EQUIVARIANT RESOLUTION, COHOMOLOGY AND INDEX
JOINT WORK WITH PIERRE ALBIN
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Abstract. In this talk I describe the resolution to unique isotropy type of the smooth action of a compact Lie group as contained in joint work with Pierre Albin, [2], [3]. The resulting 'resolution tower' leads to resolved and reduced models for equivariant cohomology, including the delocalized cohomology of Baum, Brylinski and MacPherson, and K-theory. Combining these constructions with a lifting map gives a families version of the pseudodifferential equivariant index theorem of Atiyah and Singer with corresponding representations of the Chern character.

Contents

1. Group actions

Let me start by reminding you of the basic properties of smooth group actions. Here, X will be a compact manifold – later with corners but initially without boundary – and G is a compact, possibly not connected, Lie group. A smooth action of G on X is a smooth map which induces a smooth homomorphism of G into the diffeomorphism group of X :

$$A : G \times X \longrightarrow X,\ G \longrightarrow \mathrm{Dfo}(X). \tag{1.1}$$

The action of an element of G is typically just denoted $g \in \mathrm{Dfo}(X)$, so $g : X \ni x \longmapsto gx \in X$. The differential of the map at the identity induces a map of Lie algebras, $\mathfrak{g} = T_{\mathrm{Id}}G$,

$$\mathfrak{g} \longrightarrow \mathcal{C}^\infty(X; TX) \tag{1.2}$$

which can be converted into an element

$$\mu \in \mathcal{C}^\infty(X; TX) \otimes \mathfrak{g}^* \tag{1.3}$$

capturing the infinitesmal action.

Although such a group action is smooth, by definition, it is in general rather seriously 'non-uniform' in the sense that the orbits, necessarily individually smooth, change dimension locally. This is encoded in the *isotropy* (also called stabilizer) groups. Namely for $x \in X$ set

$$G_x = \{g \in G; gx = x\}\,. \tag{1.4}$$

This is a Lie subgroup and long-established local regularity theory shows that if $K \subset G$ is a Lie subgroup then

$$X^K = \{x \in X; G_x = K\} \tag{1.5}$$

is a smooth, but in general non-closed, submanifold. Since $G_{gx} = g^{-1}(G_x)g$, the conjugate of an isotropy group is an isotropy group and if $[K]$ is the class of subgroups conjugate to K then

$$X^{[K]} = \{x \in X; G_x = gKg^{-1} \text{ for some } g \in G\} \tag{1.6}$$

is also a smooth submanifold of X. The conjugacy classes are partially ordered by inclusion and the *isotypes* $X^{[K]} \subset X$ give a stratification of X.

In this talk I describe the canonical resolution of X from [2] corresponding to a smooth action, which resolves these isotypes (which are necessarily of iterated conic type), and some consequences and developments of it. I will start with applications to equivariant cohomology and try to use these to motivate the resolution.

2. Equivariant cohomology and K-theory

The basic topological invariants of a group action are the equivariant cohomology and K groups. Since what I discuss here is rather elementary, let me start with sketched definitions.

In Cartan's formulation of equivariant cohomology, the chain space is

$$\left(\mathcal{C}^\infty(X; \Lambda^* X) \otimes S(\mathfrak{g}^*)\right)^G, \tag{2.1}$$

the subspace of the polynomials on $\mathfrak{g}$, identified with the symmetric part of the full tensor algebra of the dual, with coefficients in smooth forms on X which are (jointly) invariant under the action of G. Here G acts through pull-back on forms and the adjoint action on $\mathfrak{g}$, the Lie algebra of G. The deRham differential acts on the forms and combines with contraction with μ above (and symmetrization into $S(\mathfrak{g}^*)$) to give the equivariant differential $d_G = d - \mu$ which satisfies $d_G^2 = 0$. The $\mathbb{Z}_2$-graded equivariant cohomology groups are denoted $H^*_G(X)$, where the degree is equal modulo 2 to the sum of the form degree and twice the polynomial degree.

There are several equivalent definitions of the equivariant K-group, $K_G(X)$, the most direct is perhaps through equivariant G-bundles – smooth complex vector bundles over X with a fibre-linear G-action covering the G-action on X. Grothendieck's construction gives $K_G(X)$ – the equivalence classes of pairs of G-equivariant bundles under G-equivariant isomorphism and stabilization.

