COMPACTIFICATION OF SEMI-SIMPLE LIE GROUPS

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Dedicated to the memory of Hans Duistermaat

Abstract. We discuss the 'hd-compactification' of a semi-simple Lie group to a manifold with corners; it is the real analog of the wonderful compactification of De Concini and Procesi. There is a 1-1 correspondence between the boundary faces of the compactification and conjugacy classes of parabolic subgroups with the boundary face fibering over two copies of the corresponding flag variety with fiber modeled on the (compactification of the) reductive part. On the hd-compactification Harish-Chandra's Schwartz space is identified with a space of conormal functions of rapid-logarithmic decay relative to square-integrable functions.

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Introduction

In this note we describe a systematic compactification procedure for Lie groups. Although semi-simple Lie groups are the primary focus of interest for well known 'iterative' reasons we work in the setting of real reductive groups with compact centers. The 'hd-compactification' of such a group, G, is a compact manifold with corners, $\overline{G}$, with interior identified with G and with additional properties described below. The hd-compactification is closely related to (derivable from) the wonderful compactification of De Concini and Procesi, [7], in the case of complex adjoint groups, and to the (maximal) Satake compactification, and more especially the Oshima compactification, [14], of the homogeneous space G/K for a maximal compact subgroup K; see also [9].

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By a semi-simple group we shall mean a connected Lie group whuch has a semi-simple Lie algebra and which can be realized as a closed subgroup of a matrix group. More generally a real reductive group is taken to be a Lie group which is a central extension of a semi-simple group by a compact abelian group, so giving a principal bundle

$$\Theta \longrightarrow \Gamma \longrightarrow G. \tag{1}$$

A *compactification* $\overline{M}$ of a manifold M without boundary is a smooth map $I : M \hookrightarrow \overline{M}$ which is a diffeomorphism onto the interior of a compact manifold with corners $\overline{M}$. Two compactifications are *equivalent* if there is a diffeomorphism between them intertwining the inclusions into the interiors. As part of the definition we always demand that the boundary hypersurfaces of a compact manifold (by default meaning with corners) are embedded. This implies that each such hypersurface H has a global boundary defining function $\rho_H \in \mathcal{C}^\infty(M)$ such that $\rho_H > 0$ on $\overline{M} \setminus H$, $H = \{\rho = 0\}$ and $d\rho_H \neq 0$ at H. Thus $d\rho_H\big|_H$ spans the conormal bundle to H. These assumptions also imply that the components of the intersections of the hypersurfaces, the boundary faces, are all embedded and are naturally compact manifolds with corners.

In the category of manifolds with corners the arrows are required to be smooth maps $f : X' \longrightarrow X''$ in the usual sense that $f^*\mathcal{C}^\infty(X'') \subset \mathcal{C}^\infty(X')$ but also they are b-maps meaning that for each boundary hypersurface H of X''

$$f^*\rho_H = \begin{cases} \equiv 0 & \text{or} \\ a\prod_K \rho_K^{n_{KH}},\ 0 < a \in \mathcal{C}^\infty(X') & \text{or} \\ > 0. \end{cases}$$

The maps we are most interested in are interior b-maps, in which the first option does not occur. The powers, n_{KH}, in the product decomposition over the boundary hypersurfaces of X' are necessarily non-negative integers (so the last case is where these integers all vanish).

The Lie algebra of the group of diffeomorphisms of a compact manifold consists of the smooth vector fields tangent to all boundary faces, the b-vector fields. Equivalently these are the smooth vector fields satisfying $V\rho_H \in \rho_H\mathcal{C}^\infty(X)$ for all boundary defining functions ρ_H. The b-vector fields form a Lie algebroid, $\mathcal{V}_{\mathrm{b}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{b}}TX)$, where ${}^{\mathrm{b}}TX \longrightarrow TX$ has null space at each boundary point p of codimension k i.e. lying in the interior of a boundary face F of codimension k, a canonically trivial vector space ${}^{\mathrm{b}}N_pF \subset {}^{\mathrm{b}}T_pM$ which extends to a smooth subbundle over F. These spaces are spanned by the vector fields $x_i\partial_{x_i}$ in terms of local coordinates in which the x_i define the boundary hypersurfaces through p.

Definition 1. An hd-compactification of a real reductive Lie group with compact center is a compact manifold with corners $\overline{G}$ and a diffeomorphism onto the interior $G \hookrightarrow \overline{G}$ such that

- (D1) [inversion] Inversion extends to a diffeomorphism of $\overline{G}$.
- (D2) [b-normality] The right action of G extends smoothly to $\overline{G}$ with isotropy algebra at each boundary point projecting to span the b-normal space.
- (D3) [b-transitivity] The combined action of $G\times G$ on left and right is b-transitive, i.e. has Lie algebra spanning ${}^{\mathrm{b}}T\overline{G}$.
- (D4) [minimality] Near each boundary point and for each local boundary hypersurface through that point the span of the Lie algebra for the right action

contains a vector field zv where z is a defining function for the hypersurface and v is tangent to the boundary but independent of the span of the Lie algebra.

Main Theorem. *Any real reductive group with compact center has an hd-compactification which is unique up to equivalence, intertwining the right and left actions, and so defines a functor from these groups and isomorphisms to compact manifolds with corners and diffeomorphisms.*

In view of this naturality, the hd-compactification extends to spaces which are left and right principal G spaces, reduced to G by fixing a point, for a real reductive group G.

Uniqueness holds under the weaker requirement that the (rather technical) minimality condition, D4, holds near one point of maximal codimension. However it cannot be removed altogether.

Conjecture. *A compactification satisfying D1-D3 is necessarily obtained from an hd-compactification by some generalized boundary blow-up in the sense of* [11]*, under which inversion lifts to a diffeomrophism.*

It is straightforward to check that such a blow-up does preserve D1-D3.

One direct consequence of the properties demanded above is that the action, on left or right, of a maximal compact subgroup $K \subset G$ on $\overline{G}$ is necessarily free. Thus $\overline{G}/K$ defines a compactification, essentially the maximal Satake compactification, of G/K, see Borel and Ji [3].

Below we give a detailed differential-geometric description of the hd-compactification with subsequent geometric analysis in mind, see Mazzeo and Vasy [13], [12] and Parthasarathy and Ramacher [15]. Considering first the semi-simple case, the boundary faces of $\overline{G}$ are shown to be in 1-1 correspondence with the conjugacy classes of parabolic subgroups of G and hence with the subsets $S \subset D$ of the nodes of the reduced Dynkin diagram. If $\mathcal{F}_S$ is the flag variety parameterizing the parabolics associated to S then the corresponding boundary face of $\overline{G}$ can be realized as a bundle

$$\begin{tikzcd} \overline{M} \arrow[r, dash] & \overline{F}_S \arrow[d] \\ & \mathcal{F}_S \times \mathcal{F}_S \end{tikzcd} \tag{2}$$

where the model fibre is the compactification of the real-reductive group in the Langlands decomposition $\mathcal{F}_S \ni P = M_P A_P N_P$ of an associated parabolic; here we are proceeding by induction over the real rank of groups. More geometrically the fibre at (P', P) is the hd-compactification of the space of elements of G such that $M_{P'} g = g M_P$. The isotropy group for the right action of G at a point of F_S is the associated normal solvable subgroup $A_P N_P$.

We show how to assemble a compactification

$$\overline{G} = G \sqcup \bigcup_{S \subsetneq D} F_S \tag{3}$$

from the unions of the interiors of the boundary faces. To do so we define 'gluing maps' from the putative normal bundles to the F_S which have fiber at a point of F_S a partial compactification of the corresponding group A_P and show that this is an hd-compactification. Subsequently we show that any two hd-compactifications are equivalent. In the more general case of a real reductive group, as in (1), which is of course necessary for the inductive argument, the construction proceeds in the same way, giving a principal Θ-bundle over $\overline{G}$.

For analytic purposes it is of prime importance to describe the behavior of the invariant vector fields for the action of G on $\overline{G}$; this can be seen from the root space decomposition as discussed by Knapp [10], Wallach [21] and Varadarajan [20]. To this end we observe that the flag varieties carry iterated tangent structures. If $P \in \mathcal{F}_S$ is a parabolic subgroup in a fixed conjugacy class then the transitive action of K, by conjugation, on $\mathcal{F}_S$ has isotropy group $K \cap P$ at P. In the Langlands decomposition $P = M_P A_P N_P$ the Lie algebra $\mathfrak{a}_P$ of A_P has rank $s = \#(D \setminus S)$. The positive root vectors, joint eigenvectors for the conjugation action of $\mathfrak{a}_P$ on the Lie algebra span, the Lie algebra of $\mathfrak{n}_P$. The eigen-decomposition of $\mathfrak{k}$ defines a surjective map $\mathfrak{k} \longrightarrow \mathfrak{n}_P$ with null space $\mathfrak{k} \cap \mathfrak{m}_P$, the Lie algebra of the isotropy group. The decomposition of $\mathfrak{n}_P$ into eigenspaces with joint eigenvalues $\alpha \cdot \mathfrak{a}_P$ thus induces a corresponding filtration

$$E^{\alpha}_P \subset T_P \mathcal{F}_S. \tag{4}$$

The action of K extends the filtration to $T\mathcal{F}_S$ by subbundles labeled by s-multi-indices satisfying

$$\begin{gathered} E^{\alpha} \subset T\mathcal{F}_S \ \forall\ \alpha \in \mathbb{N}^s,\ E^{\alpha} = T\mathcal{F}_S \text{ for some } \alpha \\ \alpha \leq \beta \Longrightarrow E^{\alpha} \subset E^{\beta}, \\ E^{\alpha} + E^{\beta} \subset E^{\alpha+\beta}, \\ [\mathcal{V}_{\alpha}, \mathcal{V}_{\beta}] \subset \mathcal{V}_{\alpha+\beta},\ \mathcal{V}_{\alpha} = \mathcal{C}^{\infty}(\mathcal{F}_S; E^{\alpha}). \end{gathered} \tag{5}$$

The iterated structures on intersecting boundary faces are related in such a way that they induce a Lie algebroid structure $E_R \subset {}^{\mathrm{b}}T\overline{G}$ consisting of the vector fields tangent to all boundary faces, to the fibers of the left fibration to $\mathcal{F}_S$ induced by (2) for each S and with normal vanishing properties corresponding to the E^{α}.

As noted above there is a direct relationship between the wonderful compactification of the adjoint form of a complex semi-simple group and the hd-compactification.

Prop-W. *The real blow up of the exceptional divisors in the wonderful compactification of the adjoint group of a complex semi-simple group is an hd-compactification.*

At the boundary of the hd-compactification Haar measure takes the form

$$dg = \rho_*^{-\sigma} \nu_{\mathrm{b}} \tag{6}$$

where ν_{b} is a non-vanishing smooth b-measure, ρ_* is a vector of defining functions for the boundary hypersurfaces (corresponding to maximal parabolic subgroups, so to the sets $D \setminus \{n\}$) for $n \in D$) and σ is the multiindex given by the sum of the positive roots.

