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REPRESENTATIONS OF QUANTUM GROUPS AT ROOTS OF 1: REDUCTION TO THE EXCEPTIONAL CASE

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ABSTRACT

This paper is a continuation of the papers [DC–K] and [DC–K–P] on representations of quantum groups at roots of 1. Here we show that an irreducible representation of a quantum group at an odd root of 1 can be uniquely induced from an exceptional representation of a smaller quantum group. This reduces the classification of representations, the calculation of their characters and dimensions, etc, to the exceptional case.

§1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, let $\mathfrak{h}$ be its Cartan subalgebra, let $R \subset \mathfrak{h}^*$ be the set of roots, let $Q = \mathbb{Z}R$ be the root lattice, and let $W \subset Aut\ \mathfrak{h}^*$ be the Weyl group. Choose a subset of positive roots $R^+ \subset R$, let $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset R^+$ be the set of simple roots and let $s_1, \dots, s_n$ be the corresponding simple reflections generating W. Let $(.|.)$ be a W-invariant bilinear form on $\mathfrak{h}^*$ normalized by the condition that the square length of a short root equals 2. Then

$$(\alpha_i|\alpha_j) = d_i a_{ij}, \qquad i, j = 1, \dots, n,$$

where $d_1, \dots, d_n$ are relatively prime positive integers and (a_{ij}) is the Cartan matrix of $\mathfrak{g}$.

Recall that connected Lie groups with Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with lattices M containing Q such that $(\lambda|d_j^{-1}\alpha_j) \in \mathbb{Z}$ for all $j = 1, \dots, n$. Fix such a lattice M and let G be the corresponding connected Lie group (so that Center $G = M/Q$).

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Fix an odd positive integer l greater then $d := \max_j 2d_j$, and let ε be a primitive l'th root of 1.

§2. Recall that the "quantum group at ε" is the associative algebra $U = U_{M,\varepsilon}(\mathfrak{g})$ over $\mathbb{C}$ on generators $E_i, F_i (i = 1, \dots, n), K_\alpha (\alpha \in M)$ and the following defining relations $(\alpha, \beta \in M, i, j = 1, \dots, n)$:

(2.1) $K_\alpha K_\beta = K_{\alpha+\beta}, K_0 = 1,$

(2.2) $K_\alpha E_i K_{-\alpha} = \varepsilon^{(\alpha|\alpha_i)} E_i, K_\alpha F_i K_{-\alpha} = \varepsilon^{-(\alpha|\alpha_i)} F_i,$

(2.3) $E_i F_j - F_j E_i = \delta_{ij}(K_{\alpha_i} - K_{-\alpha_i})/(\varepsilon^{d_i} - \varepsilon^{-d_i}),$

(2.4) certain Chevalley-Serre type relations between the E_i and between the F_i (see e.g. [L] or [DC-K, (1.2.4 and 5)]).

Let ω be a conjugate-linear anti-automorphism of U defined by: $\omega E_i = F_i, \omega F_i = E_i, \omega K_\alpha = K_{-\alpha}$.

Let U^+, U^- and U^0 be the subalgebras of U generated by the E_i, by the $F_i (i = 1, \dots, n)$ and by the $K_\alpha (\alpha \epsilon M)$ respectively. Then multiplication defines a $\mathbb{C}$-vector space isomorphism [R]

$$U = U^- \otimes U^0 \otimes U^+. \tag{2.5}$$

§3. Recall that the braid group $\mathcal{B}_W$ (associated to W) acts by automorphisms of U defined by [L] $(i = 1, \dots, n)$:

$$T_i K_\alpha = K_{s_i(\alpha)},$$

$$T_i E_i = -F_i K_i, \quad T_i E_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} \varepsilon_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \text{ if } i \neq j,$$

$$T_i F_i = -K_i^{-1} E_i, \quad T_i F_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} \varepsilon_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \text{ if } i \neq j.$$

Here and further $E_i^{(a)}$ and $F_i^{(a)}$ stand for $E_i^a/[a]_{d_i}!$ and $F_i^a/[a]_{d_i}!$, where $[a]_d! = [a]_d[a-1]_d \dots [1]_d$ and $[a]_d = (\varepsilon^{da} - \varepsilon^{-da})/(\varepsilon^d - \varepsilon^{-d})$. Note that $T_i\omega = \omega T_i$.

