting f_{ij}, and we can therefore define

$$[X; \mathcal{R}] = \bigsqcup_i V_i' / \sim$$

where $p \sim q \iff f_{ij}'(p) = q$ for some pair (i, j).

Since they commute with the transition maps, f_{ij}', the local blow-down maps $\beta_i : V_i' \longrightarrow V_i$ patch together to define the global blow-down map

$$\beta : [X; \mathcal{R}] \longrightarrow X.$$

Clearly $[X; \mathcal{R}]$ is paracompact. To verify that it is Hausdorff, let p and q be distinct points. If $\beta(p) \neq \beta(q)$, then they can be separated by sets of the form $\beta^{-1}(O_i)$ where O_i for $i = 1, 2$ are open sets in X separating $\beta(p)$ and $\beta(q)$. On the other hand, if $\beta(p) = \beta(q)$, then p and q can be separated inside some set V_i', as in the proof of Proposition 3.1.

For a fixed $F \in \mathcal{M}(X)$, it follows from Proposition 3.1 that each V_i' for which $F_i = F$ has boundary faces $(F_i)_\sigma$ in correspondence with monoids $\sigma \in \mathcal{R}(F)$. These are connected in $[X; \mathcal{R}]$ for adjacent pairs such that $V_i \cap V_j \cap F \neq \emptyset$ since the diffeomorphisms f_{ij}' preserve the identification of boundary hypersurfaces with monoids in $\mathcal{R}(F)$. Thus for each $\sigma \in \mathcal{R}$ there is a unique boundary face $F_\sigma \in$

$\mathcal{M}([X; \mathcal{R}])$ given by the quotient of the union of the local boundary faces $(F_i)_\sigma$, and hence the blow-down maps give an identification

$$\mathcal{P}_{[X;\mathcal{R}]} \cong \mathcal{R}. \qquad \qquad \square$$

In fact the blow-up can be defined globally near each boundary face since $F \in \mathcal{M}_k(X)$ has a neighborhood in X which is diffeomorphic to $\mathbb{R}^k_+ \times F$ and then the preimage of this open set in $[X; \mathcal{R}]$ is diffeomorphic to $[\mathbb{R}^k_+; \mathcal{R}(F)] \times F$ obtained by localizing the resolution to boundary faces containing F.

If $f : X \longrightarrow Y$ is a b-map and $\beta : [Y; \mathcal{R}] \longrightarrow Y$ is a generalized blow-down map we say f is *compatible with* β if the morphism $f_\natural : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$ factors through a morphism $\phi : \mathcal{P}_X \longrightarrow \mathcal{R}$:

(6.4)

$$\begin{array}{ccc} & & \mathcal{R} \\ & \overset{\phi}{\nearrow} & \downarrow \\ \mathcal{P}_X & \longrightarrow & \mathcal{P}_Y \end{array}$$

Theorem 6.4 (Lifting b-maps)**.** *If $f : X \longrightarrow Y$ is an interior b-map compatible with a generalized blow-up in the sense of* (6.4) *then f lifts uniquely to a b-map $f' : X \longrightarrow [Y; \mathcal{R}]$ such that*

$$\begin{array}{ccc} & & [Y; \mathcal{R}] \\ & \overset{f'}{\nearrow} & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes, and such that $f'_\natural = \phi : \mathcal{P}_X \longrightarrow \mathcal{R} \cong \mathcal{P}_{[Y;\mathcal{R}]}$.

Proof. Again, the construction is local. We consider a covering of Y by coordinate charts $V_i \cong \mathbb{R}^{k(i)}_+ \times \mathbb{R}^{n-k(i)}$ and, refining if necessary, a covering of X by coordinate charts $W_j \cong \mathbb{R}^{l(j)}_+ \times \mathbb{R}^{m-l(j)}$ such that for all j, $f(W_j) \subset V_i$ for some i. As in the previous proof, for each i there is a maximal codimension face $F_i \in \mathcal{M}_{k(i)}(Y)$ such that $F_i \cap V_i \neq \emptyset$ and similarly for each j a face $G_j \in \mathcal{M}_{l(j)}(X)$ such that $W_j \cap G_j \neq \emptyset$. Shrinking the coordinate charts if necessary, we can assume without loss of generality that $f_\#(G_j) = F_i$.

Locally, f has the form

$$f_j = f_{|W_j} : \mathbb{R}^{l(j)}_+ \times \mathbb{R}^{m-l(j)} \longrightarrow \mathbb{R}^{k(i)}_+ \times \mathbb{R}^{n-k(i)}$$

which lifts by Corollary 3.7 to a b-map

$$f'_j : W_j \cong \mathbb{R}^{l(j)}_+ \times \mathbb{R}^{m-l(j)} \longrightarrow V'_i \cong [\mathbb{R}^{k(i)}_+; \mathcal{R}(F_i)] \times \mathbb{R}^{n-k(i)}.$$

It follows from the functoriality of these local lifted maps that they are compatible with the transition maps $g'_{ij} : O'_{ij} \longrightarrow O'_{ji}$ used to construct $[Y; \mathcal{R}]$, and so the f'_j patch together to form a b-map

$$f' : X \longrightarrow [Y; \mathcal{R}]$$

as claimed. $\qquad \qquad \square$

Even if $f : X \longrightarrow Y$ is not necessarily compatible with the refinement giving the blow-up $[Y; \mathcal{R}]$, there are generalized blow-ups of X through which f does lift. Indeed, from Theorem 6.4 above a generalized blow-up $[X; \mathcal{S}]$ admits a map to $[Y; \mathcal{R}]$ over $f : X \longrightarrow Y$ precisely when $f_\flat \circ \beta_\flat : \mathcal{S} \longrightarrow \mathcal{P}_Y$ factors through $\mathcal{R} \longrightarrow \mathcal{P}_Y$. Such a blow-up always exits.

Theorem 6.5 (Blowing up the domain). *Let $f : X \longrightarrow Y$ be an interior b-map and $[Y; \mathcal{R}] \longrightarrow Y$ a generalized blow-up. Then there exists a generalized blow-up $[X; \mathcal{S}] \longrightarrow X$ and a map $f' : [X; \mathcal{S}] \longrightarrow [Y; \mathcal{R}]$ such that*

$$
\begin{array}{ccc}
[X; \mathcal{S}] & \xrightarrow{\;f'\;} & [Y; \mathcal{R}] \\
\downarrow{\scriptstyle \beta_\mathcal{S}} & & \downarrow{\scriptstyle \beta_\mathcal{R}} \\
X & \xrightarrow{\;f\;} & Y
\end{array}
$$

commutes.

Proof. First we consider $\mathcal{P}_X \times_{\mathcal{P}_Y} \mathcal{R} \longrightarrow \mathcal{P}_X$, which is a refinement by Proposition 5.4. If it is a smooth refinement, then we take $\mathcal{S} = \mathcal{P}_X \times_{\mathcal{P}_Y} \mathcal{R}$ and we are done. Note that in this case $[X; \mathcal{S}]$ is the unique "minimal" blow-up, meaning it is universal among blow-ups of X which lift f: any other blow-up which lifts f must factor through $[X; \mathcal{S}]$ by the universality of fiber products of monoidal complexes and Theorem 6.4. In general however, $\mathcal{P}_X \times_{\mathcal{P}_Y} \mathcal{R}$ is not smooth. We let $\mathcal{S}$ be any smooth refinement of $\mathcal{P}_X \times_{\mathcal{P}_Y} \mathcal{R}$, for instance the natural smooth refinement $\mathrm{NS}(\mathcal{P}_X \times_{\mathcal{P}_Y} \mathcal{R})$ of Theorem 5.5. There are many other choices, none of which is universal. $\square$

Note that this includes Theorem 6.4 as a special case, as follows from the fact that $f_\flat : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$ is compatible with $\mathcal{R} \longrightarrow \mathcal{P}_Y$ if and only if $\mathcal{P}_X \times_{\mathcal{P}_Y} \mathcal{R} \longrightarrow \mathcal{P}_X$ is an isomorphism, which we leave as an exercise for the reader.

7. Ordinary blow-up and examples

Recall the ordinary blow-up of $[X; F]$ of a boundary face $F \in \mathcal{M}_k(X)$. As a set this is

$$[X; F] = X \setminus F \; \cup \; SN_+ F$$

where $SN_+ F \xrightarrow{\pi} F$ denotes the inward-pointing spherical normal bundle, and the blow-down map $\beta : [X; F] \longrightarrow X$ is given by the identity on $X \setminus F$ and by π on $SN_+ F$ (see Figure 4). The smooth structure on $[X; F]$ is generated by $\beta^* C^\infty(X)$ as well as the quotients x_i/x_j (where they are finite) of boundary defining functions for the boundary hypersurfaces H_i such that F is a component of $\bigcap_i H_i$.

Proposition 7.1. *The ordinary blow-up of X at the boundary face F is the generalized blow-up corresponding to the star subdivision along the sum over generators of σ_F. That is,*

$$[X; F] \cong [X; \mathcal{S}(\mathcal{P}_X, v_F)],$$

where $v_F = \sum_i x_i \partial_{x_i}$ and $\sigma_F = \mathbb{Z}_+ \langle x_i \partial_{x_i} \rangle_{1 \leq i \leq \mathrm{codim}(F)}$.

Proof. With the smooth structure above, the "front face" $SN_+ F \in \mathcal{M}([X; F])$ is a boundary hypersurface which is fibered over F, whose fiber is a $(k-1)$-simplex where $k = \mathrm{codim}(F)$. In fact, for any coordinate chart $(x, x', y) : U \cong \mathbb{R}_+^k \times \mathbb{R}_+^{l-k} \times \mathbb{R}^{n-l}$

in X such that $F \cap U \cong \{x_1 = \cdots = x_k = 0\}$, there are k coordinate charts $\widetilde{U}_i$, $i = 1, \ldots, k$ covering $\beta^{-1}(U)$ in $[X; F]$:

$$(t_i, x', y) : \widetilde{U}_i \cong \mathbb{R}_+^k \times \mathbb{R}_+^{l-k} \times \mathbb{R}^{n-l}$$

where

$$t_{i,j} = \begin{cases} x_i & \text{if } i = j, \\ x_j/x_i & \text{otherwise.} \end{cases}$$

Thus $\beta : \widetilde{U}_i \longrightarrow U$ has the form

$$\beta : (t_i, x', y) \longmapsto (t_i^{\mu_i}, x', y) = (x, x', y)$$

where μ_i is the identity matrix with its ith row replaced by ones:

$$\mu_i = \begin{pmatrix} 1 & 0 & \ldots\ldots\ldots & 0 \\ 0 & 1 & \ldots\ldots\ldots & 0 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & 1 \quad 1 \quad \cdots & 1 \\ \vdots & & & \ddots \quad \vdots \\ 0 & & \ldots\ldots\ldots\ldots & 1 \end{pmatrix}.$$

Observe that μ_i^{T} represents the monoid inclusion

$$\mathbb{Z}_+ \left\langle x_1 \partial_{x_1}, \ldots, x_{i-1} \partial_{x_{i-1}}, v_F, \ldots, x_k \partial_{x_k} \right\rangle \longrightarrow \mathbb{Z}_+ \left\langle x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k} \right\rangle,$$

where $v_F = x_1 \partial_{x_1} + \cdots + x_k \partial_{x_k}$. The collection of these μ_i^{T}, $i = 1, \ldots, k$ therefore give the monoid homomorphisms from the maximal dimension $\tau \in \mathcal{S}(\mathcal{P}, v_F)(\sigma_F)$ to σ_F.

