sics" (J. Lepovsky

8 (C. Domb and J.L.

nn HE-86-02

nn HE-86-03

Lecture Notes in Physics 261 (1986), 345-371

UNITARIZABLE HIGHEST WEIGHT REPRESENTATIONS OF THE VIRASORO, NEVEU-SCHWARZ AND RAMOND ALGEBRAS

Victor G. Kac
IHES and MIT

Minoru Wakimoto
Hiroshima University

§0. The Virasoro algebra Vir is the universal central extension of the complexified Lie algebra of vector fields on the circle with finite Fourier series. Its (irreducible) highest weight representations $\sigma_{z,h}$ are parametrized by two numbers, the central charge z, and the minimal eigenvalue h of the energy operator ℓ_o. These representations play a fundamental rôle in statistical mechanics [1,5,6] and string theory [16].

The study of representations $\sigma_{z,h}$ was started by the first author [8], [9] with the computation of the determinant of the contravariant Hermitian form lifted to the corresponding "Verma module", on each eigenspace of ℓ_o. This led to a criterion of inclusions of Verma modules and the computation of the characters $\operatorname{tr} q^{\ell_o}$ in some cases, in particular, for the critical value $z = 1$ [9]. Feigin and Fuchs [3] succeeded in proving the fundamental fact (conjectured in [10]) that Verma modules over Vir are multiplicity-free, which led them, in particular, to the computation of the characters of all representations $\sigma_{z,h}$.

Using the determinantal formula, it is not difficult to show that $\sigma_{z,h}$ is unitarizable (i.e. the contravariant Hermitian form is positive definite) for $z \geq 1$ and $h \geq 0$ [10]. It is obvious that $V(z,h)$ is not unitarizable if $z < 0$ or $h < 0$. The case $0 \leq z < 1$ was analysed, using the determinantal formula, by Friedan-Qiu-Shenker [5]. They found the remarkable fact that the only possible places of unitarity in this region are $(z_m, h^{(m)}_{r,s})$, where

$$(0.1)\quad z_m = 1 - \frac{6}{(m+2)(m+3)}\ ;\quad h^{(m)}_{r,s} = \frac{[(m+3)r-(m+2)s]^2-1}{4(m+2)(m+3)}\ .$$

Here $m,r,s \in \mathbb{Z}_+ = \{0,1,2,\dots\}$ and $1 \leq s \leq r \leq m+1$. (Actually, the series (0,1) was discovered by Belavin-Polyakov-Zamolodchikov [1].)

On the other hand, according to the Goddard-Kent-Olive (GKO) construction [7], Vir acts on the tensor product of two unitarizable highest weight representations of an affine (Kac-Moody) Lie algebra $\hat{\mathfrak{g}}'$ commuting with $\hat{\mathfrak{g}}'$. This construction was applied in [7] to the tensor product of the basic representation with a highest weight representation of level m of $\hat{s\ell}'_2$ to show that all the z_m indeed occur as central charges of unitarizable representations of Vir.

In the present paper we show that the "discrete series" representations $\sigma_{z,h}$ of Vir described by (0.1) appear with multiplicity one in the space of highest weight vectors of the tensor product of the basic representation and the sum of all unitarizable highest weight representations of $\hat{s\ell}_2'$, and hence are unitarizable. This is derived by a simple calculation with the Weyl-Kac character formula for $\hat{s\ell}_2'$ (see e.g. [11, Chapter 12]) and the Feigin-Fuchs character formula for Vir [3].

A similar result for the Neveu-Schwarz and Ramond superalgebras is obtained by applying the same argument to the super-symmetric extensions of $\hat{s\ell}_2'$ and their minimal representations (in place of the basic representation) constructed in [13]. (The list analogous to (0.1) was found in [6], and it was shown in [13] that all corresponding central charges indeed occur).

All the discrete series unitarizable representations $\sigma_{z,h}$ are degenerate (i.e. correspond to the zeros of the determinant). The only other degenerate unitarizable representations (apart from the "non-interesting" case $z > 1$, $h = 0$) are $\sigma_{1,m^2/4}$, where $m \in \mathbb{Z}_+$, and all of them appear with multiplicity one on the space of highest weight vectors for $s\ell_2$ in the sum of (two) fundamental representations of $\hat{s\ell}_2'$ [9]. We show that a similar result holds in the super case as well.

Finally, the above construction of the discrete series representations, allowed us to give a very simple proof of all determinantal formulas (cf. [2], [6], [9], [17]).

Geometrically, the main result of the paper concerning Vir can be stated as follows. Let G be the "minimal" group associated to $\hat{s\ell}_2'$ and let U_+ and U_- be the "opposite maximal unipotent" subgroups of G [19]. Let V be the space of the basic representation of G. Then Vir acts on the space of regular U_+-equivariant maps $\mathrm{Map}_{U_+}(U_-\backslash G, V)$, and all its unitarizable representations $\sigma_{z,h}$ with $z < 1$ appear with multiplicity 1.

The first author acknowledges the support of IHES and the NSF grant DMS-8508953.

After this work was completed, we received two preprints, "Unitary representations of the Virasoro algebra" by A. Tsuchiya and Y. Kanie, and "Unitary representations of the Virasoro and super Virasoro algebras" by P. Goddard, A. Kent and D. Olive, which overlap considerably with the present paper.

We added several Appendices to the paper. Appendix 1 provides a simple self-contained proof of the determinantal formulas for the Neveu-Schwarz and Ramond superalgebras Vir_ε. Appendix 2 contains multiplicative formulas for characters of Vir and Vir_ε; we hope that these formulas will provide a clue to more explicit constructions of the discrete series representations of Vir and Vir_ε (cf. Remark 8.2). Finally, in Appendix 3 we uncover a mysterious connection between "exceptional"

Lie algebras E_8, E_7, A_2 and E_6, and the representations of Vir corresponding to the following two dimensional models : Ising, tricritical Ising, 3-state Potts and tricritical 3-state Potts respectively (see Remark 8.3).

The first author acknowledges the hospitality of TIFR.

§1. Here we recall some necessary facts about affine Kac-Moody algebras in the simplest case of $\widehat{s\ell}_2$.

Let $g = s\ell_2(\mathbb{C})$ be the Lie algebra of complex traceless 2×2-matrices, and let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be its standard basis.

Let $\mathbb{C}[t,t^{-1}]$ be the algebra of Laurent polynomials over $\mathbb{C}$ in an indeterminate t . We regard the loop algebra $\tilde{g} = s\ell_2(\mathbb{C}[t,t^{-1}])$ as an (infinite-dimensional) complex Lie algebra. It has a central extension $\hat{g}' = \tilde{g} \oplus \mathbb{C}c$ by a 1-dimensional center $\mathbb{C}c$ with the bracket

$$(1.1) \qquad [x,y] = xy-yx+(\mathrm{Res}_{t=o}\ \mathrm{tr}\ \frac{dx}{dt}\ y)c$$

for $x,y \in \tilde{g}$. One includes $\hat{g}'$ as a subalgebra of codimension 1 in a larger algebra $\hat{g} = \hat{g}' \oplus \mathbb{C}d$, where

$$(1.2) \qquad [d,x] = t\frac{dx}{dt} \ \text{ for } x \in \tilde{g} \ ; \ [d,c] = 0 .$$

The Lie algebra $\hat{g}$ (and often its subalgebra $\hat{g}'$) with bracket defined by (1.1) and (1.2) is called an affine (Kac-Moody) Lie algebra associated to g . This is the simplest example of an infinite-dimensional Kac-Moody algebra (cf. [11, Chapter 7]). Putting $x(k) = t^k x$ for $x \in g$ and $k \in \mathbb{Z}$, we have an equivalent form of (1.1) and (1.2) :

$$(1.3) \qquad [x(k),y(n)] = (xy-yx)(k+n)+k\delta_{k,-n}(\mathrm{tr}\ xy)c;[d,x(k)] = kx(k);[c,\hat{g}] = 0 .$$

The (commutative 3-dimensional) subalgebra $\hat{h} = \mathbb{C}\alpha+\mathbb{C}c+\mathbb{C}d$ of $\hat{g}$ is called the Cartan subalgebra. Introduce the "upper triangular" subalgebra $\hat{n} = \mathbb{C}e + \sum_{k>o} t^k g$. Define a symmetric bilinear form $(.|.)$ on $\hat{h}$ by :

$$(1.4) \qquad (\alpha|\alpha) = 2 \ ; \ (c|d) = 1; \ (\alpha|c) = (\alpha|d) = (d|d) = (c|c) = 0 .$$

(It extends to a non-degenerate invariant symmetric bilinear form on $\hat{g}$ by $(x(k)|y(n)) = \delta_{k,-n}\mathrm{tr}\, xy$, $(x(k)|c) = (x(k)|d) = 0)$. Introduce the following subsets of $\hat{h}$: $P_+^o = \{md+\frac{1}{2}n\alpha | m,n \in \mathbb{Z}_+, n \leq m\}$; $P_+ = P_+^o + \mathbb{R}c$.

Given $\lambda \in P_+$, there exists a unique (up to equivalence) irreducible representation π_λ of $\hat{g}$ on a complex vector space $L(\lambda)$ which admits a non-zero vector $v_\lambda \in L(\lambda)$ such that

$$(1.5) \quad \pi_\lambda(\hat{n})v_\lambda = 0 \ ; \ \pi_\lambda(\mu)v_\lambda = (\lambda|\mu)v_\lambda \quad \text{for all } \mu \in \hat{h} \ .$$

This is called the <u>integrable representation with highest weight</u> λ (cf. [11, chapter 10]), v_λ being called the highest weight vector. The number $m = (\lambda|c)$ is called the <u>level</u> of $L(\lambda)$; we have : $\pi_\lambda(c) = mI$. Recall that $m \in \mathbb{Z}_+$, furthermore, $m = 0$ if and only if $\dim L(\lambda) = 1$. Note that viewed as a representation of $\hat{g}'$, π_λ remains irreducible and is independent of the c-component of λ .

All representations π_λ are <u>unitarizable</u> in the sense that there exists a positive definite Hermitian form $\langle .|.\rangle$ on $L(\lambda)$ such that (cf. [11, Theorem 11.7b]) :

$$(1.6) \quad \langle \pi_\lambda(x(k))u|v\rangle = \langle u|\pi_\lambda({}^t\bar{x}(-k))v\rangle \quad \text{for all } u,v \in L(\lambda) \ .$$

(Actually, property (1.6) together with $\langle v_\lambda|v_\lambda\rangle = 1$ determines the Hermitian form uniquely; a Hermitian form satisfying (1.6) exists for any $\lambda \in \hat{h}$, but is positive definite only for $\lambda \in P_+$) .