Choosing a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_N}$ of the longest element of $W (N = |R^+|)$, we get a total ordering of R^+:

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \dots, \beta_N = s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N},$$

and the corresponding root vectors $(k = 1, \dots, N)$:

$$E_{\beta_k} = T_{i_1} \dots T_{i_{k-1}} E_{i_k}, \quad F_{\beta_k} = T_{i_1} \dots T_{i_{k-1}} F_{i_k} = \omega E_{\beta_k}$$

(they depend on the choice of the reduced expression).

For $k = (k_1, \dots, k_N) \in \mathbb{Z}_+^N$ we let

$$E^k = E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N}, \quad F^k = \omega E^k.$$

Lemma 3.1. *[L] (a) Elements E^k (resp F^k), $k \in \mathbb{Z}_+^N$, form a basis of U^+ (resp. U^-) over $\mathbb{C}$.*

(b) Elements $F^k K_\alpha E^r$, where $k, r \in \mathbb{Z}_+^N$, $\alpha \in M$, form a basis of U over $\mathbb{C}$. □

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Lemma 3.2. *[L–S] For $i < j$ one has:*

$$E_{\beta_j}E_{\beta_i} - \varepsilon^{(\beta_i|\beta_j)}E_{\beta_i}E_{\beta_j} = \sum_{k\in\mathbb{Z}_+^N} c_k E^k, \tag{3.1}$$

where $c_k \in \mathbb{C}$ and $c_k \neq 0$ only when $k = (k_1, \ldots, k_N)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$. □

§4. Let Z denote the center of the algebra U.

Lemma 4.1. *[DC–K] Elements E_α^l, F_α^l and K_β^l ($\alpha \in R^+, \beta \in M$) lie in Z.* □

Let Z_0 (resp. Z_0^0 or Z_0^+ or Z_0^-) be the subalgebra of Z generated by all the elements E_α^l, F_α^l and K_β^l (resp. K_β^l or E_α^l or F_α^l). By (2.5) we have:

$$Z_0 = Z_0^- \otimes Z_0^0 \otimes Z_0^+.$$

Now Lemma 3.1 implies

Lemma 4.2. *[DC–K] The algebra U is a free Z_0–module on the basis $\{F^k K_\alpha E^r\}$, where $k = (k_1, \ldots, k_N)$ and $r = (r_1, \ldots, r_N)$ are such that $0 \leq k_i < l, 0 \leq r_i < l$ and α runs over a basis of M mod lM.* □

Given a homomorphism $\chi : Z_0 \to \mathbb{C}$, let

$$U_\chi = U/(z - \chi(z), \text{ where } z \in Z_0).$$

Corollary 4.1. *U_χ is an algebra of dimension $l^{\dim \mathfrak{g}}$ with a basis over $\mathbb{C}$ described by Lemma 4.2.* □

§5. Let $\mathcal{A}$ be the algebra of rational functions in q that have no poles at ε. Let $U_\mathcal{A}$ be the algebra over $\mathcal{A}$ on generators E_i, F_i and K_α and defining relations (2.1)-(2.4) where ε is replaced by q, so that $U = U_\mathcal{A}/(q-\varepsilon)$. Suppose that we have an element $b \in U_\mathcal{A}$ with the property that $[b, a] \in (q-\varepsilon)U_\mathcal{A}$ for all $a \in U_\mathcal{A}$. Then of course the image of b in U is central. Moreover one can also define a derivation P_b of U by

$$P_b(a) = (q-\varepsilon)^{-1}[b, \hat{a}] \mod(q-\varepsilon),$$

where $\hat{a}$ is a preimage of a in $U_\mathcal{A}$. In particular, we have derivations e_i and f_i of U given by [DC-K] (in a slightly different normalization):

$$e_i = P_{E_i^l}, \quad f_i = P_{F_i^l}.$$

It was shown in [DC-K] that the series $\exp te_i$ and $\exp tf_i (t \in \mathbb{C})$ converge to analytic automorphisms of certain analytic completion $\hat{U}$ of the algebra U. Denote by $\tilde{G}$ the group of automorphisms of $\hat{U}$ generated by all these 1-parameter groups.