All coordinate charts meeting F are blown-up in this way, according to the star subdivision $\mathcal{S}(\mathcal{P}_X, v_F)$, and this is precisely the construction of the generalized blow-up $[X; \mathcal{S}(\mathcal{P}_X, v_F)] \longrightarrow X$. $\qquad\square$

The *inhomogeneous blow-up* of $F \in \mathcal{M}(X)$ is similar to ordinary blow-up but more general. It consists again of the set $X \setminus F \cup SN_+F$, but the smooth structure is generated over $\beta^* C^\infty(X)$ by quotients of the form $x_i^{1/n(i)} / x_j^{1/n(j)}$, where the $n(i) \in \mathbb{Z}_+$ are consistently associated with the boundary hypersurfaces through F.

The proof of Proposition 7.1 can be modified in a straightforward manner to give the following result.

Proposition 7.2. *Let $n : \{H \in \mathcal{M}_1(X) \, ; \, F \subseteq H\} \longrightarrow \mathbb{Z}_+$ be an assignment of integer roots to the boundary hypersurfaces through F. Then the inhomogeneous blow-up of F with respect to n is realized by the generalized blow-up by the weighted star subdivision $[X; \mathcal{S}(\mathcal{P}_X, v_{F,n})]$, where*

$$v_{F,n} = \sum_i n(i) x_i \partial_{x_i}$$

is the corresponding weighted sum of the generators of $\sigma_F = \mathbb{Z}_+ \left\langle x_1 \partial_{x_1}, \ldots, x_n \partial_{x_n} \right\rangle$.

Iterating either of these constructions, we find that iterated boundary blow-up is also a special case of generalized blow-up. Recall that the *lift* (or *proper transform*)
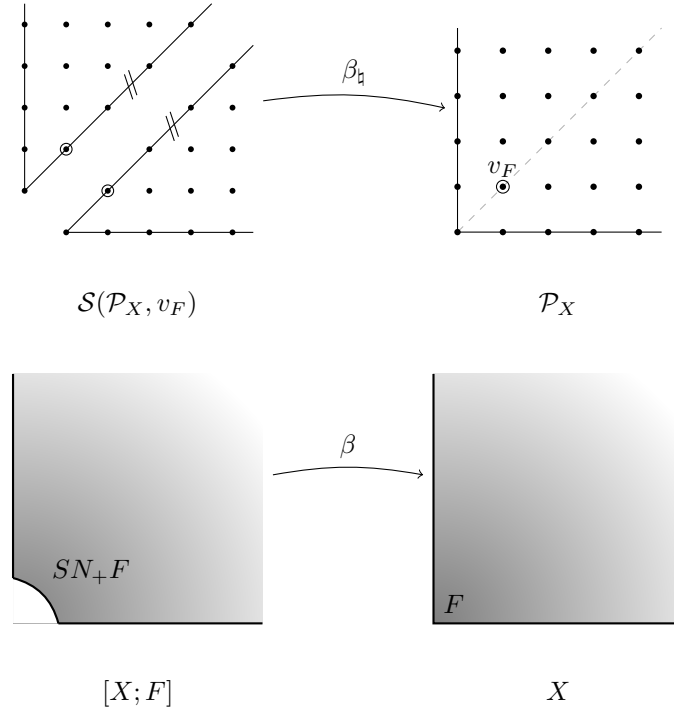
FIGURE 4. The ordinary blow-up of F in X, and the associated star subdivision. Only the monoids of maximal dimension are pictured.

of a submanifold $Y \subset X$ under a blow-up $\beta : [X; F] \longrightarrow X$ is the set

$$\beta^{\#}(Y) = \begin{cases} \beta^{-1}(Y) & \text{if } Y \subseteq F, \\ \text{clos}\left(\beta^{-1}\left(Y \setminus F\right)\right), & \text{otherwise.} \end{cases}$$

The *iterated boundary blow-up* $[X; F_N, \ldots, F_1]$, $F_i \in \mathcal{M}(X)$ is defined by successive lifting:

$$[X; F_1, \ldots, F_N] = [\cdots [[X; F_1], \beta_1^{\#}(F_2)], \cdots, \beta_1^{\#} \circ \cdots \circ \beta_{N-1}^{\#}(F_N)]$$

where $\beta_i : [X; F_1, \ldots, F_i] \longrightarrow [X; F_1, \ldots, F_{i-1}]$, and can be extended similarly to the case of inhomogeneous blow-up.

Observe that the lift of a boundary face $G \in \mathcal{M}(X)$ to the blow-up $\beta : [X; F] \longrightarrow X$ is again a boundary face of $[X; F]$. In light of the identification of boundary faces of a generalized blow-up with monoids in the refinement, it follows directly that iterated boundary blow-up is realized by iterated star subdivisions.

Corollary 7.3. *Let $F_1, \ldots, F_N \in \mathcal{M}(X)$, and let $\mathcal{R}_1, \ldots, \mathcal{R}_N$ be the sequence of refinements of $\mathcal{P}_X$ obtained by iteratively defining $\mathcal{R}_j = \mathcal{S}(\mathcal{R}_{j-1}, v_{\widetilde{F}_j})$, where $\widetilde{F}_j = \beta_1^{\#} \circ \cdots \circ \beta_{j-1}^{\#}(F_j)$.*

$$[X; F_N, \ldots, F_1] = [X; \mathcal{R}_N].$$

One might wonder at this point if there are generalized blow-ups of X which are not of the above type. In fact we will prove in Section 8 that generalized blow-ups are determined uniquely up to diffeomorphism by their refinements. It follows that any smooth refinement of $\mathcal{P}_X$ which cannot be obtained by iterated (possibly weighted) star subdivision gives a generalized blow-up which is not of this classical type. Examples of these are easy to construct provided X has corners with codimension at least 3.

8. Characterization of generalized blow-down maps

In this section we complete our treatment of generalized blow-up, showing that blow-down maps from a generalized blow-up of the target are characterized among b-maps in general by two properties.

Definition 8.1. A *generalized blow-down map* between manifolds with corners, $\beta : X \longrightarrow Y$, is a proper b-map which is a diffeomorphism of the interiors and which has b-differential, $^{\mathrm{b}}\beta_* : {^{\mathrm{b}}T_x X} \longrightarrow {^{\mathrm{b}}T_{\beta(x)} Y}$, an isomorphism for each $x \in X$.

The range of a continuous proper map is closed, so it follows directly from the definition that a generalized blow-down map is surjective. It is convenient, and no restriction, to assume in the subsequent discussion that X, and hence Y, is connected. We continue to follow the convention for local coordinates (x, y), where $x_i \in \mathbb{R}_+$ are local boundary defining functions and $y_i \in \mathbb{R}$ are tangential variables.

First we find a local normal form for a generalized blow-down map.

Lemma 8.2. *Let $\beta : X \longrightarrow Y$ be a generalized blow-down map between compact manifolds, $p \in G \setminus \partial G$ a point in the interior of a boundary face $G \in \mathcal{M}_k(X)$, and $(x', y') = (x'_1, \ldots, x'_{k'}, y'_1, \ldots, y'_{n-k'})$ local coordinates near $q = f(p) \in F = \beta_\#(G) \in \mathcal{M}_{k'}(Y)$. Then there exist local coordinates (x, y, z) near p, with the $z_i > 0$, such that, after perhaps renumbering the x' coordinates near q, β has the local form*

$$(8.1) \quad \beta(x, z, y) = (x^{\nu_1}, \ldots, x^{\nu_k}, z_1 x^{\nu_{k+1}}, \ldots, z_{k'-k} x^{\nu_{k'}}, y_1, \ldots, y_{n-k'}) = (x', y'),$$

where the first k of the $\nu_i \in \mathbb{Z}^k$ are linearly independent.

Proof. Let $\overline{x}_j$, $j = 1, \ldots, k$ be local boundary defining functions for hypersurfaces through p. Then, since β is an interior b-map,

$$(8.2) \qquad\qquad \beta^*(x'_i) = a_i \, \overline{x}^{\nu_i} = a_i \prod_{j=1}^{k} \overline{x}_j^{\nu_{ji}}, \; 0 < a_i \in C^\infty(X)$$

where $\nu_{ji} \in \mathbb{Z}_+$. Indeed, $\nu^{\mathrm{T}} \in \mathrm{Mat}(k' \times k, \mathbb{Z}_+)$ is the matrix representing $^{\mathrm{b}}\beta_* : {^{\mathrm{b}}NG} \to {^{\mathrm{b}}NF}$ with respect to the bases $\{\overline{x}_i \partial_{\overline{x}_i}\}$ and $\{x'_j \partial_{x'_j}\}$.

Since ν must have full rank (or else $^{\mathrm{b}}\beta_*$ could not be bijective), relabeling the x'_i appropriately ensures that the $k \times k$ matrix formed by the first k entries in the ν_{ji} is invertible and these give the vectors $\nu_i := (\nu_{1i}, \ldots, \nu_{ki})$ for $i = 1, \ldots, k$.

Changing the $\overline{x}_j$ by positive smooth factors, $x_j = b_j \overline{x}_j$, $0 < b_j \in C^\infty(X)$, multiplies the coefficient functions a_i in (8.2) by the monomials b^{ν_i}. The independence of ν_i, $i \le k$ means that the b_j can be chosen so each $a_i \equiv 1$ for $i \le k$. This gives the first k equations in (8.1).

The tangential coordinates y'_i pull back under β to be smooth and independent at p, so we take $y_i = \beta^* y'_i$ for $i = 1, \ldots, n - k'$, ensuring the last $n - k'$ equations in (8.1), without affecting the first k.

Finally then consider the pull-back of the last $k' - k$ boundary defining functions. The logarithmic differentials, $^b\beta^*(dx_i'/x_i')$ must be independent at p, and be independent of the dx_j/x_j, $j = 1, \ldots k$, and dy_l, $l = 1, \ldots, n - k'$. In view of (8.2) this means precisely that the smooth differentials da_i/a_i must be linearly independent at p for $i = k + 1, \ldots, k'$ and since the $a_i > 0$ this in turn is equivalent to the independence of the corresponding da_i, so $z_i := a_{k-i}$, $i = 1, \ldots, k - k'$, can be introduced as additional tangential coordinates giving (8.1). $\square$

In fact, one can take the x_i' to be globally defined boundary defining functions on a neighborhood of $F \setminus \partial F$, and then it follows from the proof that the x_i can be taken to be global on a neighborhood of $G \setminus \partial G$. It follows similarly that the y_i and z_j are globally defined on each fiber $\beta^{-1}(q') \cap G \setminus \partial G$ for q' in a neighborhood of q, and it follows that these fibers are contractible. In fact, more is true.

Lemma 8.3. *The functions z_i, $i = 1, \ldots, k' - k$ in (8.1) are globally defined on the fibers $\beta^{-1}(q') \cap G \setminus \partial G$ for q' in a neighborhood of q, and the map*

$$(8.3) \qquad z : \beta^{-1}(q) \cap G \setminus \partial G \longrightarrow (0, \infty)^{k'-k}$$

is surjective.

Proof. As noted, z is a globally defined map on the fiber in light of (8.2) and the fact that $z_i = a_{k-i}$. Suppose then that (8.3) is not surjective, and consider a point in the closure of the image under z of a component of $\beta^{-1}(q) \cap G \setminus \partial G$. By compactness of G, this must be the image of a point p' in ∂G, in the interior of a boundary face $H \subseteq G$ with $H \in \mathcal{M}_r(X)$, say. The assumption that z is not surjective means that $z(p')$ lies inside $(0, \infty)^{k-k'}$, so that $0 < z_i(p') < \infty$ for all $i \le k - k'$.