As already noted in [1] the utility of the hd-compactification is illustrated by the fact that Harish-Chandra's Schwartz space takes a relatively simple form with respect to the hd-compactification.

Prop-H-C. *Harish-Chandra's Schwartz space is the space of conormal functions with respect to the boundary of $\overline{G}$ of log-rapid decay with respect to $L^2(G)$.*

This was proved for $\mathrm{SL}(2, \mathbb{R})$ in [1] but the argument is sufficiently general to apply here.

In §1 general properties of groups actions on manifolds with corners are discussed and the characterization of Harish-Chandra's Schwartz space is recalled from [1]. The special case of $\mathrm{SL}(n,\mathbb{R})$ is treated in §2 where the hd-compactification is obtained by blow-up of the sphere in the linear space of $n \times n$ real matrices. For the closely related case of $\mathrm{SL}(n,\mathbb{C})$ both the wonderful and the hd-compactification are obtained by similar methods in §3. The construction in the general case is contained in §4 and the relationship with the wonderful compactification is described in §5. Uniqueness is shown in §6.

1. Group actions on manifolds with corners

First consider the smooth action of a compact Lie group on a compact manifold with corners; the fact that the boundaries are 'one-sided' imposes constraints on such actions since the isotropy algebra at boundary points must act trivially on the normal bundle to the boundary.

Lemma 1. *If a compact Lie group acts smoothly on a connected compact manifold with corners and acts trivially on one boundary face then it acts trivially.*

Proof. This is a consequence of linearizability of compact group actions. Linearizing the action around a point in the interior of a boundary hypersurface shows that the local action is the same as that on the spherical normal bundle. If the action on the hypersurface is trivial then, from the one-sided nature of the boundary, there is no $\mathbb{Z}_2$ action between the two normal directions so the action on the normal bundle is also trivial and hence the action is trivial nearby. Again the triviality of the action on an open set implies that the action is trivial on that component of the space. Applying this argument iteratively, triviality on any one boundary face implies triviality on all faces of which it is a boundary hypersurface, and hence triviality on the full manifold. □

In particular a smooth action of a connected compact group on a connected compact manifold with corners of codimension up the dimension, so including points, is necessarily trivial. This includes simplexes.

Conversely there is a similar inheritance of freedom of an action.

Proposition 1. *If a compact group acts smoothly on a compact manifold with corners with the action free on the interior then it acts freely.*

Proof. Consider the isotropy group at any boundary point, which is always in the interior of some boundary face. By linearizability, the fact that the action is free in the interior implies that the isotropy group must act freely and linearly on the fibres of the inward-pointing part of the spherical normal bundle; since this is a simplex the action must preserve all boundary faces and so is trivial and hence so is the isotropy group and the full action must be free. □

One of the most fundamental requirements that we place on the compactification of a group $G \hookrightarrow \overline{G}$ as the interior of a compact manifold with corners in Definition 1 is that the left and right actions of G extend smoothly to $\overline{G}$. The existence of such a smooth extension of the action is equivalent to the condition that the left- and right-invariant vector fields extend from the interior to be b-vector fields on $\overline{G}$. Thus each action gives a Lie algebra homomorphism

$$\mathfrak{g} \longrightarrow \mathcal{V}_{\mathrm{b}}(\overline{G}) = \mathcal{C}^\infty(\overline{G}; {}^{\mathrm{b}}T). \tag{1.1}$$

A compactification for which the actions extends smoothly is by no means unique since one can make 'transcendental' changes to $\overline{G}$ under which the smooth action of the vector fields lifts. Suppose one has such a (weak) compactification. Take a defining function for one of the boundary hypersurfaces, x, and replace it by

$$t = \operatorname{ilog} x = \frac{1}{\log \frac{1}{x}},\ \ x = \exp(-\frac{1}{t}). \tag{1.2}$$

The manifold with new $\mathcal{C}^\infty$ structure generated near this boundary hypersurface by t, in addition to x and all other smooth functions, is homeomorphic to the original $\overline{G}$ and maps smoothly back to it. Moreover, a simple computation shows that the tangential vector fields lift to be smooth and tangential on the manifold, since $x\partial_x$ lifts to $t^2\partial_t$. Thus the 'blown up' manifold is still a weak compactification but is, in a sense, much larger and certainly not equivalent.

The condition D3 in the definition of an hd-compactification in the Introduction prevents this extreme sort of 'enlargement'.

Definition 2. The action of a Lie group on a compact manifold with corners is *b-normal* if at each boundary point the b-normal space is spanned by elements of the Lie algebra of the isotropy group.

In fact this still does not prevent 'blow-up' indeterminacy, as discussed above, from arising. However, the transformations can no longer be transcendental. Still, the introduction of $s = x^{\frac{1}{k}}$ in place of x at a boundary hypersurface, for any positive integer k, again defines a smooth map from the 'resolved' space which is a homeomorphism and under which the b-vector fields lift to smooth b-vector fields – so a smooth action lifts to be smooth. In this case the b-normal $x\partial_x$ is replaced by $k^{-1}t\partial_t$. Thus b-normality is also preserved. In consequence we need to impose the further 'minimality' condition, (1), in the definition of an hd-compactification, to ensure uniqueness.

The action of G on itself is of course transitive hence so is the combined right and left action of $G \times G$. In (1) we impose a condition which, in a weakened sense, extends this to the compactification.

Definition 3. The action of a Lie group on a manifold with corners is *b-transitive* if its Lie algebra generates the b-vector fields as a $\mathcal{C}^\infty$ module.

A *b-compactification* $G \hookrightarrow \overline{G}$ is an inclusion as the interior of a compact manifold with corners such that the right and left actions extend smoothly and are b-normal and the combined action $G \times G$ is b-transitive.

Since the interiors of all boundary faces are (by assumption) connected, a b-transitive action is transitive on these interiors.

Lemma 2. *For a b-compactification the isotropy groups of the left and right actions at the interior are conjugate over the interior of any boundary face.*

Proof. Since the right and left actions commute the orbits through points in the interior of any one boundary face are equivariantly diffeomorphic. □

The proof of the analogous result in [1] only uses properties holding for a b-compactification and (6) so yields Prop-H-C of the Introduction.

2. $\mathrm{SL}(n,\mathbb{R})$

Before proceeding to the construction of an hd-compactification in general we consider the case of $\mathrm{SL}(n,\mathbb{R})$ (and subsequently $\mathrm{SL}(n,\mathbb{C})$). For these standard groups we give an explicit construction of an hd-compactification which illustrates and guides the more abstract construction below. For later convenience in place of $\mathbb{R}^n$ consider an oriented real Euclidean vector space, V, of dimension n. Then consider the inclusions

$$\mathrm{SO}(V) \subset \mathrm{SL}(V) \subset \mathrm{GL}(V) \subset \mathrm{Hom}(V), \tag{2.1}$$

where $\mathrm{Hom}(V) = \mathrm{Hom}(V, V)$ is the space of linear maps. We give $\mathrm{Hom}(V)$ the usual Hilbert-Schmidt norm $\|e\| = \mathrm{Tr}(e^*e)^{\frac{1}{2}}$.

Let $\mathbb{SH}(V) \subset \mathrm{Hom}(V)$ be the unit sphere and consider the map given by (positive) radial scaling

$$\mathrm{SL}(V) \longrightarrow \mathbb{SI}(V) \subset \mathbb{SH}(V) \tag{2.2}$$

where $\mathbb{SI}(V) = \mathbb{SH}(V) \cap \mathrm{GL}(V)$ is the open subset consisting of the invertible homomorphisms of norm one.

Lemma 3. *Radial scaling* (2.2) *is a diffeomorphism onto its range* $\mathbb{SI}_+(V) \subset \mathbb{SI}(V)$, *the open subset of* $\mathbb{SH}(V)$ *where the determinant is positive.*

Proof. The inverse of (2.2) is

$$\mathbb{SI}_+(V) \ni e \longrightarrow e/(\det(e))^{\frac{1}{n}} \in \mathrm{SL}(V) \tag{2.3}$$

so this map is a diffeomorphism. □

The hd-compactification of $\mathrm{SL}(V)$ is obtained by compactifying the range $\mathbb{SI}_+(V)$ of (2.2) and this in turn is accomplished through blow-up of $\mathbb{SH}(V)$. Again for use in inductive arguments below we proceed slightly more generally. Suppose W_1 and W_2 are two oriented real Euclidean vector spaces of the same dimension, k. Then consider the unit sphere (always in the Hilbert-Schmidt norm) $\mathbb{SH}(W_1, W_2) \subset \mathrm{Hom}(W_1, W_2)$. The orthogonal groups $\mathrm{SO}(W_i)$ act on $\mathbb{SH}(W_1, W_2)$ on the right and left and both these actions are free on $\mathbb{SI}(W_1, W_2) \subset \mathbb{SH}(W_1, W_2)$, the open set of invertible homomorphisms. Following [2] these actions may be resolved by successive blow-up of the isotropy types which are the

$$S_q(W_1, W_2) = \{e \in \mathbb{SH}(W_1, W_2); e \text{ has corank } q\},\ 1 \le q \le k-1; \tag{2.4}$$

here q is the 'depth' of the stratum.

Denote the resolution, (obtained by blow up in order of increasing dimension or decreasing depth),

$$\overline{\mathbb{SH}}(W_1, W_2) = [\mathbb{SH}(W_1, W_2), S_*]. \tag{2.5}$$

In case $W_1 = W_2 = W$ we set $\overline{\mathbb{SH}}(W, W) = \overline{\mathbb{SH}}(W)$ and for $W = \mathbb{R}^k$, $\overline{\mathbb{SH}}(W) = \overline{\mathbb{SH}}(k)$, etc.

Diffeomorphisms fixing the center lift under blow-up so the orthogonal actions lift and Euclidean isomorphisms $I_i : W_i \longrightarrow \mathbb{C}^k$ $i = 1, 2$ result in a commutative square covering the blow-down maps

$$\begin{tikzcd} \overline{\mathbb{SH}}(W_1, W_2) \arrow[r, leftrightarrow] \arrow[d, "\beta"'] & \overline{\mathbb{SH}}(k) \arrow[d, "\beta"] \\ \mathbb{SH}(W_1, W_2) \arrow[r, leftrightarrow] & \mathbb{SH}(k). \end{tikzcd} \tag{2.6}$$

Proposition 2. *The resolution* (2.5) *is a compact manifold with corners up to codimension $k-1$ with the boundary hypersurface corresponding to S_q fibering over the double Grassmannian with fiber modeled (inductively) by resolved spaces*

$$\begin{tikzcd} \overline{\mathbb{SH}}(k-q) \times \overline{\mathbb{SH}}(q) \arrow[r, dash] & H_q \arrow[d] \\ & \mathrm{Gr}(W_1, k-q) \times \mathrm{Gr}(W_2, k-q) \end{tikzcd} \tag{2.7}$$

where the fiber above (U_1, U_2) is $\overline{\mathbb{SH}}(U_1, U_2) \times \overline{\mathbb{SH}}(U_1^\perp, U_2^\perp)$.