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The group $\tilde{G}$ leaves the completion of Z_0 invariant [DC-K]. Hence it acts on Spec Z_0 by $(\tilde{g}\chi)(z) = \chi(\tilde{g}^{-1}(z)), \tilde{g} \in \tilde{G}$, and we have an isomorphism of algebras:

$$\tilde{g} : U_\chi \xrightarrow{\sim} U_{\tilde{g}(\chi)}, \quad \tilde{g} \in \tilde{G}. \tag{5.1}$$

This induces a canonical bijection (for the definition of Spec see below)

$$\tilde{g} : \operatorname{Spec} U_\chi \to \operatorname{Spec} U_{\tilde{g}(\chi)}, \tag{5.2}$$

where $(\tilde{g}\sigma)(u) := \sigma(\tilde{g}^{-1}u), u \in U_\chi$.

§6. Let G' be the connected cover of G with fundamental group $\pi_1(G') = \pi_1(G)/\pi_1(G)^2$. Denote by Spec A the set of all equivalence classes of irreducible finite-dimensional representations of an algebra A. Recall that we have the following sequence of canonical maps:

$$\operatorname{Spec} U \xrightarrow{X} \operatorname{Spec} Z \xrightarrow{\tau} \operatorname{Spec} Z_0 \xrightarrow{\pi} G'. \tag{6.1}$$

Here X is the map of taking central characters, τ is the restriction map and π is a map constructed in [DC-K-P]. The maps X and τ are surjective, the map χ is bijective over a Zariski open dense subset of Spec Z and has finite fibers, the map τ is finite with fibers of order $\leq l^n$, which are explicitly described ([DC-K],[DC-K-P]). Note also that a representation $\sigma \in \operatorname{Spec} U$ with $\chi = X(\sigma)$ is actually a representation of the algebra U_χ.

In order to describe properties of the map π which will be needed in the sequel, introduce some notation. Let T (resp. T') be the maximal torus of G (resp. G') corresponding to $\mathfrak{h} \subset \mathfrak{g}$, and let N_- and N_+ be maximal unipotent subgroups of G' corresponding to $-R^+$ and R^+ respectively. We shall identify Spec Z_0^0 with T via the isomorphism $M \xrightarrow{\sim} lM$ given by multiplication by l. Recall that multiplication in G' defines a biregular isomorphism $N_- \times T' \times N_+ \xrightarrow{\sim} N_- T' N_+ = G'^0$, where G'^0 is a Zariski open dense subset of G' (called the big cell of G'). Given a conjugacy class $\mathcal{O}$ of G' we let $\mathcal{O}^0 = \mathcal{O} \cap G'^0$; this is a Zariski open dense subset of $\mathcal{O}$.

Lemma 6.1. *[DC-K-P] (a) We have:*

$$\pi = \pi^- \times \pi^0 \times \pi^+ : \quad \operatorname{Spec} Z_0^- \times \operatorname{Spec} Z_0^0 \times \operatorname{Spec} Z_0^+ \xrightarrow{\sim} N_- \times T' \times N_+ \simeq G'^0 \subset G',$$

where $\pi^\pm$: Spec $Z_0^\pm \to N_\pm$ is a biregular isomorphism and $\pi^0 : T \to T'$ is a homomorphism given by the square map.
(b) The set F of fixed points of $\tilde{G}$ in Spec Z_0 is $(\pi^0)^{-1}$ (Center G') $\subset T =$ Spec Z_0^0.
(c) If $\mathcal{O}$ is a conjugacy class of a non-central element of G', then $\pi^{-1}(\mathcal{O}^0)$ is a single $\tilde{G}$-orbit and (Spec $Z_0)\backslash F$ is a union of these G-orbits.
(d) If $\chi_- \in$ Spec Z_0^- and $\chi_0 \in$ Spec Z_0^0 are such that $\pi^-(\chi_-)$ and $\pi^0(\chi_0)$ are commuting elements of G' and $\chi_0(K_\alpha^{2l}) \neq 1$ for some $\alpha \in R^+$, then $\chi_-(F_\alpha^l) = 0$. □

§7. We call a semisimple element g of the algebraic group G' exceptional if its centralizer in G' has a finite center. All semisimple exceptional elements are classified by the following lemma which can be easily deduced from [K, Chapter 8]:

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Lemma 7.1. *(a) Let $\theta = \sum_{i=1}^{n} a_i\alpha_i$ be the highest root in R^+. Define elements $\omega_m^\vee \in \mathfrak{h}$ $(m = 1, \cdots, n)$ by*

$$\langle \alpha_j, \omega_m^\vee \rangle = \delta_{jm}, \qquad j = 1, \ldots, n.$$

Then elements $s_m := \exp(2\pi i \omega_m^\vee / a_m) \in T' \subset G'$ and $s_0 = 1$ are exceptional semisimple elements and any exceptional semisimple element is conjugate to one of the $s_m (m = 0, 1, \cdots, n)$.

(b) Up to multiplication by a central element the s_m give a complete non-redundant list of representatives of exceptional semisimple elements for the following m (the numbering of simple roots is taken from [K, Chapter 4]):

A_n	$m = 0$	E_6	$3 \le m \le 6$
B_n	$1 \le m \le n$	E_7	$3 \le m \le 7$
C_n	$0 \le m \le [n/2]$	E_8, F_4, G_2	$0 \le m \le n$
D_n	$0 \le m \le [(n-1)/2]$		

□

An element g of G' is called exceptional if its semisimple part is exceptional. In other words a complete set of representatives of conjugacy classes of exceptional elements is given by $\{s_m u\}$, where u are representatives of conjugacy classes of unipotent elements in the centralizer of the s_m. Note that the number of conjugacy classes of exceptional elements in G' is finite.

§8. Let $\varphi = \pi \circ \tau \circ X : \operatorname{Spec} U \to G'$ be the composition of maps of the sequence (6.1). A finite-dimensional irreducible representation of U is called exceptional if its image in G' under the map φ is an exceptional element.

Suppose now that σ is a non-exceptional finite-dimensional irreducible representation of the algebra U in a vector space V, and let $\chi = X(\sigma) \in \operatorname{Spec} Z$ so that $\sigma \in \operatorname{Spec} U_\chi$. Since the element $\varphi(\sigma)$ is not exceptional, its conjugacy class in G' contains an element g with the following properties:

(8.1) $$g_s \in T', g_u \in N_-,$$

where g_s and g_u denote the semisimple and unipotent parts of g;

(8.2) $$\mathfrak{h}_g := \text{Lie}(\text{center of Centralizer }_{G'}(g_s)) \neq 0;$$

(8.3) $$R' := \{\alpha \in R \mid \alpha \text{ vanishes on } \mathfrak{h}_g\} = M' \cap R,$$

where $M' = \mathbb{Z}\Pi'$ is a sublattice of M spanned by a subset Π' of Π different from Π.

By Lemma 6.1c, there exists an element $\tilde{g} \in \tilde{G}$ such that $\varphi(\tilde{g}(\sigma)) = g$. Replacing σ by $\tilde{g}(\sigma)$ and χ by $\tilde{g}(\chi)$, we may assume that σ is an irreducible representation of the algebra U_χ in the vector space V, such that $g := \varphi(\sigma)$ satisfies (8.1)–(8.3).

Let U' be the subalgebra of U generated by U^0 and all the elements E_i and F_i such that $\alpha_i \in \Pi'$, and let $U'_\chi = U'/(z - \chi(z)$, where $z \in Z_0 \cap U')$. Let $U^{\mathfrak{p}} = U'U^+$ and $U^{\mathfrak{p}}_\chi = U^{\mathfrak{p}}/(z - \chi(z)$, where $z \in Z_0 \cap U^{\mathfrak{p}})$ be the corresponding "parabolic" subalgebras.

Now we are in a position to state the main theorem (Theorem 2 from [W-K] may be viewed as an "infinitesimal" analogue of this theorem).

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Theorem. *(a) The U_χ-module V contains a unique irreducible $U_\chi^{\mathfrak{g}}$-submodule V', which is in fact a U'_χ-module.*
(b) The U_χ-module V is induced from the $U_\chi^{\mathfrak{g}}$-module V', i.e.

$$V = U_\chi \otimes_{U_\chi^{\mathfrak{g}}} V',$$

with the action of U_χ on V defined by left multiplication on U_χ. In particular, $\dim V = l^t \dim V'$, *where* $2t = |R \setminus R'|$.
(c) The map $V \to V'$ thus obtained establishes a bijection: $\operatorname{Spec} U_\chi \to \operatorname{Spec} U'_\chi$.

