TALK TO PROSPECTIVE GRADUATE STUDENTS 2012

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What sort of problems am I interested in and would suggest to students to work on? Mostly these concern analysis on (smoothly) singular spaces including non-compact

- Non-compact spaces with structure at infinity
- spectral theory
- scattering theory
- quotients by group actions
- compactified moduli spaces
- rings of pseudodifferential operators
- index theory
- hyperbolic equations.

Rather than give a general introduction to these sorts of topics, let me talk mostly about a relatively elementary example which may give you some of the flavor. Namely, the Dirichlet problem for triangles. More specifically, consider the eigenvalue problem

(1)
$$(\partial_x^2 + \partial_y^2)u = -\lambda u \text{ in the interior of T}$$
$$u = 0 \text{ on } \partial T$$

for a planar triangle T. There are countably many eigenvalues and corresponding eigenfunctions,

$$\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \lambda_j \to \infty, \ u_i \in \dot{H}^1(T)$$

each associated with a finite-dimensional space of eigenfunctions (so they are repeated with multiplicity).

On a smoothly bounded region the eigenfunctions are smooth, here they are not quite smooth at the corners. To make them smooth we need to replace the triangle by a 'resolved' version which includes idealized points for each direction of approach to a corner.

Now, consider a slightly more sophisticated question, namely who do things vary if the triangle is varied?

T $\longleftarrow$ u_i smooth
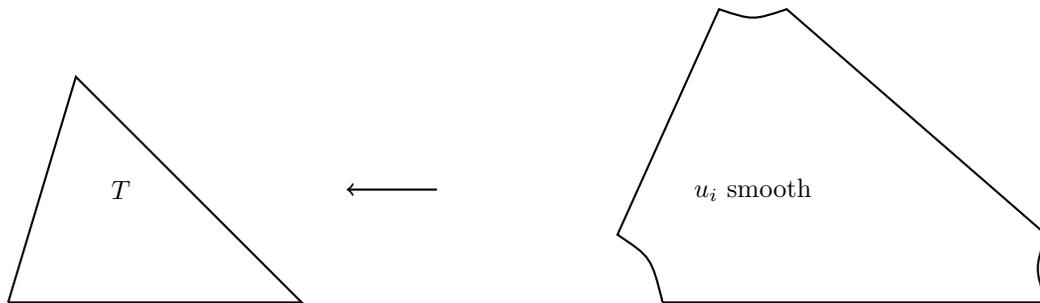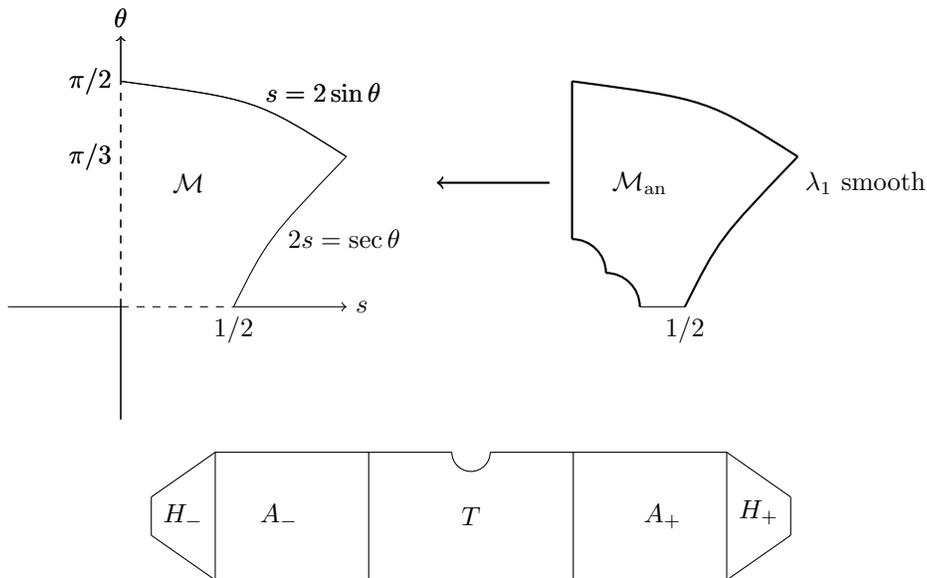
The space of triangles is an open, algebraic, region in the space, $(\mathbb{R}^2)^3$ of the corners. However, modulo Euclidean motions and radial scaling (so this is an example of a group quotient), the moduli space, can be reduced to a two-dimensional region – in terms of one angle and height it looks like the left figure above.

A point here represents a triangle in special position with base of length one and shortest side on the left. The top and right sides are included in D – they represent the isosceles triangles (there are two types, depending on whether the equal sides are longer or shorter than the third). The corner is the equilateral triangle.

A compactification of D means adding some 'ideal points' perhaps representing 'idealized triangles' to get a compact parameter space. How do these arise? We can ask a reasonable question, such as what sort of a function is the smallest Dirichlet eigenvalue? Smooth in the interior, but what happens as we approach the boundary? Need to add 'idealized' limiting triangles where height and angle are related by $\theta \simeq h^2$ and $\theta^2 \simeq h$. Then it becomes smooth. Essentially the same is true of the eigenfunctions – which one needs to construct to prove this – on an even more idealized triangle. The lower figure is a picture of the idealized triangle added for a point where $\theta = 0$ but $s > 0$. The corners are in the middle of the top and at the ends. So the 'limiting' eigenfunctions are smooth on this space.

More realistic problems:-

- Adiabatic limit problems, collapse like the triangles above but with respect to a fibration.
- Equation of quantum gravity – generalizing Poincaré-Einstein metrics.
- Moduli spaces for magnetic monopoles, including the struture of the intrinsic hyper-Kähler metric on it.
- Boundary problems on manifolds with corners for Dirac operators and their global invariants.

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