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SOME REMARKS ON REPRESENTATIONS OF QUIVERS AND INFINITE ROOT SYSTEMS

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This is an addendum to my paper [4]. The purpose of it is to give simpler proofs of the main results of [4] in a more general situation. In [4] properties of the infinite root systems are used in the representation theory of quivers. Here properties of the root systems (and their existence, which in [4] is deduced from the theory of the Kac-Moody Lie algebras) are obtained in the framework of the representation theory of quivers. We do not exclude edges-loops from our consideration. This makes us introduce a more general notion of infinite root system than the one in [4].

In the remainder of the article some remarks on related topics are made and some open problems are discussed. They include:

a) an "abstract" definition of an infinite root system (i.e., a definition which does not depend on the basis);

b) multiplicities of roots and ζ-functions of quivers;

c) a connection with the problem of classification of prehomogeneous linear groups.

We keep the notations of [4]. The base field $\mathbb{F}$ is arbitrary unless otherwise stated.

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1. (Generalized) infinite root systems.

An $(n \times n)$ square matrix $A = (a_{ij})$ with integral entries is called a (generalized) Cartan matrix if

(C1) $a_{ii} \leq 2$ and even;

(C2) $a_{ij} \leq 0$ for $i \neq j$;

(C3) $a_{ij} = 0$ implies $a_{ji} = 0$, $i,j = 1,\ldots,n$.

Notice that Lemmas 1.2 and 1.3 of [4] hold in this more general situation. The lists of Cartan matrices of positive and zero type is almost the same as Tables P and Z in [4]: one should only add to Table Z the (1×1) zero matrix which we denote by $A_0^{(1)}$. The Dynkin diagram of a Cartan matrix A is defined in the same way as in [4] with additional $\frac{1}{2}(2 - a_{ii})$ edges-loops to a vertex p_i.

Let A be a (generalized) Cartan $(n \times n)$-matrix, let Γ be a free abelian group with free generators $\alpha_1,\ldots,\alpha_n$ and let Γ_+ be the set of all non-zero elements in Γ of the form $\alpha = k_1\alpha_1 + \ldots + k_n\alpha_n$ with $k_i \geq 0$, $i = 1,\ldots,n$. For $\alpha = \Sigma k_i\alpha_i \in \Gamma$ we call the support of α the subdiagram of the Dynkin diagram of A, consisting of those vertices p_i, for which $k_i \neq 0$, and all the edges joining these vertices.

The set $\Pi = \{\alpha_i | a_{ii} = 2\}$ is called the set of simple roots. We define the positive root system $\Delta_+ = \Delta_+(A)$, associated with A, by the properties:

(R1) $\{\alpha_1, \ldots, \alpha_n\} \subset \Delta_+ \subset \Gamma_+$; $2\alpha_i \notin \Gamma_+$ if $\alpha_i \in \Pi$;

(R2) if $\alpha = \sum k_j \alpha_j \in \Delta_+$, $\alpha_i \in \Pi$ and $\alpha \neq \alpha_i$, then $\alpha + k\alpha_i \in \Delta_+$ if and only if $-p \leq k \leq q$, $k \in \mathbb{Z}$, where p and q are some non-negative integers satisfying $p - q = \sum_j a_{ij} k_j$;

(R3) if $\alpha \in \Delta_+ \setminus \Pi$, then $\alpha - \alpha_i \in \Delta_+$ for some $\alpha_i \in \Pi$.[1)]

(R4) if $\alpha \in \Delta_+$, $\alpha_i \notin \Pi$ and the vertex p_i is joined by an edge with a vertex from the support of α, then $\alpha + \alpha_i \in \Delta_+$.