With respect to $\pi_\lambda(d)$ we have the eigenspace decomposition :

$$(1.7) \quad L(\lambda) = \bigoplus_{k \in \mathbb{Z}_+} L^{((\lambda|d)-k)} \ , \quad \text{where } \dim L^{((\lambda|d)-k)} < \infty \ .$$

Consider the domain $D = \{z\alpha + \tau d + uc \in \hat{h} \ | \tau,u,z \in \mathbb{C} \text{ and } \mathrm{Im}\,\tau > 0\}$. Define the <u>character</u> of the representation π_λ by :

$$\mathrm{ch}_\lambda(\tau,z,u) = \sum_{k \in \mathbb{Z}_+} \mathrm{tr}\, \exp 2\pi i(\pi_\lambda(\tfrac{1}{2}z\alpha - \tau d + uc))|_{L^{((\lambda|d)-k)}} \ .$$

This is an absolutely convergent series defining a holomorphic function on D . It can be written in terms of elliptic theta functions $\Theta_{n,m}$ as follows [11, Chapter 12]. For a positive integer m and an integer n put

$$\Theta_{n,m}(\tau,z,u) = e^{2\pi imu} \sum_{k \in \mathbb{Z}+\frac{n}{2m}} q^{mk^2} e^{2\pi imkz} \ .$$

Here and further on, $q = e^{2\pi i\tau}$. For $\lambda \in P_+$, $\lambda = md + \frac{1}{2}n\alpha + rc$, $r \in \mathbb{R}$, put

$$s_\lambda = \frac{(n+1)^2}{4(m+2)} - \frac{1}{8} + r.$$

Then we have

(1.8) ch_λ =

In the f
(cf. [12, p.2

(1.9a) ch_d = (

(1.9b) $\varphi(q)$ =

(1.10a) ch_{2d}

(1.10b) $\varphi_{1/2}$(

(1.11a) ch_{2d+}

(1.11b) $\varphi_0(q)$

§2. We now r
Let $\{u_i\}$ and
$\lambda,\lambda' \in P_+$ of l
space $L(\lambda) \otimes$

(2.1) $L_k = \frac{\cdot}{m}$

+ $\Big\{\frac{1}{2(}$

+ $\Big\{\frac{1}{2(m}$

Let Ω be the
will need only
on which Ω a

(2.2) $\pi(\Omega)v$ =

Here and furthe

The proof
[18]) :

Then we have the following special case of the Weyl-Kac character formula :

$$(1.8)\quad ch_\lambda = q^{-s_\lambda}(\Theta_{n+1,m+2}-\Theta_{-n-1,m+2}) \,/\, (\Theta_{1,2}-\Theta_{-1,2}) \,.$$

In the following three simplest cases there are simpler formulas (cf. [12, p.218]) :

$$(1.9a)\quad ch_d = \Theta_{0,1}/\varphi(q) \,, \quad \text{where}$$

$$(1.9b)\quad \varphi(q) = \prod_{k=1}^{\infty} (1-q^k) \,;$$

$$(1.10a)\quad ch_{2d} + q^{1/2}\, ch_{2d+\alpha} = (\Theta_{0,2}+\Theta_{2,2})/\varphi_{1/2}(q) \,, \quad \text{where}$$

$$(1.10b)\quad \varphi_{1/2}(q) = \varphi(q^{1/2})\varphi(q^2)/\varphi(q) \,;$$

$$(1.11a)\quad ch_{2d+1/2\alpha} = q^{-1/8}(\Theta_{1,2}+\Theta_{-1,2})/\varphi_0(q) \,, \quad \text{where}$$

$$(1.11b)\quad \varphi_0(q) = \varphi(q)^2/\varphi(q^2) \,.$$

§2. We now recall a special case of the Goddard-Kent-Olive construction [7]. Let $\{u_i\}$ and $\{u^i\}$ be dual bases of $\mathfrak{g}$, i.e. tr $u_i u^i = \delta_{ij} (i,j = 1,2,3)$. Pick $\lambda, \lambda' \in P_+$ of levels m and m' and define the following operators L_k on the space $L(\lambda) \otimes L(\lambda')$ $(k \in \mathbb{Z})$:

$$(2.1)\quad L_k = \frac{-1}{m+m'+2} \sum_{j\in\mathbb{Z}} \sum_i \pi_\lambda(u_i(-j)) \otimes \pi_{\lambda'}(u^i(j+k))$$
$$+ \left(\frac{1}{2(m+2)} - \frac{1}{2(m+m'+2)}\right) \sum_{j\in\mathbb{Z}} \sum_i \pi_\lambda\,(:u_i(-j)u^i(j+k):) \otimes 1$$
$$+ \left(\frac{1}{2(m'+2)} - \frac{1}{2(m+m'+2)}\right) \sum_{j\in\mathbb{Z}} \sum_i 1 \otimes \pi_{\lambda'}(:u_i(-j)u^i(j+k):)$$

Let Ω be the Casimir element of $\hat{\mathfrak{g}}$ (cf. [11, Chapter 2 and Exercise 7.16]). We will need only the following property of Ω . If (π,V) is a representation of $\hat{\mathfrak{g}}$ on which Ω acts and $v \in V^{\hat{\mathfrak{n}}}$, then

$$(2.2)\quad \pi(\Omega)v = \pi(2(c+2)d + \tfrac{1}{2}\alpha^2 + \alpha)v \,.$$

Here and further on $V^{\mathfrak{a}}$ stands for $\{v \in V \mid \pi(a)v = 0 \text{ for all } a \in \mathfrak{a}\}$.

The proof of the following formulas is straightforward (cf. [12, §2.5] or [18]) :

(2.3a) $[L_k, L_n] = (k-n)L_{k+n} + 3\delta_{k,-n}\frac{k^3-k}{12}\,p(m,m')$, where

(2.3b) $p(m,m') = \frac{m}{m+2} + \frac{m'}{m'+2} - \frac{m+m'}{m+m'+2}$

(2.4) $L_o = \frac{1}{2}\left(\frac{(\lambda|\lambda+\alpha)}{m+2} + \frac{(\lambda'|\lambda'+\alpha)}{m'+2} - \frac{\Omega}{m+m'+2}\right)$

(2.5) $[L_k, \mathfrak{g}'] = 0$,

i.e. the L_k are intertwining operators for the representation $\pi_\lambda \otimes \pi_{\lambda'}$ of $\mathfrak{g}'$.

<u>Remark</u>. Formulas (2.3-5) hold for all non-twisted affine algebras $\mathfrak{g}$ with the following changes : $m+2$, $m'+2$ and $m+m'+2$ are replaced by $m+g$, $m'+g$ and $m+m'+g$, where g is the dual Coxeter number [11, Chapter 6] , the coefficient 3 is replaced by dim g, and α is replaced by 2ρ . In the twisted case, formulas are somewhat more complicated (see Appendix 3).

§3. Now we turn to the Virasoro algebra Vir. Recall that this is a complex Lie algebra with a basis $\{\tilde{c}; \ell_j, j \in \mathbb{Z}\}$ with commutation relations

(3.1) $[\ell_i, \ell_j] = (i-j)\ell_{i+j} + \frac{1}{12}(i^3-i)\delta_{i,-j}\tilde{c}; [\tilde{c}, \ell_j] = 0$.

Given two numbers z and h , there exists a unique irreducible representation $\sigma_{z,h}$ of Vir on a complex vector space $V(z,h)$ which admits a non-zero vector $v = v_{z,h}$ such that

(3.2) $\sigma_{z,h}(\ell_j)v = 0$ for $j > 0$; $\sigma_{z,h}(\ell_o)v = hv$; $\sigma_{z,h}(\tilde{c}) = zI$.

Note an analogy of this definition with that of highest weight representation of $\hat{\mathfrak{g}}$. Similarly, provided that z and h are real numbers, $V(z,h)$ carries a unique Hermitian form $\langle .|.\rangle$ such that $\langle v_{z,h}|v_{z,h}\rangle = 1$ and

(3.3) $\langle \sigma_{z,h}(\ell_j)u|v\rangle = \langle u|\sigma_{z,h}(\ell_{-j})v\rangle$ for all $u,v \in V(z,h)$.

The representation $\sigma_{z,h}$ is called <u>unitarizable</u> if this Hermitian form is positive definite.

With respect to $\sigma_{z,h}(\ell_o)$ we have the eigenspace decomposition

(3.4) $V(z,h) = \bigoplus_{k \in h+\mathbb{Z}_+} V(z,h)_k$, where $\dim V(z,h)_k < \infty$.

We define the character of the representation $\sigma_{z,h}$ by

$$((3.5) \quad ch_{z,h} = \sum_{k \in h+\mathbb{Z}_+} (\dim V(z,h)_k) q^k \; (= \operatorname{tr} q^{\ell_o}) \; .$$

Note that putting (cf. §2) :

$$(3.6) \quad \pi(\ell_j) = L_j \quad , \quad \pi(\tilde{c}) = 3p(m,m')I ,$$

we obtain a unitarizable representation of the Virasoro algebra on the space $L(\lambda') \otimes L(\lambda)$. It decomposes into a direct sum of unitarizable highest weight representations of Vir with "central charge" $3p(m,m')$. Note that the central charge z_m (defined by (0.1)) occurs if one takes $\lambda' = d$ and λ of level m [7]. In the next section we show that all $h^{(m)}_{r,s}$ from (0.1) occur in this construction as well and, moreover, we "locate" the corresponding representations of Vir .

§4. Fix $\lambda = md + \frac{1}{2} n\alpha \in P^o_+$, and put $J_\lambda = \{k \in \mathbb{Z} \mid -\frac{1}{2}(m+1-n) \leq k \leq \frac{1}{2} n\}$. Define the following subspace for $k \in J_\lambda$:

$$U_{\lambda,k} = \{v \in (L(d) \otimes L(\lambda))^{\mathfrak{n}} \mid (\pi_d \otimes \pi_\lambda)(\alpha)v = (n-2k)v\} \quad .$$

Note that this is the subspace spanned by highest weight vectors of $\mathfrak{g}'$ in $L(d) \otimes L(\lambda)$ with weight $d+\lambda-k\alpha$. In particular, $(L(d) \otimes L(\lambda))^{\mathfrak{n}}$ decomposes into a direct sum of the $U_{\lambda,k}$. Furthermore, $U_{\lambda,k}$ is invariant with respect to d and hence decomposes into a direct sum of its eigenspaces $U^{(j)}_{\lambda,k}$ (with eigenvalue $j \in \mathbb{Z}$) . Note that every non-zero vector of $U^{(j)}_{\lambda,k}$ is a highest weight vector for $\mathfrak{g}$ with highest weight $d+\lambda-k\alpha+jc$. In other words, $\dim U^{(j)}_{\lambda,k}$ is the multiplicity of occurence of $L(d+\lambda-k\alpha+jc)$ in $L(d) \otimes L(\lambda)$. Here and further on we use the fact that all representations in question are completely reducible with respect to $\mathfrak{g}$ and Vir (since they are unitarizable).

Putting $m_{\lambda,k}(q) = \sum_j (\dim U^{(j)}_{\lambda,k}) q^{-j}$, we have :

$$(4.1) \quad ch_d \, ch_\lambda = \sum_{k \in J_\lambda} m_{\lambda,k} \, ch_{d+\lambda-k\alpha} \quad .$$

To compute the $m_{\lambda,k}$ we multiply formulas (1.9) and (1.8) and use the following multiplication formula of theta functions [12, p.188] :

$$(4.2) \quad \Theta_{n,m}\Theta_{n',m'} = \sum_{j \in \mathbb{Z} \bmod (m+m')\mathbb{Z}} d_j^{(m,m',n,n')} \, \Theta_{n+n'+2mj,m+m'} \text{ , where}$$

$$d_j^{(m,m',n,n')}(q) = \Theta_{m'n-mn'+2jmm',mm'(m+m')}(\tau,0,0) \; .$$

We obtain :

$$(4.3)\quad m_{\lambda,k} = \varphi(q)^{-1}(f_k^{(m,n)} - f_{n+1-k}^{(m,n)}) \quad ,$$

where

$$(4.3a)\quad f_k^{(m,n)} = \sum_{j\in\mathbb{Z}} q^{(m+2)(m+3)j^2+((n+1)+2k(m+2))j+k^2} \ .$$

(Formula (4.3) may be also derived from [4]).

On the other hand, it follows from (2.5) that the subspace $U_{\lambda,k}$ is invariant with respect to Vir and thus carries a unitary representation of Vir . Putting $m' = 1$ in (3.6) and (2.3) we find (as GKO did) that the central charge of this representation is z_m (see (0.1)). Furthermore, it is clear from (4.3) that the minimal eigenvalue of $-d$ on $U_{\lambda,k}$ is k^2 . But we have by (2.4) and (2.2) :

$$(4.4)\quad L_o = -d + \frac{n(n+2)}{4(m+2)} - \frac{(n-2k)(n-2k+2)}{4(m+3)} \quad \text{on } U_{\lambda,k} \ .$$

Defining numbers r_λ and $s_{\lambda,k}$ by $r_\lambda = n+1$, $s_{\lambda,k} = n+1-2k$ if $k \geq 0$ and $r_\lambda = m-n+1$, $s_{\lambda,k} = m-n+2+2k$ if $k < 0$, we arrive at the following

<u>Lemma 4.1.</u> The minimal eigenvalue of L_o on $U_{\lambda,k}$ is $h^{(m)}_{r_\lambda,s_{\lambda,k}}$.

