Remarks added in proof:

1) The conjecture in 3.1 is wrong as Dr. Werner Meyer pointed out:

Take arrangements A_1 and A_2 of k_1 and k_2 lines respectively which are in general position to each other. Consider the combined arrangement $A_1 \cup A_2$. Then the numerical characteristics t_r are additive except for t_2 which satisfies

$$t_2(A_1 \cup A_2) = t_2(A_1) + t_2(A_2) + k_1 k_2$$

Let F_1, F_2, F be the quadratic polynomials for $A_1, A_2, A_1 \cup A_2$. Then

$$F(x) = F_1(x) + F_2(x) + k_1 k_2 x^2$$

If we take for A_2 a pencil with k_2 large with respect to k_1, then $F(x)$ becomes indefinite. Maybe the conjecture remains true if one assumes that the arragement "does not contain large pencils".

2) The inequality (3) in 3.1 can be improved by using results of F. Sakai (Semi-Stable Curves on Algebraic Surfaces and Logarithmic Pluricanonical Maps, Math. Ann 254, p. 89–120 (1980)). We have

$$t_2 + \frac{3}{4} t_3 \geq k + t_5 + 2t_6 + 3t_7 + \cdots$$

This is sharp for $A_1(6), A_1(9), A_3^0(3), A_3^0(4)$ and the arrangements with S–T numbers 24 and 25.

3) B. Grünbaum has written to me that the arrangements $A_2(17)$ and $A_7(17)$ of his list are isomorphic.

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Regular Functions on Certain Infinite-dimensional Groups

Victor G. Kac and Dale H. Peterson

To Igor Rostislavovich Shafarevich on his 60th birthday

§0. Introduction

In the paper [18], we began a detailed study of the "smallest" group G associated to a Kac-Moody algebra $\mathfrak{g}(A)$ and of the (in general infinite-dimensional) flag varieties $\mathbf{P}\mathcal{V}_\Lambda$ associated to G. In the present paper we introduce and study the algebra $\mathbf{F}[G]$ of "strongly regular" functions on G. We establish a Peter-Weyl-type decomposition of $\mathbf{F}[G]$ with respect to the natural action of $G \times G$ (Theorem 1) and prove that $\mathbf{F}[G]$ is a unique factorization domain (Theorem 3).

These considerations are intimately related to the study of the algebra $\mathbf{F}[\mathcal{V}_\Lambda]$ of polynomial functions on the variety $\mathcal{V}_\Lambda$ (Theorem 2) and the so-called Bruhat and Birkhoff decompositions of $\mathcal{V}_\Lambda$.

The group G is a (possibly infinite-dimentional) algebraic group in the sense of Shafarevich [20], and belongs to one of the following three classes (we assume A to be indecomposable):

1) Finite type groups. These are connected simply-connected split simple finite-dimensional algebraic groups. In this case almost all the results of the paper are well-known.

2) Affine type groups. Such a G is an $\mathbf{F}^*$-extension of the group of regular maps from $\mathbf{F}^*$ to a group of finite type, or a "twisted" analogue. The simplest flag variety may be regarded as the space of based polynomial loops on a compact Lie group (in the case $\mathbf{F} = \mathbf{C}$).

3) "Wild" type groups. No "concrete" realization of these groups or their flag varieties is known.

The study of the groups G and the varieties $\mathcal{V}_\Lambda$ in the affine case is of particular importance because of applications to topology [2], [8], analysis [1], [9], soliton equations [4], etc.

Throughout the paper the base field $\mathbf{F}$ is of characteristic zero.

We thank M. Hochster for numerous consultations in commutative algebra.

§1. Kac-Moody Algebras and Associated Groups. Integrable Representations

1A) A *symmetrizable generalized Cartan matrix* $A = (a_{ij})_{i,j\in I}$ indexed by a nonempty finite set I is a matrix of integers satisfying: $a_{ii} = 2$ for all i; $a_{ij} \leq 0$ if $i \neq j$; DA is symmetric for some nondegenerate diagonal matrix D. We fix such a matrix A, assumed for simplicity to be indecomposable.

Choose a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, unique up to isomorphism, where $\mathfrak{h}$ is a vector space over $\mathbf{F}$ of dimension $|I| + \operatorname{corank} A$, and $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, $\Pi^\vee = \{h_i\}_{i\in I} \subset \mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$.