By continuity p' is mapped by β to q. Consider the construction of the x_i and z_i in the proof of Lemma 8.2, where we now begin with additional boundary defining functions $\overline{x}_i'$, $i = k + 1, \ldots, r \le k'$ for H. The equations (8.2) become

$$\beta^*(x_i') = a_i \, \overline{x}^{\nu_j} = a_i' \, \overline{x}'^{\gamma_i} \, \overline{x}^{\nu_i}, \ 0 < a_i',$$

where not all the $\gamma_j \in \mathbb{Z}_+^{r-k}$ can vanish without violating the isomorphism condition on $^b\beta_*$. It follows by going through the construction in the proof that $z_i = e_i \overline{x}_i'^{\mu}$, where $\mu_i \in \mathbb{Q}^r$ cannot all vanish (though μ_{ji} may be non-integral and/or negative since the b_j may now involve the $\overline{x}_i'$), so at least one z_i tends to either zero or infinity at H, giving a contradiction. $\square$

From the normal form (8.1) we derive the following path lifting result, which determines in which face $G \in \beta^{-1}(F)$ a lifted path will hit the boundary of X from its b-tangent at $t = 0$.

Lemma 8.4. *Let $\beta : X \longrightarrow Y$ be a generalized blow-down, $G \in \mathcal{M}(X)$, $F = \beta_\#(G) \in \mathcal{M}(Y)$ and $q \in F \setminus \partial F$, with coordinates (x', y') centered at q. If*

$$\gamma : [0, \epsilon) \ni t \longmapsto (t^\kappa, 0) \in Y$$

is a path with endpoint at q and initial b-tangent vector $\kappa = \sum \kappa_i x_i' \partial_{x_i'} \in \, ^bN_+F$, with all $\kappa_i > 0$ then the lift $\mathrm{clos}(\beta^{-1}(\gamma((0, \epsilon)))$ of the image of γ to X meets $\beta^{-1}(q) \cap G \setminus \partial G$ if and only if there exists $\lambda = \sum_i \lambda_i x_i \partial_{x_i} \in \, ^bN_+G$ such that $^bf_*(\lambda) = \kappa$.

Proof. Since β is a diffeomorphism on the interiors, the smooth path $\gamma : (0, \epsilon) \longrightarrow Y \setminus \partial Y$ lifts to $X \setminus \partial X$. If such a $\lambda \in {}^b N_+ G$ does exist, then near a component of $\beta^{-1}(q) \cap G \setminus \partial G$ it follows directly that γ has a lift

$$[0, \epsilon) \ni t \longmapsto (t^\lambda, 0, 1) = (x, y, z)$$

extending to $t = 0$, with endpoint $p = (0, 0, 1) \in G \setminus \partial G$. (Observe that p lies in the domain of the coordinates (x, y, z) in light of Lemma 8.3.)

If no such λ exists, then every point p of $\beta^{-1}(q) \cap G \setminus \partial G$ has a neighborhood

$$D(\epsilon, p) = \{(x, y, z) \; ; \; x_i < \epsilon, i = 1, \ldots, k, |y| < \epsilon\}, \; \epsilon > 0$$

which does not meet the lift of γ to $X \setminus \partial X$.

Indeed, the image of such a neighborhood under β contains a point of γ if and only if

$$(t^{\kappa_1}, \ldots, t^{\kappa_{k'}}, 0) = (x^{\nu_1}, \ldots, x^{\nu_k}, z_1 x^{\nu_{k+1}}, \ldots, z_{k'-k} x^{\nu_{k'}}, y).$$

Taking the logarithm of the first k conditions gives

$$(8.4) \qquad\qquad \kappa_i \log t = \sum_j \nu_{ji} \log x_j = ({}^b \beta_* \log x)_i$$

since ν^{T} is the matrix representing ${}^b \beta_*$. Now, in $D(\epsilon, p)$, the vector $(\log x) \in (-A, -\infty)^k$ where $A = -\log \epsilon > 0$. Thus, right side of (8.4) lies in $-{}^b \beta_* ({}^b N_+ G)$, but by assumption $\kappa \notin {}^b \beta_* ({}^b N_+ G)$ so (8.4) can have no solution with $0 < t < 1$. $\quad\square$

In fact, though we do not use this directly below, Lemmas 8.2 and 8.4 show that for all G such that $\beta_\#(G) = F$, the map $\beta : G \setminus \partial G \longrightarrow F \setminus \partial F$ is a fibration, with fibers diffeomorphic to $(0, \infty)^{k'-k}$.

To see this note that since $\beta : G \longrightarrow F$ is an interior b-map it follows from (8.1) that

$$(8.5) \qquad\qquad \beta : G \setminus \partial G \longrightarrow F \setminus \partial F$$

is a submersion. By compactness of G, any limit point of the range of (8.5) must be the image of a point in G, so it must be surjective, or else an interior point of F would only meet the boundary of G which is inconsistent with the defining properties of an interior b-map. By Lemma 8.2 it follows that $z : \beta^{-1}(q) \cap G \setminus \partial G \longrightarrow (0, \infty)^{k'-k}$ is a covering map, hence the fibers of (8.5) are a disjoint union of components diffeomorphic to the latter space, but by Lemma 8.4 these fibers must be connected, since otherwise there would be multiple preimages of $\gamma(t)$ for small t.

We now proceed to show that for a generalized blow-down $\beta : X \longrightarrow Y$, the image $\beta_\flat(\mathcal{P}_X)$ is a well-defined smooth refinement of $\mathcal{P}_Y$.

Proposition 8.5. *If $\beta : X \longrightarrow Y$ is a generalized blow-down map then the morphism of monoidal complexes*

$$\beta_\flat : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$$

is a smooth refinement.

Proof. Since ${}^b \beta_*$ is bijective, $\beta_\flat$ is necessarily injective, and it suffices to verify that the cones

$$\{\operatorname{supp}(\beta_\flat(\sigma_G)) = \beta_*({}^b N_+ G) \; ; \; \beta_\#(G) = F\} \subset \operatorname{supp}(\sigma_F) = {}^b N_+ F$$

have union equal to ${}^b N_+ F$, and have no common interior vectors.

First observe that the union of these cones is indeed ${}^{b}N_{+}F$. If not, the complement, which is open, would contain an interior point of ${}^{b}N_{+}F$. Thus, proceeding by contradiction, we can suppose that there is a vector $\kappa = (\kappa_1, \ldots, \kappa_k) \in {}^{b}N_{+}F$, with positive integer entries, which is disjoint from all the $\beta_*({}^{b}N_{+}G)$. By Lemma 8.4 there is a path $\gamma(t) \in Y$ with initial b-differential equal to κ and endpoint at $q \in F \setminus \partial F$ whose lift to X does not meet any G with $\beta_\#(G) = F$. On the other hand, by the properness of β there is a sequence $0 < t_j \to 0$ such that the points $\beta^{-1}(\gamma(t_j))$ converge in X. By continuity, the limit must be in $\beta^{-1}(q)$. However, since q lies in the interior of F such a point must lie in the interior of one of the G with $\beta_\#(G) = F$ which is a contradiction.

Next consider two of the cones $\beta_*({}^{b}N_{+}G_i)$, $i = 1, 2$ which contain an interior vector κ of ${}^{b}N_{+}F$ in both their relative interiors, i.e. this point is the image of an interior vector λ_i of each of the ${}^{b}N_{+}G_i$. Lemma 8.4 applies to both faces, hence the curve γ with endpoint $q \in F \setminus \partial F$ and initial b-tangent vector κ has a lift with endpoint in the interior of each boundary face. The assumption that β is a diffeomorphism in the interior therefore ensures that these two boundary faces have interiors which intersect and they are therefore equal.

We conclude that $\beta_\natural : \mathcal{P}_X \to \mathcal{P}_Y$ is a refinement, which must be smooth since each $\sigma_G \in \mathcal{P}_X$ is smooth. $\qquad\square$

Proposition 8.6. *A generalized blow-down map $\beta : X \longrightarrow Y$ is a diffeomorphism if and only if*

$$\beta_\natural : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$$

is invertible.

Proof. That $\mathcal{P}_Y \cong \beta_\natural(\mathcal{P}_X)$ if β is a diffeomorphism is clear, since $\beta_\natural^{-1}$ furnishes an inverse.

Assume then that $\beta_\natural : \mathcal{P}_X \cong \mathcal{P}_Y$. For any $F \in \mathcal{M}_{k'}(Y)$, there is therefore a unique $G \in \mathcal{M}_k(X)$ with $\beta_\#(G) = F$ and $\beta_\natural : \sigma_G \cong \sigma_F$ (it follows that $k = k'$). Since the ν_j, $j = 1, \ldots, k$ in Lemma 8.2 are precisely the coordinates for the generators of σ_G in terms of those of σ_F, we can arrange that $\nu = \mathrm{Id}$, and there are therefore local coordinates as in (8.1) near each point in which $\beta = \mathrm{Id}$. $\qquad\square$

Theorem 8.7. *If $\beta : X \longrightarrow Y$ is a generalized blow-down map then*

$$X \cong [Y; \mathcal{P}_X]$$

with respect to the simplicial refinement $\beta_\natural : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$.

Proof. Let $Y_0 = [Y; \mathcal{P}_X]$, and let $\beta_0 : Y_0 \longrightarrow Y$ be the blow-down. The morphism $\beta_\natural : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$ is tautologically compatible with the refinement $\mathcal{P}_X \longrightarrow \mathcal{P}_Y$, since $\beta_\natural$ factors through the identity morphism $\mathrm{Id} : \mathcal{P}_X \longrightarrow \mathcal{P}_X$.

From Theorem 6.4 then, β lifts to a b-map

$$\beta' : X \longrightarrow Y_0$$

which is easily seen to be a generalized blow-down since $\beta : X \setminus \partial X \longrightarrow Y \setminus \partial Y$ factors through $\beta' : X \setminus \partial X \longrightarrow Y_0 \setminus \partial Y_0$, which must therefore be a diffeomorphism. Since the lifted map on monoidal complexes is just the identity,

$$\beta'_\natural = \mathrm{Id} : \mathcal{P}_X \longrightarrow \mathcal{P}_X,$$

$X \cong Y_0$ by Proposition 8.6. $\qquad\square$

As noted earlier one consequence of this is that the blow-up with respect to a refinement determines the refinement.

Corollary 8.8. *Let* $\mathcal{R}_i$, $i = 1, 2$ *be two refinements of* $\mathcal{P}_X$, *then* $[X; \mathcal{R}_1] \cong [X; \mathcal{R}_2]$ *if and only if* $\mathcal{R}_1 = \mathcal{R}_2$ *as refinements of* $\mathcal{P}_X$.

9. Binomial subvarieties

We consider subvarieties of a manifold which near the boundary have the local form

$$(9.1) \qquad a_i x^{\alpha_i} = b_i x^{\beta_i}, \ y_j = 0, \ a_i, \ b_i > 0,$$

which is to say they are given by the vanishing of some binomial equations with smooth positive coefficients in the boundary defining variables, and the vanishing of some interior variables. Such objects occur naturally in the setting of fiber products and in other contexts, for instance the embedding under a b-map of one manifold into another.

After verifying that such subvarieties are well-defined and have a standard form, we show that, although they are not in general smoothly embedded manifolds with corners, they have enough structure to support the machinery of monoidal complexes and b-maps. This allow us to develop their resolution theory in the next section. While it would be possible to define an intrinsic category of abstract 'binomial varieties', since it suffices in our later treatment of fiber products, attention here is restricted to the case of a binomial variety explicitly embedded in a manifold.

Observe in (9.1) that by dividing by $b_i x^{\beta_i}$ and also exponentiating the interior coordinates, the equations take the unified form $a_i' x^{\gamma_i} = 1$, with $\gamma_i = \alpha_i - \beta_i$, $a_i' = a_i/b_i > 0$ in the first case and $\gamma_i = 0$, $a_j = \exp(y_j)$ in the second. This may involve the cancellation of factors of x_i which appear on both sides of the equation and hence the loss of some solutions contained entirely in the boundary. In view of this we will define binomial structures on sets which are the closure of their intersection with the interior as in (9.4) below.