One of the issues I want to address here is the well-known 'defect' in the induced Chern character in this setting. By general abstract constructions the Chern character is a natural transformation

$$\mathrm{Ch} : K_G(X) \longrightarrow H_G^{\mathrm{even}}(X). \tag{2.2}$$

The 'problem' is that in general the Atiyah-Hirzebruch theorem does not extend to this case, namely the Abelianized map

$$\mathrm{Ch} : K_G(X) \otimes \mathbb{C} \longrightarrow H_G^{\mathrm{even}}(X) \tag{2.3}$$

is not in general an isomorphism.

This issue was addressed by Baum, Brylinski and MacPherson, [4], who introduced 'delocalized' equivariant cohomology groups, $H^{\mathrm{even}}_{G,\mathrm{dl}}(X)$, in the case of an Abelian group action, such that the Chern character factorizes

$$\begin{array}{ccccc} \mathrm{Ch} : K_G(X) & \longrightarrow & H^{\mathrm{even}}_{G,\mathrm{dl}}(X) & \xrightarrow{L} & H^{\mathrm{even}}_G(X) \\ & \searrow^{\otimes\mathbb{C}} & \nearrow_{\simeq} & & \\ & & K_G(X)\otimes\mathbb{C} & & \end{array} \tag{2.4}$$

where the first map induces an isomorphism in place of (2.3) and the localization map L is discussed below. One consequence of the resolution procedure is that the extension of the groups $H^{\mathrm{even}}_{G,\mathrm{dl}}(X)$ to the non-Abelian case becomes transparent.

3. Model cases

3.1. **Trivial actions.** If a Lie group acts trivially on a space then the chain space becomes $(S(\mathfrak{g}^*))^G \otimes \mathcal{C}^\infty(X; \Lambda^*)$, the differential reduces to the untwisted deRham differential and the polynomial coefficients commute with the differential. It follows that the equivariant cohomology group is given by the finite tensor product

$$H^*_G(X) = (S(\mathfrak{g}^*))^G \otimes H^*(X). \tag{3.1}$$

Similarly for K-theory, the group acts on the individual fibres of a G-equivariant bundle which decomposes in terms of representations. Let $\hat{G}$ be set of equivalence classes of representations of G and $\mathcal{R}(G)$ the ring of virtual representations given by Grothendieck's construction applied to $\hat{G}$. The ring operations are direct sum and tensor product and $\mathcal{R}(G)$ is spanned over $\mathbb{Z}$ by the irreducible representations. Then, again with finite tensor products, over $\mathbb{Z}$,

$$K_G(X) = \mathcal{R}(G) \otimes K(X). \tag{3.2}$$

In this case we can see the structure of the Chern character explicitly. Namely, from naturality considerations, it has to reduce to the ordinary Chern character at the identity and have appropriate G-equivariance. Thus in fact the equivariant Chern character decomposes as a tensor product

$$\mathrm{Ch}_G = L \otimes \mathrm{Ch} : \mathcal{R}(G) \otimes K(X) \longrightarrow (S(\mathfrak{g}^*))^G \otimes H(X) \tag{3.3}$$

where the localization map L is given by identifying a representation with its character in $\mathcal{C}^\infty(G)$, with value at g the trace of the action of g, and then mapping it to the invariant polynomial on the Lie algebra determined by its Taylor series at the identity.

This also reveals the problem with the Chern character under Abelianization. Namely if G is not connected, then there may well be a representation with all tensor powers non-trivial – for instance the standard representation of a copy of $\mathbb{Z}_2$ – but with trace which is constant on the component of the identity. The formal difference between this representation and the trivial one-dimensional representation survives in the tensor product with $\mathbb{C}$ but is annihilated by the localization map. Even for a connected group, this phenomenon may arise from isotropy groups.

3.2. **Finite quotients.** In the case of a Lie subgroup with finite quotient, $G_0 \subset G$, there is a direct relationship between the G_0- and the G-equivariant cohomologies of a space on which G acts. Namely, the differential is determined by the infinitesmal action whereas the chain spaces are related by

$$(\mathcal{C}^\infty(X;\Lambda^*X)\otimes S(\mathfrak{g}^*))^G = \Big((\mathcal{C}^\infty(X;\Lambda^*X)\otimes S(\mathfrak{g}^*))^{G_0}\Big)^{G/G_0} \tag{3.4}$$

which descends to the cohomology

$$H_G(X) = (H_{G_0}(X))^{G/G_0} \tag{3.5}$$

where there is an induced G-action on $H_{G_0}(X)$ in which G_0 acts trivially.

