3 Homework Solutions

18.335 - Fall 2004

3.1 Trefethen 10.1

(a) $H = I - 2vv^*$ where $\|v\| = 1$. If $v^*u = 0$ (u is perpendicular to v), then $Hu = u - 2vv^*u = u$. So 1 is an eigenvalue with multiplicity $n-1$ (there are $n-1$ linearly independent eigenvectors perpendicular to v). Also $Hv = v - 2vv^*v = v - 2v = -v$, so -1 is an eigenvalue of H. The geometric interpretation is given by the fact that reflection of v is $-v$, and reflection of any vector perpendicular to v is v itself.

(b) $\det H = \prod_{i=1}^{n} \lambda_i = (-1)\, 1^{n-1} = -1.$

(c) $H^*H = (I - 2vv^*)^* (I - 2vv^*) = I - 4vv^* + 4vv^*vv^* = I$. So the singular values are all 1's.

3.2 Let B be an $n \times n$ upper bidiagonal matrix. Describe an algorithm for computing the condition number of B measured in the infinity norm in $\mathcal{O}(n)$ time.

The condition number of B in the infinity norm is defined as:

$$\kappa_\infty(B) = \left\|B^{-1}\right\|_\infty \|B\|_\infty$$

We have to compute these two matrix norms separately. For $\|B\|_\infty$, the operation count is $\mathcal{O}(n)$ since only $n-1$ operations (corresponding to row sums for the first $n-1$ rows) are required. In order to calculate $\left\|B^{-1}\right\|_\infty$ we need to compute B^{-1} first. To do so, let $C = B^{-1}$. Performing the matrix multiplication $BC = I$, one can see that the inverse is an upper triangular matrix whose entries are given by:

$$c_{i,j} = \begin{cases} 0 & , i > j \\ \dfrac{1}{b_{i,i}} & , i = j \\ \dfrac{1}{b_{i,i}} \displaystyle\prod_{k=i}^{j-1} \left(-\frac{b_{k,k+1}}{b_{k+1,k+1}} \right) & , i < j \end{cases}$$

This enables us to compute the infinity norm of B^{-1} :

$$\left\|B^{-1}\right\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |c_{i,j}| = \max_i \underbrace{\frac{1}{|b_{i,i}|} \left(1 + \sum_{j=i+1}^{n} \left| \prod_{k=i}^{j-1} \left(-\frac{b_{k,k+1}}{b_{k+1,k+1}} \right) \right| \right)}_{P_i}$$

Normally, doing this directly would require $\mathcal{O}\left(n^2\right)$ flops. We can avoid this many flops by making some simplifications. Let:

$$d_k = \left|\frac{b_{k,k+1}}{b_{k+1,k+1}}\right|$$

and notice that the row sum, P_i can be written as:

$$\begin{aligned}
P_i &= \frac{1}{|b_{i,i}|}\left(1 + \sum_{j=i+1}^{n}\prod_{k=i}^{j-1} d_k\right) = \frac{1}{|b_{i,i}|}\left(1 + d_i + \sum_{j=i+2}^{n}\prod_{k=i}^{j-1} d_k\right) \\
&= \frac{1}{|b_{i,i}|}\left[1 + d_i\left(1 + \sum_{j=i+2}^{n}\prod_{k=i+1}^{j-1} d_k\right)\right] = \frac{1}{|b_{i,i}|}\left(1 + d_i\left|b_{i+1,i+1}\right| P_{i+1}\right) \\
&= \frac{1}{|b_{i,i}|}\left(1 + \left|b_{i,i+1}\right| P_{i+1}\right)
\end{aligned}$$

Thus knowing P_{i+1} we can calculate P_i in 5 operations. Hence we have to start from the n-th row and proceed backwards. So our algorithm to compute $\left\|B^{-1}\right\|_\infty$ is:

$$\left.P_n = \frac{1}{|b_{n,n}|} = \left\|B^{-1}\right\|_\infty \qquad\right\}\text{2 flops}$$

$$\left.\begin{array}{ll}
\textbf{for} & \text{i=n-1 } \textbf{to } 1 \\
 & P_i = \dfrac{1 + \left|b_{i,i+1}\right| P_{i+1}}{|b_{i,i}|} \\
 & \left\|B^{-1}\right\|_\infty = \max\left(P_i, P_{i+1}\right) \\
\textbf{end} &
\end{array}\right\}\mathcal{O}\left(n\right)\ \text{flops}$$

Since both $\left\|B^{-1}\right\|_\infty$ and $\|B\|_\infty$ require $\mathcal{O}\left(n\right)$ flops, so does $\kappa_\infty\left(B\right)$.