Thus, $U_{\lambda,k}$ contains the unitary representation of Vir, which we denote by σ for short, with highest weight $(z_m,h^{(m)}_{r_\lambda,s_{\lambda,k}})$. But actually it coincides with this representation. Indeed $\operatorname{tr} q^{L_o}$ on $U_{\lambda,k}$ is equal to $m_{\lambda,k}(q)$ (given by (4.3)) multiplied by a power of q equal to the constant in the right-hand side of (4.4). Comparing this with the Feigin-Fuchs character formula for σ [3] (see [15] for an exposition of their results) we find that the character of σ coincides with $\operatorname{tr} q^{L_o}$ on $U_{\lambda,k}$!

We summarize the results obtained in the following theorem.

<u>Theorem 4.1.</u> (a) All highest weight representations of the Virasoro algebra with highest weights $(z_m,h^{(m)}_{r,s})$ given by (0.1) are unitary. Moreover, all these representations appear with multiplicity 1 in $\bigoplus_{\lambda\in P_+^o} (L(d)\otimes L(\lambda))^{\mathfrak{n}}$.

(b) With respect to the direct sum of $\mathfrak{g}'$ and Vir , we have the following decomposition, for $\lambda \in P_+^o$ of level m :

$$L(d)\otimes L(\lambda) = \bigoplus_{k\in J_\lambda} (L(d+\lambda-k\alpha)\otimes V(z_m,h^{(m)}_{r_\lambda,s_{\lambda,k}})) \ .$$

<u>Remark 4.1.</u> The characters $ch_{z_m,h^{(m)}_{r,s}}$ become holomorphic modular forms in τ of weight 0 on the upper half-plane when multiplied by a suitable power of q . Since

they coincide with $m_{\lambda,k}$ multiplied by a power of q, it follows from [12, p.243] that the linear span of these "corrected" characters for fixed m and all $h^{(m)}_{r,s}$ from (0.1) form an $(m+1)(m+2)/2$-dimensional space invariant with respect to the usual action of $SL_2(\mathbb{Z})$ $(f(\tau) \mapsto f((a\tau+b)/(c\tau+d)))$.

<u>Remark 4.2</u>. Theorem 4.1(a) gives us what is called a <u>model</u> (i.e. a space where each representation of a given family appears once) for all unitary representations of the Virasoro algebra with $z < 1$. A model for all degenerate representations with $z = 1$ was constructed in [9]. Namely, the space $(L(d) \oplus L(d+\frac{1}{2}\alpha))^e$ contains exactly once all representations $V(1, \frac{m^2}{4})$, $m \in \mathbb{Z}_+$, so that with respect to the direct sum of $\mathfrak{g}$ and Vir we have [9] :

$$L(d) \oplus L(d+\frac{1}{2}\alpha) = \bigoplus_{m \in \mathbb{Z}_+} (T_{m+1} \otimes V(1, \frac{m^2}{4})) ,$$

where T_m denotes the m-dimensional irreducible representation of $\mathfrak{g} = s\ell_2(\mathbb{C})$.

§5. We now turn to the supersymmetric extensions of the above results. The terminology and conventions of Lie superalgebra theory adopted here are that of [14, §1.1].

Fix $\varepsilon = \frac{1}{2}$ or 0. Take the <u>superloop algebra</u> $\tilde{\mathfrak{g}}_\varepsilon = s\ell_2(\mathbb{C}[t,t^{-1},\theta])$, where $\theta^2 = 0$, and put $x(k+\varepsilon)' = t^k\theta x$ for $x \in \mathfrak{g}$ and $k \in \mathbb{Z}$. Define the <u>affine superalgebra</u> [13] $\hat{\mathfrak{g}}_\varepsilon = \tilde{\mathfrak{g}}_\varepsilon \oplus \mathbb{C}c \oplus \mathbb{C}d$ with the (super)bracket defined by (1.3) and

(5.1a) $[x(k)',y(n)']_+ = \delta_{k,-n}(\operatorname{tr} xy)c$ for $k,n \in \varepsilon+\mathbb{Z}$;

(5.1b) $[x(k),y(n)'] = (xy-yx)(k+n)'$ for $k \in \mathbb{Z}$, $n \in \varepsilon+\mathbb{Z}$;

(5.1c) $[d,x(k)'] = kx(k)'$ for $k \in \varepsilon+\mathbb{Z}$; $[c,\hat{\mathfrak{g}}_\varepsilon] = 0$.

The Lie superalgebra $\hat{\mathfrak{g}}_\varepsilon$ contains $\hat{\mathfrak{g}}$ as the even part and $\hat{h}$ is called the Cartan subalgebra of $\hat{\mathfrak{g}}_\varepsilon$. Also, $\hat{\mathfrak{g}}_\varepsilon' = \tilde{\mathfrak{g}}_\varepsilon+\mathbb{C}c$ is a subalgebra of $\hat{\mathfrak{g}}_\varepsilon$. Put $\hat{n}_{1/2} = \hat{n}+\sum_{k>0}\theta t^k\mathfrak{g}$ and $\hat{n}_0 = \hat{n}+\mathbb{C}\theta e+\sum_{k>0}\theta t^k\mathfrak{g}$. For $\lambda \in \hat{h}$ define the ($\mathbb{Z}_2$-graded) irreducible highest weight representation $(\pi_{\lambda;\varepsilon},L_\varepsilon(\lambda))$ of $\hat{\mathfrak{g}}_\varepsilon$ by the property (1.5) where $\hat{n}$ is replaced by $\hat{n}_\varepsilon$. Unitarizability of $\pi_{\lambda;\varepsilon}$ and its character $ch_{\lambda;\varepsilon}$ are defined in the same way as for π_λ [13].

The representation of $\hat{\mathfrak{g}}_\varepsilon$ with highest weight $\lambda_\varepsilon = 2d+(\frac{1}{2}-\varepsilon)\alpha$ is called <u>minimal</u> [13]. With respect to $\hat{\mathfrak{g}}$ it decomposes as follows :

(5.2) $L_{1/2}(\lambda_{1/2}) = L(2d) \oplus L(2d+\alpha-\frac{1}{2}c)$; $L_o(\lambda_o) = L(\lambda_o)\oplus L(\lambda_o)$.

Denote the right-hand sides of (5.2) by F_ε. Given a representation (π,V) of $\hat{\mathfrak{g}}$, one can construct its "supersymmetrization" $(\pi^\varepsilon,V^\varepsilon)$ [13], which with respect to $\hat{\mathfrak{g}}$

is just $F_\varepsilon \otimes V$. It is shown in [13] that all unitarizable highest weight representations of $\hat{\mathfrak{g}}_\varepsilon$ are of the form $\pi_{\lambda+\lambda_\varepsilon;\varepsilon}$, $\lambda \in P_+$, and that $\pi_{\lambda+\lambda_\varepsilon;\varepsilon} = \pi_\lambda^\varepsilon$. It follows that with respect to $\hat{\mathfrak{g}}$ we have :

(5.3) $\quad L_\varepsilon(\lambda_\varepsilon) \otimes L_\varepsilon(\lambda+\lambda_\varepsilon) \simeq (F_\varepsilon \otimes L(\lambda))^\varepsilon \quad , \lambda \in P_+$.

We denote by Vir_ε the complex Lie superalgebra with a basis $\{\tilde{c}; \ell_j, \ j \in \mathbb{Z}$, and g_j , $j \in \varepsilon + \mathbb{Z}\}$ with commutation relations (3.1) and

(5.4a) $\quad [g_m, \ell_n] = (m - \frac{n}{2}) g_{m+n} \quad ; \quad [g_m, \tilde{c}] = 0$;

(5.4b) $\quad [g_m, g_n]_+ = 2\ell_{m+n} + \frac{1}{3}(m^2 - \frac{1}{4})\delta_{m,-n}\tilde{c}$.

(For $\varepsilon = \frac{1}{2}$ or 0 , Vir_ε is called the Neveu-Schwarz and Ramond superalgebras, respectively). The highest weight representation $(\sigma_{z,h;\varepsilon}, V_\varepsilon(z,h))$ of Vir_ε is defined by (3.2) and $\sigma_{z,h;\varepsilon}(g_j)v_{z,h} = 0$ for $j > 0$. Its unitarizability and character $\mathrm{ch}_{z,h;\varepsilon}$ are defined in the same way as for $\sigma_{z,h}$ in §3.

The analysis of the unitarizability of the representations $\sigma_{z,h;\varepsilon}$ is similar to that of $\sigma_{z,h}$ [5], [6], [9], [10], [13]. It turned out that these representations are unitarizable for $z \geq \frac{3}{2}$ and $h \geq 0$ [6], [10]. (Note that $\mathrm{ch}_{z,h;\varepsilon} = (2-2\varepsilon)q^h/\varphi_\varepsilon(q)$, the character of the Verma module, if $z > \frac{3}{2}$ and $h \geq 0$). Furthermore, the only other possible places of unitarity are $(z_{m;\varepsilon}, h^{(m)}_{r,s;\varepsilon})$, where [5], [6] :

(5.5) $\quad z_{m;\varepsilon} = \frac{3}{2}\left(1 - \frac{8}{(m+2)(m+4)}\right) \; ; \; h^{(m)}_{r,s;\varepsilon} = \frac{((m+4)r-(m+2)s)^2-4}{8(m+2)(m+4)} + \frac{1}{8}\left(\frac{1}{2} - \varepsilon\right)$.

Here $m,r,s \in \mathbb{Z}_+$, $1 \leq s \leq r+1-2\varepsilon \leq m+2-2\varepsilon$ and $r-s \in 2\varepsilon+1+2\mathbb{Z}$, $r \neq 0$.

Let $\lambda, \lambda' \in P_+$ be of level m and m' . In the same way as in §2, one can construct intertwining operators $L_j^{(\varepsilon)}$ and $G_j^{(\varepsilon)}$ on the space $L_\varepsilon(\lambda+\lambda_\varepsilon) \otimes L_\varepsilon(\lambda'+\lambda_\varepsilon)$ (see [13]) which satisfy commutation relations (5.4) with central charge

(5.6) $\quad 3\left(\frac{m}{m+2} + \frac{m'}{m'+2} - \frac{m+m'+2}{m+m'+4}\right) + \frac{3}{2}$

and with the following expression for $L_0^{(\varepsilon)}$ on the kernel of $\mathfrak{n}_\varepsilon$:

(5.7) $\quad \frac{1}{2}\left(\frac{(\lambda|\lambda+\alpha)}{m+2} + \frac{(\lambda'|\lambda'+\alpha)}{m'+2} - \frac{\frac{1}{2}\alpha^2+\alpha}{m+m'+4}\right) - d + \frac{3}{8}\left(\frac{1}{2} - \varepsilon\right)$.

Now take $\lambda' = 0$ (so that $m' = 0$) and $\lambda = md + \frac{1}{2}n\alpha \in P_+^o$. Then (as pointed out in [13]) , we get all the central charges $z_{m;\varepsilon}$. We proceed as for the Virasoro algebra, to show that all the h's from (5.5) occur as well. Put $J_{\lambda;\varepsilon} = \{k \in \mathbb{Z} \mid -\frac{m-n+1}{2} - \varepsilon \le k \le \frac{n+1}{2} - \varepsilon\}$ and, for $k \in J_{\lambda;\varepsilon}$, put

$$U_{\lambda,k;\varepsilon} = \{v \in (F_\varepsilon \otimes L(\lambda))^{\hat{\mathfrak{h}}} \mid (\pi_{\lambda_\varepsilon} \otimes \pi_\lambda)(\alpha)v = (n-2k+1-2\varepsilon)v\} \quad .$$

Then the subspace spanned by all highest weight vectors of $\hat{\mathfrak{g}}'_\varepsilon$ in $L_\varepsilon(\lambda_\varepsilon) \otimes L_\varepsilon(\lambda+\lambda_\varepsilon)$ of weight $2\lambda_\varepsilon+\lambda-k\alpha$ coincides with $v_{\lambda_\varepsilon} \otimes U_{\lambda,k;\varepsilon}$, where v_{λ_ε} is the highest weight vector os F_ε (see (5.3)), and $(L_\varepsilon(\lambda_\varepsilon) \otimes L_\varepsilon(\lambda+\lambda_\varepsilon))^{\hat{\mathfrak{h}}_\varepsilon}$ decomposes into a direct sum of these subspaces with $k \in J_{\lambda;\varepsilon}$. Each subspace $U_{\lambda,k;\varepsilon}$ decomposes with respect to d into a direct sum of eigenspaces $U^{(j)}_{\lambda,k;\varepsilon}$ with eigenvalue $j \in \varepsilon + \mathbb{Z}$. Putting $m_{\lambda,k;\varepsilon} = \sum_j (\dim U^{(j)}_{\lambda,k;\varepsilon}) q^{-j}$, we have

$$(5.8) \quad ch_{\lambda_\varepsilon;\varepsilon} ch_{\lambda+\lambda_\varepsilon;\varepsilon} = \sum_{k \in J_{\lambda;\varepsilon}} m_{\lambda,k;\varepsilon} ch_{\lambda+2\lambda_\varepsilon-k\alpha;\varepsilon} \quad .$$

To compute the $m_{\lambda,k;\varepsilon}$, we multiply formulas (1.10) (resp. (1.11)) and (1.8) and use (4.2).