With this motivation in mind, on any manifold consider the set of functions

$$\mathcal{G}(X) = \bigcup_{\gamma:\mathcal{M}_1(X)\to\mathbb{Z}} \mathcal{G}_\gamma(X), \text{ where}$$
$$(9.2)$$
$$\mathcal{G}_\gamma(X) = \{u \in C^\infty(X \setminus \partial X) \ ; \ u = a\rho^\gamma, \ 0 < a \in C^\infty(X)\};$$

here $\rho = (\rho_H)_{H\in\mathcal{M}_1(X)}$ is a collection of global boundary defining functions. Observe that $\gamma : \mathcal{M}_1(X) \longrightarrow \mathbb{Z}$ induces a functional ${}^b\gamma : {}^bNG \longrightarrow \mathbb{C}$ for any $G \in \mathcal{M}(X)$ by setting

$$(9.3) \qquad {}^b\gamma = \sum_{\mathcal{M}_1(X)\ni H\supseteq G} \gamma(H)\frac{d\rho_H}{\rho_H}.$$

The $\mathcal{G}_\gamma(X)$ are independent of the defining functions used in (9.2) and pull back under any interior b-map $f : Z \longrightarrow Y$ giving an inclusion $f^*\mathcal{G}(Y) \subset \mathcal{G}(X)$. Clearly $\mathcal{G}(X)$ is an abelian group under pointwise multiplication, and $\mathcal{G}_{\gamma_1}(X) \cdot \mathcal{G}_{\gamma_2}(X) \subset \mathcal{G}_{\gamma_1+\gamma_2}(X)$.

A function $f \in \mathcal{G}(X)$ can be extended by continuity to points in ∂X at which it has finite limits but the logarithmic differential, df/f, extends by continuity to a

smooth global section of $^{b}T^{*}X$ since near the boundary it reduces to

$$\frac{df}{f} = d\log a + \sum_{H} \gamma(H) \frac{d\rho_{H}}{\rho_{H}}.$$

Definition 9.1. A *local binomial structure* on a closed subset $D \subset X$ near a point $p \in D$ consists of a coordinate neighborhood $U \ni p$ and functions $f_{i} \in \mathcal{G}_{\gamma_{i}}(U)$, $i = 1, \ldots, d$ which have independent logarithmic differentials $df_{i}/f_{i} \in {}^{b}T_{q}^{*}U$ at each $q \in D \cap U$, and which define D locally in the sense that

(9.4) $D \cap U = \mathrm{clos}_{U}\{q \in U \setminus \partial U \ ; \ f_{i}(q) = 1, \ i = 1, \ldots, d\}.$

The *codimension* of D at p is d.

An exponent vector $\gamma \in \mathcal{M}_{1}(X) \longrightarrow \mathbb{Z}$ is said to be *non-negative* (resp. *non-positive*) if $\gamma(H) \geq 0$ (resp. ≤ 0) for all $H \in \mathcal{M}_{1}(X)$ and γ is *indefinite* if it is nonzero, and neither non-negative nor non-positive, so an indefinite γ must have at least one positive and at least one negative coefficient. More locally, γ is non-negative *with respect to* $F \in \mathcal{M}(X)$ if $\gamma(H) \geq 0$ for all $H \in \mathcal{M}_{1}(X)$ such that $F \subseteq H$ and non-positivity and indefiniteness with respect to F are defined similarly.

For a local binomial structure U on $D \subset X$, the boundary faces of $X \cap U$ which are met by D can be seen by examining the indefiniteness of the exponent vectors.

Lemma 9.2. *If U, $f_{i} \in \mathcal{G}_{\gamma_{i}}(U)$ is a local binomial structure on D near p and U only meets boundary hypersurfaces which pass through p then every exponent vector γ_{i} is either zero or indefinite. Similarly if D meets the interior of $F \in \mathcal{M}(U)$ then each γ_{i} is either zero or indefinite with respect to F.*

Proof. By assumption all the local boundary defining functions vanish at p so if γ_{i} is non-zero and has all entries of a fixed sign then $a_{i}x^{\gamma_{i}}$ cannot be equal to 1 near p. The same argument applies to other boundary points of D where fewer of the boundary defining functions vanish. $\square$

Next we establish a local normal form for local binomial structures.

Lemma 9.3. *If U, $f_{i} \in \mathcal{G}_{\delta_{i}}(U)$ is a local binomial structure on D then near any boundary point $q \in D$ in U there are local coordinates (x, y) in a (possibly smaller) neighborhood U' of q in terms of which*

(9.5) $D \cap U' = \{x^{\gamma_{i}} = 1, \ i = 1, \ldots, d', \ y_{j} = 0, \ j = d' + 1, \ldots, d\}$

where the γ_{i} are linearly independent and indefinite vectors with respect to the maximal codimension boundary face through q.

Proof. By definition of local binomial structure, $D \cap U$ is the closure of its intersection with the interior, so there are interior points of D near q. Let F be the boundary face of maximal codimension containing q; so q lies in the interior of F. The b-cotangent space $^{b}T_{q}^{*}U$, of which the $d\log f_{i}$ are sections, has a 'smooth subspace' $^{b}N_{q}^{\perp}F$ consisting of the differentials of smooth functions; it is the image in $^{b}T_{q}^{*}U$ of the natural map $T_{q}^{*}U \longrightarrow {}^{b}T_{q}^{*}U$, and is the annihilator of the b-normal space $^{b}N_{q}F$.

The assumed independence of the logarithmic differentials df_{i}/f_{i} in U ensures that they span a linear space of dimension d at q,

(9.6) $A_{q}(D) := \mathrm{span}\{df_{i}/f_{i}\} \subset {}^{b}T_{q}^{*}U.$

Within this space consider the intersection

$$A_q^{\mathrm{sm}}(D) = A_q(D) \cap {}^{\mathrm{b}}N_q^\perp F$$

with the smooth subspace and set $d - d' = \dim(A_q^{\mathrm{sm}}(D))$. In terms of local coordinates (x, y) at q, the functions $f_i = a_i\, x^{\delta_i'}$ where the δ_i' are the restrictions of the δ_i to the boundary hypersurfaces through q, and ${}^{\mathrm{b}}N_q^\perp F$ is the span of the dy_k. Thus $A_q^{\mathrm{sm}}(D)$ is spanned by those linear combinations of logarithmic differentials

$$\sum_i t_i df_i/f_i = \sum_{i,j} t_i \delta_{ij}' dx_j/x_j + \sum_i t_i d\log a_i$$

for which $\sum_i t_i \delta_i' \equiv 0$.

Thus, after some renumbering, an independent set of the δ_i' can be chosen and renamed γ_i, $i = 1, \ldots, d'$. The remaining f_j, $j = d' + 1, \ldots, d$ can be replaced by products $f_j' = \prod_i f_i^{t_i}$ corresponding to nontrivial independent relations $\sum_i t_i \delta_i' = 0$. Then the set $\{df_i/f_i, df_j'/f_j'\}$ is independent and spans $A_q(D)$ and D is given locally by the equations $f_i = f_j' = 1$, but the $f_j' = a_j'\, x^0$ are smooth and positive.

The functions $\log a_j'$, $j = d' + 1, \ldots, d$ can then be introduced in place of some of the tangential variables y_j and their differentials span $A_q^{\mathrm{sm}}(D)$. Since the γ_i are now linearly independent, after changing the boundary variables from x_k to $g_k x_k$ with $g_k > 0$ and renumbering, $a_i\, x^{\gamma_i}$ is reduced to the desired form x^{γ_i} giving (9.5). □

Definition 9.4. A connected, closed subset $D \subset X$ of a manifold with corners is an *interior binomial subvariety* if it has a covering by local binomial structures.

The codimension of D is well-defined as the local codimension by connectivity, and $D \cap (X \setminus \partial X)$ is a smooth manifold of dimension $\dim(D) = \dim(X) - \mathrm{codim}(D)$.

It follows from the proof of Lemma 9.3 that the 'b-conormal spaces' $A_q(D) \subset {}^{\mathrm{b}}T_q^* X$ in (9.6) are well-defined at each point $q \in D$ and independent of the local binomial structure used – for points in the interior $A_q(D)$ is just the ordinary conormal space to the smooth manifold $D \cap X \setminus \partial X$, and its extension by continuity to $D \cap \partial X$ is unique. So we may proceed as for a smooth submanifold of a manifold and set

$$\begin{aligned}
(9.7) \qquad {}^{\mathrm{b}}T_q D &= (A_q(D) \subset {}^{\mathrm{b}}T_q^* X)^\perp \subset {}^{\mathrm{b}}T_q X, \\
{}^{\mathrm{b}}N_q D_G &= {}^{\mathrm{b}}T_q D \cap {}^{\mathrm{b}}N_q G, \quad p \in D \cap G, \ G \in \mathcal{M}(X).
\end{aligned}$$

The b-tangent bundle ${}^{\mathrm{b}}TD \longrightarrow D$ is actually independent of the 'binomial embedding' of D in X once this is understood correctly, but here we persist with the extrinsic discussion. Observe that each ${}^{\mathrm{b}}N_q D_G$ is just the nullspace of the maps ${}^{\mathrm{b}}\gamma_i$ as in (9.3) for the γ_i occuring in any local binomial structure for D near p. In particular it is invariant with respect to replacing the γ_i by linear combinations with the same span.

From the local normal form (9.5) it follows that the intersection of D with a boundary face through p is again a binomial subvariety, as we now show. By way of motivation, notice that the behavior of the b-tangent spaces on passage to a boundary face of X: for $p \in G \in \mathcal{M}(X)$, ${}^{\mathrm{b}}T_p G = {}^{\mathrm{b}}T_p X / {}^{\mathrm{b}}N_p G$.

Lemma 9.5. *Suppose $D \subset X$ is an interior binomial variety with codimension d, then for any $G \in \mathcal{M}(X)$, such that $D \cap (G \setminus \partial G) \neq \emptyset$, a component G' of the closure*

$D_G := \mathrm{clos}\,(D \cap G \setminus \partial G)$ *is an interior binomial subvariety of* G *with codimension* $d - \dim({}^{\mathrm{b}}N_p D_G)$, *for any* $p \in G'$ *and*

$$(9.8) \qquad\qquad {}^{\mathrm{b}}T_p D_G = {}^{\mathrm{b}}T_p D / {}^{\mathrm{b}}N_p D_G.$$

It may well happen that $D \cap (G \setminus \partial G) = \emptyset$ but $D \cap G \neq \emptyset$ in which case we do not regard $D \cap G$ as an *interior* binomial subvariety of G since it lies only in the boundary.

Proof. Let p be a limit point of $G' \cap G \setminus \partial G$, and consider the local form for D near p given on a neighborhood U' by (9.5); we can assume without loss of generality that U' does not meet any other component of $D \cap G$. Thus the x's are local defining functions for the boundary face $F \subseteq G$ of maximal codimension through p. Divide them as $x = (x', x'')$ where the x'' define G locally.