Proof. Consider the deepest stratum $S_{k-1} \subset \mathbb{SH}(W_1, W_2)$ with maximal isotropy group. Each element of S_{k-1} has one-dimensional range and $k-1$ dimensional null space. So these are determined precisely by the pair of lines U_1, the range of the adjoint, and the range U_2 and the element itself lies in the two-point space $\mathbb{SH}(U_1, U_2) \ni e$.

Thus S_{k-1} is a double cover of the product $\mathbb{P}W_2 \times \mathbb{P}W_1$ of the real projective spaces with fiber $\mathbb{SH}(U_1, U_2)$. By the collar neighbourhood theorem, a neighbourhood of S_{k-1} in $\mathbb{SH}(W_1, W_2)$ fibers over S_{k-1} and can be taken to be form

$$(1 - \|f\|^2)^{\frac{1}{2}} e + f,\ f \in \mathrm{Hom}(U_1^\perp, U_2^\perp),\ \|f\| < \epsilon. \tag{2.8}$$

Indeed this follows from the left and right polar decomposition of homomorphisms close to e. Both polar parts must have only one eigenvalue near 1, hence with smooth eigenspace, and this gives the projection onto S_{k-1} with the remainder in $\mathrm{Hom}(U_1^\perp, U_2^\perp)$. This identifies the normal bundle to S_{k-1} and hence the elements of the unit normal sphere bundle, which is the face H'_{k-1} produced by blow-up of S_{k-1} (before the other blow-ups) as a fiber bundle

$$\begin{tikzcd} \mathbb{SH}(k-1) \arrow[r, dash] & H'_{k-1} \arrow[d] \\ & S_{k-1} \end{tikzcd} \tag{2.9}$$

with fibre $\mathbb{SH}(U_1^\perp, U_2^\perp)$ at (U_1, U_2).

The naturality of blow-up shows that the subsequent steps in the resolution of $\mathbb{SH}(W_1, W_2)$ restrict to H'_{k-1} as the corresponding resolution of $\mathbb{SH}(U_1^\perp, U_2^\perp)$. Thus the final hypersurface $H_{k-1} \subset \overline{\mathbb{SH}}(W_1, W_2)$ becomes the fiber bundle

$$\begin{tikzcd} \overline{\mathbb{SH}}(k-1) \arrow[r, dash] & H_{k-1} \arrow[d] & \text{or} & \overline{\mathbb{SH}}(1) \times \overline{\mathbb{SH}}(k-1) \arrow[r, dash] & H_{k-1} \arrow[d] \\ & S_{k-1} & & & \mathbb{P}W_2 \times \mathbb{P}W_1 \end{tikzcd} \tag{2.10}$$

with total fiber in the second case $\overline{\mathbb{SH}}(U_1, U_2) \times \overline{\mathbb{SH}}(U_1^\perp, U_2^\perp)$. This is (2.7) for $q = 1$.

The structure of the other faces follows iteratively. Namely after the blow-up of the faces S_{k-j} for $j < q$, the stratum S_{k-q} in (2.4) is resolved to the bundle

$$\begin{tikzcd} \overline{\mathbb{SH}}(k-q) \arrow[r, dash] & S'_{k-q} \arrow[d] \\ & \mathrm{Gr}(W_2, k-q) \times \mathrm{Gr}(W_1, k-q) \end{tikzcd} \tag{2.11}$$

where the fibre is $\overline{\mathbb{SH}}(U_1, U_2)$ over the pair (U_2, U_1) in the Grassmannian. The fiber of the normal bundle is precisely $\mathbb{SH}(U_1^\perp, U_2^\perp)$ by essentially the same argument in terms of the two polar decompositions. The subsequent blow-ups resolve the fibers to give (2.7). $\square$

We define a compactification of $\mathrm{SL}(V)$ by taking the closure of the image of (2.2):

$$\overline{\mathrm{SL}}(V) = \text{Closure of } \mathbb{SI}_+(V) \text{ in } \overline{\mathbb{SH}}(V). \tag{2.12}$$

In fact this is one of the two components of $\overline{\mathbb{SH}}(V)$; $\overline{\mathbb{SH}}(V)$ is an hd-compactification of $\mathbb{Z}_2 \times \mathrm{SL}(V)$ and from this we deduce that

Proposition 3. *The compact manifold* (2.12) *is an hd-compactification of* $\mathrm{SL}(V)$ *for an oriented Euclidean space* V.

Proof. The description of the iterative blow-up allows the blow-down map to be described near a point of (exactly) codimension r in $\overline{\mathbb{SH}}(W_1, W_2)$. This corresponds to a subset

$$\bar{q} \subset \{1, \dots, k-1\},\ \#(\bar{q}) = r,\ q_i \in \bar{q},\ 1 \le q_1 < q_2 < \cdots < q_r \le k-1. \tag{2.13}$$

Then the corresponding boundary face $F_{\bar{q}} = H_{q_1} \cap H_{q_2} \cdots \cap H_{q_r}$ fibres over the product of two copies of the flag manifold

$$\mathcal{F}(W_j, \bar{q}) = \{U_{j,\bar{q}}\},\ U_{i,q_r} \subset U_{i,q_{r-1}} \subset \dots U_{q_1} \subset W_i,\ \dim(U_{j,q_j}) = k - q_j. \tag{2.14}$$

We set $q_{r+1} = k$ and $q_0 = 0$ for notational convenience and $U_{j,q_{r+1}} = W_j$, $U_{j,q_0} = \{0\}$. Then the flag defines a sequence of orthogonal projections,

$$\pi_{j,q} \text{ onto } V_{j,q_i} = U_{j,q_i} \ominus U_{j,q_{i+1}}.$$

There are r local defining functions $\tau_{q_i} \in [0, \epsilon)$ and an element of $\overline{\mathbb{SH}}(W_1, W_2)$ projecting to this neighbourhood is of the form

$$\gamma = \sum_{i=1}^{r} (\prod_{i'>i} \tau_{q_{i'}}) \pi_{2,q_i} e_{q_i} \pi_{1,q_i},\ e_{q_i} \in \mathbb{SI}(V_{1,q_i}, V_{2,q_i}). \tag{2.15}$$

This point, after radial rescaling, lies in $\overline{\mathrm{SL}}(V)$ if all the $\tau_q \ge 0$ and the determinant at $\tau_q = 1$ is positive. From (2.3) the corresponding point in $\mathrm{SL}(n, \mathbb{R})$, for all $\tau_i > 0$, is

$$g = a \left(\prod_{i=1}^{r} \tau_i^{-d_i/n} \right) \sum_{i=1}^{r} (\prod_{i'>i} \tau_{q_{i'}}) \pi_{2,q_i} e_{q_i} \pi_{1,q_i} \tag{2.16}$$

where $a > 0$ is locally smooth. The inverse is therefore of the form

$$g^{-1} = a^{-1} \left(\prod_{i=1}^{r} \tau_i^{(d_i-n)/n} \right) \sum_{i=1}^{r} (\prod_{i'\le i} \tau_{q_{i'}}) \pi_{1,q_i} e_{q_i}^{-1} \pi_{2,q_i}. \tag{2.17}$$

From this it follows that inversion is a diffeomorphism on $\overline{\mathrm{SL}}(n,\mathbb{R})$. That the right and left actions extend smoothly follows from the construction since these actions are smooth after projection to $\mathbb{S}\mathrm{H}$ and fix the centers of blow-up, so lift smoothly to $\overline{\mathrm{SL}}$.

A point in the flag variety determines a parabolic subgroup although this involves a consistent choice of Weyl chamber. Consider the standard flag with kth subspace spanned by the first k elements of the standard basis of $\mathbb{R}^n$. This fixes an Iwasawa decomposition

$$\mathrm{SL}(n) = \mathrm{SO}(n)AN \tag{2.18}$$

where A is the subgroup of positive diagonal matrices and N, corresponding to a choice of positive roots, consists of the upper diagonal matrices with all diagonal entries one. Thus the Lie algebra of N is spanned by the elementary matrices E_{ij} with one non-zero entry, 1 at (i,j) for $j > i$. The Lie algebra, $\mathfrak{a}$, of A consists of the diagonal matrices, with diagonal entries α_i satisfying the trace condition

$$\sum \alpha_i = 0. \tag{2.19}$$

The E_{ij} are joint eigenvectors for the adjoint action of $\mathfrak{a}$ with collective eigenvalues, the positive roots, being the elements $\alpha_i - \alpha_j$ of $\mathfrak{a}'$. The primitive roots are the $\alpha_i - \alpha_{j+1}$ for $1 \le i \le n-1$. The dual basis of $\mathfrak{a}$, the coroots, are the diagonal matrices with first k diagonal entries $1 - \frac{k}{n}$ and remaining entries $-\frac{k}{n}$. This decomposes $\mathfrak{a}$ into a sum of one-dimensional spaces, with basis elements, and hence decomposes A into a product of half-lines A_i with coordinates (the coweights) t_i so that the coroots become $t_i\partial_{t_i}$. In terms of these coordinates on A an element of A_i is of the form

$$\mathrm{diag}(t_i^{1-\frac{k}{n}}, \dots, t_i^{1-\frac{k}{n}}, t_i^{-\frac{k}{n}}, \dots, t_i^{-\frac{k}{n}}) \tag{2.20}$$

Observe that as $t_i \to \infty$ the projection of this 1-parameter group into $\overline{\mathrm{SL}}(n,\mathbb{R})$ is smooth up to the boundary in terms of the parameter $\tau_i = t_i^{-1}$, where it meets H_i as a normal vector field with z_i extending to a local boundary defining function.

The other Borel subgroups $B \subset \mathrm{SL}(n,\mathbb{R})$ are conjugate, under the action of $\mathrm{SO}(n)$, to this basic choice, so the flag variety is identified as

$$\mathcal{F}(\mathbb{R}^n, \{1, \dots, n-1\}) = \mathrm{SL}(n)/\,\mathrm{SO}(n), \tag{2.21}$$

and this action conjugates the Iwasawa decomposition as well, giving the corresponding groups A_B and N_B replete with their root space decompositions.

From this we conclude that in the decomposition (2.15) near the 'Borel' face, of the maximal codimension $n-1$, the parameters τ_i correspond (at least to first order at the boundary) to the action on the right by A_B or on the left by $A_{B'}$ on the fiber of the boundary above $(B, B') \in \mathrm{SL}(n)/\,\mathrm{SO}(n) \times \mathrm{SL}(n)/\,\mathrm{SO}(n)$. This certainly shows the b-normality of the extended action since it persists nearby and then extends to all boundary points using the left and right actions of $\mathrm{SO}(n)$.

