18.335 Practice Midterm

1. (5 points) Let A be real symmetric and positive semidefinite, i.e. $x^T Ax \geq 0$ for all $x \neq 0$. Show that if the diagonal of A is zero, then A is zero.

2. (5 points) Show that if
$$Y = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}$$
then $\kappa_F(Y) = 2n + \|Z\|_F^2$.

3. Let
$$T = \begin{bmatrix} a_1 & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix}$$
be a real, n-by-n, nonsymmetric tridiagonal matrix where $c_i b_i > 0$ for all $1 \leq i \leq n-1$. Show that the eigenvalues of T are real (5 points) and distinct (5 points).

   Hint: Find a diagonal matrix D such that $C = DTD^{-1}$ is symmetric. Then argue about the rank of $C - \lambda I$.

4. (5 points) Let A be symmetric positive definite matrix with Cholesky factor C, i.e. $A = C^T C$. Show that $\|A\|_2 = \|C\|_2^2$.

5. (5 points) If A and B are real symmetric positive definite matrices then decide whether the following are true, justifying your results:

   - $A + B$ is symmetric positive definite.
   - $A \cdot B$ is symmetric positive definite.

6. (5 points) Prove that $\det(I + xy^T) = 1 + x^T y$.