We obtain :

$$(5.9) \quad m_{\lambda,k;\varepsilon} = (2-2\varepsilon)\varphi_\varepsilon(q)^{-1}\left(f^{(m,n)}_{k,\varepsilon} - f^{(m,n)}_{n+1-k,\varepsilon}\right) \quad ,$$

where

$$(5.9a) \quad f^{(m,n)}_{k,\varepsilon} = \sum_{j \in \mathbb{Z}} q^{\frac{1}{2}(m+2)(m+4)j^2+((n+1)+(k+\varepsilon-\frac{1}{2})(m+2))j+\frac{1}{2}k^2-(\frac{1}{2}-\varepsilon)k} \quad .$$

Using (5.7) and (5.9), we find that the lowest eigenvalue of $L^{(\varepsilon)}_o$ on $U_{\lambda,k;\varepsilon}$ is

$$(5.10) \quad \frac{[(n+1)+(k+\varepsilon-1/2)(m+2)]^2-1}{2(m+2)(m+4)} + \frac{1}{8}\left(\frac{1}{2}-\varepsilon\right) \quad .$$

Define numbers r_λ and $s_{\lambda,k}$ by $r_\lambda = n+1$, $s_{\lambda,k} = n+2-2\varepsilon-2k$ if $k \ge 0$, and $r_\lambda = m-n+1$, $s_{\lambda,k} = m-n+2k+2+2\varepsilon$ if $k < 0$. We arrive at the following theorem.

<u>Theorem 5.1</u>. (a) All highest weight representations of the Neveu-Schwarz and Ramond superalgebras Vir_ε with highest weights (5.5) are unitary. All these representations appear in

$$\bigoplus_{\lambda \in P^o_+} (L_\varepsilon(\lambda_\varepsilon) \otimes L_\varepsilon(\lambda+\lambda_\varepsilon))^{\hat{\mathfrak{h}}_\varepsilon}$$

with multiplicity one, except for $(z_{m;\varepsilon}, h^{(m)}_{r+2\varepsilon,r+1;\varepsilon})$ with $m \ne 2r$, which appears twice.

(b) Given $\lambda \in P^o_+$ of level m, we have the following decomposition with respect to the direct sum of $\hat{\mathfrak{g}}'_\varepsilon$ and Vir_ε :

$$L_\varepsilon(\lambda_\varepsilon) \otimes L_\varepsilon(\lambda+\lambda_\varepsilon) = \bigoplus_{k\in J_{\lambda;\varepsilon}} L_\varepsilon(\lambda+2\lambda_\varepsilon-k\alpha) \otimes V(z_{m;\varepsilon}, h^{(m)}_{r_\lambda, s_{\lambda,k};\varepsilon}).$$

Remark 5.1. The proof of Theorem 5.1 (b) and the part of 5.1(a) concerning multiplicities require showing that, up to multiplication by a suitable power of q, we have the following equality :

$$(5.11) \quad ch_{z_{m;\varepsilon}, h^{(m)}_{r_\lambda, s_{\lambda,k};\varepsilon}} = m_{\lambda,k;\varepsilon} \quad \text{(given by (5.9))}.$$

This can be done by applying the Feigin-Fuchs analysis [3] to Vir_ε. Let us say that a number from the set $\{h^{(m)}_{r_\lambda, s_{\lambda,k};\varepsilon} | k \in J_{\lambda;\varepsilon}\}$ is good if adding to it a positive integer never gives a number from this set. It follows from (5.5) and (5.9) that for $(z_{m;\varepsilon}, h)$ with good h, (5.11) holds automatically. This observation proves (5.11) in most of the cases (but not in all of them). Similar remark holds, of course, for Vir.

Remark 5.2. Taking integral and half-integral powers of q in $m_{\lambda,k;1/2}$ gives the characters of the even and odd part for the Neveu-Schwarz superalgebra. For the Ramond superalgebra these two characters are both equal to the half of $m_{\lambda,k;o}$, since g_o is invertible and hence permutes the even and odd parts of all representations in question (since $g_o^2 = \ell_o - \frac{1}{24}\tilde{c}$ and the spectrum of ℓ_o on all unitarizable representations from (5.5) with $\varepsilon = 0$ is greater than $\frac{1}{24}$).

Remark 5.3. Vir_ε acts on $L_\varepsilon(\lambda_\varepsilon)$, commuting with $g(\subset \mathfrak{g}_\varepsilon)$, hence on $L_\varepsilon(\lambda_\varepsilon)^e$, with central charge $z = \frac{3}{2}$ [13]. It is not difficult to show that $L_\varepsilon(\lambda_\varepsilon)^e$ is a model for degenerate highest weight representations of Vir_ε with $z = \frac{3}{2}$. More precizely, with respect to the direct sum of g and Vir_ε we have the following decomposition :

$$L_\varepsilon(\lambda_\varepsilon) = \sum_{k\in \mathbb{Z}_+} T_{2k+2-2\varepsilon} \otimes V_\varepsilon\left(\frac{3}{2}, \frac{k^2+(1-2\varepsilon)k}{2} + \frac{3}{8}\left(\frac{1}{2}-\varepsilon\right)\right).$$

Remark 5.4. Using the above construction, we can give a very simple proof of the formulas for $\det_n(z,h)$ of the determinant of the contravariant form on the subspace of elements of degree n of the Verma module with highest weight (z,h) (cf. [9], [2], [3], [6], [17],...). Consider, for example, the case of Vir (the argument for Vir_ε is exactly the same). It follows from (4.3) and the fact that Vir acts on $U_{\lambda,k}$, that

$$ch_{z_m, h^{(m)}_{r,s}} \leq q^{h^{(m)}_{r,s}} \varphi(q)^{-1}(1-q^{rs}-q^{(m+2-r)(m+3-s)} + \dots).$$

Hence the kernel of the contravariant form on the Verma module with highest weight

$(z_m, h^{(m)}_{r,s})$ contains non-zero vectors of degree rs and $(m+2-r)(m+3-s)$. Hence, $h = h^{(m)}_{r,s}$ are roots of $\det_{rs}(z_m, h)$ for all $r,s > 0$. So, as a polynomial in two variables, $\det_{rs}(z,h)$ vanishes at infinitely many points of the curve $\phi_{r,s}(z,h) = 0$, where $\phi_{r,s}$ is defined by $\phi_{r,s}(z_m, h) = (h-h^{(m)}_{r,s})(h-h^{(m)}_{s,r})$. Thus, $\det_{rs}(z,h)$ is divisible by $\phi_{r,s}(z,h)$ if $r \neq s$ or by its square root if $r = s$. An easy induction on n, as in [2, §4.2], completes the proof of the formula [8], [9]:

$$(\det_n(z,h))^2 = \text{const} \prod_{a=1}^{n} \prod_{j|a} \phi_{j,a/j}(z,h)^{p(n-a)},$$

where const $\neq 0$ depends only on the choice of basis. The argument for Vir_ε is given in Appendix 1.

Appendix 1. A proof of the determinantal formulas.

We give here, for the convenience of the reader, a selfcontained proof of the determinantal formulas for Vir_ε.

Given numbers z and h, there exists a unique ($\mathbb{Z}_2$-graded) module $M_\varepsilon(z,h)$ over Vir_ε, called Verma module, which admits a non-zero vector $v_{z,h}$, such that $\ell_0 v_{z,h} = h v_{z,h}$, $\tilde{c} v_{z,h} = z v_{z,h}$ and the vectors

$$v(i_1,\dots,i_\alpha; j_1,\dots,j_\beta) = g_{-j_\beta} \cdots g_{-j_1} \ell_{-i_\alpha} \cdots \ell_{-i_1} v_{z,h}$$

with $0 < i_1 \leq \dots \leq i_\alpha$ and $0 \leq j_1 < \dots < j_\beta$ form a basis of $M_\varepsilon(z,h)$ (in particular, $\ell_j z_{v,h} = 0$ and $g_j v_{z,h} = 0$ for $j > 0$). The space $M_\varepsilon(z,h)$ carries a unique Hermitian form $\langle \cdot | \cdot \rangle$ such that the norm of $v_{z,h}$ is 1 and $\ell_j^* = \ell_{-j}$, $g_j^* = g_{-j}$, called the contravariant Hermitian form.1) With respect to ℓ_0, $M_\varepsilon(z,h)$ decomposes into an orthogonal direct sum of eigenspaces $M_\varepsilon(z,h)_n$ with eigenvalues $h + n$, where $n \in (1-\varepsilon)\mathbb{Z}_+$. We say that vectors from $M_\varepsilon(z,h)_n$ have degree n. Let $M_\varepsilon(z,h)^+_n$ and $M_\varepsilon(z,h)^-_n$ denote the even (resp. odd) part of $M(z,h)_n$. We have : $M_{\frac12}(z,h)_n = M_{\frac12}(z,h)^+_n$ (resp. $= M_{\frac12}(z,h)^-_n$) if $n \in \mathbb{Z}_+$ (resp. $n \in \frac12 + \mathbb{Z}_+$) and $M_0(z,h)_n$ is an orthogonal direct sum of subspaces $M_0(z,h)^+_n$ and $M_0(z,h)^-_n$. Let $p_\varepsilon(n)$ be the coefficient of q^n in the power series expansion of $\varphi_\varepsilon(q)^{-1}$. Note that

(6.1) $\quad \dim M_{\frac12}(z,h)_n = p_{\frac12}(n)$; $\dim M_0(z,h)^{\pm}_n = p_0(n)$.

We put $p_0^{\pm}(n) = p_0(n) \mp \delta_{n,0}$, and

$$\phi_{r,s;\varepsilon}(z_{m;\varepsilon}, h) = (h-h^{(m)}_{r,s;\varepsilon})(h-h^{(m)}_{s,r;\varepsilon}).$$

---

1) We assume that the even and odd subspaces are orthogonal (this is not satisfied automatically if $\varepsilon = 0$).

Note that $\phi_{r,s;\varepsilon}(z,h)$ is a polynomial (of degree 2) in h and z. Given $n \in (2-2\varepsilon)\mathbb{Z}_+$, let $\det^{\pm}_{\frac{1}{2}n}(z,h)_\varepsilon$ denotes the determinant of the contravariant Hermitian form on $M_\varepsilon(z,h)^{\pm}_{n}$. The aim of this appendix is to prove the following formula (cf. [9] and [6]) :

$$(6.2)\quad \det{}^{\pm}_{\frac{1}{2}n}(z,h)^2_\varepsilon = \mathrm{const}\left(h-\frac{1}{24}z\right)^{(1-2\varepsilon)p_0^{\pm}(\frac{1}{2}n)} \prod_{\substack{a,b \in \mathbb{Z}_+ \\ 1 \le ab \le n \\ a-b \in 2\varepsilon+1+2\mathbb{Z}}} \phi_{a,b;\varepsilon}(z,h)^{p_\varepsilon(\frac{1}{2}(n-ab))}$$

where const. is a non-zero constant, independent of z and h.