The b-transversality also follows directly from this analysis. At the maximal codimension face the Lie algebra of the right action projects surjectively to the tangent space of the right flag variety (2.21) and the left action projects surjectively to the left factor. The b-normal space is in the range of either action and the fiber is discrete. By smoothness, the surjectivity of the map from $\overline{\mathrm{SL}}(n) \times \mathfrak{g} \times \mathfrak{g}$ to ${}^{\mathrm{b}}T\,\overline{\mathrm{SL}}(n)$ extends to an open neighborhood. Every orbit of the $G \times G$ action intersects such an open neighborhood of this face so b-transitivity extends to the whole space. In fact this is also clear from the analysis below of the behavior near a general boundary face.

Finally, it remains to show that the minimality condition holds. Again it suffices to compute the behavior of the Lie algebra for the right action near one point in the boundary face of maximal codimension and then extend the result using the action of $G \times G$. Choosing the base point to be the standard flag in (2.16) with the $e_i = 1$ in this case the action of the element $E_{ij} \in \mathfrak{n}$, $j > 1$, on the right acts on the flag on the right as $\frac{1}{2}(E_{ij} - E_{ji}) \in K$. The infinitesmal action of $\mathfrak{n}$ on the right also shifts the left flag in the same way, in this case after conjugation by γ. Thus the right action of E_{ij} projects on the left flag to

$$\frac{1}{2}\left(\prod_{k \geq j-1} \tau_k\right) \kappa_{ij},\ \kappa_{ij} = (E_{ij} - E_{ji}),\ j > i. \tag{2.22}$$

It follows that nearby the projection of the right action of the Lie algebra onto the left flag manifold vanishes at the boundary and there is an element (with $j = i+1$) vanishing simply on a given hypersurface. This the minimality condition is also satisfied. □

From the analysis of the action on the right of the Lie algebra leading to (2.22) it follows that this compactification has the properties discussed in the Introduction. Thus on the flag variety corresponding to Borel subgroups, $\mathcal{F} = \mathrm{SO}(n)/Z$, the tangent bundle is stratified by one-dimensional subbundles E^α where α runs over the $(n-1)$-multiindices with entries 1 forming an interval and all others vanishing. The commutators of these subbundles are contained in the sums unless the two intervals are non-overlapping but contiguous, in which case the commutator, modulo the sum, span the bundle corresponding to the union.

3. $\mathrm{SL}(n, \mathbb{C})$

For the complex group, $\mathrm{SL}(n, \mathbb{C})$, there are two distinct generalization of the compactification via the resolution of the projective image as carried out above. In the first we proceed by passing systematically to the complex category and in the second by proceeding by direct analogy. The first approach yields the wonderful compactification of the adjoint group $\mathrm{SL}(n, \mathbb{C})/Z$ where the center is the multiplicative group of nth roots of unity. The second approach gives an hd-compactification.

Consider an Hermitian vector space V of complex dimension n then in place of (2.1)

$$\mathrm{SU}(V) \subset \mathrm{SL}(V) \subset \mathrm{GL}(V) \subset \hom(V) \tag{3.1}$$

where the last three spaces are complex. Let $\mathbb{P}\mathrm{h}(V) = (\hom(V) \setminus \{0\})/\mathbb{C}^\times$ be the projective space of homomorphisms. Projection gives

$$\mathrm{SL}(V) \longrightarrow \mathbb{P}\mathrm{I}(V) \subset \mathbb{P}\mathrm{h}(V) \tag{3.2}$$

where the center is mapped to the image of the identity and the range, $\mathbb{P}\mathrm{I}(V)$ is an open dense subset of $\mathbb{P}\mathrm{h}(V)$. The complement $\mathbb{P}\mathrm{h}(V) \setminus \mathbb{P}\mathrm{I}(V)$ is the image of the non-vanishing, non-invertible, matrices and so is again stratified by corank,

$$\mathbb{P}\mathrm{h}(V) \setminus \mathbb{P}\mathrm{I}(V) = \bigcup_{1 \leq k < n} S_k. \tag{3.3}$$

These are all complex submanifolds, although only S_{n-1} is compact. As in the real case they are the isotropy types for the action of $\mathrm{SU}(V)/Z$ which acts freely on $\mathbb{P}\mathrm{I}(V)$.

Again in this case we may blow these submanifolds up, iteratively, in the complex sense, in order of increasing dimension to produce the resolved space which we denote tentatively as

$$\overline{\mathbb{P}\mathrm{h}}_{\mathrm{CP}}(V) = [\mathbb{P}\mathrm{h}(V); S_*]_{\mathbb{C}}. \tag{3.4}$$

To see that this is well-defined as a compact complex manifold note that an element of S_{n-1} is the projective image of a complex homomorphism of rank one, so

$$S_{n-1} = \mathbb{P}V \times \mathbb{P}V \tag{3.5}$$

where the first point is the orthcomplement of the null space and the second the range. The normal bundle to S_{n-1} can then be identified with the bundle with fibre at a point the homomorphisms from the null space to the orthocomplement. Blowing up replaces S_{n-1} be its projectivized normal bundle D'_{n-1}

$$\begin{tikzcd} \mathbb{P}\mathrm{h}(\mathbb{C}^{n-1}) \arrow[r, dash] & D'_{n-1} \arrow[d] \\ & \mathbb{P}V \times \mathbb{P}V. \end{tikzcd} \tag{3.6}$$

In a small neighbourhood of this exceptional divisor the resolved manifold is a bundle over $\mathbb{P}V \times \mathbb{P}V$ with fiber a neighborhood of the zero section of the blow-up of the origin in the homomorphism bundle from null space to orthocomplement of range. The closures of the other strata meet this bundle in the corresponding strata modeled on $S_j(\mathbb{P}\mathrm{h}(\mathbb{C}^{n-1}))$. Proceeding by induction as in the real case these lower depth strata can be blown up in order and replace the divisor (3.6) by its resolution giving the bundle

$$\begin{tikzcd} \overline{\mathbb{P}\mathrm{h}}(\mathbb{C}^{n-1}) \arrow[r, dash] & D_{n-1} \arrow[d] \\ & \mathbb{P}V \times \mathbb{P}V. \end{tikzcd} \tag{3.7}$$

It follows that the full resolution is possible with the stratum S_j replaced by a divisor which is a complex bundle over the product of two copies of the Grassmannian

$$\begin{tikzcd} \overline{\mathbb{P}\mathrm{h}}(\mathbb{C}^{j}) \times \overline{\mathbb{P}\mathrm{h}}(\mathbb{C}^{n-j}) \arrow[r, dash] & D_j \arrow[d] \\ & \mathrm{Gr}(V, n-j) \times \mathrm{Gr}(V, n-j). \end{tikzcd} \tag{3.8}$$

Proposition 4. *This resolution, $\overline{\mathbb{P}\mathrm{h}}(V)$, of $\mathbb{P}\mathrm{h}(V)$ is the wonderful compactification of the adjoint group $\mathrm{SL}(n, \mathbb{C})/Z$.*

This is consequence of the characterization of the wonderful compactification as the unique regular compactification with a single compact orbit, see Uma [19] and Brion [4] and references therein.

Real and complex blow-up of a complex submanifold are related in a simple way, namely the real blow-up is the complex blow-up followed by the real blow up of the resulting divisor, as a real submanifold of codimension two. For normally intersecting divisors this carries over iteratively – the real blow-ups can all be performed after the complex ones because of transversality.

Now Prop-W in the Introduction can be stated more explicitly.

Proposition 5. *The blow-up, in the real sense, of all the divisors in* (3.4) *gives an hd-compactification of the adjoint group* $\mathrm{SL}(n,\mathbb{C})/Z$:

$$\overline{\mathrm{SL}(n,\mathbb{C})/Z} = [\overline{\mathbb{P}\mathrm{h}}\, V; S_*]_{\mathbb{R}}. \tag{3.9}$$

Rather than prove this immediately we examine the compactification of $\mathrm{SL}(n,\mathbb{C})$ obtained by following the procedure for $\mathrm{SL}(n,\mathbb{R})$ more closely. That is, first project radially into the sphere in the Hermitian vector space

$$\mathrm{SL}(V;\mathbb{C}) \longrightarrow \mathbb{S}\mathrm{h}(V). \tag{3.10}$$

Again this is a diffeomorphism but now onto the smooth hypersurface, $\mathbb{S}\mathrm{h}_+(V)$, in $\mathbb{S}\mathrm{h}(V)$ where the determinant is positive.

The actions of $\mathrm{SU}(V)$ on $\mathbb{S}\mathrm{h}(V)$, left and right, fix $\mathbb{S}\mathrm{h}_+(V)$ and act freely on it. The isotropy types for the action on $\mathbb{S}\mathrm{h}(V)$ are again the submanifolds $S_q \subset \mathbb{S}\mathrm{h}(V)$ of corank q. We define a resolution of $\mathrm{SL}(V)$ by taking the closure in the (real) resolution

$$\overline{\mathrm{SL}}(V) \subset [\mathbb{S}\mathrm{h}(V), S_*]. \tag{3.11}$$

The determinant is well-defined and non-zero on $\mathbb{S}\mathrm{I}(V) \subset \mathbb{S}\mathrm{H}(V)$ and we define a normalized version by

$$\widehat{\det} : \mathbb{S}\mathrm{I}(V) \ni g \longmapsto \frac{\det g}{|\det g|} \in \mathbb{T}. \tag{3.12}$$

Now the $\mathrm{SU}(V)$ actions extend to actions of $\mathrm{U}(V)$ with the extended actions free on $\mathbb{S}\mathrm{I}(V)$. The isotropy types are the same as for the $\mathrm{SU}(V)$ action, i.e. the S_q. Thus in the case $W_1 = W_2 = V$ the resolution (3.11) also resolves the $\mathrm{U}(V)$ actions, which therefore become free on $\overline{\mathbb{S}\mathrm{H}}(V)$. The normalized determinant factors through the action of $\mathrm{U}(V)$,

$$\widehat{\det}(ug) = \det(u)\widehat{\det}(g) \tag{3.13}$$

and it follows from this that $\widehat{\det}$ extends smoothly to $\overline{\mathbb{S}\mathrm{H}}(V)$ and has non-vanishing differential everywhere. In fact the differential must be independent of all conormals as well, so the level surfaces are well-defined p-submanifolds of $\overline{\mathbb{S}\mathrm{H}}(V)$.

