CONTACT DEGREE

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Let X be a compact contact manifold, with oriented contact line $L \subset T^*X$. Let $\mathcal{M}(X)$ be the contact mapping class group of X; i.e. the group of components of the contact diffeomorphisms. We construct a homomorphism which we are calling the "contact degree"

$$\text{c-deg} : \mathcal{M}(X) \longrightarrow \mathbb{Z}$$

using analytic methods related to the algebra of Heisenberg operators. This homomorphism is directly related to a question of Weinstein (see [8]) concerning the index of Fourier integral operators.

Theorem. *If Y is a compact manifold and $X = S^*Y$ is its cosphere bundle then for any contact diffeomorphism, ϕ, of X*

$$\text{c-deg}(\phi) = \text{ind}(F_\phi)$$

where F_ϕ is a Fourier integral operator associated to ϕ and with symbol 1 *(*[6]*) hence Fredholm on $L^2(Y)$.*

Let Z_ϕ be the mapping cylinder (or 'torus') of ϕ, i.e. $X \times [-1, 1]$ with the ends identified by ϕ. The contact structure on X gives Z_ϕ a natural Spin-$\mathbb{C}$ structure. Let $\eth_\phi$ be the associated Dirac operator then the extension of Weinstein's question is:

Conjecture. $\text{c-deg}(\phi) = \text{ind}(\eth_\phi)$.

This note is written to ask the following questions:

1. Is this conjectural index formula already known?
2. Is there a case when either side is non-zero?

The construction

We rely on the books of Boutet de Monvel and Guillemin [4], Blackadar [2] and of Beals and Greiner [1] (see also the book of Taylor ([7]). The properties of the Heisenberg algebra will be discussed more fully in a forthcoming paper of G. Mendoza and the present authors.

Let $\Psi^0_{\text{He}}(X)$ be the Heisenberg algebra, of 'parabolic' pseudodifferential operators associated to the contact structure on X. This has a natural ideal $\mathcal{I}^0_{\text{He}}(X) \subset \Psi^0_{\text{He}}(X)$ consisting of the operators with full symbols trivial in the lower half of the cotangent bundle. The (non-commutative) symbol map for the Heisenberg calculus gives a short exact sequence of algebras

$$0 \longrightarrow \mathcal{I}^{-1}_{\text{He}}(X) \longrightarrow \mathcal{I}^0_{\text{He}}(X) \longrightarrow \mathcal{S}(\tilde{W}) \longrightarrow 0. \tag{1}$$

Here $\tilde{W}$ is a vector bundle isomorphic to T^*X/L. The product on the Schwartz space $\mathcal{S}(\tilde{W})$ is fibre-wise the usual 'pseudodifferential' product given by the differential of a contact form

$$a\#b = e^{id\alpha(D)} a \otimes b\big|_{\mathrm{Diag}}.$$

This is isomorphic to the operator product on $\mathcal{S}(\mathbb{R}^{2n})$ as kernels of operators on $\mathbb{R}^n$.

The choice of a positive almost complex structure and admissible metric on T^*X/L induces harmonic oscillators on the fibres of $\tilde{W}$. Let $s \in \mathcal{S}(\tilde{W})$ be the projection onto the ground state. The set of these projections, for different choices, is connected. Boutet de Monvel and Guillemin show that s can be lifted to a projection in $\mathcal{I}^0_{\mathrm{He}}(X)$, a generalized Szegő projection, or 'quantized contact structure.'

For any two such projections S, S' (possibly with different choices for symbols s) the composite SS' is Fredholm as a mapping from the range of S to the range of S'; so we may define the relative index

$$\operatorname{ind}(S, S') = \operatorname{ind}(S'S).$$

This can be understood more topologically in terms of the K-theory of the completions of these algebras. Namely $\mathcal{I}^{-1}_{\mathrm{He}}(X)$ is dense in the compact operators on $L^2(X)$ and the completion of the quotient algebra is isomorphic to the continuous functions on X with values in the compact operators on a Hilbert space. The short exact sequence (1) extends by continuity to a short exact sequence of C^* algebras

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\mathcal{I}^0_{\mathrm{He}}(X)} \longrightarrow C^0(X; \mathcal{K}) \longrightarrow 0.$$

In the corresponding 6-term long exact sequence in K-theory there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{e} K(\overline{\mathcal{I}^0_{\mathrm{He}}(X)}) \xrightarrow{\sigma} K(X) \longrightarrow 0.$$

Then

$$e(\operatorname{ind}(S, S')) = [S] - [S'] \text{ if } \sigma([S']) = \sigma([S]) = [s].$$

Using somewhat different, though related, techniques the first author had earlier defined this relative index for a pair of Szegő projectors induced by a pair of embeddable, strictly pseudoconvex CR–structures with the given underlying contact structure, see [5].

Now if ϕ is a contact diffeomorphism of X and $S \in \mathcal{I}^0_{\mathrm{He}}(X)$ is any choice of generalized Szegő projection then $(\phi^*)^{-1} S \phi^*$ is another such choice. The 'contact degree' defined by

$$\text{c-deg}(\phi) = \operatorname{ind}(S, (\phi^*)^{-1} S \phi^*)$$

is independent of the choice of S and is an homotopy invariant of ϕ, hence defined on $\mathcal{M}(X)$.

References

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