Remark 8.1. The representation of U'_χ in V' remains irreducible when restricted to the subalgebra U''_χ of U'_χ generated by the E_i and F_i such that $\alpha_i \in \Pi'$ and by the K_β such that $\beta \in M'$. This representation of U''_χ is in fact an exceptional representation of the quantum group $U_{M',\varepsilon}(\mathfrak{g}')$, where $\mathfrak{g}'$ is the subalgebra of $\mathfrak{g}$ generated by the Chevalley generators corresponding to $\alpha_i \in \Pi'$.

§9. The proof of this theorem is similar to that of Theorem 2 from [W-K] on irreducible representations of simple Lie algebras of characteristic p. It is based on several lemmas that we prove in this section.

Consider the root system R'. Let R'^+ be the corresponding subset of positive roots. Let w'_0 be a reduced expression of the longest element of the Weyl group W' of R'. We complete w'_0 to a reduced expression of the longest element of W:

(9.1) $$w_0 = w'_0 s_{i_1} \cdots s_{i_t}.$$

Let

(9.2) $$\gamma_1 = \alpha_{i_1}, \gamma_2 = s_{i_1}(\alpha_{i_2}), \ldots, \gamma_t = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}).$$

Let $R^+_{(k)} = s_{i_1} \cdots s_{i_k} R^+ (k = 1, \ldots, t)$.

Lemma 9.1. *(a)* $R^+ \setminus R'^+ = \{\gamma_1, \ldots, \gamma_t\}$.
(b) γ_k is a simple root of $R^+_{(k)}$ and $\gamma_j \in -R^+_{(k)}$ for $j < k$.

Proof. It is clear that $\{\gamma_1, \ldots, \gamma_t\} \subset R^+$ and that $w'_0\{\gamma_1, \ldots, \gamma_t\} \subset R^+$. This implies that $\{\gamma_1, \ldots, \gamma_s\} \subset R^+ \setminus R'^+$. Since these two sets have equal cardinality, this proves (a).

It is clear by definition that γ_k is simple in $R^+_{(k)}$. Since $(s_{i_1} \cdots s_{i_{k-1}})^{-1} s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$ $= s_{i_{k_1}} \cdots s_{i_j} \alpha_{i_j} \in -R^+$, (b) follows. □

Note that we have the following important properties of the γ_i :

(9.3) $$K^{2l}_{\gamma_i} \neq 1, \qquad i = 1, \ldots, t,$$

hence, by Lemma 6.1d,

(9.4) $$F^l_{\gamma_i} = 0, \qquad i = 1, \ldots, t.$$

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Let B be the subalgebra of U_χ generated by the K_α ($\alpha \in M$) and E_i ($i = 1, \ldots n$). Given $m \in \{1, 2, \ldots, n\}$, let P_m denote the subalgebra of U_χ generated by B and F_m. (In the sequel, we shall take $m = i_1$.) Taking a reduced expression of w_0 which starts with s_m, consider the corresponding root vectors $E_{\beta_1} = E_m, E_{\beta_2}, \ldots, E_{\beta_N}$. Denote by N_m the subalgebra of U_χ generated by $E_{\beta_2}, \ldots, E_{\beta_N}$ and let $\overline{N}_m$ be its 2-sided ideal generated by $E_{\beta_2}, \ldots, E_{\beta_N}$.

Lemma 9.2 *(a) $F_m E_\beta - \varepsilon^{(\alpha_m|\beta)} E_\beta F_m \in \overline{N}_m$ for $\beta = \beta_2, \ldots, \beta_N$.*
(b) N_m is independent of the choice of the reduced expression (which starts with s_m).

Proof. (a) follows from formula (3.1) for E_m and $E_{s_m(\beta)}$ by applying T_m to both sides.