The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra over $\mathbf{F}$ generated by $\mathfrak{h}$ and symbols e_i and $f_i (i \in I)$ with defining relations:

$$\text{(1.1)} \qquad \begin{aligned} &[\mathfrak{h}, \mathfrak{h}] = (0); \quad [e_i, f_j] = \delta_{ij} h_i; \\ &[h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i \quad (h \in \mathfrak{h}); \end{aligned}$$

$$\text{(1.2)} \qquad (ad\, e_i)^{1-a_{ij}}(e_j) = 0, \quad (ad\, f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j).$$

We have the canonical embedding $\mathfrak{h} \subset \mathfrak{g}$ and linearly independent *Chevalley generators* $e_i, f_i (i \in I)$ for the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$. The center $\mathfrak{c}$ of $\mathfrak{g}$ lies in $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{g}' = \sum \mathbf{F} h_i$. Every ideal of $\mathfrak{g}$ contains $\mathfrak{g}'$ or is contained in $\mathfrak{c}$ [7].

Define an involution ω of $\mathfrak{g}$ by requiring: $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h) = -h (h \in \mathfrak{h})$. Let $\mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}$ generated by the $e_i (i \in I)$, and put $\mathfrak{n}_- = \omega(\mathfrak{n}_+)$. We have the vector space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

We have the *root space decomposition* $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. Put $Q = \sum_{i\in I} \mathbf{Z}\alpha_i$, $Q_+ = \sum_{i\in I} \mathbf{Z}_+\alpha_i$ (where $\mathbf{Z}_+ = \{0, 1, \ldots\}$), and define a partial order on $\mathfrak{h}^*$ by: $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. A *root* (resp. *positive root*) is an element of $\Delta := \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq (0)\}$ (resp. $\Delta_+ := \Delta \cap Q_+$). We have: $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}$. For $\alpha = \sum k_i\alpha_i \in \Delta$, we write $ht\, \alpha = \sum k_i$.

Define *fundamental reflections* $r_i \in \operatorname{Aut}_{\mathbf{F}}(\mathfrak{h})$, $i \in I$, by $r_i(h) = h - \alpha_i(h) h_i$. They generate the *Weyl group* W, which is a Coxeter group on $\{r_i\}_{i\in I}$, with length function $w \mapsto l(w)$. W preserves the root system Δ. A *real root* is an element of $\Delta^{re} := \{w(\alpha) \mid w \in W, \alpha \in \Pi\}$. If $\alpha \in \Delta^{re}$, then $\dim \mathfrak{g}_\alpha = 1$. Put $\Delta^{re}_+ = \Delta^{re} \cap \Delta_+$. For $\alpha \in \Delta^{re}$, write $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in I$; then $r_\alpha := w r_i w^{-1}$ depends only on α.

We choose a nondegenerate $\mathfrak{g}$-invariant symmetric $\mathbf{F}$-bilinear form $(.|.)$ on $\mathfrak{g}$ such that $(h_i \mid h_i)$ is positive rational for all $i \in I$. $(.|.)$ is nondegenerate and W-invariant on $\mathfrak{h}$, and hence induces an isomorphism $\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ and a form $(.|.)$ on $\mathfrak{h}^*$ [10, Chapter II].

1B) Consider a $\mathfrak{g}'$-module V, or (V, π), where $\pi : \mathfrak{g}' \longrightarrow \operatorname{End}_{\mathbf{F}}(V)$. Let $V_{fin} = \{v \in V \mid \text{for every } \alpha \in \Delta^{re} \text{ there exists } N \text{ such that } \pi(\mathfrak{g}_\alpha)^N(v) = 0\}$ ($\alpha \in \pm\Pi$ suffices). V_{fin} is a $\mathfrak{g}'$-submodule of V; the $\mathfrak{g}'$-module V is called *integrable* if $V = V_{fin}$. $(\mathfrak{g}, ad)$ is an integrable $\mathfrak{g}'$-module.

Remark 1.1. We feel that the functor $V \mapsto V_{fin}$ from the category of all $\mathfrak{g}'$-modules to the category of integrable $\mathfrak{g}'$-modules is important.