For K-theory the reduction is similar. The action of G means that pull-back by $g \in G$ maps a G_0-equivariant into a G_0-equivariant bundle with the conjugate G_0 action. This preserves the equivalence conditions, so G acts on $K_{G_0}(X)$, and again

$$K_G(X) = (K_{G_0}(X))^{G/G_0} \tag{3.6}$$

in the sense that the G_0 action is trivial.

3.3. **Free actions.** The opposite extreme of the trivial case, studied extensively by Cartan, arises when G acts freely, i.e. without fixed points, so $gx = x$ implies $g = \mathrm{Id}$. The assumed smoothness and compactness shows that the action corresponds to a principal G-bundle,

$$\begin{array}{ccc} G & \text{———} & X \\ & & \downarrow \\ & & Z = G\backslash X. \end{array} \tag{3.7}$$

In this case Borel showed that not only is the quotient of the action smooth but

$$G \text{ acts freely} \Longrightarrow H^*_G(X) = H^*(Z). \tag{3.8}$$

This indeed is one justification for the definition of equivariant cohomology.

The isomorphism in (3.8) was analyzed explicitly by Cartan at the chain level, in terms of a connection on X as a principal bundle over X (see [1]). Thus, if θ is a connection on the principal bundle then its curvature, ω is a 2-form with values in the tensor product of the Lie algebra with itself. The formally infinite sum $\exp(\omega/2\pi i)$ can therefore be paired with an element of the finite part of the symmetric tensor product $S(\mathfrak{g}^*)$; the G-invariant part of the resulting form descends to the quotient and gives a map at the form level

$$(\mathcal{C}^\infty(X;\Lambda^*X)\otimes S(\mathfrak{g}^*))^G \ni u \longmapsto (\exp(\omega/2\pi i)\cdot u)^G \in \mathcal{C}^\infty(G\backslash X;\Lambda^*) \tag{3.9}$$

which induces the isomorphism (3.8).

In this free case the equivariant K-theory is also immediately computable. A G-equivariant vector bundle over X is equivariantly isomorphic to the pull-back of a vector bundle over Z and in consequence

$$G \text{ acts freely} \Longrightarrow K_G(X) = K(Z). \tag{3.10}$$

In this setting the equivariant Chern character reduces to the standard Chern character on the quotient and in particular the equivariant form of the Atiyah-Hirzebruch isomorphism does hold.

3.4. **Pairs of free actions.** If there are commuting actions by two compact Lie groups which act freely, individually,

$$G_1 \times X \longrightarrow X,\ G_2 \times X \longrightarrow X \tag{3.11}$$

(with the second action written on the right to emphasize commutativity) then there are isomorphisms given by the isomorphism in the free case, including maps at the level of forms,

$$H_{G_1}(X/G_2) = H_{G_1\times G_2}(X) = H_{G_2}(G_1\backslash X). \tag{3.12}$$

Again for K-theory the analogous conclusion holds:

$$K_{G_1}(X/G_2) = K_{G_1\times G_2}(X) = K_{G_2}(G_1\backslash X). \tag{3.13}$$

3.5. **Fixed, normal, isotropy group.** The free case corresponds to $G_x = \{\mathrm{Id}\}$. Perhaps the next most regular case is where $G_x = K$ is a fixed group. Necessarily K is normal in this case:

$$G_x = K\ \forall\ x \in X \Longrightarrow K \text{ normal in } G. \tag{3.14}$$

The action of G then factors through the free action of the quotient group $Q = G/K$. Thus X is the total space of a principal Q-bundle. Choosing an adjoint-invariant metric on $\mathfrak{g}$, the orthocomplement is also a Lie subalgebra, $\mathfrak{q}$, which may be identified with the Lie algebra of Q. Furthermore, the conjugation action of G on its Lie algebra, $\mathfrak{g}$, preserves both $\mathfrak{k}$ (since it corresponds to a normal subgroup) and $\mathfrak{q}$. Indeed, the normality of K implies that its adjoint action on $\mathfrak{q}$ is trivial as is the action of the component of the identity Q_0 of Q on $\mathfrak{k}$. Then if we let G' be the preimage of Q_0 in G under the projection, K is normal in G' with quotient Q_0. Using the invariant metric to decompose the polynomial algebra

$$S(\mathfrak{g}^*) = S(\mathfrak{k}^*) \otimes S(\mathfrak{q}^*) \tag{3.15}$$

it follows that the K invariant part of Cartan's form bundle is

$$(S(\mathfrak{k}^*))^K \otimes S(\mathfrak{q}^*) \otimes \mathcal{C}^\infty(X;\Lambda^*) \tag{3.16}$$