As in Remark 5.4, it follows from (5.9) and the fact that Vir_ε acts on $U_{\lambda,k;\varepsilon}$, that

$$\mathrm{ch}_{z_{m;\varepsilon},h^{(m)}_{r,s;\varepsilon}}(q) \le q^{h^{(m)}_{r,s;\varepsilon}}\varphi_\varepsilon(q)^{-1}(1-q^{\frac{1}{2}rs}-q^{\frac{1}{2}(m+2-r)(m+4-s)}+\dots).$$

Since $L_\varepsilon(z,h)$ is the quotient of $M_\varepsilon(z,h)$ by the kernel of the contravariant form, it follows that for $M(z_{m;\varepsilon},h^{(m)}_{r,s;\varepsilon})$ this kernel contains non-zero vectors of degree $\frac{1}{2}rs$ and $\frac{1}{2}(m+2-r)(m+2-s)$. It follows that for all a and b as in (6.2), $\det_{\frac{1}{2}ab}(z,h)_\varepsilon$ is divisible by $\phi_{a,b;\varepsilon}(z,h)$ if $a \neq b$ or by its square root if $a=b$. Furthermore, it is clear that $g_0 v_{z+h}$ is in the kernel of $\langle\cdot|\cdot\rangle$ if $h = \frac{1}{24}z$ (and $\varepsilon = 0$); also g_0 is invertible on $M_0(z,h)$ if $h > \frac{1}{24}z$. It follows that for all a and b as in (6.2), $\det^{\pm}_{\frac{1}{2}ab}(z,h)_0$ is divisible by $\phi_{a,b;0}(z,h)$ and that $\det^{-}_{0}(z,h)$ is divisible by $h-\frac{1}{24}z$. An induction on n, using (6.1) and well-known elementary properties of Verma modules, proves that the left-hand side of (6.2) is divisible by its right-hand side.

We will show that, for a fixed z, the degree of $Q^{\pm}_{n;\varepsilon}(h) = \det^{\pm}_{\frac{1}{2}n}(z,h)_\varepsilon$, viewed as a polynomial in h, is exactly the half of the degree of the polynomial on the right of (6.2). Recall that the vectors $v(i_1,\dots,i_\alpha;j_1,\dots,j_\beta)$ with $i_1+\dots+i_\alpha+j_1+\dots+j_\beta = n$ and β even (resp. odd) form a basis of $M_\varepsilon(z,h)^{+}_{n}$ (resp. $M_\varepsilon(z,h)^{-}_{n}$), so that $Q^{\pm}_{n;\varepsilon}(h)$ is the determinant of the matrix of the inner products of these vectors. It is clear that only the product of the diagonal entries of this matrix gives a non-zero contribution to the highest power of h, and that $\langle v(i_1,\dots,i_\alpha;j_1,\dots,j_\beta)|v(i_1,\dots,i_\alpha;j_1,\dots,j_\beta)\rangle$ has degree $\alpha+\beta$ in h. It is easy to deduce now that :

$$\deg Q_{n;\frac{1}{2}}(h) = \sum_{\substack{s>0 \\ s \text{ even}}} \sum_{m>0} p_{\frac{1}{2}}\left(n-\frac{1}{2}ms\right) + \sum_{\substack{s>0 \\ s \text{ odd}}} \sum_{m>0} (-1)^{m+1} p_{\frac{1}{2}}\left(n-\frac{1}{2}ms\right),$$

$$\deg Q^{\pm}_{n;0}(h) = \frac{1}{2}\Big(p^{\pm}_0(n) + \sum_{s>0}\sum_{m>0}(p_0(n-ms) + (-1)^{m+1}p_0(n-ms))\Big),$$

where s and m are integers. This completes the proof of (6.2).

Appendix 2. Multiplicative formulas for characters.

We present here formulas connecting the characters of discrete series representations of Vir and Vir_ε with specialized characters of affine Kac-Moody algebras of type $A_1^{(1)}$ and $A_2^{(2)}$. In many cases this gives simple product decompositions of characters of Vir and Vir_ε. In what follows we use freely notation and results of the book [11].

Let A be the generalized Cartan matrix of type $A_1^{(1)}$ or $A_2^{(2)}$. Let $g(A)$ be the associated Kac-Moody algebra. Let Δ_+ be the set of positive roots and let α_0, α_1 be simple roots (in the case $A_1^{(1)}$, $\alpha_0 = c-\alpha$ and $\alpha_1 = \alpha$). Let Λ_0, Λ_1 be fundamental weights (in the case $A_1^{(1)}$, $\Lambda_0 = d$ and $\Lambda_1 = d+\frac{1}{2}\alpha$) and let $P_+^0 = \{k_0\Lambda_0+k_1\Lambda_1 \mid k_i \in \mathbb{Z}_+\}$. Given $\Lambda = k_0\Lambda_0+k_1\Lambda_1 \in P_+^0$, which is usually written as $\Lambda = (k_0,k_1)$, we have the integrable representation $L(\Lambda;A)$ of $g(A)$ with highest weight Λ.

Let $W(A)$ be the Weyl group and let $\rho = \Lambda_0 + \Lambda_1$. Given $\lambda \in \rho + P_+^0$, put

$$N_\lambda^{(A)} = \sum_{w \in W(A)} \mathrm{sgn}(w)\, e^{w\cdot\lambda-\lambda} .$$

Then the Weyl-Kac character and denominator formulas read [11, Chapter 10] :

$$e^{-\Lambda}\,\mathrm{ch}\, L(\Lambda;A) = N_{\Lambda+\rho}^{(A)}/N_\rho^{(A)} \; ; \tag{7.1}$$

$$N_\rho^{(A)} = \prod_{\alpha\in\Delta_+} (1-e^{-\alpha}) \; . \tag{7.2}$$

(Note that in the case $A_1^{(1)}$, formula (7.1) is another form of formula (1.8); note that in our cases, $\mathrm{mult}\,\alpha = 1$ for all $\alpha \in \Delta_+$).

Given a pair of positive integers $t = (t_0,t_1)$, the algebra homomorphism $F_t^{(A)} : \mathbb{C}[[e^{-\alpha_0}, e^{-\alpha_1}]] \longrightarrow \mathbb{C}[[q]]$ defined by $F_t^{(A)}(e^{-\alpha_i}) = q^{t_i}$ $(i = 0,1)$ is called the specialization of type t. In what follows we shall often write 1 and 2 in place of $A_1^{(1)}$ and $A_2^{(2)}$ respectively.

Fix $\Lambda = (M-1,N-1)$, where M and N are positive integers. Using that $W(A) = \{(r_0r_1)^n, (r_0r_1)^n r_0;\ n \in \mathbb{Z}\}$, one easily deduces the following formulas :

$$F_t^{(1)}(N_{\Lambda+\rho}^{(1)}) = \sum_{j\in\mathbb{Z}} q^{|t|(M+N)j^2+(|t|N-t_1(M+N))j} - \sum_{j\in\mathbb{Z}} q^{|t|(M+N)j^2+(|t|N+t_1(M+N))j+t_1N} \tag{7.3}$$

$$F_t^{(2)}(N_{\Lambda+\rho}^{(2)}) = \sum_{j\in\mathbb{Z}} q^{\frac{1}{2}\|t\|(M+2N)j^2+\frac{1}{2}(2\|t\|N-t_1(M+2N))j} - \sum_{j\in\mathbb{Z}} q^{\frac{1}{2}\|t\|(M+2N)j^2+\frac{1}{2}(2\|t\|N+t_1(M+2N))j+t_1N} , \tag{7.4}$$

where $|t| = t_0+t_1$ and $\|t\| = 2t_0+t_1$.

One knows the following general product decomposition [11, Chapter 10] :

$$F^{(A)}_{(1,1)}(N^{(A)}_{\Lambda+\rho}) = F^{(A)}_{(M,N)}(N^{(A)}_{\rho}). \tag{7.5}$$

Furthermore, there are the following special product decompositions [20] :

$$F^{(1)}_{(1,2)}(N^{(1)}_{\Lambda+\rho}) = F^{(2)}_{(M,2N)}(N^{(2)}_{\rho}) \tag{7.6a}$$

$$F^{(1)}_{(2,1)}(N^{(1)}_{\Lambda+\rho}) = F^{(2)}_{(N,2M)}(N^{(2)}_{\rho}) \tag{7.6b}$$

$$F^{(1)}_{t}(N^{(1)}_{(n,2n)}) = F^{(2)}_{(nt_0,2nt_1)}(N^{(2)}_{\rho}) \tag{7.6c}$$

$$F^{(1)}_{t}(N^{(1)}_{(2n,n)}) = F^{(2)}_{(nt_1,2nt_0)}(N^{(2)}_{\rho}). \tag{7.6d}$$

We put

$$d^{(t;A)}_{\Lambda}(q) = F^{(A)}_{t}(e^{-\Lambda}\,\mathrm{ch}\,L(\Lambda;A)).$$

In the case $t = \mathbb{1} = (1,1)$, $d^{(\mathbb{1};A)}_{\Lambda}(q)$ is called the q-dimension of $L(\Lambda;A)$; due to (7.5), it has a product decomposition.

We turn now to the product decompositions of the characters of the Virasoro algebra. For the sake of simplicity, we put

$$\chi^{(m)}_{r,s} = q^{-h^{(m)}_{r,s}}\,\mathrm{ch}_{z_m,h^{(m)}_{r,s}}(q).$$

Comparing formula (4.3) (which gives the character of a discrete series representation of Vir) with (7.3) and using (7.1) and (7.2), we arrive at the following beautiful formula.

<u>Proposition 7.1</u>. Take $1 \le s \le r \le m+1$, and put $\Lambda = (m+2-s,s-1)$ and $t = (m+2-r,r)$ (or $\Lambda = (m+1-r,r-1)$ and $t = (m+3-s,s)$ respectively). Then

$$\chi^{(m)}_{r,s}(q) = d^{(t;1)}_{\Lambda}(q) \prod_{\substack{j\ge 1 \\ j\not\equiv 0,\pm r \bmod (m+2)}} (1-q^j)^{-1}. \quad (\text{or } 0,\ \pm s \bmod (m+3) \text{ resp.}) \tag{7.7}$$

(If $2r=m+2$ (or $2s=m+3$ resp.), the product on the right should be interpreted in a usual way).

<u>Remark 7.1</u>. Formula (7.7) shows that $V(z_m,h^{(m)}_{r,s})$ is a tensor product of the $(m+2-r,r)$-graded space $L(m+2-s,s-1;A_1^{(1)})$ and $(1,1)$-graded space $L(m+1-r,r-1;A_1^{(1)})^{s_+}$, where s_+ is the "positive part" of the principal Heisenberg subalgebra of $\hat{s\ell}_2$. This suggests that there may be some more explicit constructions of the discrete series representations of the Virasoro algebra.

Using formulas (7.6), we can obtain, in some cases, from (7.7) multiplicative formulas. They are collected in Table 1, where, for simplicity, we use the abbreviated product symbol

$$\prod_j (1-q^{uj\pm v}) = \prod_{j\geq 0} (1-q^{uj+v}) \prod_{j\geq 1} (1-q^{uj-v}),$$

and similarly for "-" replaced by "+". If r and s do not satisfy the condition $1 \leq s \leq r \leq m+1$, it is assumed further on that they are brought to this form by transformation $r' = k(m+2)\pm r$, $s' = k(m+3)\pm s$, with some $k \in \mathbb{Z}$ (which leave $h^{(m)}_{r,s}$ unchanged).