The set $\Delta = \Delta_+ \cup (-\Delta_+)$ is called the root system. For $\alpha_i \in \Pi$ we define a reflection r_i by

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \; j = 1, \ldots, n,$$

and call the group generated by all these reflections the Weyl group. We call the fundamental set the following subset in Γ_+:

$$K = \{\alpha = \sum_j k_j \alpha_j \in \Gamma_+ | \sum_j a_{ij} k_j \leq 0 \text{ if } \alpha_i \in \Pi; \text{ support } \alpha \text{ is connected}\}.$$

Notice that properties (R1) - (R3) define Δ_+ uniquely; the existence and other properties of Δ_+ will be deduced from the representation theory of quivers.

We call $\alpha \in \Delta$ a nil root if the support of α is one of the diagrams of zero type and $\alpha = k\sum_i a_i \alpha_i$, a_i's being the labels of the Dynkin diagram ($a_1 = 1$ for $A_0^{(1)}$), and $k \in \mathbb{Z} \setminus \{0\}$.

1) In [4], p. 63 and 69, I missed (R3). I am grateful to J. Morita, who pointed out on this.

Note that the set Δ is W-invariant. The roots from $\Delta^{re} = \bigcup_{w \in W} w(\Pi)$ are called real roots and from $\Delta^{im} = \Delta \setminus \Delta^{re}$ are called imaginary roots.

2. Dimensions of indecomposable representations of quivers.

We recall that a quiver is an oriented graph (S,Ω) (we admit edges-loops), where S is a connected graph with n vertices $S_o = \{p_1,\ldots,p_n\}$ and Ω is an orientation of S. Denote by S_1 the set of edges of S. We associate with S a symmetric Cartan matrix $A = (a_{ij})$ as follows: $-a_{ij}$ is the number of edges, connecting p_i and p_j in S if $i \neq j$ and $a_{ii} = 2 - 2\#$ (loops-edges in p_i), $i,j = 1,\ldots,n$. This is a bijection between the finite connected graphs and the indecomposable symmetric (generalized) Cartan matrices, S being the Dynkin diagram of A. We define a bilinear form (,) on Γ by $(\alpha_i,\alpha_j) = \frac{1}{2}a_{ij}$. This form is W-invariant. It is also clear that $(\alpha,\alpha) \leq 0$ for $\alpha \in K$.

We recall the definition of the category $\mathcal{M}(S,\Omega)$. An object is a collection (U,φ) of finite-dimensional vector spaces U_p, $p \in S_o$, and linear maps $\varphi_\ell : U_{i(\ell)} \to U_{f(\ell)}$ for any size $\ell \in S_1$ ($i(\ell)$ and $f(\ell)$ denote the initial and finite vertices of the oriented edge ℓ). A morphism $\Psi: (U,\varphi) \to (U',\varphi')$ is a collection of linear maps $\Psi_p : U_p \to U_p'$, $p \in S_o$, such that $\Psi_{f(\ell)}\varphi_\ell = \varphi_\ell'\Psi_{i(\ell)}$. A class of equivalence of isomorphic objects of $\mathcal{M}(S,\Omega)$ is called a representation of the quiver (S,Ω). The element $\sum_i (\dim U_{p_i})\alpha_i \in \Gamma_+$ is called the dimension of the representation.

Denote by $d(S,\Omega)$ the set of dimensions of indecomposable representations of the quiver (S,Ω). The problem we are concerned with is to describe this set.

The following lemma is trivial.

Lemma 1. *The set $d(S,\Omega)$ satisfies the properties (R1) and (R3) of a positive root system. Any $\alpha \in d(S,\Omega)$ has a connected support.*

Lemma 2. *Suppose that $\mathbb{F}$ is infinite. Then the set $d(S,\Omega)$ contains the fundamental set K. Moreover, if $\alpha \in K$ is not a nil root and U is a representation of dimension α with minimal possible dimension of End U, then U is absolutely indecomposable; if char $\mathbb{F} = 0$, then End $U = \mathbb{F}$. In particular,[1] $\mu_\alpha \geq 1 - (\alpha,\alpha)$.*

Proof is exactly the same as that of Lemmas 2.5 and 2.7 in [4]. The only additional remark we need is that $\sum_j a_{ij}k_j \leq 0$ if $\alpha_i \notin \Pi$ and $\alpha = \sum_j k_j\alpha_j \in \Gamma_+$.