and hence its G'-invariant part is

$$(S(\mathfrak{k}^*))^K \otimes (S(\mathfrak{q}^*) \otimes \mathcal{C}^\infty(X;\Lambda^*))^{Q_0} \tag{3.17}$$

in view of the triviality of the action of K on $\mathfrak{q}$. The differential reduces to that of the Q_0 action so, using the result for principal bundles discussed above,

$$H^*_{G'}(X) = (S(\mathfrak{k}^*)^K \otimes H^*(G'\backslash X). \tag{3.18}$$

The induced action of G on this space, in which G' acts trivially, factors through the action on

$$(S(\mathfrak{k}^*))^K \otimes \mathcal{C}^\infty(G'\backslash X;\Lambda^*). \tag{3.19}$$

Now, G acts on the second factor through the finite group Q/Q_0 realized as diffeomorphisms on $G'\backslash X = Q_0\backslash X$. However, it may also act on the first factor, since although Q_0 acts trivially on $\mathfrak{k}$, the finite quotient may not. Instead of realizing the equivariant cohomology as

$$\Big((S(\mathfrak{k}^*))^K \otimes \mathcal{C}^\infty(G'\backslash X;\Lambda^*)\Big)^{Q/Q_0} \tag{3.20}$$

it is convenient to interpret the action of Q/Q_0 on $(S(\mathfrak{k}^*))^K$, as a trivial bundle over $Q_0\backslash X$, as defining a flat bundle over $G\backslash X = Q\backslash X$.

Definition 1. The *Borel bundle* over $Z = G\backslash X$ for the action of G with fixed, and normal, stabilizer group K is the flat bundle which is the quotient by the action of Q/Q_0 on $(S(\mathfrak{k}^*))^K$ as a trivial bundle $Z' = Q_0\backslash X$, where Q_0, is the component of the identity in $Q = G/K$.

Even though this bundle is infinite dimensional, it is the direct sum of its homogeneous components, so there is no difficulty in defining the twisted deRham cohomology of $G\backslash X$ with coefficients in B. Then the discussion above of finite quotients leads to the conclusion that

$$H^*_G(X) = H^*(G\backslash X; B) \tag{3.21}$$

is the twisted cohomology on the quotient with coefficients in the Borel bundle. Thus the Borel bundle is the only 'remnant' of the G action remaining on the quotient that is needed to compute the equivariant cohomology.

There is a similar reduction for equivariant K-theory in this case of a single isotropy group. Namely K necessarily acts on the fibres of a G-equivariant vector bundle over X, which can then be decomposed into subbundles tensored with representations of K, we may think of an equivariant bundle as defining a bundle over X with coefficients in $\hat{K}$, the (discrete) representation ring of K. Then an equivariant K-class is represented by a pair of equivariant bundles, or a bundle with coefficients in the representation ring $\mathcal{R}(K)$. Again the quotient group Q acts, by conjugation, on $\mathcal{R}(K)$ with Q_0 acting trivially and we may identify

$$K_G(X) = (\mathcal{R}(K) \otimes K(Z'))^{Q/Q_0} \text{ or } K_G = K(Z;\mathcal{R}) \tag{3.22}$$

where the second, more geometric, form of the identification is as the K-theory with coefficients in the flat 'representation bundle' over the quotient, which we denote $\mathcal{R}$ – it is just the quotient of the trivial $\mathcal{R}$-bundle over Z'.

Each element of $\mathcal{R}(K)$ is represented by an integral combination of representations of K. The localization map L in (2.4) induces a map at the bundle level

$$L : \mathcal{C}^\infty(G\backslash X;\mathcal{R}) \longrightarrow \mathcal{C}^\infty(G\backslash X; B). \tag{3.23}$$

The equivariant Chern character is induced by the usual Chern character, at the level of forms, over Z', with this localization map acting on the coefficients

$$\mathrm{Ch}_G = \mathrm{Ch} \otimes L : K(Z;\mathcal{R}) \longrightarrow H^{\mathrm{even}}(Z; B). \tag{3.24}$$

From this example we can see what is required to construct a cohomology theory which is the natural target of the Chern character. Namely, the coefficient bundle B should be replaced by the coefficient bundle $\mathcal{R}$.

Definition 2. The 'delocalized' equivariant cohomology in case of a unique, normal, isotropy group is the cohomology with coefficients in the representation bundle $\mathcal{R}$, over the quotient.