Thus the definition (3.11) does lead to a compact manifold with corners, with extended actions of $\mathrm{SL}(V)$, since the initial action preserves the S_q. That this is an hd-compactification now follows as in the real case discussed above. Moreover there is a simple relation between the real and complex cases:

Proposition 6. *The closure of* $\mathrm{SL}(n,\mathbb{R})$ *in the compactification* $\overline{\mathrm{SL}}(n,\mathbb{C})$ *is a p-submanifold which is an hd-compactification.*

4. Direct construction

In the construction of an hd-compactification of G, a real reductive group with compact center, we use the root space decomposition, see for example [10], which we proceed to summarize briefly. First we make the choice of a 'real Cartan subgroup' of G, a maximal product of multiplicative half-lines, A with Lie algebra $\mathfrak{a}$ and compatible Cartan involution. The adjoint action of $\mathfrak{a}$ on the full Lie algebra, $\mathfrak{g}$, is symmetric and decomposes $\mathfrak{g}$ into the joint null space and a direct sum of eigenspaces with corresponding joint eigenvectors, the roots, in $\mathfrak{a}^*$. The choice of a positive subspace, R_+, of the roots induces an Iwasawa decomposition

$$G = KAN \tag{4.1}$$

where K is a maximal compact subgroup and N is the unipotent group generated by the span of the strictly positive root (eigen)vectors. This also induces a decomposition

$$A = \prod_{i\in D}^{r} A_i,\ \tau_i : A_i \simeq \mathbb{R}^+,\ \#(D) = r, \tag{4.2}$$

as a product of half-lines labeled by the nodes, D, of the Dynkin diagram. The parameters in the factors which may be identified with the coweights. Thus $\mathfrak{a}$ is spanned by the $\tau_i\partial_{\tau_i}$ and each of the root vectors spanning the Lie algebra $\mathfrak{n}$ of N has non-negative homogeneity under the adjoint action A_i (with at least one positive exponent). This decomposes $\mathfrak{n}$ as a graded algebra under the conjugation action of the A_i;

$$\begin{gathered}\mathrm{Ad}(a)n = \tau^\alpha n,\ n \in \mathfrak{n}_\alpha,\ a \in A,\ \mathfrak{n} = \bigoplus_{\alpha\in R_+} \mathfrak{n}_\alpha \\ \mathfrak{n}_{\alpha+\beta} = [\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \text{ if } \alpha,\ \beta,\ \alpha+\beta \in R_+.\end{gathered} \tag{4.3}$$

These choices correspond to a particular minimal parabolic subgroup

$$P = MAN \tag{4.4}$$

where in this case the real-reductive part is compact. The other minimal parabolics are conjugate to this one, under the action of K, and so have a similar decomposition (4.3) and (4.4).

Each conjugacy class of parabolic sugroups corresponds to a subset $S \subset D$ where the minimal case corresponds to the empty set. To S we associate the group

$$A_S = \prod_{i\notin S} A_i \subset A. \tag{4.5}$$

The normalizer of A_S in G contains a maximal real-reductive subgroup M_S with compact center fixed by the Cartan involution. The corresponding parabolic subgroup has Langlands decomposition

$$P_S = M_S A_S N_S \tag{4.6}$$

where the unipotent group N_S has Lie algebra

$$\mathfrak{n}_S = \bigoplus_{\gamma\in R_+^S} n_\gamma. \tag{4.7}$$

Here R_+^S is the subset of the positive roots which have a positive value on one of the basis elements of the Lie algebra of A_S. This induces an action of A_S of the form (4.3) for these restricted roots.

Again the conjugation action of the maximal compact subgroup in the initial Iwasawa decomposition induces similar decompositions of all the conjugate parabolics $P = M_P A_P N_P$ independent of the indeterminacy in conjugation.

Let $\mathcal{F}_S$ be the flag variety of parabolic subgroups conjugate to P_S. For a pair $(P', P) \in \mathcal{F}_S \times \mathcal{F}_S$ consider the space

$$F(P, P') = \{g \in K \cdot M_P; M_{P'} g = g M_P\}. \tag{4.8}$$

Lemma 4. *For each $S \subset D$, the spaces $F(P', P)$ have principal M_P actions on the right and $M_{P'}$ actions on the left and form a smooth bundle with total space F_S*

$$\begin{tikzcd} M_S \arrow[r, dash] & F_S \arrow[d] \\ & \mathcal{F}_S \times \mathcal{F}_S \end{tikzcd} \tag{4.9}$$

and fiber modeled on M_S.

Proof. The action of K on $\mathcal{F}_S$, by conjugation on G, is transitive with isotropy group at $P \in \mathcal{F}_S$ the subgroup $K \cap M_P$. It follows that if $k \in K$ and $kPk^{-1} = P'$ then $F(P, P') = kM_P$ from which the result follows. □

The properties of the putative compactification functor imply that it extends to principal spaces such as $F(P, P')$ so, proceeding inductively over dimension or real rank, we can define the *closed* boundary face corresponding to each conjugacy class of parabolics in G as the fiberwise compactification of the smooth bundle (4.9) with fibre the compactification of $F(P', P) \simeq M_S$

$$\begin{tikzcd} \overline{M}_S \arrow[r, dash] & \overline{F}_S \arrow[d] \\ & \mathcal{F}_S \times \mathcal{F}_S. \end{tikzcd} \tag{4.10}$$

We define a partial compactification

$$A_S \hookrightarrow \overline{A}_S \tag{4.11}$$

of each A_S by adding a point at infinity to each factor, with smooth structure given by the parameter τ_i^{-1} where τ_i is the multiplicative variable in (4.2). Thus $\overline{A}_S$ is a product of $\#(D \setminus S)$ half-closed intervals. The conjugation action of K induces corresponding compactifications $\overline{A}_P$ of each A_P and so defines a bundle over $\mathcal{F}_S$ and hence another level on the fiber bundles (4.10):

$$\begin{tikzcd} \overline{A}_S \arrow[r, dash] & N_+ \overline{F}_S \arrow[d] \\ \overline{M}_S \arrow[r, dash] & \overline{F}_S \arrow[d] \\ & \mathcal{F}_S \times \mathcal{F}_S. \end{tikzcd} \tag{4.12}$$

The fibre of $N_+\overline{F}_S$ is $\overline{A}_P$ over (P, P') but may also be naturally identified with $\overline{A}_{P'}$. The presumptuous notation $N_+\overline{F}_S$ presages the identification with the inward-point normal bundle to $\overline{F}_S$ as a boundary face of $\overline{G}$.

To construct $\overline{G}$ as a smooth space we define gluing maps, the most fundamental one (recalling that we proceed by induction) is for the minimal face corresponding to (4.12) with $S = \emptyset$; in this case we drop the subscript. Then M is already compact. The partially compactified fibre at each point $(B, B') \in \mathcal{F} \times \mathcal{F}$ has a point of maximal codimension, corresponding to passage to infinity in all factors $A_i \subset A$. This defines a section, realizing the zero section of N_+F; the bundle F in this case is already compact.

Proposition 7. *The map*

$$\gamma : N_+F \ni (m, a) \longrightarrow ma \in G,\ m \in F(P, P'),\ (P', P) \in \mathcal{F} \times \mathcal{F} \tag{4.13}$$

is smooth and if $\overline{O} \subset N_+F$ is a small (relatively) open neighborhood of the zero section of N_+F, corresponding to minimal parabolic subgroups, then γ is a diffeomorphism of O, the intersection of $\overline{O}$ with the interior of N_+F as a manifold with corners, onto an open subset of G.

In fact we can take $\overline{O}$ to be invariant under the left and right actions of K.

Proof. The map is well-defined and smooth. In the discussion above we have chosen a base minimal parabolic and this induces an identification $\mathcal{F} = K/Z$. The total space of the bundle F, the boundary face here is $K \times_Z K$ and the map becomes

$$g = k_1 a k_2,\ k_i \in K,\ a \in A \tag{4.14}$$

which descends to $K \times_Z K \times A$. This is the Cartan decomposition. In the open set O, where all the components of a are large, the decomposition is unique, up to the indeterminacy in the center. The factors may be chosen smoothly, locally, which shows (4.13) to be a smooth bijection onto its open image with smooth local inverses, i.e. a diffeomorphism. □

To complete the construction of $\overline{G}$ we extend the map (4.14) to the bundles corresponding to the other parabolics and then to similar maps between these bundles. For each $S \subset D$ the total space of F_S may be identified with

$$F_S \simeq K \times_{K_S} M_S \times_{K_S} K,\ K_S = K \cap M_S, P_S = M_S A_S N_S. \tag{4.15}$$

Then consider the smooth map

$$\gamma_S : K \times M_S \times K \times A_S \ni (k_1, m, k_2, a) \longmapsto k_1 m a k_2 \in G. \tag{4.16}$$

This descends to F_S, using (4.15).

Theorem 1. *The total spaces of the fibrations* (4.10) *form the boundary stratification of an hd-compactification*

$$\overline{G} = G \sqcup \bigcup_{S \subsetneq D} \overline{F}_S \tag{4.17}$$

where the $\mathcal{C}^\infty$ structure, near each boundary face, is fixed by the gluing maps (4.16).

Proof. As noted above, we proceed by induction over the dimension of G, allowing for real-reductive groups with compact centers. In particular this means that we take as given the compactification in (4.10), and hence all the $\overline{F}_S$ are well-defined when we come to construct $\overline{G}$.

The partial order on the $S \subset D$ by inclusion induces a partial order on the $\overline{F}_S$ which corresponds to inclusion as boundary faces. Consider two subsets $S_1 \subsetneq S_2 \subsetneq D$ – so excluding the interior which corresponds to D. Set $M_i = M_{S_i}$ etc, so in particular $K_i = K \cap M_i$. The partially compactified 'normal' spaces are products

$$\overline{A}_1 = \overline{A}_2 \times \overline{A}_{12}. \tag{4.18}$$

There is a corresponding 'lifted' gluing map

$$\begin{aligned} \gamma_{12} : K \times K_2 \times M_1 \times K_2 \times K \times A_1 \ni (k', k_1', m, k_1'', k'', a_2 a_{12}) \\ \longmapsto (k', k_1' m a_{12} k_1'', k'', a_2) \end{aligned} \tag{4.19}$$

which descends to a map

$$N_+F_1 = K \times_{K_1} M_1 \times_{K_1} K \times A_1 \longrightarrow K \times_{K_2} M_2 \times_{K_2} \times A_2 = N_+F_2. \tag{4.20}$$

Since these are proper boundary faces, corresponding to lower dimensional groups M_S, the inductive hypothesis means that the maps extend to smooth maps on the compactified spaces

$$\gamma_{12} : N_+\overline{F}_1 \longrightarrow N_+\overline{F}_2. \tag{4.21}$$

Moreover the gluing maps (4.16), γ_i to G for the two boundary faces factor

$$\gamma_2 = \gamma_1 \circ \gamma_{12}. \tag{4.22}$$

If $U_S \subset M_S$ is an open subset with compact closure and $\overline{O}_S$ is a sufficiently small, relatively open neighborhood of the point at infinity in $\overline{A}_S$, corresponding to the zero section of the normal bundle, then the map γ_S restricted to the image of the set $K \times U_S \times K \times O_S$ is a diffeomorphism onto its range in G.