Lemma 1.1. *Let (V, π) be a locally-finite $sl_2(\mathbf{F})$-module and let $\{e, f, h\}$ be the standard basis of $sl_2(\mathbf{F})$. Let $x \in \operatorname{End}_{\mathbf{F}}(V)$ satisfy*

$$\text{(1.3)} \qquad [\pi(f), x] = 0 \text{ and } [\pi(h), x] = ax, \quad \text{where } -a \in \mathbf{Z}_+.$$

Then $\left(ad\pi(e)\right)^{1-a} x = 0$.

Proof. The $sl_2(\mathbf{F})$-module V decomposes into a direct sum of finite-dimensional submodules: $V = \bigoplus_i V_i$. Then x has a "block decomposition": $\sum_{i,j} x_{ij}$, where $x_{ij} \in \operatorname{Hom}_{\mathbf{F}}(V_i, V_j)$. (1.3) holds for each x_{ij}, and hence we have $\left(ad\, \pi(e)\right)^{1-a} x_{ij} = 0$ for all i, j by the finite-dimensional representation theory of sl_2. Q.E.D.

One immediately deduces the following corollary, which allows us to "differentiate" integrable G-modules (i.e., modules such that the $U_\alpha (\alpha \in \Delta^{re})$ act locally unipotently).

Corollary 1.1. *Let $\tilde{\mathfrak{g}}'$ be the Lie algebra on generators e_i, f_i, h_i $(i \in I)$ with defining relations (1.1), with $\mathfrak{h}$ replaced by $\sum \mathbf{F} h_i$. Let (V, π) be a $\tilde{\mathfrak{g}}'$-module such that all $\pi(e_i)$, $\pi(f_i)$ are locally nilpotent. Then $\pi(e_i)$ and $\pi(f_i)$ satisfy relations (1.2), so that we may regard (V, π) as an integrable $\mathfrak{g}'$-module.*

1C) We now recall the construction of the group G associated to the Lie algebra $\mathfrak{g}'$ [18].

Let G^* be the free product of the additive groups $\mathfrak{g}_\alpha$, $\alpha \in \Delta^{re}$, with canonical inclusions $i_\alpha: \mathfrak{g}_\alpha \longrightarrow G^*$. For any integrable $\mathfrak{g}'$-module (V, π), define a homomorphism $\pi^*: G^* \longrightarrow \mathrm{Aut}_{\mathbf{F}}(V)$ by $\pi^*(i_\alpha(e)) = \exp \pi(e)$. Let N^* be the intersection of all $\mathrm{Ker}(\pi^*)$, put $G = G^*/N^*$, and let $q: G^* \longrightarrow G$ be the canonical homomorphism. For $e \in \mathfrak{g}_\alpha(\alpha \in \Delta^{re})$, put $\exp e = q(i_\alpha(e))$, so that $U_\alpha := \exp \mathfrak{g}_\alpha$ is an additive one-parameter subgroup of G. The $U_\alpha(\alpha \in \pm\Pi)$ generate G, and G is its own derived group. Denote by U_+ (resp. U_-) the subgroup of G generated by the U_α (resp. $U_{-\alpha}$), $\alpha \in \Delta_+^{re}$.

Example 1.1. a) Let A be the Cartan matrix of a split simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbf{F}$. Then the group G associated to $\mathfrak{g} \cong \mathfrak{g}'(A)$ is the group $\underline{G}(\mathbf{F})$ of $\mathbf{F}$-valued points of the connected simply-connected algebraic group $\underline{G}$ associated to $\mathfrak{g}$, and $U_+ a \underline{U}(\mathbf{F})$ for some maximal unipotent subgroup $\underline{U}$ of $\underline{G}$. These groups G are called *groups of finite type.*

b) Let $\mathfrak{g}$ be as in a), and let $\tilde{A}$ be the extended Cartan matrix of $\mathfrak{g}$. Then the group G associated to $\mathfrak{g}'(\tilde{A})$ is a central extension by $\mathbf{F}^*$ of $\underline{G}(\mathbf{F}[z, z^{-1}])$, and

$$U_+ \cong \{g \in \underline{G}(\mathbf{F}[z]) \mid g|_{z=0} \in \underline{U}(\mathbf{F})\}.$$