Table 1

$$\chi^{(2r-2)}_{r,s}(q) = \frac{\varphi(q^{r(2r+1)})}{\varphi(q)} \prod_j (1-q^{r(2r+1)j\pm rs})$$

$$\chi^{(3r-2)}_{r,s}(q) = \frac{\varphi(q^{2r(3r+1)})}{\varphi(q)} \prod_j (1-q^{r(3r+1)j\pm rs})$$

$$\times \prod_{j=\text{odd}} (1+q^{r(3r+1)j\pm rs})$$

$$\chi^{(3r-2)}_{2r,s}(q) = \frac{\varphi(q^{2r(3r+1)})}{\varphi(q)} \prod_j (1-q^{r(3r+1)j\pm rs})$$

$$\times \prod_{j=\text{even}} (1+q^{r(3r+1)j\pm rs})$$

$$\chi^{(2s-3)}_{r,s}(q) = \frac{\varphi(q^{s(2s-1)})}{\varphi(q)} \prod_j (1-q^{s(2s-1)j\pm rs})$$

$$\chi^{(3s-3)}_{r,s}(q) = \frac{\varphi(q^{2s(3s-1)})}{\varphi(q)} \prod_j (1-q^{s(3s-1)j\pm rs})$$

$$\times \prod_{j=\text{odd}} (1+q^{s(3s-1)j\pm rs})$$

$$\chi^{(3s-3)}_{r,2s}(q) = \frac{\varphi(q^{2s(3s-1)})}{\varphi(q)} \prod_j (1-q^{s(3s-1)j\pm rs})$$

$$\times \prod_{j=\text{even}} (1+q^{s(3s-1)j\pm rs})$$

Next, we put

$$\psi^{(m)\pm}_{r,s} = q^{-h^{(m)}_{r,s}} (\mathrm{ch}_{z_m, h^{(m)}_{r,s}}(q) \pm \mathrm{ch}_{z_m, h^{(m)}_{m+2-r,s}}(q)) .$$

Then, in a similar way, we obtain the following table :

Table 2

$$\psi_{r,s}^{(4r-2)\pm}(q) = \frac{1}{\varphi(q)} \prod_j (1-(\mp 1)^j q^{\frac{r(4r+1)}{2} j})$$

$$\times \prod_j (1-(\mp 1)^j q^{\frac{r(4r+1)}{2} j \pm rs})$$

$$\psi_{r,s}^{(4s-3)\pm}(q) = \frac{1}{\varphi(q)} \prod_j (1-(\mp 1)^j q^{\frac{s(4s-1)}{2} j})$$

$$\times \prod_j (1-(\mp 1)^j q^{\frac{s(4s-1)}{2} j \pm rs})$$

$$\psi_{r,s}^{(3r-2)-}(q) = \frac{\varphi(q^{\frac{r(3r+1)}{2}})}{\varphi(q)} \prod_j (1-q^{\frac{r(3r+1)}{4} j \pm \frac{rs}{2}})$$

$$\times \prod_j (1+q^{\frac{r(3r+1)}{2} j \pm \frac{rs}{2}})$$

$$\psi_{r,s}^{(6r-2)-}(q) = \frac{\varphi(q^{r(6r+1)})}{\varphi(q)} \prod_j (1-q^{r(6r+1)j \pm rs})$$

$$\times \prod_{j=odd} (1-q^{r(6r+1)j \pm 2rs})$$

$$\psi_{r,s}^{(3s-3)-}(q) = \frac{\varphi(q^{\frac{s(3s-1)}{2}})}{\varphi(q)} \prod_j (1-q^{\frac{s(3s-1)}{4} j \pm \frac{rs}{2}})$$

$$\times \prod_j (1+q^{\frac{s(3s-1)}{2} j \pm \frac{rs}{2}})$$

$$\psi_{r,s}^{(6s-3)-}(q) = \frac{\varphi(q^{s(6s-1)})}{\varphi(q)} \prod_j (1-q^{s(6s-1)j \pm rs})$$

$$\times \prod_{j=odd} (1-q^{s(6s-1)j \pm 2rs})$$

Note that formulas from Tables 1 and 2 cover all cases for small m. The case $m = 1$ is well-known; the case $m = 2$ was worked out in [15].

In a similar way, one finds product decompositions for the characters of Vir_ε. Put [2)]

$$\chi^{(m)}_{r,s;\varepsilon}(q) = \frac{1}{2-2\varepsilon}\, q^{-h^{(m)}_{r,s;\varepsilon}}\, ch_{z_{m;\varepsilon},h^{(m)}_{r,s;\varepsilon}}(q)\ .$$

Then we have

$$(7.8)\quad \chi^{(m)}_{r,s;\varepsilon}(q) = \frac{1}{\varphi_\varepsilon(q)}\, d^{(t;1)}_{\Lambda}(q^{\frac12}) \prod_{\substack{j\geq 1\\ j\equiv 0,\pm r \bmod (m+2)}} (1-q^{j/2})\ ,$$

where $\Lambda = (m+3-s, s-1)$, $t = (m+2-r, r)$.

There are other formulas, similar to (7,8), which involve only integral j, and also, in some cases, multiplicative formulas for Vir_ε, similar to that from Tables 1 and 2 for Vir. We present some of these formulas in Tables 3 and 4.

Table 3

$$\chi^{(2r-2)}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{r(r+1)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{r(r+1)j\pm rs/2})$$

$$\chi^{(3r-2)}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{r(3r+2)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{r(3r+2)j\pm rs/2}) \times \prod_{j=odd} (1-q^{r(3r+2)j\pm rs})$$

$$\chi^{(3r-2)}_{2r,s;\varepsilon}(q) = \frac{\varphi(q^{r(3r+2)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{2r(3r+2)j\pm rs}) \times \prod_{j=odd} (1-q^{(r(3r+2)/2)j\pm rs/2})$$

$$\chi^{(2s-4)}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{s(s-1)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{s(s-1)j\pm rs/2})$$

$$\chi^{(3s-4)}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{s(3s-2)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{s(3s-2)j\pm rs/2}) \times \prod_{j=odd} (1-q^{s(3s-2)j\pm rs})$$

$$\chi^{(3s-4)}_{r,2s;\varepsilon}(q) = \frac{\varphi(q^{s(3s-2)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{2s(3s-2)j\pm rs}) \times \prod_{j=odd} (1-q^{(s(3s-2)/2)j\pm rs/2})$$

Table 4

$$\psi^{(4r-2)-}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{r(r+\frac12)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{r(r+\frac12)j\pm rs/2})$$

$$\psi^{(6r-2)-}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{r(3r+1)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{r(3r+1)j\pm rs/2}) \times \prod_{j=odd} (1-q^{r(3r+1)j\pm rs})$$

$$\psi^{(6r-2)-}_{2r,s;\varepsilon}(q) = \frac{\varphi(q^{r(3r+1)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{(r(3r+1)/2)j\pm rs/2}) \times \prod_j (1+q^{r(3r+1)j\pm rs/2})$$

$$\psi^{(4s-4)-}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{s(s-\frac12)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{s(s-\frac12)j\pm rs/2})$$

---

2) The definition of $\psi^{(m)\pm}_{r,s;\varepsilon}$ is completely similar to that of $\psi^{(m)\pm}_{r,s}$.

$$\psi^{(6s-4)-}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{s(3s-1)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{s(3s-1)j\pm rs/2}) \times \prod_{j=odd} (1-q^{s(3s-1)j\pm rs})$$

$$\psi^{(6s-4)-}_{r,2s;\varepsilon}(q) = \frac{\varphi(q^{s(3s-1)})}{\varphi_\varepsilon(q)} \prod_j (1-q^{(s(3s-1)/2)j\pm rs/2}) \times \prod_j (1+q^{s(3s-1)j\pm rs/2})$$

$$\psi^{(4r-2)+}_{r,s;\varepsilon}(q) = \frac{1}{\varphi_\varepsilon(q)} \prod_j (1-(-1)^j q^{r(r+\frac{1}{2})j}) \times \prod_j (1-(-1)^j q^{r(r+\frac{1}{2})j\pm rs/2})$$

$$\psi^{(4s-4)+}_{r,s;\varepsilon}(q) = \frac{1}{\varphi_\varepsilon(q)} \prod_j (1-(-1)^j q^{s(s-\frac{1}{2})j}) \times \prod_j (1-(-1)^j q^{s(s-\frac{1}{2})j\pm rs/2})$$

Remark 7.2. It is always possible to write $\chi^{(m)}_{r,s}$ and $\chi^{(m)}_{r,s;\varepsilon}$ as a sum of two infinite products (using the Jacobi triple product identity) :

(7.9a) $$\chi^{(m)}_{r,s}(q) = \frac{\varphi(q^{2(m+2)(m+3)})}{\varphi(q)} \times \Big[\prod_{\substack{j\geq 1\\ j=odd}} (1+q^{(m+2)(m+3)j\pm((m+3)r-(m+2)s)}) - q^{rs}\prod_{\substack{j\geq 1\\ j=odd}} (1+q^{(m+2)(m+3)j\pm((m+3)r+(m+2)s)})\Big]$$

(7.9b) $$\chi^{(m)}_{r,s;\varepsilon}(q) = \frac{\varphi(q^{(m+2)(m+4)})}{\varphi_\varepsilon(q)} \times \Big[\prod_{\substack{j\geq 1\\ j=odd}} (1+q^{\frac{(m+2)(m+4)}{2}j\pm\frac{(m+4)r-(m+2)s}{2}}) - q^{\frac{rs}{2}}\prod_{\substack{j\geq 1\\ j=odd}} (1+q^{\frac{(m+2)(m+4)}{2}j\pm\frac{(m+4)r+(m+2)s}{2}})\Big]$$

Appendix 3. An application to the decomposition of tensor products of two level 1 representations of exceptional affine algebras.

In this appendix we will show that the affine Lie algebras $E_8^{(1)}$, $E_7^{(1)}$, $A_2^{(1)}$ and $A_2^{(2)}$, $E_6^{(1)}$ and $E_6^{(2)}$ provide a model for discrete series representations of the Vivasoro algebra with central charge z_m, where $m = 1,2,3,4$ respectively. Namely we will prove the following remarkable fact : taking tensor products of the basic representation with all level 1 fundamental representations of the affine algebras listed above, one gets (in the space of highest weight vectors) all discrete series representations of Vir for $m = 1,2,3,4$ and exactly once. Turning the point of view, "generalized string functions" [12, § 4.9] of the tensor product of two level 1 fundamental representations of the above affine algebras turn out to be nothing else but the characters of the corresponding discrete series representations of Vir.

As in Appendix 2, we will use freely the notation, conventions and results of the book [11]. In particular, the enumeration of the vertices of the Dynkin diagrams of affine algebras adopted here is that of [11, Chapters 4 and 6].

First, we will prove a few facts about Kac-Moody algebras which are used later on.

<u>Lemma 8.1</u>. Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with a symmetrizable Cartan matrix. Let $\Lambda, \Lambda' \in P_+$ and $\sigma \in W$ be such that $M = \sigma\cdot\Lambda + \Lambda' \in P_+$. Then

(a) $\mathrm{mult}_\Lambda\,(M+\rho-w(\Lambda'+\rho))$ is 1 if $w = 1$ and is 0 if $w \in W$, $w \neq 1$.

(b) The multiplicity of $L(M)$ in $L(\Lambda) \otimes L(\Lambda')$ is 1.

<u>Proof</u>. Claim (a) for $w = 1$ is clear. If $w \neq 1$, then $(M+\rho|\Lambda'+\rho-w(\Lambda'+\rho)) > 0$, and we have :

$$|M+\rho-w(\Lambda'+\rho)|^2-|\Lambda|^2=|M+\rho|^2+|w(\Lambda'+\rho)|^2-2(M+\rho|w(\Lambda'+\rho)-(\Lambda'+\rho))-2(M+\rho|\Lambda'+\rho)-|\Lambda|^2$$
$$>|M+\rho|^2+|\Lambda'+\rho|^2-2(M+\rho|\Lambda'+\rho)-|\Lambda|^2=|M+\rho-(\Lambda'+\rho)|^2-|\Lambda|^2=|\sigma\cdot\Lambda|^2-|\Lambda|^2=0.$$

Thus, $|M+\rho-w(\Lambda'+\rho)|^2-|\Lambda|^2 > 0$ and hence (by [11, Proposition 11.4]), $M+\rho-w(\Lambda'+\rho)$ is not a weight of $L(\Lambda)$, which completes the proof of (a). Claim (b) follows from (a) and the Racah "outer multiplicity" formula (cf. [4]) : the multiplicity of $L(M)$ in $L(\Lambda) \otimes L(\Lambda')$ is $\sum_{w\in W} \varepsilon(w)\,\mathrm{mult}_\Lambda\,(M+\rho-w(\Lambda'+\rho))$.

Further on, S^2V and Λ^2V stand for the symmetric and antisymmetric square of the space V, respectively.