The following lemma follows from the existence of a reflection functor in the case of an admissible vertex p_i of (S,Ω) (i.e., a source or a sink).

Lemma 3. *If p_i is an admissible vertex of the quiver (S,Ω) and $\alpha \in d(S,\Omega)$, $\alpha \neq \alpha_i$, then[2] $r_i(\alpha) \in d(S,\tilde{r}_i(\Omega))$. Moreover, $\mu_\alpha = \mu_{r_i(\alpha)}$ and in the case of a finite base field $\mathbb{F}$ the numbers of indecomposable (or absolutely indecomposable) representations of dimensions α and $r_i(\alpha)$ are equal.*

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[1] μ_α is the "number of parameters" of the set of indecomposable representations of dimension α of the quiver (S,Ω) (see [4] for a precise definition).

[2] $\tilde{r}_i(\Omega)$ is an orientation of the graph S obtained from Ω by reversing the direction of arrows along all the edges containing p_i.

Lemma 4. Provided that $\mathbb{F}$ is algebraically closed, the set $d(S,\Omega)$ does not depend on the orientation Ω of the graph S; moreover, μ_α does not depend on Ω. In the case of a finite base field $\mathbb{F}$ the number of indecomposable (or absolutely indecomposable) representations of dimension α does not depend on the orientation Ω.

Proof. Let $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in \Gamma_+$ and $V_1, \ldots, V_n$ be vector spaces of dimensions $k_1, \ldots, k_n$. Recall that the classification of the representations of a quiver (S,Ω) is equivalent to the classification of the orbits of the linear group $G^\alpha(\mathbb{F}) = GL_{k_1}(\mathbb{F}) \times \ldots \times GL_{k_n}(\mathbb{F})$ operating in the space

$$(1) \qquad \mathfrak{M}^\alpha(S,\Omega) = \bigoplus_{\ell \in S_1} \operatorname{Hom}_{\mathbb{F}} (V_{i(\ell)}, V_{f(\ell)}).$$

The reversing of the direction of an arrow of the quiver (S,Ω) gives a new quiver (S,Ω_1) and is equivalent to the replacement of the corresponding summand in (1) by a contragredient representation of the group G^α.

Suppose now that $\mathbb{F}$ is a finite field. Recall that by a theorem of Brauer, for any linear finite group G operating in a vector space $V \simeq \mathbb{F}^k$ the numbers of orbits in V and V* are equal (see [4], Lemma 2.10 for the proof). This implies that if $U \simeq \mathbb{F}^m$ is the space of another representation of G, then the numbers of orbits in $U \oplus V$ and $U \oplus V^*$ are equal (one should apply the Brauer theorem to all the linear groups G_x, $x \in U$, operating in V and V*).

These two remarks imply immediately that the number of all representations of dimension α does not depend on the orientation of the quiver.

Now we obtain immediately by induction on the height α that the number of indecomposable (over the finite field $\mathbb{F}$) representations of dimension α does not depend on Ω (we use the uniqueness of the decomposition of a representation into direct sum of indecomposable representations).

The fact that the number of absolutely indecomposable representations of dimension α does not depend on Ω is also proven by induction on height α for any finite field $\mathbb{F}$. The proof is more delicate. It uses the fact that any indecomposable representation over $\mathbb{F}$ is an essentially unique absolutely indecomposable representation over a bigger finite field $\mathbb{F}' \supset \mathbb{F}$, considered over $\mathbb{F}$. The details can be found in Appendix to [4].