Following the proof of Lemma 9.3 above, a maximal subset of the γ_i may be chosen so that their restrictions to the boundary faces containing G are independent, i.e. the projections $\gamma_i = (\gamma_i', \gamma_i'') \longmapsto \gamma_i''$ onto the subspace corresponding to the x'' are independent. The remaining γ_j may be replaced by independent linear combinations $\delta_j = \sum_i t_i \gamma_i$ which are non-zero only with respect to the boundary hypersurfaces which do not contain G, i.e. their projections $\delta_j = (\delta_j', \delta_j'') \longmapsto \delta_j''$ vanish. The defining conditions (9.5) may therefore be rewritten, after renumbering,

$$(9.9) \qquad \begin{array}{c} (x')^{\gamma_i'}(x'')^{\gamma_i''} = 1, \ i = 1, \ldots, k, \ (x')^{\delta_i'} = 1, \ i = k+1, \ldots, d', \\[4pt] y_j = 0, \ j = d'+1, \ldots, d, \end{array}$$

where the γ_i'' and δ_i' are separately independent, and the $\gamma_i'' dx_i''/x_i''$ span ${}^{\mathrm{b}}N_p D_G$.
   Then

$$(9.10) \qquad G' \cap U' = \left\{ (x')^{\gamma_i'} = 1, i = k+1, \ldots, d', \ y_j = 0, \ j = d'+1, \ldots, d \right\}$$

is a local binomial structure on G'. Certainly the right side of (9.10) is included in the left. To see the converse, observe that, by the assumption that D meets the interior of G near p, the system (9.9) must have a sequence of solutions with $x'' \to 0$ and all entries of x' positive but small.

Writing out first set of equations as the linear system $\gamma_i'' \cdot \log x'' = c_i$ for the vector $\log x''$, with entries the logarithms of the x'', we see that this system must, for each $N \in \mathbb{R}$, have a solution with all entries less than N with the c_i bounded. Since the γ_i'' are linearly independent, it follows that the same is true for *any* c_i. Thus in fact any solution of the equations in (9.10) corresponds to a point in G'. $\square$

In light of this result, we define the *boundary faces* of D to be the components G' of the nonempty $D_G = \mathrm{clos}(D \cap G \setminus \partial G)$. The linear space ${}^{\mathrm{b}}N_p G' := {}^{\mathrm{b}}N_p D_G = \mathrm{Null}\left\{ {}^{\mathrm{b}}\gamma_i \right\} \subset {}^{\mathrm{b}}N_p G$ as in (9.7) can then be identified as the 'b-normal space to G' as a boundary face of D,' consistent with the smooth case. The codimension of a boundary face $G' \subset D_G$ with respect to D is given by $\dim({}^{\mathrm{b}}NG')$ as expected, and we let $\mathcal{M}_k(D)$ be the boundary faces of codimension k in this sense, equipping $\mathcal{M}(D) = \bigcup_k \mathcal{M}_k(D)$ with the order of reverse inclusion.
   The intersection of the rational subspaces ${}^{\mathrm{b}}NG'$, $G' \in \mathcal{M}(D)$ with the monoids σ_G, $G \in \mathcal{M}(X)$ gives the 'basic monoidal complex' of D.

Proposition 9.6. *If* $D \subset X$ *is an interior binomial subvariety the monoids*

$$(9.11) \qquad\qquad \sigma_{G'} = {}^{\mathrm{b}}NG' \cap \sigma_G, \ G' \in \mathcal{M}(D)$$

where G' is a component of $\mathrm{clos}(D \cap G \backslash \partial G)$ *form a (not necessarily smooth) complex* $\mathcal{P}_D$ *over* $(\mathcal{M}(D), \leq)$ *and there is a natural, injective morphism of complexes*

$$(9.12) \qquad\qquad i_\natural : \mathcal{P}_D \longrightarrow \mathcal{P}_X$$

over $i_\# : \mathcal{M}(D) \longrightarrow \mathcal{M}(X)$, *where* $i_\#(G') = G$ *such that* $G' \subset D_G$ *as above.*

Proof. The monoids $\sigma_{G'}$ are clearly well-defined and toric; it suffices to verify that they form a complex. If $F' \subseteq G' \in \mathcal{M}(D)$ and $p \in F'$, then $F \subseteq G \in \mathcal{M}(X)$. Since ${}^bN_pG' = {}^bT_pD \cap {}^bN_pG$ and similarly for ${}^bN_pF'$, and since ${}^bN_pG \subseteq {}^bN_pF$, it follows that ${}^bN_pG' \subseteq {}^bN_pF'$, giving an inclusion $\sigma_{G'} \subset \sigma_{F'}$, which must be an isomorphism onto a face since it is the intersection of σ_F with a subspace. Thus

$$\mathcal{P}_D = \{\sigma_G \ ; \ G \in \mathcal{M}(D)\}$$

is a (complete, reduced) monoidal complex over $\mathcal{M}(D)$, and the inclusions $\sigma_G \subset \sigma_{i_\#(G)}$ produce a necessarily injective morphism $i : \mathcal{P}_D \longrightarrow \mathcal{P}_X$. $\qquad\square$

A very special instance of an interior binomial subvariety is a 'product-' or *p-submanifold* (see [Mel] for background). This is a smooth submanifold $D \subset X$ which meets all boundary faces of X transversally, and which is covered by coordinate neighborhoods $\big(U, (x, y)\big)$ in X such that

$$D \cap U = \{y_j = 0 \ ; \ j = 1, \ldots, \mathrm{codim}(Y)\}.$$

For a p-submanifold, it is evident that ${}^bN_pG' \equiv {}^bN_pG$ whenever $G' \in \mathcal{M}(D)$ is a component of D_G, $G \in \mathcal{M}(X)$, and so the morphism $i_\natural : \mathcal{P}_D \longrightarrow \mathcal{P}_X$ consists of monoid isomorphisms $\sigma_{G'} \xrightarrow{\cong} \sigma_G$. In other words $i_\natural$ is a *local* isomorphism, though it need not be a global one, since the D_G may consist of multiple components, and therefore $i_\# : \mathcal{M}(D) \longrightarrow \mathcal{M}(X)$ need not be injective.

In fact this global issue of the failure of $i_\# : \mathcal{M}(D) \longrightarrow \mathcal{M}(X)$ to be injective arises as a technical obstruction to the resolution of a general interior binomial variety $D \subset X$ by generalized blow-up of X in the next section. Indeed, if there are multiple components $G'_i \subset D_G$, there may be no way to refine $\sigma_G \in \mathcal{P}_X$ in a way which appropriately resolves the images $i_\natural(\sigma_{G'_i})$. Fortunately one can always pass to a 'collar neighborhood' $X' \supset D$ in X for which this obstruction does not arise. The following lemma guarantees the existence of such a neighborhood; the proof follows directly from the local normal form (9.5).

Lemma 9.7. *If $D \subset X$ is an interior binomial subvariety then there exists an open submanifold $X' \subset X$ containing D such that $i_\# : \mathcal{M}(D) \longrightarrow \mathcal{M}(X')$ is injective so for each $G \in \mathcal{M}(X')$ there is at most one connected component of D_G.*

Finally, observe that other concepts can be extended from the 'smooth' case of manifolds to binomial subvarieties.

Definition 9.8. If $D \subset X$ is a binomial subvariety, and Y a manifold with corners then a map $f : Y \longrightarrow D$ is a b-map if it is a b-map in the smooth sense, i.e. as a map $f : Y \longrightarrow X$ with $f(Y) \subset D$.

For such a b-map, the range of the b-differential ${}^bf_*$ will lie in ${}^bTD$ and if $G' \in \mathcal{M}(D)$ is the highest codimension face such that $f(F) \subset G'$ it follows that

$$(9.13) \qquad\qquad {}^bf_* : {}^bN_pF \longrightarrow {}^bN_{f(p)}G'.$$

Just as in Proposition 6.2 there is an induced map

$$(9.14) \qquad\qquad f_\natural : \mathcal{P}_Y \longrightarrow \mathcal{P}_D$$

of monoidal complexes.

10. Resolution of binomial subvarieties

In this section we use show that a smooth refinement of the monoidal complex of the ambient manifold X which also refines the monoidal complex of an interior binomial subvariety $D \subset X$ leads to a blow-up under which D lifts to p-submanifold, in particular this resolves D. The resolution of D so obtained depends essentially only on the choice of the smooth refinement of the monoidal complex $\mathcal{P}_D$.

The notion of a resolution is analogous to that of a generalized blow-up defined earlier.

Definition 10.1. If $D \subset X$ is an interior binomial subvariety then a manifold Y with a b-map $f : Y \longrightarrow X$ with $f(Y) \subset D$ is a *resolution of D* if ${}^{\mathrm{b}}f_* : {}^{\mathrm{b}}T_p Y \longrightarrow {}^{\mathrm{b}}T_{f(p)}D$ is a bijection for all $p \in Y$ and $f : Y \setminus \partial Y \longrightarrow D \setminus \partial D$ is a diffeomorphism.

First observe that the lift of D under any blow-up of X is again an interior binomial subvariety.

Proposition 10.2. *If $\beta : X_1 \longrightarrow X$ is a generalized blow-down map between manifolds and $D \subset X$ is an interior binomial subvariety then the lift (or proper transform)*

$$(10.1) \qquad\qquad \beta^{\#}(D) = \mathrm{clos}_{X_1}\big(\beta^{-1}(D \setminus \partial X)\big)$$

is an interior binomial subvariety of X_1.

Proof. Certainly $\beta^{\#}(D) \subset X_1$ is a closed subset. It is a smooth embedded submanifold in the interior, so it suffices to show that it has a local binomial structure at each boundary point. If $p \in \partial X_1 \cap \beta^{\#}(D)$ then by definition, $\beta(p) \in \partial X \cap D$ and D has a local binomial structure given by $f_i \in \mathcal{G}(X)$. The pull-backs $f^* f_i \in \mathcal{G}(X_1)$ define $\beta^{\#}(D)$ locally and their logarithmic differentials are independent since the b-differential of β is an isomorphism at each point. $\qquad\square$

Let $\mathcal{R} \longrightarrow \mathcal{P}_X$ be a smooth refinement. If $D \subset X$ is an interior binomial subvariety then $\mathcal{R}$ is *compatible with $\mathcal{P}_D$* if there is a subcomplex $\mathcal{R}_D \subset \mathcal{R}$ such that $\mathcal{R}_D \longrightarrow \mathcal{P}_X$ factors through $\mathcal{P}_D$ giving a commutative diagram

$$(10.2)$$

$$\begin{array}{ccc} \mathcal{R}_D & \xrightarrow{\;\subset\;} & \mathcal{R} \\ \downarrow & & \downarrow \\ \mathcal{P}_D & \xrightarrow{\;i_\natural\;} & \mathcal{P}_X \end{array}$$

where the vertical arrows are smooth refinements and the top is the inclusion of a subcomplex.

Proposition 10.3. *If $\mathcal{R} \longrightarrow \mathcal{P}_X$ is a smooth refinement which is compatible with an interior binomial subvariety D in the sense of (10.2) then D lifts to a p-submanifold $\widetilde{D} = \beta^{\#}(D)$ of the generalized blow-up*

$$\beta : [X; \mathcal{R}] \longrightarrow X.$$

In particular $\beta : \widetilde{D} \longrightarrow D$ is a resolution of D and $\beta_\natural : \mathcal{P}_{\widetilde{D}} \longrightarrow \mathcal{P}_D$ factors through an isomorphism $\mathcal{P}_{\widetilde{D}} \cong \mathcal{R}_D$ of monoidal complexes.

Proof. Consider an arbitrary $p' \in \widetilde{D} = \beta^\#(D)$, and let $p = \beta(p') \in D$. There is a unique face $G' \in \mathcal{M}(D)$ such that p lies in the interior of G', and we set $G = i_\#(G') \in \mathcal{M}(X)$. Since $i_\natural : \mathcal{P}_D \to \mathcal{P}_X$ in (9.12) and $\beta_\natural : \mathcal{R} \longrightarrow \mathcal{P}_X$ are both injective, we identify monoids $\sigma_{G'}$ and $\tau \in \mathcal{R}(G)$ with their respective images in σ_G.