<u>Lemma 8.2.</u> Let $\mathfrak{g}(A)$ be an affine algebra of A-D-E type all of whose exponents are odd, and let $\Lambda \in P_+$ be of level 1. Suppose that $L(M)$ occurs in $L(\Lambda) \otimes L(\Lambda)$. Then $L(M) \subset S^2L(\Lambda)$ (resp. $\subset \Lambda^2L(\Lambda)$) if and only if $\mathrm{ht}(2\Lambda-M)$ is even (resp. odd).

<u>Proof</u>. Using a diagram automorphism of $\mathfrak{g}(A)$, we may assume that $\Lambda = \Lambda_0$. The (basic) representation $L(\Lambda_0)$ of $\mathfrak{g}(A)$ is realized on the space of polynomials $\mathbb{C}[u_j; j \in E_+]$, where $E_+ = \mathbb{Z}_+ \cap E$ and E is the set of exponents of $\mathfrak{g}(A)$, so that the principal gradation is given by $\deg u_j = j$, and $u_j \in n_-$ and $\frac{\partial}{\partial u_j} \in n_+ (\subset \mathfrak{g}(A))$, $j \in E_+$ (cf. [11], Chapter 14). But then

$$L(\Lambda_0) \otimes L(\Lambda_0) = \mathbb{C}[u_j^{(1)}, u_j^{(2)};\ j \in E_+] = \mathbb{C}[x_j, y_j; j \in E_+],$$

where we put $x_j = u_j^{(1)}+u_j^{(2)}$ and $y_j = u_j^{(1)}-u_j^{(2)}$, so that $x_j \in n_-$ and $\frac{\partial}{\partial x_j} \in n_+$. Thus a highest weight vector of $L(M)$ is a polynomial in y_j's whose principal degree is equal to $\mathrm{ht}(2\Lambda_0-M)$.

Since E_+ consists of odd numbers, we deduce that

$$S^2L(\Lambda_0) = \mathbb{C}[x] \otimes \mathbb{C}_{even}[y]; \ \Lambda^2L(\Lambda_0) = \mathbb{C}[x] \otimes \mathbb{C}_{odd}[y],$$

where $\mathbb{C}_{even}[y]$ (resp. $\mathbb{C}_{odd}[y]$) denotes the subspace spanned by all monomials in y_j's of even (resp. odd) principal degree. This completes the proof of the lemma.

Let now A be an affine generalized Cartan matrix of type $X_N^{(k)}$, let $g(A)$ be the corresponding affine (Kac-Moody) algebra and let $d = \dim g(X_N)$ be the dimension of the "underlying" simple finite dimensional Lie algebra. Let $L(\Lambda')$ and $L(\Lambda'')$ be two highest weight representations of levels $m' = \Lambda'(c)$ and $m'' = \Lambda''(c)$, such that m', m'' and $m'+m'' \neq -g$, where g is the dual Coxeter number. Then (as has been mentioned in § 2), Vir acts on $L(\Lambda') \otimes L(\Lambda'')$ commuting with $g'(A)$, and formulas, corresponding to (2.3 a,b) and (2.4) generalize as follows (cf. [12],[18]) :

(8.1a) the central charge = $dp(m',m'')$, where

(8.1b) $p(m',m'') = \frac{m'}{m'+g} + \frac{m''}{m''+g} - \frac{m'+m''}{m'+m''+g}$

(8.2) $L_0 = \frac{1}{2k}\left[\frac{(\Lambda'|\Lambda'+2\rho)}{m'+g} + \frac{(\Lambda''|\Lambda''+2\rho)}{m''+g} - \frac{\Omega}{m'+m''+g}\right] + \left[\frac{d}{24} - \frac{|\rho|^2}{2gk}\right] p(m',m'')$.

Note that the second term on the right in (8.2) vanishes if $k = 1$ due to the Frendenthal-de Vries strange formula, whereas in case $k > 1$ it is "alive" and will play an important role.

The main result of this Appendix is the following theorem.

Theorem 8.1. One has the following decompositions with respect to the direct sum of $g'(A)$ and Vir :

1) $A = E_8^{(1)}$:

$S^2L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{1}{2},0) + L(\Lambda_7) \otimes V(\frac{1}{2},\frac{1}{2})$, $\Lambda^2L(\Lambda_0) = L(\Lambda_1) \otimes V(\frac{1}{2},\frac{1}{16})$.

2) $A = E_7^{(1)}$:

$S^2L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{7}{10},0) + L(\Lambda_5) \otimes V(\frac{7}{10},\frac{3}{5})$,

$\Lambda^2L(\Lambda_0) = L(2\Lambda_6) \otimes V(\frac{7}{10},\frac{3}{2}) + L(\Lambda_1) \otimes V(\frac{7}{10},\frac{1}{10})$,

$L(\Lambda_0) \otimes L(\Lambda_6) = L(\Lambda_0+\Lambda_6) \otimes V(\frac{7}{10},\frac{3}{80}) + L(\Lambda_7) \otimes V(\frac{7}{10},\frac{7}{16})$.

3) $A = A_2^{(1)}$:

$S^2L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{4}{5},0) + L(\Lambda_1+\Lambda_2) \otimes V(\frac{4}{5},\frac{7}{5})$,

$\Lambda^2L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{4}{5},3) + L(\Lambda_1+\Lambda_2) \otimes V(\frac{4}{5},\frac{2}{5})$,

$L(\Lambda_0) \otimes L(\Lambda_1) = L(2\Lambda_2) \otimes V(\frac{4}{5},\frac{2}{3}) + L(\Lambda_0+\Lambda_1) \otimes V(\frac{4}{5},\frac{1}{15})$.

$A = A_2^{(2)}$:

$S^2L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{4}{5},\frac{1}{40}) + L(\Lambda_1) \otimes V(\frac{4}{5},\frac{13}{8})$,

$\Lambda^2 L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{4}{5},\frac{21}{40}) + L(\Lambda_1) \otimes V(\frac{4}{5},\frac{1}{8})$.

4) $A = E_6^{(1)}$:

$S^2 L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{6}{7},0) + L(\Lambda_1+\Lambda_5) \otimes V(\frac{6}{7},\frac{5}{7}) + L(\Lambda_6) \otimes V(\frac{6}{7},\frac{22}{7})$,

$\Lambda^2 L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{6}{7},5) + L(\Lambda_1+\Lambda_5) \otimes V(\frac{6}{7},\frac{12}{7}) + L(\Lambda_6) \otimes V(\frac{6}{7},\frac{1}{7})$,

$L(\Lambda_0) \otimes L(\Lambda_1) = L(2\Lambda_5) \otimes V(\frac{6}{7},\frac{4}{3}) + L(\Lambda_0+\Lambda_1) \otimes V(\frac{6}{7},\frac{1}{21}) + L(\Lambda_4) \otimes V(\frac{6}{7},\frac{10}{21})$.

$A = E_6^{(2)}$:

$S^2 L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{6}{7},\frac{1}{56}) + L(\Lambda_1) \otimes V(\frac{6}{7},\frac{33}{56}) + L(\Lambda_4) \otimes V(\frac{6}{7},\frac{23}{8})$,

$\Lambda^2 L(\Lambda_0) = L(2\Lambda_0) \otimes V(\frac{6}{7},\frac{85}{56}) + L(\Lambda_1) \otimes V(\frac{6}{7},\frac{5}{56}) + L(\Lambda_4) \otimes V(\frac{6}{7},\frac{3}{8})$.

The proof of the theorem is based on the following observations. Let $\Lambda \in P_+$ be of level 1 and let $M \in P_+$ be such that $L(M)$ occurs in $L=L(\Lambda_0) \otimes L(\Lambda)$. Note that M has level 2 and $M \in \Lambda_0 + \Lambda + Q$, where Q is the root lattice of $\mathfrak{g}(A)$. Let U_M denote the sum of all subrepresentations in L of the form $L(M+s\delta)$, $s \in \mathbb{Z}$. Then L decomposes into a direct sum of subspaces of the form U_M. Vir acts on $U_M^{\mathfrak{n}_+}$ with central charge z_m, where $m = 1,2,3$ or 4 is the number of claim of Theorem 8.1, and with respect to the direct sum of $\mathfrak{g}'(A)$ and Vir we have : $L = \bigoplus_{M \bmod \mathbb{C}\delta} (L(M) \otimes U_M^{\mathfrak{n}_+})$. The eigenvalues of L_0 on $U_M^{\mathfrak{n}_+}$ are, due to (8.2), of the form $h_M^{(\Lambda)} + \frac{1}{k}\mathbb{Z}$, where

$$h_M^{(\Lambda)} = \frac{1}{2k}\left[\frac{(\Lambda|\Lambda+2\rho)}{g+1} - \frac{(M|M+2\rho)}{g+2}\right] + \left[\frac{d}{24} - \frac{|\rho|^2}{2gk}\right]p(1,1). \tag{8.3}$$

On the other hand, since the representation of $\mathfrak{g}(A)$ on L is unitary, so is the representation of Vir on $U_M^{\mathfrak{n}_+}$, hence the eigenvalues of L_0 on $U_M^{\mathfrak{n}_+}$ are of the form $h_{r,s}^{(m)} + \mathbb{Z}$.

The values of $h_M^{(\Lambda)} \bmod \frac{1}{k}\mathbb{Z}$ for all $\Lambda \in P_+$ of level 1 and all $M \in P_+$ of level 2 such that $M \in \Lambda_0 + \Lambda + Q$ are listed in the Table M below.

The proof of Theorem 8.1 in all cases, except for the representation $L(\Lambda_0) \otimes L(\Lambda_0)$ of $E_6^{(1)}, A_2^{(2)}$ and $A_2^{(1)}$, is obtained now directly by making use of Lemmas 8.1 and 8.2.

The remaining cases require more calculations. We shall demonstrate them in the case of $A_2^{(1)}$. From Table M we see that $L(\Lambda_0) \otimes L(\Lambda_0)$ for $A_2^{(1)}$ decomposes as follows :

$$L(\Lambda_0) \otimes L(\Lambda_0) = L(2\Lambda_0) \otimes V(\tfrac{4}{5},0) + b_1 L(2\Lambda_0) \otimes V(\tfrac{4}{5},3) + b_2 L(\Lambda_1+\Lambda_2) \otimes V(\tfrac{4}{5},\tfrac{7}{5}) + b_3 L(\Lambda_1+\Lambda_2) \otimes V(\tfrac{4}{5},\tfrac{2}{5}), \tag{8.4}$$

where $b_i \in \mathbb{Z}_+$.

In order to show that $b_i = 1$ and to distribute each term in the right hand side of (8.4) to the symmetric or the skew-symmetric part, we compute the q-dimension of each component. In doing this, it suffices to know only coefficients of q^i for

$0 \leq j \leq 9$, since the lowest among leading weights $2\Lambda_0$, $2\Lambda_0-3\delta$, $\Lambda_1+\Lambda_2$, $\Lambda_1+\Lambda_2-\delta$ is $2\Lambda_0-3\delta$ and $ht(3\delta)$ is equal to 9. The coefficients of q^j of q-dimensions are listed on the following Table Q, where $\psi(q) = \varphi(q)/\varphi(q^3)$. They are computed using [11, Proposition 10.10].