3.6. **Unique isotropy type.** A more general version of the preceding case occurs if one assumes that there is a unique isotropy type, meaning that all the isotropy subgroups are conjugate in G. This case was also studied by Borel who showed that the quotient is smooth. In fact, if one selects an isotropy group, K, the the set of points

$$X^K = \{x \in X; G_x = K\} \tag{3.25}$$

is a smooth submanifold. The normalizer $N(K)$ of K in G acts on X^K. The product $G \times X^K$ has a free left action by G and a free diagonal action by $N(K)$ and Borel observed that the unique map giving a commutative diagram with the quotient with respect to the $N(K)$ action

$$\begin{array}{ccccc} X^K & \xleftarrow{G\backslash} & G \times X^K & \xrightarrow{A} & X \\ & & {\scriptstyle /N(K)}\downarrow & \nearrow{\scriptstyle \simeq} & \\ & & G \times_{N(K)} X^K & & \end{array} \tag{3.26}$$

is a G-equivariant diffeomorphism. Thus, applying the discussion above on commuting free actions it follows that

$$H^*_G(X) = H^*_G(G \times_{N(K)} X^K) = H^*_{N(K)}(X^K). \tag{3.27}$$

Thus in fact the analysis of the case of a single normal isotropy group applies here and allows us to conclude from (3.21) that

$$H^*_G(X) = H^*(G\backslash X; B). \tag{3.28}$$

The Borel bundle over $G\backslash X$ is identified with the Borel bundle over $N(K)\backslash X^K$ for K as a subgroup of $N(K)$ in Definition 2; different choices of isotropy groups given naturally isomorphic bundles, so the result is well-defined.

K-theory behaves similarly,

$$K_G(X) = K_{N(K)}(X^K) = K(Z; \mathcal{R}). \tag{3.29}$$

Again one can introduce the delocalized cohomology groups by replacing B by $\mathcal{R}$ in (3.28).

4. Equivariant fibrations

Suppose that Y_1 and Y_2 are two compact manifolds (in the application below they will be manifolds with corners, but his makes little difference) each with a smooth G action with unique isotropy type, and with a smooth map

$$f : Y_1 \longrightarrow Y_2 \tag{4.1}$$

which is both G-equivariant and a fibration.

Proposition 1. *A G-equivariant fibration between compact manifolds with G actions with unique isotropy type, projects to a fibration of the quotients inducing well defined pull-back maps on the Borel bundles, the representation bundles and the corresponding chain spaces for equivariant and delocalized equivariant cohomology and for bundles with representation bundle coefficients.*

A G-equivariant fibration as in (4.1) maps orbits to orbits and hence descends to a map on the quotients $\gamma_f : Z_1 \longrightarrow Z_2$, $Z_i = G\backslash Y_i$, $i = 1, 2$; smoothness of γ_f follows from the fact that G-invariant smooth functions on the base pull back to smooth G-invariant functions, this also shows that $(\gamma_f)^*$ is injective and hence that γ_f is a fibration.

The definition of the Borel bundle over the quotient above is tailored to showing that it is flat, but it can also be viewed as having fibre at a point on the quotient (for an action with unique isotropy type) as the 'push-forward' of the bundle over the corresponding orbit with fibres $(S(\mathfrak{k}_x^*))^{K_x}$ where K_x is the isotropy group at x. The adjoint action of G makes this bundle G-equivariant over the orbit. Restricting the fibration to an orbit of the G action on Y_1 gives a fibration over the corresponding G orbit in Y_2 and there is a natural pull-back map between the fibres

$$f^{\#} : \Big(S((\mathfrak{k}^{(2)}_{f(p)})^*)\Big)^{K^{(2)}_{f(p)}} \longrightarrow \Big(S((\mathfrak{k}^{(1)}_{p})^*)\Big)^{K^{(1)}_{p}} \tag{4.2}$$

given by restriction (of polynomials) to the the subspace $\mathfrak{k}^{(1)}_p \subset \mathfrak{k}^{(2)}_{f(p)}$. This descends to the desired pull-back map $\gamma_f^{\#}$ on the Borel bundles over the quotients.

Together with pull-back of forms this generates a pull-back map for the 'reduced' deRham spaces defining the equivariant cohomology

$$\gamma_f^{\#} : \mathcal{C}^\infty(G\backslash Z_2; B \otimes \Lambda^*) \longrightarrow \mathcal{C}^\infty(G\backslash Z_1; B \otimes \Lambda^*) \tag{4.3}$$

which commutes with the differential an so in turn induces the pull-back map for equivariant cohomology.