The fact that $d(S,\Omega)$ and μ_α do not depend on Ω follows from the preceding result by the following

Proposition 1. Let A be a finite dimensional algebra and α be an element from the Grotendique ring $K_o(A)$. If the base field is $\mathbb{F}_q$, $q = p^s$, then the number $m_t^\alpha(A)$ of absolutely indecomposable representations of A of "dimension" α over field $\mathbb{F}_{q^t}$ is given by the following formula:

$$(2) \quad m_t^\alpha(A) = rq^{Nt} + \lambda_2^t + \dots + \lambda_k^t - \mu_1^t - \dots - \mu_s^t,$$

where r and N are positive integers and $\lambda_2,\dots,\mu_s$ are complex numbers (not depending on t) such that $|\lambda_i|,|\mu_j|$ are non-negative half-integral powers of q smaller than q^N. The number N is equal to the number of parameters and r to the number of irreducible components of maximal dimension of the set of indecomposable representations of A over an algebraically closed field of characteristic p.

If the base field $\mathbb{F}$ is algebraically closed and of characteristic 0, then for all but a finite number of primes p for a reduction mod P the numbers N and r are again the number of parameters and number of irreducible components of maximal dimension of the set of indecomposable representations of A.

Proof. The set of representations of A of dimension α is the set of orbits of an algebraic group G operating on an algebraic variety M, the subset of absolutely indecomposable representations being a constructible G-invariant subset $X \subset M$.

By Rosenblicht's theorem, we can represent X as a union of G-invariant algebraic varieties $X = \bigcup_{i=1}^{s} X_i$, such that each X_i/G is again an algebraic variety.

Since G_x is connected for any $x \in M$ (as the group of units in the endomorphism ring), we obtain bijections between the set of $G(\mathbb{F}_q)$-rational orbits on $M(\mathbb{F}_q)$, the set of $\mathbb{F}_q$-rational points on $\bigcup_i X_i/G$ and the set of absolutely indecomposable representations defined over $\mathbb{F}_q$ (see Appendix to [4] for details).

Recent general results of Deligne [9] give now formula (2). A standard reduction mod P argument proves the last statement.

An immediate consequence of Lemmas 3 and 4 is:

Lemma 5. Suppose that the base field $\mathbb{F}$ is finite or algebraically closed. Then the set $d(S,\Omega)\setminus\{\alpha_i\}$ is r_i-invariant (and, therefore, $d(S,\Omega) \cup (-d(S,\Omega))$ is W-invariant). Moreover, over a finite base field the numbers of indecomposable (or absolutely indecomposable) representations of dimension α and $w(\alpha)$, $w \in W$, are equal;

over an algebraically closed field one has: $\mu_\alpha = \mu_{w(\alpha)}$, $w \in W$.

Now we are able to prove the final:

Lemma 6. For an algebraically closed base field, the set $d(S,\Omega)$ is exactly the set of positive roots $\Delta_+(A)$, where A is the Cartan matrix of the graph S.

Proof. We will prove that the set $d = d(S,\Omega)$ satisfies properties (R1)-(R3) of $\Delta_+ = \Delta_+(A)$. The properties (R1) and (R3) of Δ_+ are satisfied by Lemma 1. By Lemma 5, $\Delta_+^{re} \subset d$ and since the support of any $\alpha \in d$ is connected we obtain that $d = \Delta_+^{re} \cup (\bigcup_{w \in W} w(K))$, where K is the fundamental set (since for any $\alpha \in d \setminus (K \cup \{\alpha_1,\dots,\alpha_n\})$ there is a reflection r_i such that height $r_i(\alpha)$ < height α).

Now we prove (R2) for any $\alpha \in d$. If $\alpha = \alpha_j$, this property obviously holds. Therefore, this property holds also for any root $\alpha \in \Delta_+^{re} \subset d$. If $\alpha \in d \setminus \Delta_+^{re}$, then $\alpha \in M = \bigcup_{w \in W} w(K)$. I claim that the set M is convex (i.e., if $\beta,\gamma \in M$, then any $\delta \in [\beta,\gamma] \cap \Gamma$ also lies in M). Indeed, let $\hat{M}$ and $\hat{K}$ be the open kernels of the convex hulls of M and K in the space $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$; $\hat{M}$ is a convex cone. We introduce the canonical Riemanian metric on $\hat{M}$ (see e.g. [10]). This metric is W-invariant and W operates discretely on the Riemanian manifold M since W is a discrete subgroup in GL(V)). Therefore, any segment $[\alpha,w(\alpha)]$, $\alpha \in \hat{M}$, $w \in W$, intersects only a finite number of hyperplanes of reflections, say, $r_{\beta_1},\dots,r_{\beta_s} \in W$. But then $[\alpha,w(\alpha)] \subset \bigcup_i r_{\beta_i} \dots r_{\beta_1}\hat{K}$. Clearly, this implies that M is convex.