Table M

A	Λ	M	$h_M^{(\Lambda)} \bmod \frac{1}{k}\mathbb{Z}$	1'st level S^2	Λ^2	A	Λ	M	$h_M^{(\Lambda)} \bmod \frac{1}{k}\mathbb{Z}$	1'st level S^2	Λ^2
$E_8^{(1)}$	Λ_0	$2\Lambda_0$	0	0		$A_2^{(2)}$	Λ_0	$2\Lambda_0$	$1/40 \equiv 21/40$	0	3
$m=1$	Λ_0	Λ_1	1/16		1	$m=3$	Λ_0	Λ_1	$1/8 \equiv 13/8$	10	1
$z_1=\frac{1}{2}$	Λ_0	Λ_7	1/2	14		$E_6^{(1)}$	Λ_0	$2\Lambda_0$	$0 \equiv 5$	0	60
$E_7^{(1)}$	Λ_0	$2\Lambda_0$	0	0		$m=4$	Λ_0	$\Lambda_1+\Lambda_5$	$5/7 \equiv 12/7$	8	20
$m=2$	Λ_0	$2\Lambda_6$	3/2		27	$z_4=\frac{6}{7}$	Λ_0	Λ_6	$1/7 \equiv 22/7$	37	1
$z_2=\frac{7}{10}$	Λ_0	Λ_1	1/10		1		Λ_1	$2\Lambda_5$	4/3	16	
	Λ_0	Λ_5	3/5	10			Λ_1	$\Lambda_0+\Lambda_1$	1/21	0	
	Λ_6	$\Lambda_0+\Lambda_6$	3/80	0			Λ_1	Λ_4	10/21	5	
	Λ_6	Λ_7	7/16	7		$E_6^{(2)}$	Λ_0	$2\Lambda_0$	$1/56 \equiv 85/56$	0	27
$A_2^{(1)}$	Λ_0	$2\Lambda_0$	$0 \equiv 3$	0	9	$m=4$	Λ_0	Λ_1	$5/56 \equiv 33/56$	10	1
$m=3$	Λ_0	$\Lambda_1+\Lambda_2$	$2/5 \equiv 7/5$	4	1		Λ_0	Λ_4	$3/8 \equiv 23/8$	52	7
$z_3=\frac{4}{5}$	Λ_1	$2\Lambda_2$	2/3	2							
	Λ_1	$\Lambda_0+\Lambda_1$	1/15	0							

Table Q

	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9
$\psi(q)\dim_q S^2L(\Lambda_0)$	1	0	1	1	2	2	4	4	7	8
$\psi(q)\dim_q \Lambda^2L(\Lambda_0)$	0	1	1	1	2	3	3	5	6	8
$\psi(q)\dim_q L(2\Lambda_0)\cdot\chi_{1,1}^{(3)}(q^3)$	1	0	1	1	1	1	3	2	4	5
$q^9\psi(q)\dim_q L(2\Lambda_0)\cdot\chi_{4,1}^{(3)}(q^3)$	0	0	0	0	0	0	0	0	0	1
$q\psi(q)\dim_q L(\Lambda_1+\Lambda_2)\cdot\chi_{2,1}^{(3)}(q^3)$	0	1	1	1	2	3	3	5	6	7
$q^4\psi(q)\dim_q L(\Lambda_1+\Lambda_2)\cdot\chi_{3,1}^{(3)}(q^3)$	0	0	0	0	1	1	1	2	3	3

In Table Q, $\chi_{r,s}^{(m)}(x)$ is as defined in Appendix 2, and we put $x = q^3$ since ht $\delta = 3$. The statements for $A_2^{(1)}$ in Theorem 8.1 follow immediately from Table Q. A similar proof works also for $A_2^{(2)}$ and $E_6^{(1)}$; one has to compute the concerned q-dimensions up to the 10-th and 60-th power of q respectively.

Remark 8.1. Theorem 8.1 covers all cases when tensor products of level 1 representations of affine algebras produce representations of Vir with $z < 1$, except for $A_1^{(1)}$, covered by Theorem 4.1, $G_2^{(1)}$ and $F_4^{(1)}$. Specifically, for $A_1^{(1)}$ we have :

$$S^2L(\Lambda_0) = L(2\Lambda_0) \otimes V(\tfrac{1}{2},0),\ \Lambda^2L(\Lambda_0) = L(2\Lambda_1) \otimes V(\tfrac{1}{2},\tfrac{1}{2}),$$

$$L(\Lambda_0) \otimes L(\Lambda_1) = L(\Lambda_0+\Lambda_1) \otimes V(\tfrac{1}{2},\tfrac{1}{16}).$$

For $G_2^{(1)}$ the central charge is $z_7 = \frac{14}{15}$; putting $U_1 = S^2L(\Lambda_0)$, $U_3 = S^2L(\Lambda_2)$, $U_5 = L(\Lambda_0) \otimes L(\Lambda_2)$, $U_7 = \Lambda^2L(\Lambda_2)$, $U_9 = \Lambda^2L(\Lambda_0)$, and $L_1 = L(2\Lambda_0)$, $L_3 = L(2\Lambda_2)$, $L_5 = L(\Lambda_0+\Lambda_2)$, $L_7 = L(\Lambda_1)$, we have:

$$U_s = \sum_{r=1,3,5,7} L_r \otimes V(z_7,h_{r,s}^{(7)}) .$$

For $F_4^{(1)}$ the central charge is $z_8 = \frac{52}{55}$; putting $U_1 = S^2L(\Lambda_0)$, $U_3 = S^2L(\Lambda_4)$, $U_5 = L(\Lambda_0) \otimes L(\Lambda_4)$, $U_7 = \Lambda^2L(\Lambda_4)$, $U_9 = \Lambda^2L(\Lambda_0)$, and $L_1 = L(2\Lambda_0)$, $L_3 = L(2\Lambda_4)$, $L_5 = L(\Lambda_0+\Lambda_4)$, $L_7 = L(\Lambda_3)$, $L_9 = L(\Lambda_1)$, we have:

$$U_r = \sum_{s=1,3,5,7,9} L_s \otimes V(z_8,h_{r,s}^{(8)}) .$$

Theorem 8.1 can be written in a similar compact form.

Remark 8.2. It is fairly well-known that all unitarizable representations of Vir with $z = \frac{1}{2}$ can be constructed as follows. Fix $\varepsilon = 0$ or $\frac{1}{2}$. Consider the "superoscillator" algebra A_ε on generators ψ_m, $m \in \varepsilon + \mathbb{Z}$, and defining relations

$$[\psi_m,\psi_n]_+ = \delta_{n,-m} .$$

Let $V_\varepsilon = \Lambda[\xi_j \mid j \geq 0,\ j \in \varepsilon+\mathbb{Z}]$ be a Grassmann algebra. Define a representation of A_ε on V_ε by $(n > 0)$:

$$\psi_n \to \frac{\partial}{\partial \xi_n},\ \psi_{-n} \to \xi_n ,\quad \psi_0 \to \frac{1}{\sqrt{2}}(\xi_0 + \frac{\partial}{\partial \xi_0}) .$$

Define a Hermitian form on V_ε by taking monomials for an orthonormal basis. Let V_ε^+ (resp. V_ε^-) denote the subspace of V_ε spanned by monomials of even (resp. odd) degree, where $\deg \xi_j = 1$, all j. Put

$$L_0 = \frac{1}{8}(\frac{1}{2} - \varepsilon) + \sum_{j \in \varepsilon+\mathbb{Z}_+} j\psi_{-j}\psi_j ,$$

$$L_n = \frac{1}{4} \sum_{j \in \varepsilon+\mathbb{Z}} (2j-n)\, \psi_{-j+n}\psi_j \quad \text{for } n \neq 0.$$

This gives irreducible representations of Vir with $z = \frac{1}{2}$ on $V_\varepsilon^{\pm}$. Explicitly :

$$V_{\frac{1}{2}}^{+} = V(\tfrac{1}{2},0),\ V_{\frac{1}{2}}^{-} = V(\tfrac{1}{2},\tfrac{1}{2}),\ V_{0}^{+} = V_{0}^{-} = V(\tfrac{1}{2},\tfrac{1}{16}).$$

No such simple construction is known (so far) for other discrete series representations of Vir.

Remark 8.3. Note the following remarkable coincidence. Let g be a simple Lie algebra of type E_8, E_7, A_2 or E_6 and let $\hat{g}$ be the associated affine algebra. Then all highest weights of the representations of Vir that occur in all pairwise tensor products of all level 1 representations of $\hat{g}$ are of the form (z_m, h), where $m = 1,2,3$ or 4 respectively and h is precisely one of the critical exponents of the Ising, tricritical Ising, 3-state Potts and tricritical 3-state Potts models respectively (cf. [5]). In other words, the $h_{r,s}^{(m)}$ that occur in 2-dimensional statistical models are precisely those which correspond to non-twisted affine algebras.

Remark 8.4. The same argument as above can be applied to the study of the problem of restriction of a unitary highest weight representation of an affine algebra $\hat{g}$ to an affine subalgebra $\hat{p}$, where p is a reductive subalgebra of reductive algebra g. In our next publication we will classify the pairs (g,p) for which the central charge of the Virasoro algebra is less than 1 and calculate the corresponding generalized string functions.

Références

[1] Belavin A.A., Polyakov A.M., Zamolodchikov A.B., Infinite conformal symmetry of critical fluctuations in two dimensions, J. Stat. Phys. 34 (1984), 763-774. Infinite conformal symmetry in two dimensional quantum field theory, Nucl. Physics B241 (1984), 333-380.

[2] Feigin B.L., Fuchs D.B., Skew-symmetric invariant differential operators on a line and Verma modules over the Virasoro algebra, Funct. Anal. Appl. 16 (1982), no.2, 47-63 (in Russian).

[3] Feigin B.L., Fuchs D.B., Verma modules over the Virasoro algebra, Funct. Anal. Appl., 17 (1983), 91-92 (in Russian). Representations of the Virasoro algebra (1983), preprint.

[4] Feingold A.J., Tensor products of certain modules for the generalized Cartan matrix Lie algebra $A_1^{(1)}$, Comm. in Alg., 9 (1981), 1323-1361.

[5] Friedan D., Qiu Z., Shenker S., Conformal invariance, unitarity and two dimensional critical exponents, MSRI Publ., 3 (1985), 419-449.

[6] Friedan D., Qiu Z., Shenker S., Superconformal invariance in two dimensions and the tricritical Ising model, Phys. Lett. 151B (1985) 37-43.

[7] Goddard P., Kent A., Olive D., Virasoro algebras and coset space models, Phys. Lett. 152B (1985), 88-93.

[8] Kac V.G., Highest weight representations of infinite-dimensional Lie algebras, Proceedings of ICM, 299-304, Helsinki, 1978.

[9] Kac V.G., C
gebras, Lec

[10] Kac V.G., S
tations, Le

[11] Kac V.G., I
Boston, 198

[12] Kac V.G. an
and modular

[13] Kac V.G., T
sentations,

[14] Kac V.G., L

[15] Rocha-Carid
Publ., 3 (1

[16] Schwarz J.H

[17] Thorn C.B.,
about the n

[18] Wakimoto M.
roku 503 (1
preprint.

[19] Kac V.G., P
groups, in
Boston, 198

[20] Wakimoto M.
(1983), pre

[9] Kac V.G., Contravariant form for infinite dimensional Lie algebras and superalgebras, Lect. Notes in Phys., 94 (1979), 441-445.

[10] Kac V.G., Some problems in infinite dimensional Lie algebras and their representations, Lect. Notes in Math., 933 (1982), 117-126.

[11] Kac V.G., Infinite Dimensional Lie Algebras, Progress in Math. 44, Birkhäuser, Boston, 1983. Second edition : Cambridge University Press, 1985.

[12] Kac V.G. and Peterson D.H., Infinite-dimensional Lie algebras, theta functions and modular forms, Advances in Math., 53 (1984), 125-264.

[13] Kac V.G., Todorov I.T., Superconformal current algebras and their unitary representations, Comm. Math. Phys 102(1985), 337 - 347.

[14] Kac V.G., Lie superalgebras, Adv. Math. 26 (1977), 8-96.

[15] Rocha-Caridi A., Vacuum vector representations of the Virasoro algebra, MSRI Publ., 3 (1985), 451-473.

[16] Schwarz J.H., Superstring theory, Physics Rep. 83 (1982), 223-322.

[17] Thorn C.B., Computing the Kac determinant using dual model techniques and more about the no-ghost theorem, Nucl. Phys. B248 (1984), 551-569.

[18] Wakimoto M., Basic representations of extended affine Lie algebras, RIMS-Kokyuroku 503 (1983), 36-46. Affine Lie algebras and the Virasoro algebra I (1984), preprint.

[19] Kac V.G., Peterson D.H., Regular functions on certain infinite dimensional groups, in Arithmetics and Geometry, 141-166, Progress in Math. 36, Birkhaüser, Boston, 1983.

[20] Wakimoto M., Two formulae for specialized characters of Kac-Moody Lie algebras (1983), preprint.