The situation is similar for the representation bundle (of rings) and hence for the 'reduced' chain spaces delocalized (i.e. over the quotient) for equivariant cohomology and for the reduced model for equivariant K-theory.

These pull-back maps also allow the introduction of relative cohomology and K-theory groups. The situation that arises inductively below corresponds to Y_1 being a boundary face of a manifold (with corners) Y; for simplicity here suppose that Y is a manifold with boundary. Then the relative theory in cohomology for the pair of quotient spaces, is fixed by the chain spaces

$$\big\{(u, v) \in \mathcal{C}^\infty(Z; B \otimes \Lambda^*) \times \mathcal{C}^\infty(Z_2; B \otimes \Lambda^*); i^*_{Z_1} u = \gamma^{\#} v\big\} \tag{4.4}$$

with the diagonal differential; here $i^*_{Z_1}$ is the map induced by restriction to the boundary.

Similar considerations apply to delocalized cohomology and K-theory.

5. Resolution tower

The general approach to delocalized equivariant cohomology proceeds through reduction, in an appropriate iterated sense, to the case of unique isotropy type discussed above. This reduction is through the resolution of the successive isotropy types of a general compact group action. Such resolutions have been discussed in the literature although with various restrictions (and errors). Here it is essential to retain the iterative structure which arises in the resolution – which we call the 'resolution tower'.

The basic principle is straightforward. We need the concept of real, or radial, blow-up. In contrast to the projective blow up, which is perhaps closer in spirit to the complex case, this operation introduces a boundary hypersurface. Precisely because of this, radial blow up can resolve reflections across a hypersurface whereas projective resolution cannot.

Given an embedded closed submanifold of a manifold, $S \subset X$, we can introduce a new manifold, now with boundary denoted, $[X;S]$, which is interpreted as 'X blown up along S'. Locally this is given by introducing polar coordinates in the normal variables, the defining functions, for S. Thus in a neighborhood U of each point $p \in S$, there are k, the codimension of S, local (real-valued, independent) defining functions $s_1, \dots, s_k$ such that $S \cap U = \{s_1 = \cdots = s_k\}$. Thinking of these as coordinates in a normal plane we replace this plane by its blow-up at the origin

$$\beta : [0,\infty) \times \mathbb{S}^{k-1} \ni (r,\omega) \longmapsto s = r\omega \in \mathbb{R}^k. \tag{5.1}$$

This map has good invariance properties under change of variables which results in the blow-up being well-defined. As a set one can take

$$[X;S] = \mathbb{S}NS \cup (X \setminus S) \tag{5.2}$$

where the normal bundle of S in X is $NS = T_S X/TS$ and the spherical normal bundle is the quotient $\mathbb{S}NS = (NS \setminus 0_S)/\mathbb{R}^+$. Local polar coordinates as described above introduce a natural $\mathcal{C}^\infty$ structure on $[X;S]$ as a compact manifold with boundary (assuming of course that X is compact).

Since blow up introduces a boundary face, and this will be done iteratively, we actually need to work directly in the context of manifolds with corners. I will not go into the general treatment of these spaces but only comment on the salient points. The 'coordinate covering' definition of a compact manifold with corners is the same as that of a manifold in the usual sense, except that the coordinate patches are taken as relatively open subsets of $[0,\infty)^n$ and the smoothness of transition maps is in the sense of extension to open subsets of $\mathbb{R}^n$. Then each point has a definite codimension, and the boundary hypersurfaces are, by definition, the closures of components of the codimension one part of the boundary. In fact we insist, as part of the definition of a manifold with corners, that all boundary hypersurfaces be embedded, meaning they do not have higher codimension self-intersections. This has the effect of ensuring that all boundary faces, closures of components of the higher codimension boundary are embedded and hence are themselves manifolds with corners. The definition of blow up outlined above can then be freely applied to the blow up of any boundary face, replacing it by a boundary hypersurface in the new manifold.

In practice the blow up construction needs to be applied to more general submanifolds than boundary faces and the crucial condition is that they be embedded in the sense that they have collar neighborhoods; these are called 'p-submanifolds' for 'product-' since the local condition is that there be a product decomposition of the manifold into intervals and half intervals in which the submanifold is a product.