The point $p' \in \widetilde{D} \subset [X; \mathcal{R}]$ lies in some coordinate chart $U_\tau \subset \mathbb{R}_+^k \times (0, \infty)^{n-k}$ with coordinates $(t, y) = (t_1, \ldots, t_k, y_1, \ldots, y_{n-k})$, for some $\tau \in \mathcal{R}(G)$, and we can assume that the coordinates y are pulled back identically from those on X in which D has the local normal form (9.5). Thus the equations $\{y_j = 0 \; ; \; j = d' + 1, \ldots, d\}$ are the same in U_τ. We consider the lift of the other equations, $x^{\gamma_i} = 1$ under β. On U_τ, β has the form $\beta : t \longrightarrow t^\mu = x$, and we obtain

$$\beta^* (x^{\gamma_i}) = (t^\mu)^{\gamma_i} = t^{\beta_i} = 1,$$

where $\beta_i = \mu \gamma_i$. Thus near p', $\widetilde{D} \subset [X; \mathcal{R}]$ has the local binomial structure

$$\widetilde{D} = \left\{ t^{\beta_i} = 1, y_j = 0 \right\}.$$

We will show that the β_i are each non-negative or non-positive.

Indeed, the compatibility assumption implies that, for all $\tau \in \mathcal{R}(G)$, either $\tau \in \mathcal{R}_D(G')$ and hence $\tau \subset \sigma_{G'}$, or $\tau \in \mathcal{R} \setminus \mathcal{R}_D$; in either case the intersection $\tau \cap \sigma_{G'}$ must be a face of τ. Since $\sigma_{G'} = \sigma_G \cap \mathrm{Null}\left\{{}^{\mathrm{b}}\gamma_i\right\}$, it follows that

$$\langle \gamma_i dx/x, \tau \rangle \geq 0 \text{ or } \langle \gamma_i dx/x, \tau \rangle \leq 0, \quad \text{for each } \gamma_i.$$

In other words, each vector $\gamma_i dx/x$ is either non-positive or non-negative with respect to $\tau \in \mathcal{R}(G)$. Let us assume non-negative; the other case is similar.

From this it follows that no β_i is indefinite, since for any $a \in \mathbb{R}^k$, $a_i > 0$,

$$\langle \beta_i, a \rangle = \langle \beta_i \, dt/t, a \, t\partial_t \rangle = \langle \mu \, \gamma_i \, dx/x, a \, t\partial_t \rangle = \left\langle \gamma_i dx/x, \mu^{\mathrm{T}}(a \, t\partial_t) \right\rangle \geq 0$$

as $\mu^{\mathrm{T}}(at\partial_t) \in \tau$. In light of Lemma 9.2, $\widetilde{D}$ only meets boundary faces with respect to which the β_i are zero, hence $\widetilde{D}$ has a covering by binomial structures such that

$$\widetilde{D} = \{y_j = 0 \; ; \; j = 1, \ldots, d\}$$

and is therefore a p-submanifold.

Since $\widetilde{D}$ is a p-submanifold, for any $\widetilde{G} \in \mathcal{M}(\widetilde{D})$, $\sigma_{\widetilde{G}} \cong \tau$ for some $\tau \in \mathcal{R} \cong \mathcal{P}_{[X;\mathcal{R}]}$. Finally, it follows from the fact that $\widetilde{D}$ is the lift of D that $\sigma_{\widetilde{G}} = \tau$ is actually in $\mathcal{R}_D$; hence $\mathcal{P}_{\widetilde{D}} \longrightarrow \mathcal{R}_D$ is a local isomorphism onto its image and $\beta : \widetilde{D} \longrightarrow D$ is a resolution since it is a diffeomorphism on interiors. $\square$

The resolution of binomial ideals in polynomial and power series rings by toric methods are well-known, see [Tei] for a good overview. The previous proposition can be seen as an extension of this theory to the interior binomial subvarieties we have been discussing.

Next we show that we can obtain a unique resolution of D realizing any smooth refinement $\mathcal{R}_D \longrightarrow \mathcal{P}_D$; in particular the resolution so obtained is essentially independent of the ambient manifold X.

Theorem 10.4 (Resolution of binomial varieties)**.** *If $D \subset X$ is an interior binomial subvariety, then for every smooth refinement $\mathcal{R}_D \longrightarrow \mathcal{P}_D$ there exists a resolution*

$$\beta : [D; \mathcal{R}_D] \longrightarrow D$$

which realizes the refinement in the sense that

$$\beta_\natural : \mathcal{P}_{[D;\mathcal{R}_D]} \cong \mathcal{R}_D \longrightarrow \mathcal{P}_D,$$

and $[D; \mathcal{R}_D]$ is unique up to diffeomorphism.

If $f : Y \longrightarrow D$ is a b-map from a smooth manifold, and if $f_\natural : \mathcal{P}_Y \longrightarrow \mathcal{P}_D$ factors through $\mathcal{R}_D$, then f factors through a unique b-map $\widetilde{f} : Y \longrightarrow [D; \mathcal{R}_D]$.

Remark. The notation is meant to suggest that this is in some sense the generalized blow-up in the category of (differentiable) 'binomial varieties' where the objects are treated intrinsically. Indeed, we believe that such a category exists and that generalized blow-up extends to include arbitrary refinements, not necessarily smooth. This would give essentially a realization of Kato's theory of toric singularities [Kat94] in a category of differentiable spaces with corners.

Proof. Assume that $i_\# : \mathcal{M}(D) \longrightarrow \mathcal{M}(X)$ is injective, passing if necessary to a collar neighborhood $X' \subset D$ as in Lemma 9.7. By Proposition 10.3 it suffices to show that, given $\mathcal{R}_D \longrightarrow \mathcal{P}_D$, there exists a refinement $\mathcal{R}_X \longrightarrow \mathcal{P}_X$ extending $\mathcal{R}_D$ (that is, containing $\mathcal{R}_D$ as a subcomplex), and that the resulting resolution $\beta^\#(D) \subset [X; \mathcal{R}_X]$ is well-defined, independent of the choice of such extension.

For the first step the planar refinement of Proposition 4.10 $\mathcal{S}(\mathcal{P}_X, \mathcal{P}_D) \longrightarrow \mathcal{P}_X$ is a (not necessarily smooth) refinement containing $\mathcal{P}_D$ as a subcomplex. Then by Lemma 5.7 the refinement $\mathcal{R}_D \longrightarrow \mathcal{P}_D$ can be extended to a smooth refinement $\mathcal{R}_X \longrightarrow \mathcal{S}(\mathcal{P}_X, \mathcal{P}_D)$, and the composition $\mathcal{R}_X \to \mathcal{S}(\mathcal{P}_X, \mathcal{P}_D) \to \mathcal{P}_X$ is therefore an extension of $\mathcal{R}_D$.

Next suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are smooth refinements of $\mathcal{P}_X$ extending $\mathcal{R}_D$, and set

$$\widetilde{D}_i := \beta_i^\#(D) \subset [X; \mathcal{R}_i] \quad i = 1, 2.$$

The blow-down $\beta_1 : \widetilde{D}_1 \longrightarrow D$, considered as a map to X, lifts by Theorem 6.4 to a b-map to $[X; \mathcal{R}_2]$ whose range lies in $\widetilde{D}_2$, and vice versa. Thus we obtain b-maps

$$\widetilde{D}_1 \leftrightarrow \widetilde{D}_2$$

which are generalized blow-down maps and in fact diffeomorphisms by Proposition 8.6 since $\mathcal{P}_{\widetilde{D}_1} \cong \mathcal{R}_D \cong \mathcal{P}_{\widetilde{D}_2}$.

Thus the lift $\widetilde{D} \subset [X; \mathcal{R}_X]$ is independent of the extension $\mathcal{R}_X$ of $\mathcal{R}_D$ up to diffeomorphism, and we define $[D; \mathcal{R}_D] = \widetilde{D}$ to be any such lift.

Recall that a b-map $f : Y \longrightarrow D$ induces a monoidal complex morphism $f_\natural : \mathcal{P}_Y \longrightarrow \mathcal{P}_X$ which factors through $\mathcal{P}_D$ as in (9.14). If in addition $f_\natural$ factors through $\mathcal{R}_D \longrightarrow \mathcal{P}_D$, then it follows that $f_\natural : \mathcal{P}_Y \longrightarrow \mathcal{P}_X$ factors through any extension $\mathcal{R}_X$, and f admits a unique lift $f' : Y \longrightarrow [X; \mathcal{R}_X]$ by Theorem 6.4. The image of $Y \setminus \partial Y$ under f' lies in $\widetilde{D} \setminus \partial[X; \mathcal{R}_X]$, and therefore by continuity $f'(Y) \subset \widetilde{D}$ so $f' : Y \longrightarrow \widetilde{D}$ is a b-map. Since the resolutions coming from different extensions are diffeomorphic, it follows from the naturality of the lifted b-maps that

$$f' : Y \longrightarrow [D; \mathcal{R}_D]$$

is well-defined, independent of the choice of extension. $\square$

Observe that if $\mathcal{P}_D$ is already smooth, then the above suggests that D is in some sense already a smooth manifold, though it may not be smoothly embedded in X. Indeed, it follows from the proof of Theorem 10.4 that there exists a refinement $\mathcal{R}_X \longrightarrow \mathcal{P}_X$ which is trivial on $\mathcal{P}_D$, and this gives a 'minimal' resolution of D which is universal in this case.

Theorem 10.5. *If $D \subset X$ is an interior binomial variety and $\mathcal{P}_D$ is smooth, then there exists a* universal *resolution $[D; \mathcal{P}_D] \longrightarrow D$, with the property that any b-map $f : Y \longrightarrow D$ factors uniquely through $[D; \mathcal{P}_D]$. In particular, any other resolution of D is a generalized blow-up of this universal resolution.*

Proof. From Theorem 10.4, there exists a unique resolution $[D; \mathcal{P}_D]$ coming from the trivial refinement Id : $\mathcal{P}_D \longrightarrow \mathcal{P}_D$. Since any b-map $f : Y \longrightarrow D$ induces a morphism $f_\natural : \mathcal{P}_Y \longrightarrow \mathcal{P}_D$ which necessarily factors through this trivial refinement, f necessarily factors through $[D; \mathcal{P}_D]$. If f itself is a resolution, it follows that the lift $f' : Y \longrightarrow [D; \mathcal{P}_D]$ is also diffeomorphic on the interiors, with bijective b-differential; in other words, it is a generalized blow-down map onto the manifold $[D; \mathcal{P}_D]$, and hence a blow-up of this space by Theorem 8.7. $\square$

In fact, though we shall not use this below, it is possible to show that in case $\mathcal{P}_D$ is smooth, the spaces D and $[D; \mathcal{P}_D]$ are actually homeomorphic. Indeed, it is straightforward to show that the map $[D; \mathcal{P}_D]$ is a diffeomorphism on the interiors of boundary faces, hence globally bijective, then since it is continuous and proper it has a continuous inverse. Thus one can regard Theorem 10.5 as giving a natural smooth structure on D itself, though this is *not* generally equal to the restriction of the smooth structure on X, as illustrated by the example $D = \left\{ x_1^2 = x_2^3 \right\} \subset \mathbb{R}_+^2$, whose universal resolution is the usual one: $[D; \mathcal{P}_D] = \mathbb{R}_+ \ni t \longmapsto (t^3, t^2) \subset \mathbb{R}_+^2$.

11. Fiber products

We finally bring the theory of the last two sections to bear on the question of fiber products of manifolds with corners. Recall that in the category of manifolds without boundary, smooth fiber products do not generally exist. A sufficient condition in this context is *transversality*; namely two smooth maps $f_i : X_i \longrightarrow Y$, $i = 1, 2$ are transversal if $(f_1)_*(T_{p_1} X_1) + (f_2)_*(T_{p_2} X_2) = T_q Y$ for all pairs $(p_1, p_2) \in X_1 \times X_2$ such that $f_1(p_1) = f_2(p_2) = q$, and then

$$(11.1) \qquad X_1 \times_Y X_2 = \{(p_1, p_2) \in X_1 \times X_2 \; ; \; f_1(p_1) = f_2(p_2)\} \subset X_1 \times X_2$$

is a smooth manifold.