So (R2) is satisfied for any $\alpha \in M$, which completes the proof of the Lemma.

We summarize the obtained results in the following two theorems (cf. [4]).

Theorem 1. *Let* (S,Ω) *be a quiver and let the base field* $\mathbb{F}$ *be a finite field* $\mathbb{F}_q$. *For* $\alpha \in \Gamma_+$ *let* $m_t^\alpha(S,\Omega)$ *denote the number of absolutely indecomposable and* $\overline{m}_t^\alpha(S,\Omega)$ *denote the number of indecomposable representations of* (S,Ω) *of dimension* α *defined over* $\mathbb{F}_{q^t}$. *Then*

a) $m_t^\alpha(S,\Omega)$ *and* $\overline{m}_t^\alpha(S,\Omega)$ *do not depend on the orientation* Ω *of* S *and the action of* W *on* α.

b) *For* $\alpha \notin \Delta_+$ *there is no indecomposable representations of* (S,Ω) *of dimension* α.

c) *For* $\alpha \in \Delta_+^{re}$ *there exists a unique indecomposable representation of* (S,Ω) *of dimension* α *which is absolutely indecomposable and is defined over the prime field.*

d) *For* $\alpha \in \Delta_+^{im}$ *there exists complex numbers* $\lambda_2,\ldots,\lambda_k,\mu_1,\ldots,\mu_s$ *(depending on* α *but not on t) and positive integers* N *and* r *such that* $|\lambda_i|,|\mu_j|$ *are non-negative half-integral powers of* q *smaller than* q^N, $N \geq 1-(\alpha,\alpha)$ *and*

$$(3) \quad m_t^\alpha(S,\Omega) = rq^{Nt} + \lambda_2^t + \ldots + \lambda_k^t - \mu_1^t - \ldots - \mu_s^t .$$

Analogous formula takes place for $\overline{m}_t^\alpha(S,\Omega)$. *One has:* $m_t^\alpha(S,\Omega) = \overline{m}_t^\alpha(S,\Omega)$ *for a non-divisible* α.

Theorem 2. *Let* (S,Ω) *be a quiver and let the base field* $\mathbb{F}$ *be algebraically closed. Let* $\Delta_+ = \Delta_+(A)$ *be the positive root system, where* A *is the Cartan matrix of the graph* S. *Then*

a) *For* $\alpha \in \Gamma_+$, α *is a dimension of an indecomposable representation of the quiver* (S,Ω) *if and only if* $\alpha \in \Delta_+$.

b) For $\alpha \in \Delta_+^{re}$ there exists a unique indecomposable representation of (S,Ω) of dimension α.

c) For $\alpha \in \Delta_+^{im}$ there exists an infinite number of indecomposable representations of (S,Ω) of dimension α. Moreover, the number of parameters of the set of indecomposable representations of dimension α is at least $1 - (\alpha,\alpha) > 0$ and does not depend on Ω and the action of W.

3. Further remarks.

a) Infinite root systems. An immediate consequence of the results of sec. 2 is

Proposition 2 (cf. [4]). Let A be a symmetric square matrix with integral entries, satisfying condition (C1)-(C3) of sec. 1. Then the associated positive root system Δ_+ (satisfying the properties (R1)-(R3)) exists. Moreover, $\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$, where $\Delta_+^{re} = \bigcup_{w \in W} (w(\Pi) \cap \Gamma_+)$ and $\Delta_+^{im} = \bigcup_{w \in W} w(K)$.