These groups G and their twisted analogues are called *groups of affine type.*

To any integrable $\mathfrak{g}'$-module (V, π) we associate the homomorphism (again denoted by) $\pi: G \longrightarrow \mathrm{Aut}_{\mathbf{F}}(V)$ satisfying $\pi(\exp e) = \exp \pi(e)$ for $e \in \mathfrak{g}_\alpha(\alpha \in \Delta^{re})$. The homomorphism associated to $(\mathfrak{g}, ad)$, denoted Ad, maps G into $\mathrm{Aut}_{\mathbf{F}}(\mathfrak{g})$. The kernel of Ad is the center C of G, and $Ad(G)$ acts faithfully on $\mathfrak{g}'/\mathfrak{c}$. We have $\pi(Ad(g)x) = \pi(g)\pi(x)\pi(g)^{-1}$ for any integrable $\mathfrak{g}'$-module (V, π) and all $g \in G$, $x \in \mathfrak{g}'$. It follows that if (V, π) is an integrable $\mathfrak{g}'$-module with $\mathrm{Ker}\,\pi \subset \mathfrak{c}$, then (on G) $\mathrm{Ker}\,\pi \subset C$.

For each $i \in I$ we have a unique homomorphism $\varphi_i: SL_2(\mathbf{F}) \longrightarrow G$ satisfying:

$$\varphi_i\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = \exp te_i, \quad \varphi_i\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} = \exp tf_i \quad (t \in \mathbf{F}).$$

Let $G_i = \varphi_i(SL_2(\mathbf{F}))$, $H_i = \varphi_i(\{\mathrm{diag}(t, t^{-1}) \mid t \in \mathbf{F}^*\})$, and let N_i be the normalizer of H_i in G_i. Let H (resp. N) be the subgroup of G generated by the H_i (resp. N_i); H is an abelian normal subgroup of N. The φ_i are monomorphisms and H is the direct product of the H_i. We have an isomorphism $\varphi: W \longrightarrow N/H$ such that $\varphi(r_i)$ is the coset $N_i H \setminus H$. We identify W and N/H using φ; this gives sense to expressions such as wH and wU_+w^{-1} occurring in the sequel. If $h \in \mathfrak{h}$, $w \in W$ and $n \in wH$, then $Ad(n)h = w(h)$. We put $B_+ = HU_+$, $B_- = HU_-$.

1D) Choose $\Lambda_i \in \mathfrak{h}^*(i \in I)$ satisfying $\Lambda_i(h_j) = \delta_{ij}(j \in I)$. Put P_+ (resp. P_{++})$= \{\sum_i k_i\Lambda_i \mid k_i \in \mathbf{Z}$ and $k_i \geq 0$ (resp. > 0)$\}$. Given $\Lambda \in P_+$ (or more generally, $\Lambda \in \mathfrak{h}^*$ such that all $\Lambda(h_i) \in \mathbf{Z}_+$), there exists an irreducible $\mathfrak{g}$-module $(L(\Lambda), \pi_\Lambda)$, unique up to isomorphism, containing a $v_\Lambda \neq 0$ satisfying: $\pi_\Lambda(\mathfrak{n}_+)v_\Lambda = (0)$; $\pi_\Lambda(h)v_\Lambda = \Lambda(h)v_\Lambda(h \in \mathfrak{h})$. $L(\Lambda)$ is an absolutely irreducible integrable $\mathfrak{g}'$-module, and we have $L(\Lambda) = \pi_\Lambda(U(\mathfrak{n}_-))v_\Lambda$, $\mathrm{End}_{\mathfrak{g}'}(L(\Lambda)) = \mathbf{F}I_{L(\Lambda)}$. The module $L(\Lambda)$ is called an *integrable module with highest weight* Λ [11]. Recall that $\bigoplus_{i\in I} L(\Lambda_i)$ is a faithful G-module [18].

We have the *weight space decomposition* $L(\Lambda) = \bigoplus_{\lambda\in\mathfrak{h}^*} L(\Lambda)_\lambda$, where $L(\Lambda)_\lambda = \{v \in L(\Lambda) \mid h(v) = \lambda(h)v$ for all $h \in \mathfrak{h}\}$. Elements of $P(\Lambda) := \{\lambda \in \mathfrak{h}^* \mid L(\Lambda)_\lambda \neq (0)\}$ are called *weights* of $L(\Lambda)$. We have $\mathbf{F}v_\Lambda = L(\Lambda)_\Lambda = \{v \in L(\Lambda) \mid \mathfrak{n}_+(v) = (0)\}$; elements of $\mathbf{F}^*v_\Lambda$ are called *highest weight vectors.* We have $P(\Lambda) \subset \Lambda - Q_+$, and $\dim L(\Lambda)_{w(\lambda)} = \dim L(\Lambda)_\lambda$ if $\lambda \in \mathfrak{h}^*$, $w \in W$; in particular, $\dim L(\Lambda)_{w(\Lambda)} = 1$.