We show that the analogous condition of b-transversality in the category of manifolds with corners implies that $X_1 \times_Y X_2 \subset X_1 \times X_2$ is a union of binomial subvarieties, each of which is interior to some product of faces. The theory we have developed then gives sufficient conditions for a fiber product (with the required universal properties) to exist in the category of manifolds with corners, and gives a coherent system of resolution by generalized blow-up even when these conditions are not satisfied.

Note that even when a fiber product does exist in the category of manifolds with corners, it is not generally equal as a set to (11.1). Indeed, one step in the construction of the fiber product is to separate the binomial subvarieties of (11.1) by taking their *disjoint* union, which is already a kind of resolution. Furthermore,

as noted in the last section, even when a binomial subvariety has a smooth monoidal complex, it need not be smoothly embedded in its ambient manifold $X_1 \times X_2$.

Definition 11.1. We say two b-maps $f_i : X_i \longrightarrow Y$, $i = 1, 2$ are *b-transversal* if for all points $p_i \in X_i$ such that $f_1(p_1) = f_2(p_2) = q \in Y$,

$$^{\mathrm{b}}(f_1)_* \left(^{\mathrm{b}}T_{p_1} X_1\right) + {}^{\mathrm{b}}(f_2)_* \left(^{\mathrm{b}}T_{p_2} X_2\right) = {}^{\mathrm{b}}T_q Y.$$

Proposition 11.2 (Iterated transversality). *If $f_i : X_i \longrightarrow Y$, $i = 1, 2$ are b-transversal, then for every pair of $F_i \in \mathcal{M}(X_i)$ such that $(f_1)_\#(F_1) = (f_2)_\#(F_2) = F \in \mathcal{M}(Y)$, the induced b-maps*

$$(f_i)_{|F_i} : F_i \longrightarrow F \quad \text{are b-transversal.}$$

Proof. Choose $p_i \in F_i$ such that $f_1(p_1) = f_2(p_2) = q$. Suppose $G_i \subseteq F_i$ are the maximal codimension boundary faces containing p_i (in particular $p_i \in G_i \setminus \partial G_i$), and set $^{\mathrm{b}}N_{p_i}(G_i; F_i) = {}^{\mathrm{b}}N_{p_i} G_i / {}^{\mathrm{b}}N_{p_i} F_i$. These are the b-normal bundles to the G_i in the manifolds F_i. The b-tangent spaces of the G_i are the quotients $^{\mathrm{b}}T_{p_i} G_i = {}^{\mathrm{b}}T_{p_i} X_i / {}^{\mathrm{b}}N_{p_i} G_i$ with respect to the natural inclusions $^{\mathrm{b}}N_{p_i} G_i \hookrightarrow {}^{\mathrm{b}}T_{p_i} X_i$, and since the p_i lie in the interior of the G_i, $^{\mathrm{b}}T_{p_i} G_i = T_{p_i} G_i$.

Using a metric to replace the quotients by orthogonal decompositions,

$$^{\mathrm{b}}T_{p_i} X_i = T_{p_i} G_i \oplus {}^{\mathrm{b}}N_{p_i}(G_i; F_i) \oplus {}^{\mathrm{b}}N_{p_i} F_i.$$

The last two factors constitute $^{\mathrm{b}}N_{p_i} G_i$, while the first two constitute $^{\mathrm{b}}T_{p_i} F_i$. The b-differentials $^{\mathrm{b}}(f_i)_*$ have the form

$$^{\mathrm{b}}(f_i)_* = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{pmatrix} : \begin{matrix} T_{p_i} G_i \\ \oplus {}^{\mathrm{b}}N_{p_i}(G_i; F_i) \\ \oplus {}^{\mathrm{b}}N_{p_i} F_i \end{matrix} \longrightarrow \begin{matrix} T_q G \\ \oplus {}^{\mathrm{b}}N_q(G; F) \\ \oplus {}^{\mathrm{b}}N_q F \end{matrix}$$

with respect to the quotients, where $G = (f_i)_\#(G_i)$, $i = 1, 2$. In particular, the only vectors with image in $^{\mathrm{b}}T_q F = T_q G \oplus {}^{\mathrm{b}}N_q(G; F)$ must lie in the first two factors, namely $T_{p_i} G_i \oplus {}^{\mathrm{b}}N_{p_i}(G_i; F_i) \equiv {}^{\mathrm{b}}T_{p_i} F_i$, thus if

$$^{\mathrm{b}}(f_1)_* + {}^{\mathrm{b}}(f_2)_* : {}^{\mathrm{b}}T_{p_1} X_1 \times {}^{\mathrm{b}}T_{p_2} X_2 \longrightarrow T_q Y$$

is surjective, then

$$^{\mathrm{b}}\left((f_1)_{|F_1}\right)_* + {}^{\mathrm{b}}\left((f_2)_{|F_2}\right)_* : {}^{\mathrm{b}}T_{p_1} F_1 \times {}^{\mathrm{b}}T_{p_2} F_2 \longrightarrow {}^{\mathrm{b}}T_q F$$

must be surjective. $\square$

Proposition 11.3. *If $f_i : X_i \longrightarrow Y$, $i = 1, 2$ be b-transverse maps then the set-theoretic fiber product (11.1) is a union of interior binomial subvarieties*

$$D(F_1, F_2) = \mathrm{clos}\left(F_1 \times_Y F_2 \setminus \partial (F_1 \times F_2)\right) \subset F_1 \times F_2 \quad F_i \in \mathcal{M}(X_i).$$

Proof. Every point in $X_1 \times_Y X_2$ lies in the interior of some $F_1 \times F_2$, hence in $D(F_1, F_2)$. It suffices to verify that the $D(F_1, F_2) \subset F_1 \times F_2$ are either empty or interior binomial subvarieties. So we may restrict attention to the case $F_i = X_i$, and assume that $D(X_1, X_2)$ is nonempty.

Suppose $X_i \ni p_i \longmapsto q \in Y$, and choose coordinates (x', y') centered at p_1, (x'', y'') centered at p_2, and $(\overline{x}, \overline{y}) = (\overline{x}_1, \ldots, \overline{x}_k, \overline{y}_{k+1}, \ldots, \overline{y}_n)$ centered at q. The maps f_i have the local form

$$f_1 : (x', y') \longmapsto \left(a_1(x', y')(x')^{\nu_1}, b_1(x', y')\right) = (\overline{x}, \overline{y}), \text{ and}$$

$$f_2 : (x'', y'') \longmapsto \left(a_2(x'', y'')(x'')^{\nu_2}, b_2(x'', y'')\right) = (\overline{x}, \overline{y}).$$

Near $(p_1, p_2) \in X_1 \times X_2$, $(x, y) = (x', x'', y', y'')$ are local coordinates in terms of which

$$D(X_1, X_2) \subset \left\{a_1(x', y') \, (x')^{\nu_1} = a_2(x'', y'') \, (x'')^{\nu_2} \, , \, b_1(x', y') = b_2(x'', y'')\right\}$$

which can be written in the form

$$D(X_1, X_2) \subset \{c_i \, x^{\gamma_i} = 1 \, ; \, i = 1, \ldots, n\}, \quad 0 < c_i \in C^\infty(X_1 \times X_2)$$

where

$$(c_i, \gamma_i) = \begin{cases} \left((a_1)_i / (a_2)_i, (\nu_1 \oplus 0)_i - (0 \oplus \nu_2)_i\right) & i = 1, \ldots, k, \\ \left(\exp\left((b_1)_i - (b_2)_i\right), 0\right) & i = k+1, \ldots, n. \end{cases}$$

Thus it only remains to check the independence of the logarithmic differentials of the $c_i x^{\gamma_i}$. Consider the b-map $f_1 \times f_2 : X_1 \times X_2 \longrightarrow Y \times Y$. There is an exact sequence

$$0 \to {}^bT_qY \xrightarrow{{}^b\Delta_*} {}^bT_{(q,q)}(Y \times Y) \to {}^bT_qY \to 0$$

where $\Delta : Y \longrightarrow Y \times Y$ is the diagonal inclusion and the subsequent map is the difference from ${}^bT_{(q,q)}(Y \times Y) = {}^bT_qY \times {}^bT_qY$. The condition of b-transversality condition means that ${}^b(f_1 \times f_2)^*$ is injective as a map

$$ {}^b(f_1 \times f_2)^* : {}^bT_q^*Y \longrightarrow {}^bT_{(p_1, p_2)}^*(X_1 \times X_2). $$

This can be identified with the map

$$ {}^bf_1^* - {}^bf_2^* : {}^bT_q^*Y \longrightarrow {}^bT_{(p_1, p_2)}^*(X_1 \times X_2) $$

which is similarly injective. Taking the coordinate basis $\{d\overline{x}_i/\overline{x}_i, d\overline{y}_j\}$ for ${}^bT_q^*Y$, we obtain that

$$ \left\{({}^bf_1^* - {}^bf_2^*)d\overline{x}_i/\overline{x}_i, ({}^bf_1^* - {}^bf_2^*)d\overline{y}_j\right\} $$

is independent. Observe however that

$$ ({}^bf_1^* - {}^bf_2^*)d\overline{x}_i/\overline{x}_i = d\log\left((a_1 x'^{\nu_1})_i\right) - d\log\left((a_2 x''^{\nu_2})_i\right) $$
$$ = d\log\left(c_i x^{\gamma_i}\right), \quad i = 1, \ldots, k $$

and

$$ ({}^bf_1^* - {}^bf_2^*)d\overline{y}_i = d(b_1)_i - d(b_2)_i $$
$$ = d\log\left(\exp\left((b_1)_i - (b_2)_i\right)\right) $$
$$ = d\log\left(c_i x^{\gamma_i}\right), \quad i = k+1, \ldots, n. $$

We conclude that $D(X_1, X_2)$ has a covering by local binomial structures and is therefore an interior binomial subvariety. $\qquad\square$

While $X_1 \times_Y X_2$ may therefore be a complicated and quite singular space, any smooth maps factoring through it must actually factor through one of the subvarieties $D(F_1, F_2)$.