In order to prove (b) suppose for example that $w_0 = w r_i r_j r_i w_1 = w r_j r_i r_j w_1$. Then the corresponding root vectors are respectively:

$$\{\ldots, T_w E_i, T_w T_i E_j, T_w T_i T_j E_i = T_w E_j, \ldots\},$$

$$\{\ldots, T_w E_j, T_w T_j E_i, T_w T_j T_i E_j = T_w E_i, \ldots\}.$$

Since $T_w(T_i E_j)$ lies in the subalgebra generated by $T_w E_i$ and $T_w E_j$, this proves (b). □

Let $B_{(1)} = B, N_{(1)} = N_m, \overline{N}_{(1)} = \overline{N}_m, P_{(1)} = P_m, F_{(1)} = F_m, K_{(1)} = K_m$, etc. For a $B_{(1)}$-module A, we let

$$A_{[1]} = \{a \in A | \overline{N}_{(1)} a = 0\}.$$

Lemma 9.3. *Let A be a $B_{(1)}$-module. Let $V = P_{(1)} \otimes_{B_{(1)}} A$ be the $P_{(1)}$-module induced from the $B_{(1)}$-module A. Then*
(a) $V_{[1]}$ is $P_{(1)}$-stable.
(b) $V_{[1]}$ lies in $\sum_{k=0}^{l-1} F_{(1)}^k A_{[1]}$.
(c) If $E_{(1)} A_{[1]} = 0$ and $K_{(1)}^{2l} \neq 1$, then any $P_{(1)}$-submodule C of $V_{[1]}$ intersects $A_{[1]}$ non-trivially.

Proof. (a) follows from Lemma 9.2a.

We shall write E and F in place of $E_{(1)}$ and $F_{(1)}$ to simplify notation. In order to prove (b), write $v \in V_{[1]}$ in the form:

$$v = \sum_{k=0}^{s} F^k x_k, \quad \text{where} \quad s \leq l - 1, x_k \in A.$$

If $\beta = \beta_2, \ldots, \beta_N$, we have:

$$0 = E_\beta v = E_\beta F^s x_s + \sum_{k=0}^{s-1} E_\alpha F^k x_k$$

$$= \varepsilon^{-s(\alpha_{(1)}|\beta)} F^s E_\beta x_s + \sum_{k=0}^{s-1} F^k y_k, \quad \text{where} \quad y_k \in A,$$

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by Lemma 9.2a. Using Corollary 4.1, it follows that $E_\beta x_s = 0$, hence $x_s \in A_{[1]}$. Since by applying a suitable power of F (here we use (a)), we can make any x_k to enter in the last term, this proves (b).

In order to prove (c) note that the subalgebra of $P_{(1)}$ generated by E, F and $K_{(1)}$ is isomorphic to $\mathrm{Mat}_l(\mathbb{C})$ (cf. [DC-K]). Hence with respect to this subalgebra, the module $V_{[1]}$ decomposes into a direct sum of l-dimensional irreducible submodules. Hence the same is true for C and therefore these exists $x \in C$ such that $E^{l-1}x \neq 0$. Write $x = \sum_{k=0}^{s} F^k x_k$, where $s \leq l-1$, $x_s \neq 0$ and $x_k \in A_{[1]}$ (by (b)). Applying E^s, we obtain:

$$E^s x = E^s F^s x_s = \mathrm{const}\, x_s, \quad \text{where} \quad \mathrm{const} \neq 0.$$

This proves (c). □

§**10.** *Proof of the theorem.* Fix the reduced expression (9.1) of the longest element of W, so that $R^+ \setminus R'^+ = \{\gamma_1, \ldots, \gamma_t\}$, where the γ_i are defined by (9.2). For $j \in \{1, \ldots, t\}$ we let:

$$\begin{aligned} &E_{(j)} = E_{\gamma_j}, \quad F_{(j)} = F_{\gamma_j}, \\ &B_{(j)} = T_{i_1} \ldots T_{i_{j-1}} B_{(j-1)}, \quad P_{(j)} = T_{i_1} \ldots T_{i_{j-1}} P_{(j-1)} \\ &N_{(j)} = T_{i_1} \ldots T_{i_{j-1}} N_{(j-1)}, \text{ etc.} \end{aligned}$$

Then Lemma 9.3 holds if the index 1 is replaced by j.