Remark. The statement that in the case of a Cartan matrix, associated with a graph without loops, any element from K is a root appears in [5] (see Theorem ; however, it seems that there is a gap in the proof of the crucial Proposition 3 - in the case k = 1).

The results of sec. 2 can be extended to the case of species (see [2], [1] for definitions) when the base field is finite. In particular, this gives a generalization of Proposition 2 for a symmetrisable A. For an arbitrary field the reduction mod p argument does not work and I can extend the results of sec. 2 only modulo the following conjecture (cf. [4]).

Conjecture (*). Let G be a linear algebraic group operating in a vector space V defined over a field $\mathbb{F}$ of characteristic 0. Then the cardinalities of the sets of the orbits with a unipotent stabilizer (or with a stabilizer such that its maximal split torus is trivial) of the group G in V and V* and the number of parameters of these sets are equal.

Now I would like to give an "abstract" definition of an (ordinary) infinite root system. Let Γ be a full lattice in a real vector space V. We recall that a reflection in a vector $\alpha \in V$ is an automorphism Γ_α of V such that its fixed point set has codimension 1, $\Gamma_\alpha(\alpha) = -\alpha$ and $\Gamma_\alpha(\Gamma) = \Gamma$.

Let Δ be a subset in Γ { }; we denote by Δ^{re} the set of vectors from Δ in which there exists a reflection preserving Δ and by W the group generated by all the reflections in vectors from Δ. The set Δ is called a root system (in general infinite) if the following conditions are satisfied:

(i) Γ is the $\mathbb{Z}$-span of Δ^{re};

(ii) For any $\beta \in \Delta$ and $w \in W$ all the points of Γ which lie on the segment $[\beta, w(\beta)]$ belong to Δ;

(iii) For $\beta \in \Delta \backslash \Delta^{re}$ the set $W(\beta)$ lies in an open half-space.

This definition includes non-reduced root systems (i.e., some of $2\alpha_i$'s may lie in Δ) which naturally appear in Lie superalgebras (see [3]), but I do not know whether they are related to representations of graphs.

Note also that one can easily show that for a finite Δ this definition is equivalent to a usual definition of a finite root system [8].

For simplicity we excluded from the abstract definition of root systems the case when the graph contains an edge-loop (see sec. 1). One can see from sec. 1 and 2 that they are also important. One can define infinite dimensional Lie algebras $\mathfrak{g}(A)$, associated with Cartan matrices introduced in sec. 1. The root system of $\mathfrak{g}(A)$ is then the system Δ. One can also define highest weight representations for these Lie algebras and prove the character formula (cf. [3]). In the simplest new case of the (1 x 1) zero matrix A the Lie algebra $\mathfrak{g}(A)$ is the infinite Heisenberg algebra.

b) <u>Representations of quivers over non-closed fields.</u>

As was mentioned in a), all the results of sec. 2 can be proven for an arbitrary base field $\mathbb{F}$ modulo conjecture (*).

The first open question is: for a root $\alpha \in \Delta_+^{re}$ is it true that the unique indecomposable representation of dimension α is defined over the prime field (this is proven in sec. 2 only in the case of fields of non-zero characteristic). It would be also interesting to give an explicit construction of these representations. Ringel has done it in [6] in the rank 2 case in terms of some generalized reflection functions.

It is easy to show that if there exists an indecomposable representation over $\mathbb{F}$ of dimension α, then either $\alpha \in \Delta_+^{im}$, or $\alpha = k\beta$, where $\beta \in \Delta_+^{re}$; if, moreover, the Brauer group of $\mathbb{F}$ is trivial, then $\alpha \in \Delta_+$.