Proposition 11.4. *If $g_i : Z \longrightarrow X_i$ are b-maps from a connected smooth manifold such that $f_1 \circ g_1 = f_2 \circ g_2$ then the maps $g_i : Z \longrightarrow X_i$ factor through a unique b-map $h : Z \longrightarrow D(F_1, F_2)$ for some $D(F_1, F_2)$:*

$$
\begin{array}{ccc}
Z & \xrightarrow{\;\;g_2\;\;} & \\
\;\;\;\downarrow h & & \\
g_1 \;\; D(F_1, F_2) & \xrightarrow{\;\pi_2\;} & X_2 \\
\downarrow \pi_1 & & \downarrow f_2 \\
X_1 & \xrightarrow{\;\;f_1\;\;} & Y.
\end{array}
$$

Proof. Let F_1, F_2 be the minimal (largest codimension) faces such that $g_i : Z \longrightarrow F_i$ are *interior* b-maps. Then $g_1 \times g_2 : Z \longrightarrow F_1 \times F_2 \subset X_1 \times X_2$ is an interior b-map, and as such

$$g_1 \times g_2 : Z \setminus \partial Z \longrightarrow F_1 \times F_2 \setminus \partial (F_1 \times F_2).$$

On the other hand, as a map of sets, $g_1 \times g_2$ factors through (the set-theoretic) $F_1 \times_Y F_2$ by the assumption that $f_1 \circ g_1 = f_2 \circ g_2$. Finally, it follows by continuity and taking the closure of the intersection with the interior of $F_1 \times F_2$ that

$$h = g_1 \times g_2 : Z \longrightarrow D(F_1, F_2) \subset X_1 \times X_2$$

which is by definition a b-map into an interior binomial subvariety. $\qquad\square$

Note that the monoidal complex $\mathcal{P}_{D(F_1, F_2)}$ consists of monoids of the form $\sigma_{G_1} \times_{\sigma_G} \sigma_{G_2}$, where $\sigma_{G_i} \in \mathcal{P}_{F_i}$, and $G = (f_1)_\#(G_1) \cap (f_2)_\#(G_2)$. Indeed, if F is a boundary face of $D(F_1, F_2)$, given by a component of $D(F_1, F_2) \cap G_1 \times G_2$ for $G_1 \times G_2 \in \mathcal{M}(X_1 \times X_2)$, then

$$\sigma_F = \sigma_{G_1 \times G_2} \cap {}^b N F = \sigma_{G_1} \times_{\sigma_G} \sigma_{G_2}.$$

It is generally *not* true that $\mathcal{P}_{D(F_1, F_2)}$ is equal to $\mathcal{P}_{F_1} \times_{\mathcal{P}_Y} \mathcal{P}_{F_2}$ as might at first be expected, since $D(F_1, F_2) \cap G_1 \times G_2$ may be empty or may have multiple components, while $\sigma_{G_1} \times_{\sigma_G} \sigma_{G_2}$ is nontrivial and appears exactly once in $\mathcal{P}_{F_1} \times_{\mathcal{P}_Y} \mathcal{P}_{F_2}$. However, this is true in some cases, as in example 11.8 below.

Theorem 11.5 (Existence of smooth fiber products). *If $f_i : X_i \longrightarrow Y$ are b-transverse maps and if in addition $\sigma_{F_1} \times_{\sigma_G} \sigma_{F_2}$ is a smooth monoid whenever $G = (f_1)_\#(F_1) \cap (f_2)_\#(F_2) \in \mathcal{M}(Y)$ then*

$$X_1 \widetilde{\times_Y} X_2 = \bigsqcup_{F_1, F_2} [D(F_1, F_2); \mathcal{P}_{D(F_1, F_2)}]$$

is a union of smooth manifold with corners and is the universal fiber product.

Remark. Observe that $X_1 \widetilde{\times_Y} X_2$ is not equal to $X_1 \times_Y X_2$. In categorical language, the forgetful functor from manifolds with corners to sets does not preserve fiber products. However there is a unique map $X_1 \widetilde{\times_Y} X_2 \longrightarrow X_1 \times_Y X_2$.

Proof. It follows from Theorem 10.5 that $X_1 \widetilde{\times_Y} X_2$ is a smooth manifold with corners, which has the requisite universal property by Proposition 11.4. Indeed, any b-maps $g_i : Z \longrightarrow X_i$ from a manifold Z which commute with the f_i factor

through a unique map $h : Z \longrightarrow D(F_1, F_2)$ for some $D(F_1, F_2)$, and this has a unique lift to the universal resolution $[D(F_1, F_2); \mathcal{P}_{D(F_1,F_2)}]$. Finally, the uniqueness of $\widetilde{X_1 \times_Y X_2}$ up to diffeomorphism follows from the universality. $\qquad \square$

In general, even for b-transverse maps, a smooth fiber product does not exist. Nevertheless, the space

$$\bigsqcup_{F_1, F_2} D(F_1, F_2)$$

serves as a fiber product in the category of interior binomial subvarieties with its attendant resolution theory, which we summarize below.

Theorem 11.6. *Let $f_i : X_i \longrightarrow Y$, $i = 1, 2$ be b-transverse maps and set $D = \bigsqcup_{F_i \in \mathcal{M}(X_i), i=1,2} D(F_1, F_2)$*

(a) *For every smooth refinement $\mathcal{R}_D \longrightarrow \mathcal{P}_D$ there is a smooth manifold with corners $[D; \mathcal{R}_D]$ with maps $h_i : [D; \mathcal{R}_D] \longrightarrow X_i$ forming a commutative square with $f_i : X_i \longrightarrow Y$:*

$$
\begin{array}{ccc}
[D; \mathcal{R}_D] & \overset{h_2}{\dashrightarrow} & X_2 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_2} \\
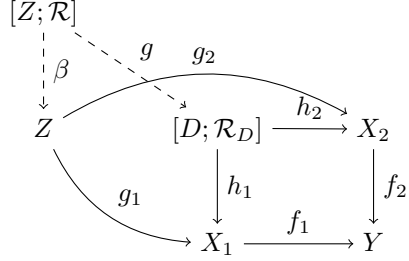X_1 & \underset{f_1}{\longrightarrow} & Y.
\end{array}
$$

(b) *Any two such resolutions $[D; \mathcal{R}_i]$ $i = 1, 2$ have a mutual smooth resolution, which is to say a third manifold $[D; \mathcal{R}_0]$ with maps $[D; \mathcal{R}_0] \longrightarrow [D; \mathcal{R}_i]$, $i = 1, 2$ forming commutative diagrams with the maps $[D; \mathcal{R}_i] \longrightarrow X_j$.*

(c) *If Z is a manifold with maps $g_i : Z \longrightarrow X_i$ such that $f_1 \circ g_1 = f_2 \circ g_2$, and if the morphism $\mathcal{P}_Z \longrightarrow \mathcal{P}_D$ factors through $\mathcal{R}_D \longrightarrow \mathcal{P}_D$, then there is a unique map $g : Z \longrightarrow [D; \mathcal{R}_D]$ such that $h_i \circ g = g_i$.*

(d) *Given a manifold Z with maps $g_i : Z \longrightarrow X_i$ such that $f_1 \circ g_1 = f_2 \circ g_2$, for any resolution $[D; \mathcal{R}_D] \longrightarrow D$, there exists a generalized blow-up $\beta : [Z; \mathcal{R}] \longrightarrow Z$ and a unique map $g : [Z; \mathcal{R}] \longrightarrow [D; \mathcal{R}_D]$ such that $h_i \circ g = g_i \circ \beta$:*

$$
\begin{array}{ccccc}
[Z;\mathcal{R}] & & & & \\
\ \downarrow \beta & \xrightarrow{\ \ g\ \ } & & \xrightarrow{\ g_2\ } & \\
Z & & [D;\mathcal{R}_D] & \xrightarrow{\ h_2\ } & X_2 \\
& g_1 \searrow & \downarrow h_1 & & \downarrow f_2 \\
& & X_1 & \xrightarrow{\ f_1\ } & Y
\end{array}
$$

Remark. For a manifold Z with maps $g_i : Z \longrightarrow X_i$, it is not generally true that there exists a resolution $[D; \mathcal{R}_D] \longrightarrow D$ through which Z factors, without blowing up the domain.

Proof. The smooth manifold $[D; \mathcal{R}_D]$ is well-defined by the results in Section 10, giving (a). (b) follows from the existence of mutual refinements, letting $\mathcal{R}_0$ be a smooth refinement of $\mathcal{R}_1 \times_{\mathcal{P}_D} \mathcal{R}_2$. (c) follows from Theorem 6.4, and (d) follows from Theorem 6.5, letting $\mathcal{R}$ be a smooth refinement of $\mathcal{P}_Z \times_{\mathcal{P}_D} \mathcal{R}_D$.   □

Some of the situations we have already considered are interesting examples of fiber products.

Example 11.7 (A blow-up and a binomial subvariety)**.** Let Y be a smooth manifold which is included in X as an interior binomial variety, and let $X_1 = [X; \mathcal{R}] \longrightarrow X$ be a generalized blow-down. Since $i : Y \hookrightarrow X$ is injective, the fiber product $X_1 \times_X Y$ can be identified with $\beta^{-1}(Y) \subset X_1$. The interior subvariety $D(Y, X_1)$ is just the lift/proper transform $\beta^{\#}(Y)$, which we showed to be an interior binomial subvariety in Section 10. Observe, however that $\beta^{-1}(Y)$ generally contains other subvarieties as well, namely $F \cap \beta^{-1}(G)$, where $F \in \mathcal{M}(X_1)$ and $G \in \mathcal{M}(D)$. See Figure 5.

Example 11.8 (Two blow-ups)**.** We leave it as an exercise for the reader to show that the fiber product of two blow-down maps $\beta_i : [X; \mathcal{R}_i] \longrightarrow X$, $i = 1, 2$ is a subvariety with the monoidal complex $\mathcal{R}_1 \times_{\mathcal{P}_X} \mathcal{R}_2$. By Proposition 5.3 this is a (not necessarily smooth) refinement of $\mathcal{P}_X$, which can be identified with the intersection complex $\mathcal{R}_1 \cap \mathcal{R}_2 = \{\sigma_{F_1} \cap \sigma_{F_2} \subset \sigma_F ; \sigma_{F_i} \in \mathcal{R}_i(F)\}$. While not smooth in general, one can construct nontrivial examples where $\mathcal{R}_1 \times_{\mathcal{P}_X} \mathcal{R}_2$ is smooth, which therefore give nontrivial instances of Theorem 11.5.

Example 11.9 (Joyce's fiber products)**.** In [Joy09], Joyce proposes a category of manifolds with corners in which the morphisms are what might be called 'simple b-maps' $f : X \longrightarrow Y$ which have the property that whenever

$$ f^* \mathcal{I}_H = \prod_{G \in \mathcal{M}_1(X)} \mathcal{I}_G^{\alpha(H,G)}, \quad H \in \mathcal{M}_1(Y) $$

the $\alpha(\cdot, \cdot)$ are either zero or one, and furthermore for every H there is at most one G such that $\alpha(H, G) \neq 0$. As a consequence the morphisms

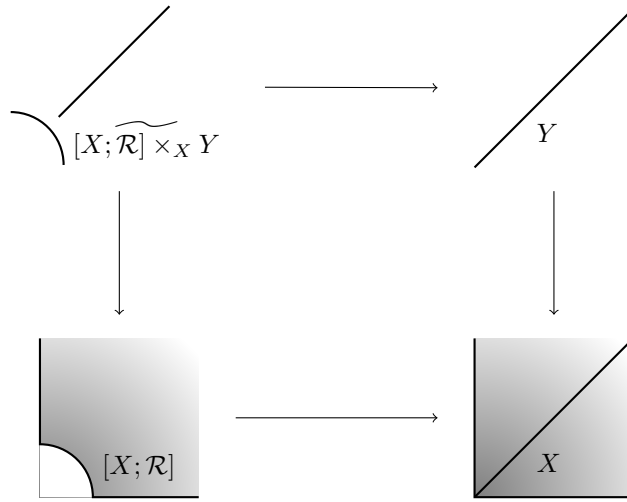$$ f_\natural : \sigma_F \longrightarrow \sigma_{f_\#(F)} $$

FIGURE 5. The fiber product of a generalized blow-down and the inclusion of a manifold as an interior binomial subvariety.

are always injective, with images which have orthogonal generators $\{v_i\}$ all of whose components are zero or one.

Joyce defines a transversality condition which is essentially equivalent to b-transversality (though he does not use the b-differential, his is an iterated transversality condition on boundary faces analogous to Proposition 11.2), and shows that for two transversal maps in his sense, the set-theoretic fiber product is a manifold with corners.

This also follows from Theorem 11.5. Indeed, for b-transversal simple b-maps, the monoids $\sigma_{F_1} \times_{\sigma_G} \sigma_{F_2}$ can be identified with the intersections $\sigma_{F_1} \cap \sigma_{F_2} \subset \sigma_G$ by injectivity, and with the properties above, these intersections are always smooth.

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