Regarded as a $\mathfrak{g}$-module under $\pi^*_\Lambda := \pi_\Lambda \circ \omega$, $L(\Lambda)$ is denoted $L^*(\Lambda)$ and v_Λ is denoted v^*_Λ. There exists a unique $\mathfrak{g}$-invariant bilinear form on $L(\Lambda) \times L^*(\Lambda)$ satisfying $\langle v_\Lambda, v^*_\Lambda\rangle = 1$; it is nondegenerate. Using $\langle\,,\,\rangle$ we regard $L^*(\Lambda)$ as a subspace of $L(\Lambda)^*$, the algebraic dual of $L(\Lambda)$.

Note that if a statement holds for $\mathfrak{n}_+, U_+$ or $L(\Lambda)$, then a similar statement holds for $\mathfrak{n}_-, U_-$ or $L^*(\Lambda)$ using ω. We keep this observation in mind in the sequel.

§2. A Peter-Weyl-Type Theorem

2A) For every real root α, we fix a non-zero element e_α of $\mathfrak{g}_\alpha$, and coordinatize U_α by putting $x_\alpha(t) = \exp te_\alpha(t \in \mathbf{F})$. Furthermore, for $\overline{\beta} = (\beta_1, \ldots, \beta_k) \in (\Delta^{re})^k$, define a map $x_{\overline{\beta}}: \mathbf{F}^k \longrightarrow G$ by

$$x_{\overline{\beta}}(t_1, \ldots, t_k) = x_{\beta_1}(t_1)\ldots x_{\beta_k}(t_k),$$

and denote by $U_{\overline{\beta}}$ the image of $x_{\overline{\beta}}$.

We call a function $f: G \longrightarrow \mathbf{F}$ *weakly regular* if $f \circ x_{\overline{\beta}}: \mathbf{F}^k \longrightarrow \mathbf{F}$ is a polynomial function for all $\overline{\beta} \in (\Delta^{re})^k$ and $k \in \mathbf{Z}_+$ ($\overline{\beta} \in (-\Pi \cup \Pi)^k$ suffices). Denote by $\mathbf{F}[G]_{w.r.}$ the algebra of all weakly regular functions.

Let V be a $\mathfrak{g}'$-module; given $v^* \in V^*$ and $v \in V_{fin}$, we get a weakly regular function $f_{v^*,v}$ on G, called a *matrix coefficient*: $f_{v^*,v}(g) = \langle g(v), v^* \rangle$. The matrix coefficients $\theta_\Lambda := f_{v^*_\Lambda, v_\Lambda}$ are especially important.

Lemma 2.1. a) *The ring* $\mathbf{F}[G]_{w.r.}$ *is an integral domain.*

b) *If* $f \in \mathbf{F}[G]_{w.r.}$ *is such that* $\bigcap_{k>0} f^k \mathbf{F}[G]_{w.r.} \neq (0)$, *then* $f \in \mathbf{F}^*$.

c) *Every unit of* $\mathbf{F}[G]_{w.r.}$ *lies in* $\mathbf{F}^*$.

d) *Any* $f \in \mathbf{F}[G]_{w.r}$ *is determined by its restriction to* $U_- H U_+$.

Proof. a) holds since any $g_1, g_2 \in G$ lie in some $U_{\overline{\beta}}$. b) and c) hold by considering pullbacks under $x_{\overline{\beta}}$. d) holds, using a), since for $\Lambda \in P_{++}$ the Birkhoff decomposition [18] gives:

$$U_- H U_+ = \{g \in G \mid \theta_\Lambda(g) \neq 0\}. \qquad \text{Q.E.D.}$$

2B) Put $\tilde{H} = \mathrm{Hom}(Q, \mathbf{F}^*)$, and define a homomorphism Ad: $\tilde{H} \longrightarrow \mathrm{Aut}_{\mathbf{F}}(\mathfrak{g})$ by $Ad(h)x = h(\alpha)x$ if $x \in \mathfrak{g}_\alpha$. Ad induces an action of $\tilde{H}$ on G, defining $\tilde{H} \ltimes G$, to which Ad extends in the obvious way. We extend the action of G on $L(\Lambda)$ to $\tilde{H} \times G$ by requiring $\tilde{H}$ to fix v_Λ.

