7 Homework Solutions

18.335 - Fall 2004

7.1 Compute the smallest eigenvalue of the 100×100 Hilbert matrix $H_{ij} = 1/(i+j-1)$. (Hint: The Hilbert matrix is also Cauchy. The determinant of a Cauchy matrix $C(i,j) = 1/(x_i + y_j)$ is $\det C = \prod_{i<j}(x_j - x_i)(y_j - y_i) / \prod_{i,j}(x_i + y_j)$. Any submatrix of a Cauchy matrix is also Cauchy. You can use Cramer's rule in order to compute accurate formulas for H^{-1} and then compute its largest eigenvalue)

We use Cramer's rule

$$H_{ij}^{-1} = (-1)^{i+j} \frac{\det(C_{ij})}{\det(H)}$$

together with the formula given for the determinant with $x_i = i$ and $y_j = j-1$ to get H_{ij}^{-1} :

$$\begin{aligned} H_{ij}^{-1} &= (-1)^{i+j} \frac{\prod\limits_{\substack{r<s \\ r\neq i, s\neq j}} (x_s - x_r)(y_s - y_r)}{\prod\limits_{r\neq i, s\neq j} (y_s + x_r)} \frac{\prod\limits_{i,j} (x_i + y_j)}{\prod\limits_{i<j} (x_j - x_i)(y_j - y_i)} \\ &= \cdots \\ &= (-1)^{i+j} (i+j-1) \binom{n+i-1}{n-j} \binom{n+j-1}{n-i} \binom{i+j-2}{i-1}^2 \end{aligned}$$

Having computed the coefficients of H^{-1} we may use any iterative scheme to estimate the largest eigenvalue which can be inverted to obtain the smallest eigenvale of H. Alternatively one could use a simple matlab command:

$$\lambda_{\min}(H) = \frac{1}{\lambda_{\max}(H^{-1})} = \texttt{1/max(eig(invhilb(100)))=5.779700862834800e-151}$$

7.2 Trefethen 30.2

- Jacobi algorithm
  Calculation of $J : \mathcal{O}(1)$ flops
  $J^T A$ alters 2 rows of A only $\Rightarrow$ 3 ops $\times$ $2m$ elements $\Rightarrow \mathcal{O}(6m)$ flops
  $\left(J^T A\right) J$ alters 2 columns $\Rightarrow \mathcal{O}(6m)$ flops.
  In total we need $\mathcal{O}(12m)$ flops for a single step of Jacobi algorithm (Half in case A is symmetric)
  In a single sweep we need $\sim m^2 \mathcal{O}(12m)/2 = \mathcal{O}\left(6m^3\right)$ flops (not counting convergence iterations).

- QR
  Requires $\mathcal{O}\left(4m^3/3\right)$, a much better algorithm!