Now, starting from the face of maximal codimension, we can successively choose such subsets which together cover the union of the boundary faces in the sense that for all S

$$M_S \subset U_S \cup \bigcup_{S' \subsetneq S} \gamma_{SS'}(K \times M_{S'} K \times K \times O_{SS'}). \tag{4.23}$$

These choices give the $\mathcal{C}^\infty$ structure on $\overline{G}$. Namely all of the maps γ_S and $\gamma_{SS'}$ are diffeomorphisms when restricted to these small domains. The conditions (4.22) mean that the maps on the components of the covering of M_S in (4.23), γ_S on U_S and the composite map $\gamma_{S'}\gamma_{SS'}^{-1}$ on the other parts, are consistent on overlaps. Each of these maps identifies an open neighborhood of the boundary face F_S with a corresponding open subset of the inward-pointing normal bundle N_+F_S with smoothness on the overlaps, thus making $\overline{G}$ into a manifold with corners.

To see that $\overline{G}$ is an hd-compactification of G we check the conditions in Definition 1. First observe the effect of inversion on the image of one of the gluing maps

$$(k_1 m a k_2)^{-1} = k_2^{-1} m^{-1} a' k_1^{-1}. \tag{4.24}$$

Here (P, P') are reversed, $m^{-1} \in F(P', P) = F(P'_-, P_-)$ and a' is a point near infinity in $A_{P'_-}$ where P_- is the opposite parabolic, which corresponds to inverting A_P and changing to the negatives of the roots. So this does indeed extend to a diffeomorphism.

Next consider the right action of $g \in G$ on the image of some γ_S. The action of K extends smoothly (and freely) up to the boundary by conjugating the 'incoming' parabolic P to P'' and over this action on $\mathcal{F}_S$ mapping A_P to $A_{P''}$ and $F(P, P')$

to $F(P'', P')$. So it suffices to consider the action of P near a boundary fiber over (P', P). In the Langlands decomposition the action of M_P on $F(P, P')$ is smooth, free and transitive and near the identity in A_P the action factors through that on $\overline{A}_P$, in particular fixing the boundary. So it remains to check the action of the unipotent group N_P. Since $a \in A_P$ acts on N_P by conjugation, as in (4.3),

$$an = n_a a,\ n_a \text{ smooth on } \overline{A}_P \tag{4.25}$$

and with n_a vanishing when a is at the point at infinity. Thus N_P fixes the boundary fibers above (P, P') for all P'. The smoothness of the action follows. The b-normality of the action follows from the fact that at a boundary fiber above (P, P') the action of A_P is through scaling on the normal fiber.

This duscussion shows that over the boundary face $F_S \longrightarrow \mathcal{F}_S \times \mathcal{F}_S$ the right action fixes the left factor of $\mathcal{F}_S$ in the base and acts transitively on the fibers. Since inversion conjugates the right to left action and reverses the factors in the base it follows that the action of $G \times G$ is b-transitive.

Finally, that the mininality condition holds can be seen from the local homogeneity in (4.3).

For the more general case of a real-reductive group, as in (1), proceed with a covering of the compactification of the quotient semi-simple group G. So again the boundary faces of $\overline{\Gamma}$ are labelled by the conjugacy classes of parabolic subroups of G. If $P \subset G$ is a closed subgroup then the preimage $\tilde{P} \subset \Gamma$ is a closed subgroup of Γ. The preimage of P has an induced decomposition from the Langlands decomposition $P = MAN$,

$$\tilde{P} = \tilde{M} \times_\Theta \tilde{A} \times_\Theta \tilde{N}. \tag{4.26}$$

This induces a bundle covering (4.9) for G :

$$\begin{array}{ccc} \tilde{M}_S & \text{———} & \tilde{F}_S \\ & & \big\downarrow \\ & & \mathcal{F}_S \times \mathcal{F}_S \end{array} \tag{4.27}$$

with fiber over (P', P) a principal Θ-bundle

$$\tilde{F}(P', P) = \{g \in \Gamma; \tilde{M}_{P'} h = h\tilde{M}_P\}. \tag{4.28}$$

The bundle formed by the $\tilde{A}$ over the right factor in the base lifts to a bundle over tF_S with fiber isomorphic to A. Now the covering 'gluing map'

$$\tilde{\gamma}(\tilde{m}, \tilde{a}) \longmapsto \tilde{m} \times_\Theta \tilde{a} \in \Gamma \tag{4.29}$$

has properties consistent with those of the gluing maps for $\overline{G}$ and assembles the compact manifold with corners $\overline{\Gamma}$ giving an hd-compactification. □

5. Relation to the wonderful compactification

The wonderful compactification of De Concini and Procesi has properties directly related to those we desire for hd-compactification in terms of the (close) analogy between log geometry in the complex realm and b-geometry in the real one.

An algebraic variety X with an action of a connected reductive group G is wonderful if:

(i) X is smooth, connected, and compact,
(ii) The G-action leaves invariant a finite number of smooth irreducible divisors $D_1, \ldots, D_r$ with strict normal crossings and nonempty intersection,
(iii) The G-orbits of the action are the subsets

$$\Big(\bigcap_{j\in J} D_j\Big) \setminus \Big(\bigcup_{j\notin J} D_j\Big), \quad J \subseteq \{1,\ldots,r\}.$$

In particular, X has 2^r G-orbits, one of which is open and dense, and exactly one of which is closed (and has codimension r). Any divisor D_i, or intersection of divisors, is again a wonderful variety. Sumihiro [18] showed that any normal G-variety with only one closed G-orbit, such as a wonderful variety, is projective. It is pointed out in [17] that the radical of G (i.e., the connected component of the identity of its maximal normal solvable subgroup) necessarily acts trivially on X, so only semisimple groups can act faithfully on their wonderful varieties.

Brion, Luna and Vust in [6] established a local structure theorem for G-varieties, which applies to a wonderful variety X near its closed G-orbit. This orbit can be identified with G/Q with Q the stabilizer of a point q and, since G/Q is projective, Q is a parabolic subgroup of G. Let P be a parabolic subgroup opposite to Q, so that $L = P \cap Q$ is a Levi subgroup of P and Q, and let P^u be its unipotent radical. There exists a locally closed affine subvariety $Z \subseteq X$ containing q that is L-stable and such that

$$\begin{aligned} P^u \times Z &\longrightarrow X \\ (g,\zeta) &\longmapsto g\zeta \end{aligned}$$

is an open P-equivariant immersion, where the action of an element $p \in P$ of the form $p = v\ell$, $v \in P^u$, $\ell \in L$, is by

$$v\ell \cdot (u,\zeta) = (v\ell u \ell^{-1}, \ell\zeta).$$

Moreover $Z \cong \mathbb{C}^r$ in such a way that

$$D_i \cap Z \leftrightarrow \{(z_1,\ldots,z_r) \in \mathbb{C}^r : z_i = 0\}$$

and the L-action is fixed by r linearly independent characters of L, $\{\sigma_1,\ldots,\sigma_r\}$,

$$\ell \cdot (z_1,\ldots,z_r) = (\sigma_1(\ell)z_1,\ldots,\sigma_r(\ell)z_r).$$

A consequence of this local theory is that the infinitesimal action of the group maps surjectively onto the vector fields tangent to all of the divisors of X, conventionally known as the logarithmic vector fields [17, Proposition 4.2], [5, Section 2]. The kernel of this map at a point p is the kernel of the action of the Lie algebra of the stabilizer of p, $\mathfrak{g}_p$ on the normal bundle to the orbit of G through p.

If G is a complex semisimple group of adjoint type, so with trivial center, De Concini and Procesi [7] have constructed a compactification of G which is wonderful, in this sense, with respect to the natural action of $G \times G$.

Theorem 2. *The wonderful compactification $G[1,\mathrm{CP}]$ of a semisimple Lie group of adjoint type G satisfies:*
i) [inversion] Inversion extends to a biholomorphism of $G[1,\mathrm{CP}]$.
ii) [log-normality] The left action of G extends smoothly to $G[1,\mathrm{CP}]$ with isotropy algebra at each boundary point projecting to span the b-normal space.
iii) [log-transitive] The combined action of $G \times G$ on left and right is b-transitive.

iv) [minimality] Near each point on $G[1, \mathrm{CP}] \setminus G$ and for each divisor through that point, the span of the Lie algebra for the right action contains a vector field xv where x is a defining function for the divisor and v is tangent to the divisor but independent of the span of the Lie algebra.

Proof. The construction of De Concini and Procesi is to embed G into the projective space of the endomorphisms of an irreducible representation, V, whose highest weight is regular and dominant,

$$\psi : G \longrightarrow \mathbb{P}(\mathrm{End}(V)), \quad \psi(g) = [g],$$

and then $G[1, \mathrm{CP}] = \overline{\psi(G)}$. This embedding is equivariant with respect to the $G \times G$ action on G by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$ and its action on $\mathbb{P}(\mathrm{End}(V))$ by $(g_1, g_2) \cdot [A] = [g_1 A g_2^{-1}]$. The map $[A] \mapsto [A^{-1}]$ on $\mathbb{P}(\mathrm{End}(V))$ is a biholomorphism and restricts to $\psi(G)$ to be the inversion map of G, so (i) is clear.

As pointed out above, (iii) holds for arbitrary wonderful varieties. This follows from the local structure theorem of Brion-Luna-Vust, which we now describe in the context of the wonderful compactification of G, following [8]. Let us fix a maximal torus $T \subseteq G$, a Borel subgroup B containing T with unipotent radical U, and the opposite Borel B^- with unipotent radical U^-. There is a closed affine variety, $Z = \overline{\psi(T)} \cong \mathbb{C}^{\dim T}$, containing the identity of G, such that

$$\begin{array}{rcl} U^- \times U \times Z & \longrightarrow & G[1, \mathrm{CP}] \\ (u, v, \zeta) & \longmapsto & u\zeta v^{-1} \end{array}$$

is an isomorphism onto its image, X_0, which is smooth, isomorphic to $\mathbb{C}^{\dim G}$, stable under the action of $U^-T \times U$, and dense in $G[1, \mathrm{CP}]$. The action of $T \cong (\mathbb{C}^*)^{\dim T}$ on $Z \cong \mathbb{C}^{\dim T}$ is by coordinatewise multiplication, so the Lie algebra of the torus $T \times \{\mathrm{id}\}$ acts via $(t_1 \partial_{t_1}, \dots, t_r \partial_{t_r})$ and (ii) holds.

The minimality condition (iv) again follows from the local homogeneity (4.3). $\square$

Corollary 1. *The real blow-up of divisors in the wonderful compactification of an adjoint group is an hd-compactification*

$$[G; \mathrm{hd}] = [G[1, \mathrm{CP}]; D_*]_{\mathbb{R}}. \tag{5.1}$$

6. Uniqueness

In this section we show that all hd-compactifications are equivalent; this mainly reduces to analysing the structure of a boundary face of maximal codimension.