Of course, all the results of sec. 2 would be extended to an arbitrary field $\mathbb{F}$ if one proves that the set $d(S,\Omega)$ does not depend on Ω over $\mathbb{F}$.

c) <u>ζ-function of a finite dimensional algebra.</u>

Let A be a finite-dimensional algebra over $\mathbb{F}_q$ and α be an element from $K_0(A)$. Denote by $m_n^{\alpha}(A)$ the number of absolutely indecomposable representations of A of "dimension" α defined over field $\mathbb{F}_{q^n}$. We set

$$\Phi_{A,\alpha}(z) = \sum_{n\geq 1} \frac{1}{n} m_n^{\alpha}(A)\, z^n$$

and define a ζ-function

$$\zeta_{A,\alpha}(z) = \exp \Phi_{A,\alpha}(z).$$

From (2) we obtain that

$$\zeta_{A,\alpha}(z) = \frac{\prod_{i=1}^{s} (1 - \mu_i z)}{(1 - q^N z)^r \prod_{i=2}^{k} (1 - \lambda_i z)}.$$

In the case of a quiver S conjecture 1 from Appendix in [4] about the multiplicity m_α of a root α can be stated as follows:

$$m_\alpha = \oint \Phi_{s,\alpha}(z)\, dz$$

where the contour of integration is any circle with the radius less than 1 and the center in 0. If Conjecture 3 from [4] is true, then Conjecture 1 can be stated as follows: m_α = multiplicity of the pole of $\zeta_{S,\alpha}(z)$ in

 $z = 1$.

d) A connection with prehomogeneous linear groups.

A prehomogeneous linear algebraic group G operating in a vector space V is a linear group, admitting dense orbit in V. For irreducible representations these groups have been classified in [7]. An essential (and the most difficult) part of the case of general reductive groups is to classify the linear groups $G^\alpha = GL_{k_1} \times \ldots \times GL_{k_n}$ operating in $\mathfrak{M}^\alpha(S,\Omega) = \bigoplus_{\ell \in S_1} \mathrm{Hom}_{\mathbb{F}}(V_{i(\ell)}, V_{f(\ell)})$, associated with a quiver (S,Ω) and $\alpha = \Sigma\, k_i\alpha_i \in \Gamma_+$, which are prehomogeneous. Of course, a necessary condition is that $(\alpha,\alpha) \geq 1$.

Let S be a connected graph. Let $\alpha \in \Gamma_+$ and let Ω be an orientation of S. Denote by (a) the following procedure: we take an admissible vertex $p_i \in S_o$ and replace α by $r_i(\alpha) + s\alpha_i$, where s is the minimal non-negative integer such that $r_i(\alpha) + s\alpha_i \in \Gamma_+$, and replace Ω by $\tilde{r}_i(\Omega)$. Denote by (b) the following procedure: we take $\ell_o \in S_1$ such that for the "generic" stabilizer H of G^α in $\bigoplus_{\ell \neq \ell_o} \mathrm{Hom}_{\mathbb{F}}(V_{i(\ell)}, V_{f(\ell)})$ the maximal dimensions of H-orbits in $\mathrm{Hom}_{\mathbb{F}}(V_{i(\ell_o)}, V_{f(\ell_o)})$ and the dual are equal, and reverse the direction of the edge ℓ_o (one has this situation, for instance, when H is reductive). Denote by $D(S,\Omega)$ (or $D_1(S,\Omega)$) the subset of those $\alpha \in \Gamma_+$ which can be transformed to 0 by iteration of the procedures (a) and (b) (resp. (a)). Clearly, if $\alpha \in D(S,\Omega)$, then G^α has a dense orbit in $\mathfrak{M}^\alpha(S,\Omega)$. It seems that the following should be true.

Conjecture. G^α has a dense orbit in $\mathfrak{M}^\alpha(S,\Omega)$ if and only if $\alpha \in D(S,\Omega)$.

Remark. I have conjectured in [4] that if G^{α} has a dense orbit in $\mathfrak{M}^{\alpha}(S,\Omega)$, then $\alpha \in D_1(S,\Omega)$. Ringel has constructed a counterexample to this conjecture. His quiver is: $0 \rightrightarrows 0 \rightarrow 0$ and $\alpha = 3\alpha_1 + 6\alpha_2 + \alpha_3$. It is easy to see that $\alpha \in D(S,\Omega)$ but $\alpha \notin D_1(S,\Omega)$.

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