Let V^0 be an irreducible $U_\chi^{\bar{g}}$-module. Note that the ideal of $U_\chi^{\bar{g}}$ generated by the E_β for $\beta \in R^+ \setminus R'$ acts on V^0 nilpotently, hence trivially. Thus V^0 is actually a U'_χ-module.

Let $\widetilde{V} = U_\chi \otimes_{U_\chi^{\bar{g}}} V^0$. We shall show that this is an irreducible U_χ-module.

Let $V^i = P_{(i)} \otimes_{B_{(i)}} V^{i-1}$ for $i \geq 1$. Since (by Lemma 9.1b) $B_{(i+1)} \subset P_{(i)}$ we have canonical inclusions:

$$V^0 \subset V^1 \subset V^2 \subset \ldots \subset V^t = \widetilde{V}.$$

Let A be a U_χ-submodule of $\widetilde{V}$ different from $\widetilde{V}$. Then $A \cap V^{i-1} = 0$ since otherwise $A \supset V^0$ and hence $A = \widetilde{V}$. Suppose that $A \cap V^{i-1} = 0$. We shall prove that $A \cap V^i = 0$, which proves the irreducibility of $\widetilde{V}$. Assuming the contrary, suppose that C is an irreducible $P_{(i)}$-submodule of $A \cap V^i$. Since $\overline{N}_{(i)}$ acts nilpotently on V^i, we conclude (using Lemma 9.2a) that $\overline{N}_i C = 0$. Hence it suffices to show that

$$E_{i+1} V^i_{[i+1]} = 0. \tag{10.1}$$

Indeed, by Lemma 9.3c (which can be used due to (9.3)) we deduce from (10.1) that $C \cap V^{i-1} \neq 0$, a contradiction with $A \cap V^{i-1} = 0$.

By Lemma 9.2a, (10.1) is an immediate consequence of

$$V^i_{[i+1]} \subset F_{(i)}^{l-1} \ldots F_{(1)}^{l-1} V^0, \tag{10.2}$$

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which we shall prove by induction. Since $F_{(i)} \in \overline{N}_{(i+2)}$ (by Lemma 9.1b), we have: $F_{(i)}V^i_{[i+1]} = 0$. Hence $V^i_{[i+1]} \subset F^{l-1}_{(i)}V^{i-1}$. We now prove by induction on $k \leq i$ that

$$V^i_{[i+1]} \subset F^{l-1}_{(i)} \dots F^{l-1}_{(k+1)} V^k, \tag{10.3}$$

By the inductive assumption, we may write any $v \in V^i_{[i+1]}$ in the form $v = F^{l-1}_{(i)} \dots F^{l-1}_{(k+1)} v_0$, where $v_0 \in V^k$. By Lemma 9.1b, $F_k v = 0$, hence

$$0 = F_{(k)} F^{l-1}_{(i)} \cdots F^{l-1}_{(k+1)} v_0 = \text{const}\, F^{l-1}_{(i)} \dots F^{l-1}_{(k+1)} F_{(k)} v_0,$$

where const $\neq 0$, by Lemma 3.2 and (9.4). Hence $F_{(k)}v_0 = 0$ and therefore $v_0 \in F^{l-1}_{(k)} V^{k-1}$ (since we are in an induced module, monomials are linearly independent due to Corollary 4.1). This completes the proof of irreducibility of the U_χ-module $\tilde{V}$.

Thus (b) is proved since the U_χ-module V is a non-zero homomorphic image of the irreducible induced module from the $U^{\bar{\mathfrak{g}}}_\chi$-module V'.

In order to complete the proof of the theorem, we need to show that V' is a unique irreducible $U^{\bar{\mathfrak{g}}}_\chi$-submodule of V. To show this we introduce a gradation $V = \bigoplus_{j \in \mathbb{Z}_+} V_j$ by letting $V_0 = V'$ and $\deg F_{(j)} = 1, j = 1, \dots, t$. Due to (9.4), $F_{(i)} V_j \subset V_{i+j}$. If now V'' is another irreducible $U^{\bar{\mathfrak{g}}}_\chi$-submodule of V, then obviously, $V' \subset \bigoplus_{j>0} V_j$, hence $V = \sum F^{s_1}_{(1)} \cdots F^{s_t}_{(t)} V'' \subset \oplus_{j>0} V_j$, a contradiction. □

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