Proposition 8. *For a b-compactification of a connected semi-simple Lie group, G, a face of maximal codimension, F, is equivariantly diffeomorphic to $K \times_Z K$ for the diagonal action of the center on a maximal compact subgroup. It has codimension equal to the real rank, l, of G and the isotropy groups for the action of $G \times G$ are the products $B \times B'$ where B, B' run over the Borel subgroups of G.*

Proof. By definition each boundary face is connected and a face of maximal codimension is necessarily a compact manifold without boundary. Thus G, acting on the right on F, must have a compact orbit at which the isotropy group is necessarily parabolic. On the other hand, since the action of a maximal compact subgroup is free, by Proposition 1, it follows that the isotropy group can only contain a finite compact subgroup so must be a Borel subgroup of G. Lemma 2 shows that all isotropy groups are conjugate, so all are Borel. The same holds for the left action of G, so for the action of $G\times G$ the isotropy groups must be of the form $B\times B'$. The assumption of b-transversality implies that $G\times G$ acts transitively on F so this face is the flag variety $K\times_Z K$. $\square$

The action of K on the left and right descends to a smooth action of $K\times_Z K$ on any b-compactification, $\overline{G}$, and as just shown this is principal on F. Thus the action remains free nearby so F has a neighbourhood basis consisting of the total spaces of principal bundles

$$\begin{tikzcd} K\times_Z K \arrow[r, dash] & \overline{O} \arrow[d] \\ & {[0,\epsilon)^l.} \end{tikzcd} \tag{6.1}$$

We proceed to analyse the action of the Lie algebra of the isotropy group at each point of F, a boundary face of maximal codimension (not yet shown to be unique). Near a point $p\in F$ consider

$$\mathcal{W}_p=\mathcal{V}_{\rm b}\cap\mathcal{I}_p\mathcal{V}, \tag{6.2}$$

composed of those smooth tangent vector fields, in a neighbourhood of p, which vanish at p as vector fields in the ordinary sense. Since the Lie action consists of elements of $\mathcal{V}_{\rm b}$ the isotropy algebra

$$\mathfrak{i}_p\subset\mathcal{W}_p. \tag{6.3}$$

Now each term in (6.2) is a Lie algebra so $\mathcal{W}_p$ is itself a Lie algebra.

The right and left actions by K are locally free near F, so we may introduce local coordinates near p, consisting of normal coordinates x_i, invariant under both actions of K, and tangential coordinates y_j, z_j where the right action of K is in the y_j variables, leaving the z_j fixed and conversely for the left action of K. In such local coordinates

$$\begin{gathered}\mathcal{W}_p\ni w=\sum_{i=1}^r\mu_ix_i\partial_{x_i}+\sum_j\gamma_j\partial_{y_j}+\sum_j\gamma'_j\partial_{z_j},\\ \gamma_j(p)=0,\ \gamma'_j(p)=0,\ \mu_i,\ \gamma_j,\ \gamma'_j\in\mathcal{C}^\infty.\end{gathered} \tag{6.4}$$

It follows that the commutator

$$[\mathcal{W}_p,\mathcal{W}_p]\subset\mathcal{I}_p\mathcal{V}_{\rm b}. \tag{6.5}$$

Indeed, the second sum in (6.4) is contained in $\mathcal{I}_p\mathcal{V}_{\rm b}$, which is a Lie algebra, and the commutators

$$\begin{aligned}[\mu_ix_i\partial_{x_i},\mu_kx_k\partial_{x_k}]&=\mu_ix_i(\partial_i\mu_k)x_k\partial_{x_k}-\mu_kx_k(\partial_k\mu_i)x_i\partial_{x_i},\\ [\mu_ix_i\partial_{x_i},\gamma_j\partial_{y_j}]&=\mu_ix_i(\partial_{x_i}\gamma_j)\partial_{y_j}-\gamma_j(\partial_{y_j}\mu_i)x_i\partial_{x_i}\\ [\mu_ix_i\partial_{x_i},\gamma'_j\partial_{z_j}]&=\mu_ix_i(\partial_{x_i}\gamma'_j)\partial_{z_j}-\gamma'_j(\partial_{z_j}\mu_i)x_i\partial_{x_i}\end{aligned} \tag{6.6}$$

are all in $\mathcal{I}_p\mathcal{V}_{\rm b}$. Now, the quotient

$$\mathcal{W}_p/\mathcal{I}_p\mathcal{V}_{\rm b}=\operatorname{sp}\{x_i\partial_{x_i}\} \tag{6.7}$$

is the b-normal Lie algebra, abelian of rank l, with the map just being evaluation of the coefficients $\mu_i(p)$.

Lemma 5. *For any b-normal compactification the maximal abelian part of the isotropy algebra $\mathfrak{i}_p$ at $p \in F$ maps surjectively to $\mathcal{W}_p/\mathcal{I}_p\mathcal{V}_{\mathrm{b}}$.*

Proof. The assumption of b-normality is that $\mathfrak{i}_p$ maps surjectively to the b-normal algebra. At a point of F, $\mathfrak{i}_p$ is (maximal) solvable with nilpotent part $\mathfrak{n}_p = [\mathfrak{i}_p, \mathfrak{i}_p]$. So, from (6.3) and (6.5), $\mathfrak{n}_p$ is mapped in $\mathcal{I}_p\mathcal{V}_{\mathrm{b}}$ and since the abelian part is a complement to $\mathfrak{n}_p$ it must map surjectively onto $\mathcal{W}_p/\mathcal{I}_p\mathcal{V}_{\mathrm{b}}$. □

Now, if we assume that a real Cartan subgroup, A, and Cartan involution, θ, have been chosen, then the Abelian part of the isotropy group is some K-conjugate of A. We shall assume without subsequent loss of generality that $p \in F$ is a point at which the isotropy group is precisely AN where N is fixed by a choice of positive Weyl chamber. Thus A is also decomposed as an explicit product of $\mathbb{R}^+$ subgroups given by the root space decomposition. Let a_i be the corresponding basis of the Lie algebra $\mathfrak{a}$ of A. Lemma 5 shows that in the right action of G these are represented by elements

$$\mathcal{W} \ni a_i \longmapsto \sum_k \alpha_{ik} x_k \partial_{x_k} \in \mathcal{W}_p/\mathcal{I}_p\mathcal{V}_{\mathrm{b}} \tag{6.8}$$

where α is an invertible matrix.

Consider the next terms in the Taylor series of the a_i, to evaluate them modulo $\mathcal{I}_p\mathcal{W}_p$. In terms of (6.4) this captures not only the values of the $\mu_i(p)$ but also the differentials of the γ_j and γ'_j at p :

$$a_i = \sum_k \alpha_{ik} x_k \partial_{x_k} + \sum_j \left(L_j(x,y)\partial_{y_j} + L'_j(x,y)\partial_{z_j} \right) \quad \mathrm{mod}\ \mathcal{I}_p\mathcal{W}_p \tag{6.9}$$

where the L_j and L'_j are linear functions of the x_i and y_j. Note that a_i is acting on the right, so commutes with the action of K on the left which implies that the coefficients are independent of the z_j (when expressed in terms of the right-invariant vector fields on K_z which reduce at a point to the basis ∂_{z_j}).

We recall properties of the reduced root decomposition of the maximal solvable subalgebras of G which we need below.

Lemma 6. *The isotropy algebra at a point $p \in F$ is $\mathfrak{i}_p = \mathfrak{a}_p \oplus \mathfrak{n}_p$ where the abelian part $\mathfrak{a}_p$ has a unique basis, up to order and sign, a_i and $\mathfrak{n}_p$ has a reduced root decompositon, namely a basis n_α where $\alpha \in \mathbb{N}$ is the associated root, so $[a_i, n_\sigma] = \alpha_i n_\sigma$; there are l simple roots, with $|\alpha| = 1$, each with a non-trivial root space and these span $\mathfrak{n}_p/[\mathfrak{n}_p, \mathfrak{n}_p]$.*

The choice of a trivialization of the principal $K \times_Z K$ bundle (6.1), and of a base-point, $p \in F$, leads to a representation of the Lie algebra

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\Psi"] \arrow[dr, "\psi"'] & & \mathcal{V}_{\mathrm{b}}(\overline{O}) \\ & \mathcal{C}^\infty(O; \mathfrak{g}) \arrow[ur] & \end{tikzcd} \tag{6.10}$$

Here the lower right map is (pointwise) evaluation via the right action. The image $\psi(v) \in \mathcal{C}^\infty(\overline{O}; \mathfrak{g})$ for $v \in \mathfrak{g}$ is determined by the conditions

(1) Evaluated at p, $\psi(v)(p) = v$
(2) $\psi(v)$ is constant on the fibres of the left action of K

(3) $\psi(v)$ transforms under the adjoint map for the right action of K.
(4) $\psi(v)$ is constant on the base.

Of course the extension off of F, forced by the last condition, is not canonical but depends on the choice of trivialization. Let $\mathcal{I}_F$ denote the ideal of smooth functions vanishing at F, generated locally by the x_i.

Lemma 7. *On the nilpotent part*

$$\Psi : \mathfrak{n}_p \longrightarrow \mathcal{I}_F \cdot \mathcal{V}_{\mathrm{b}} \tag{6.11}$$

and the leading part determines a map

$$\Psi' : \mathfrak{n}_p/[\mathfrak{n}_p,\mathfrak{n}_p] \longrightarrow \bigoplus_{i=1}^{l} \mathfrak{k}, \tag{6.12}$$
$$\Psi'(v) = 0 \Longleftrightarrow \Psi(v) = (x_1\Psi'_1,\dots,x_r\Psi'_r) + E',\ E' \in \mathcal{I}_F^2 \cdot \mathcal{V}_{\mathrm{b}}.$$

Proof. The construction of Ψ means that, in terms of the local coordinates above, at each section where y and z are constant, $\Psi(v)$ is equal to the right action of v. In particular, from (6.3) if $v \in \mathfrak{n}_p$ then

$$\Psi(v)(x) = \sum_i x_i(w_i + w'_i) + E, \tag{6.13}$$
$$w_i = \sum_j a_{ij}(x,y)\partial_{y_j},\ w'_i = \sum_j a'_{ij}(x)V_j,\ E \in \mathcal{I}_F^2 \cdot \mathcal{V}_{\mathrm{b}}$$

where the V_j are generators of the left action of K. By evaluation of the V_j at p this defines a map

$$\Psi' = (\Psi'_1,\dots,\Psi'_r) : \mathfrak{n}_p \longrightarrow \bigoplus_{i=1}^{r} \mathfrak{g}. \tag{6.14}$$

However, the commutator $[\Psi(v),\Psi(v)] = \Psi([v,v'])$ must lie in $\mathcal{I}_F^2 \cdot \mathcal{V}_{\mathrm{b}}$ so (6.14) descends to a map (6.12).