With this definition of a compact manifold with corners, there is no difficulty directly extending the notion of a smooth action by a compact Lie group, given as a smooth map $A : G \times X \longrightarrow X$ which embeds the group into the group of diffeomorphisms $G \longrightarrow \mathrm{Dfo}(X)$. In fact again we insist on a restriction, essentially by restricting the diffeomorphism group. The obvious definition of a diffeomorphism implies that it maps each boundary hypersurface into a boundary hypersurface and we consider only 'boundary free' diffeomorphisms with the property that under them the image $f(H)$ of a hypersurface H does not intersect H unless $f(H) = H$. We require all elements in the image of G be boundary free in this sense. This excludes, for example, the rotation around the centre through $\pi/2$ of a square. However, this condition can itself be recovered by appropriate resolution of the manifold.

Under the blow up of an embedded boundary face a smooth group action always lifts to a group action. More generally, if $S \subset X$ is a closed p-submanifold which is invariant under the action then again the action lifts smoothly to $[X; S]$; if, as we are insisting, the initial action is boundary free then so is the lifted action.

The conjugacy classes of closed subgroups of G are partially ordered by inclusion and a minimal isotropy type, $S = X^{[K]}$ (corresponding to a maximal isotropy subgroup) is necessarily a closed smooth p-submanifold (possibly with several components) which is invariant under the group action. Thus, the group action lifts to be smooth on the blown up manifold $[X; S]$ and crucially does not have $[K]$ as an isotropy group. Thus, we may proceed inductively, successively blowing up a minimal isotropy type in the manifold at that level of resolution.

Let Y be the manifold obtained by such a complete chain of blow ups

$$\begin{array}{c} Y = Y_n \\ \big\downarrow \beta_{[K_n]} \\ Y_{n-1} \\ \big\downarrow \beta_{[K_{n-1}]} \\ \cdots \\ \big\downarrow \beta_{[K_2]} \\ Y_1 \\ \big\downarrow \beta_{[K_1]} \\ X \end{array} \tag{5.3}$$

where at each stage a new boundary hypersurface is introduced. The resulting smooth manifold with corners is canonically determined, since the order, provided it is consistent with the partial order on isotropy groups, is immaterial. This resolved space has, by construction, a smooth G action with a unique isotropy type, arising from the 'open' isotropy type in the original manifold and which covers the original action.

At each stage of blow up, the 'centre' is a smooth p-submanifold which is replaced by a boundary hypersurface. Thus the boundary hypersurfaces of Y are labelled by the isotropy types in the original manifold (although there may be several non-intersecting boundary hypersurfaces with the same label). When it is introduced each boundary hypersurface carries a G-equivariant fibration, given by the restriction of the blow down map to it, fibering over the centre. The crucial 'iterated conic' property of group actions (as opposed to more general smoothly stratified spaces to which this resolution construction applies) is that successive centres are always transversal to the fibres of the earlier fibrations – which therefore survive as G-equivariant fibrations. Thus each boundary hypersurface of $H_{[K]} \subset Y$, labelled by an isotropy type, fibres G-equivariantly over a smooth manifold with corners

$$\beta_{[K]} : H_{[K]} \longrightarrow Y_{[K]} \tag{5.4}$$

where $Y_{[K]}$ is in fact the resolution of $X_{[K]}$, the closure of $X^{[K]}$. Furthermore these fibrations form a 'tower'. For two boundary hypersurfaces of Y to intersect, their isotropy groups must be related, one must be (conjugate to) a subgroup of the other. The smaller isotropy group corresponds to smaller fibres and the fibrations (5.4) are iteratively related:

$$
\begin{gathered}
H_{[K]} \cap H_{[J]} \neq \emptyset \Longrightarrow [K] \subset [J] \text{ or } [J] \subset [K] \text{ and} \\
[K] \subset [J] \Longrightarrow \begin{array}{ccc} H_{[K]} \cap H_{[J]} & \xrightarrow{\beta_{[J]}} & Y_{[J]} \\ \beta_{[K]} \downarrow & \nearrow \beta_{[J][K]} & \\ H_{[J]}(Y_{[K]}) & & \end{array}
\end{gathered}
\tag{5.5}
$$

where $H_{[J]}(Y_{[K]})$ is a boundary hypersurface of $Y_{[K]}$ and the fibration $\beta_{[J][K]}$ is uniquely determined by the commutativity of the diagram.