On the null space of Ψ' the leading part reduces to that for a sum of terms $x_i\Psi(k_i)$, $k_i \in \mathfrak{k}$ so (6.12) follows. □

We proceed to use this to show:

Lemma 8. *For an hd-compactification the basis a_i corresponding to the (reduced) root space decomposition of the isotropy algebra at a point of F is mapped (after reordering) to the basis $x_i\partial_{x_i}$ by* (6.8)*.*

Proof. The minimality condition in the definition of an hd-compactification can be restated as folliws. Near each boundary point of $\overline{G}$ and each boundary hypersurface through it, there exists a smooth vector field, given by a smooth map into the Lie algebra evaluated on the right action, which vanishes exactly to first order at the hypersurface as a b-vector field but has coefficient vector field outside the span of the Lie algebra at that point.

To fix notation, choose the hypersurface to be $x_1 = 0$ through p so the linear part of the vector field at p must be of the form

$$u = x_1 u',\ u' \in \mathcal{V}_{\mathrm{b}},\ u'(p) = \sum_j \left(e_j\partial_{y_j} + e'_j\partial_{z_j}\right). \tag{6.15}$$

with at least one $e'_j \neq 0$, since the other elements lie in the Lie algebra. By assumption this is given by a smooth function $\phi : U \longrightarrow \mathfrak{g}$ on some neighbourhood of p. Since u vanishes at F, $\phi(q) \in \mathfrak{n}_q$ at each $q \in F \cap U$. As noted above, an element of $\mathfrak{n}_q$ has linear part

$$\sum_{i,j} x_i(\tau_{ij}\partial_{y_j} + \tau'_{ij}\partial_{z_j}) + \sum_j (\lambda_j(y)\partial_{y_j} + \lambda'_j(y)\partial_{z_j}) \tag{6.16}$$

for linear functions λ_j, λ'_j. So for $\phi(p) \in \mathfrak{n}_p$, we must have $\tau_{ij} = \tau'_{ij} = 0$ for $i \neq 1$ but by the independence assumption, some $\tau'_{1,j} \neq 0$. This implies the same conclusion for the vector field that $\Phi(\phi(0))$ defines above, i.e.

$$\Psi'_1(\phi(p)) \neq 0,\ \Psi'_j(\phi(p)) = 0,\ j > 1. \tag{6.17}$$

From Lemma 6 it follows that in its root space decomposition

$$\phi(v)(p) = \sum_\alpha c_\alpha n_\alpha \tag{6.18}$$

there must be a non-zero multiple of a simple root.

In this case we can see the form of $[a_l, u]$ for each l. Namely it must again vanish at $x_1 = 0$ and the coefficients e_j and e'_j are simply multiplied by the one constant α_{i1}. It follows from the root space decomposition (6.18) that precisely one simple root must appear and that the corresponding a_i must have leading part $x_1\partial_{x_1}$ Applying the same procedure to the other hypersurfaces completes the proof of the Lemma. □

As noted in the proof of Lemma 5, the nilpotent part $\mathfrak{n}_p \subset \mathfrak{i}_p$ is mapped into $\mathcal{I}_p\mathcal{V}_{\mathrm{b}}$. Let n_α be the basis of root vectors discussed above. The Lie algebra of K is spanned by the $(n_\alpha + \theta n_\alpha)$ but we continue to use the indexing by j. For this basis the commutators with the a_i are

$$[a_i, k_j] = \lambda_{ij} n_j - \lambda_{ij}\theta n_j = -\lambda_{ij} k_j + 2\lambda_{ij} n_j \tag{6.19}$$

where the λ_{ij} are a relabelling of the non-negative integers from the entries of the reduced roots α. Since the K action is free on F we can choose the tangential coordinates y'_j so that $k_j = \partial_{y'_j} + k'_j$, $k'_j \in \mathcal{I}_p\mathcal{V}_{\mathrm{b}}$.

Now consider the implications of these identities for the Taylor series at p of the a_i. Instead of just the b-normal part of the a_i, as in (6.8) consider the smaller ideal

$$\mathcal{W}'_p = \mathcal{I}_F\mathcal{V}_{\mathrm{b}} + \mathcal{I}_p\mathcal{W}_p \subset \mathcal{I}_p\mathcal{V}_{\mathrm{b}}. \tag{6.20}$$

Directly from (6.4) it follows that

$$\mathcal{W}_p/\mathcal{W}'_p = \mathrm{sp}\{x_i\partial_{x_i}, y_l\partial_{y_j}\} \tag{6.21}$$

is spanned by the b-normal together with the tangential linear vector fields.

The relevance of this quotient is due to the following commutator inclusions

$$[\mathcal{W}, \mathcal{V}_{\mathrm{b}}] \subset \mathcal{V}_{\mathrm{b}},\ [\mathcal{W}', \mathcal{V}_{\mathrm{b}}] \subset \mathcal{I}_p\mathcal{V}_{\mathrm{b}},\ [\mathcal{W}, \mathcal{I}_F\mathcal{V}] \subset \mathcal{I}_F\mathcal{V}. \tag{6.22}$$

Thus for $k_j \in \mathcal{V}_{\mathrm{b}}$ the commutator $[a_i, k_j]$ is determined modulo $\mathcal{I}_p\mathcal{V}_{\mathrm{b}} + \mathcal{I}_F\mathcal{V}$ by the image

$$a_i = x_i\partial_{x_i} + L_i \mod \mathcal{W}'_p \tag{6.23}$$

where the L_i are linear vector fields in the y_j. Up to errors in $\mathcal{I}_p\mathcal{V}_{\mathrm{b}} + \mathcal{I}_F\mathcal{V}$,

$$k_j = \partial_{y_j},\ [a_i, k_j] = -\lambda_{ij} k_j = [L_i, \partial_j] \tag{6.24}$$

since $n_j \in \mathcal{I}_p\mathcal{V}_{\mathrm{b}}$. Thus in fact

$$
\begin{aligned}
(6.25)\quad [L_i, \partial_j] = -\lambda_{ij} \Longrightarrow L_i = -\sum_j \lambda_{ij} y_j \partial_{y_j} \\
\Longrightarrow a_i = x_i \partial_{x_i} - \sum_j \lambda_{ij} y_j \partial_{y_j} \quad \mod \mathcal{I}_p^2\mathcal{V}.
\end{aligned}
$$

This shows that the vector fields representing the $a_i \in \mathfrak{a}_p$ are essentially hyperbolic with one positive and many negative (plus some 'flat') directions. This allows one to construct integral curves with closure containing p. The local (in fact also global) boundary faces through F, the given maximal codimension boundary face, may be labelled by the subsets of $e \subset \{1, \dots, r\}$ so that F_e is defined by the vanishing of the x_i for $i \notin e$. Thus $F = F_\emptyset$.

Proposition 9. *For each $p \in F$ the vector field for the conjugate of a_i in the isotropy group at that point has a unique integral curve in O, near p, with p in its closure; for each singleton $\{i\}$ these give a smooth fibration of the boundary face $F_{\{i\}} \cap O$ over F.*

Proof. In (6.25) the vanishing λ_{ij} correspond the the elements of the Lie algebra in K of the stabilizer of a_i under the right action. Under the quotient by this action, and the left action of K, it follows that a_i descends to a hyperbolic vector field with one as the single positive eigenvalue and all others negative integers. There is therefore a unique integral curve in $x_i > 0$ with end-point at 0, the base point. The integrality of the linearization implies that this is smooth and lifting under the commuting group actions this gives a smooth surface lying in $F_{\{i\}}$, transversal to F and meeting it in the submanifold where the isotropy group algebra contains this a_i. Extending this under the right action of K, which is consistent on the stabilizer group, gives the smooth fibration of $F_{\{i\}} \cap O$ over F by the closure of integral curves of a_i. □

Along the fibrations of the $F_{\{i\}}$ the identities in the Lie algebra show that the remaining a_k project to the b-normal vectors $x_k\partial_{x_k}$ and the n_j with $\lambda_{ij} = 0$ remain in the isotropy algebra which is therefore the solvable part of a parabolic subgroup with maximal abelian part of rank $l - 1$.

This allows the extension to be carried out iteratively to the lower codimension boundary faces near F. The a_j along $F_{\{i\}}$ remain hyperbolic transverse to the left action and the action of their stabilizers in the right action of K, so they give a smooth fibration of $F_{\{i,j\}}$. Moreover the fibration obtained in the opposite order is the same, by commutativity.

The b-transversality of the product action on $\overline{G}$ implies that all isotropy groups are conjugate at interior points of each boundary face, thus all parabolic subgroups appear in the left and right isotropy groups. Consider the submanifold formed by the union of orbits in the interior of a boundary face where the right action has given isotropy group; so the real reductive factor, M, in the Langlands decomposition of the parabolic acts freely here. Thus the trivial fibration (6.1) is explicitly trivialized near F with the transverse fibre through each $p \in F$ being the closure of the abelian part of the isotropy group at p.

To aid in the argument by induction over real rank showing the uniqueness of an hd-compactification consider the product compactification, based on the reduced root space decomposition, of the chosen real Cartan group A :

$$\overline{A} = \prod_{i=1}^{l} [0, \infty] \tag{6.26}$$

where the coordinate near infinity in each factor corresponds to inversion of the coefficient of the corresponding a_i. Thus such a compactification is by no means natural for a contractible abelian Lie group. Nevertheless it is natural for A here since all choices are conjugate, with their bases, under the action of K.

Proposition 10. *For an hd-compactificiation the Cartan decomposition extends to a smooth surjective b-map*

$$\overline{\chi} : \overline{G} \longrightarrow \overline{A}. \tag{6.27}$$

Remark 1. A global alternative to axiom D4 is to require, in addition to D1-D3, that the conclusion of this lemma should hold.

Proof. The Cartan decomposition $G = KAK$ is not unique, but the factor in A is determined and gives a smooth map $\chi : G \longrightarrow A$. The Weyl group of G acts on A by factor exchange, so this extends smoothly to $\overline{A}$.

As seen above, the base of the fibration near a maximal codimension face, F of $\overline{G}$, maps to $\overline{A}$ since the interior maps to A and this extends smoothly to the compactification. The image must contain a corner point of $\overline{A}$ and since the Weyl group action on A is through conjugation by K all 2^l corners must appear in the base of this local fibration.

The local boundary hypersurfaces of $\overline{G}$ near F are in 1-1 correspondence with the subsets of $\{1, \dots, r\}$ corresponding to the basis elements a_i which are not in the corresponding isotropy group. Proceeding by induction over real rank, it follows that the the orbits in each boundary face map smoothly onto the corresponding boundary face of $\overline{A}$ and hence the smoothness and surjectivity of the map follows. □

Thus in fact the boundary faces of $\overline{G}$ are in 1-1 correspondence with the boundary faces of $\overline{A}$ near one corner. In particular, assuming that G is connected and semi-simple there is precisely one boundary face of maximal codimension, F, and all left and right orbits of G meet any neighbourhood of F. Near that boundary face the identification in the interior of two hd-compactifications extends uniquely to a diffeomorphism of compact spaces and using the G actions and induction this extends to a global diffeomorphism.

The characterization of the $\mathcal{C}^\infty$ span of the Lie algebra for the right action of G follows from the discussion above.

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