The fact that all the iterated fibrations in the resolution tower are G-equivariant means that the structure passes to the quotients. Thus $Z = G\backslash Y$ is a manifold with corners with boundary faces $H_{[K]}(Z) = G\backslash H_{[K]}$ again labelled by the isotropy groups in the original action, and with fibrations to the bases $Z_{[K]} = G\backslash Y_{[K]}$ (although several components may be combined in these quotients).

Thus, the quotient of the resolution (to be thought of as the resolution of the quotient) is a compact manifold with iterated boundary fibrations

$$
\psi_{[K]} : H_{[K]}(Z) \longrightarrow Z_{[K]} \tag{5.6}
$$

which form a fibration tower as in (5.5):

$$
\begin{gathered}
H_{[K]}(Z) \cap H_{[J]}(Z) \neq \emptyset \longrightarrow [K] \subset [J] \text{ or } [J] \subset [K] \text{ and} \\
[K] \subset [J] \Longrightarrow \begin{array}{ccc} H_{[K]}(Z) \cap H_{[J]}(Z) & \xrightarrow{\psi_{[J]}} & Z_{[J]} \\ \psi_{[K]} \downarrow & \nearrow \psi_{[J][K]} & \\ H_{[J]}(Z_{[K]}) & & \end{array}
\end{gathered}
\tag{5.7}
$$

In fact the manifold Z can also be denoted $Z_{[K_{\min}]}$ since it is the resolution of the open isotropy type corresponding to the minimal isotropy group (or groups if the original manifold is not connected). Then the resolution tower can be put in a more succinct form as consisting of iterated boundary fibrations, whenever $[K] \subset [J]$ there is a fibration

$$
\begin{gathered}
\psi_{[J][K]} : H_{[J]}(Y_{[K]}) \longrightarrow Y_{[J]} \text{ and where where } [K] \subset [J] \subset [L] \Longrightarrow \\
\begin{array}{ccccc} & & H_{[J]}(Y_{[K]}) \cap H_{[L]}(Y_{[K]}) & & \\ & \swarrow \psi_{[K][J]} & & \searrow \psi_{[L][K]} & \\ H_{[L]}(Y_{[J]}) & & \xrightarrow[\psi_{[J][L]}]{} & & Y_{[L]} \end{array}
\end{gathered}
\tag{5.8}
$$

commutes.

6. Reduced models for cohomology

The are reduced (and relative) cohomology theories directly associated With the iterated boundary fibrations on Z. Thus for equivariant cohomology consider the chain spaces

$$\begin{aligned}(6.1)\quad & \mathcal{C}^\infty(Z_*; B\otimes\Lambda^*)) \\ & = \Big\{ u_{[K]} \in \mathcal{C}^\infty(X_{[K]}; B\otimes\Lambda^*); i^*_{H_{[J]}} u_{[K]} = \psi^{\#}_{[K][J]} u_{[J]} \forall\ [K]\subset[J] \Big\}\end{aligned}$$

with the flat differential.

Theorem 1. *The cohomology of $\mathcal{C}^\infty(Z_*; B\otimes\Lambda^*)$ for the quotient resolution tower of the canonical resolution of a compact group action on a manifold X is naturally isomorphic to the equivariant cohomology of X.*

Similarly for the K-theory, we can consider bundles over Z with coefficients in the representation bundle. Alternatively the resolution bundles define countably sheeted covers over each of the resolved spaces $Z_{[K]}$ and then the representation-twisted K theory may be constructed from pairs of equivariant bundles over each of the these coverings with support in a finite number of components and with compatibility under the $\psi^{\#}_{[K][J]}$ over boundary hypersurfaces. The same is true of the reduced delocalized cohomology, which can be represented by forms on these covers, with compact supports (hence vanishing outside a finite number of components) and with consistency under pull-back.

Theorem 2. *For the quotient resolution tower of a compact group action, the reduced K-theory with coefficients in the representation bundles is canonically isomorphic to the equivariant K-theory of the original space and the equivariant Chern character to the reduced model for cohomology factors through the delocalized equivariant cohomology*

$$\mathrm{Ch}_G : K_G(X) \longrightarrow H^{even}_{\mathrm{dl},G}(X) \longrightarrow H^{even}_G(X) \tag{6.2}$$

with the first map inducing an isomorphism over the Abelianized spaces.

In the reduced model the Chern character is given by direct generalizations of Chern-Well theory.

7. Index

I did not have time in the lecture to discuss applications of the resolution construction to equivariant index theory but these will be forthcoming in [3].

References

[1] *Supersymmetry and equivariant de Rham theory*, Springer-Verlag, Berlin, 1999.
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