Subgroups U'_+ of U_+ and U'_- of U_- are called *large* if there exist $g_1, \dots, g_m \in G$ such that $\bigcap_{j=1}^m g_j U_\pm g_j^{-1} \subset U'_\pm$.

Lemma 2.2. a) *A subgroup* U' *of* U_+ *is large if and only of it contains the stabilizer*[1] *in* G *of some finite-dimensional subspace of* $\bigoplus_{\Lambda \in P_+} L(\Lambda)$.

b) *Let* U' *be a large subgroup of* U_+. *Then:*

(i) *for every* $\alpha \in \Delta^{re}$, *the subgroup* $\bigcap_{u \in U_\alpha} uU'u^{-1}$ *of* U_+ *is large;*

(ii) *the subgroup* $\bigcap_{h \in \tilde{H}} hU'h^{-1}$ *of* U_+ *is large;*

(iii) *there exists* $\overline{\beta}$ *such that* $U_{\overline{\beta}} U' = U_+$.

Proof. a) holds since $U_+ = \{g \in G \mid g(v_{\Lambda_i}) = v_{\Lambda_i}$ for all $i \in I\}$ [18], and since $L(\Lambda)$, $\Lambda \in P_+$, is spanned by $G(v_\Lambda)$. b (i) and (ii) hold by a). To prove b (iii), we may assume that U' is the stabilizer in U_+ of a U_+-invariant finite-dimensional subspace V of $\bigoplus L(\Lambda)$. Let π_V be the restriction of the action of U_+ to V. U_+ acts as a finite-dimensional unipotent group $U := \pi_V(U_+)$ on V. The one-parameter subgroups $\pi_V(U_\alpha)$ ($\alpha \in \Delta_+^{re}$) generate U, hence $\pi_V(U_{\overline{\beta}}) = U$ for some $\overline{\beta} \in (\Delta_+^{re})^k$, $k \in \mathbf{Z}_+$. Hence $U_+ = U_{\overline{\beta}} U'$. Q.E.D.

[1] We define the *stabilizer* (resp. *normalizer*) in G of a subset M of a G-set to be the group $\{g \in G \mid g(v) = v$ (resp. $g(v) \in M$) for all $v \in M\}$.

2C) We call a weakly regular function f *strongly regular* if there exist large subgroups $U'_\pm$ of $U_\pm$ such that $f(u_- g u_+) = f(g)$ for all $g \in G$ and $u_\pm \in U'_\pm$. Note that the matrix coefficients $f_{v^*,v}$, where $v^* \in L^*(\Lambda)$ and $v \in L(\Lambda)$, are strongly regular functions. We denote by $\mathbf{F}[G]_{s.r.}$, or $\mathbf{F}[G]$ for short, the algebra of all strongly regular functions on G. $\mathbf{F}[G]$ is a $G \times G$-module under π_{reg}, where $(\pi_{reg}(g_1, g_2)f)(g) = f(g_1^{-1} g g_2)$.

Now we can prove the following analogue of the Peter-Weyl theorem.

Theorem 1. *The linear map* ϕ: $\bigoplus_{\Lambda \in P_+} L^*(\Lambda) \otimes L(\Lambda) \longrightarrow \mathbf{F}[G]$ *defined by* $\phi(v^* \otimes v) = f_{v^*,v}$ *is an isomorphism of* $G \times G$*-modules.*

Proof. From Lemmas 2.1d and 2.2b (i and iii) it follows that every $U_\alpha \times U_\beta (\alpha, \beta \in \Delta^{re})$ acts locally unipotently on $\mathbf{F}[G]$. Hence there exist unique locally nilpotent elements of $\mathrm{End}_{\mathbf{F}} \mathbf{F}[G]$, which we denote by $\pi(e_i, 0)$, $\pi(0, e_i)$, $\pi(f_i, 0)$, $\pi(0, f_i)$, such that $\pi_{reg}(\exp te_i, 1) = \exp t\, \pi(e_i, 0)$, etc. Then Corollary 1.1 shows that there exists a unique homomorphism π: $\mathfrak{g}' \times \mathfrak{g}' \longrightarrow \mathrm{End}_{\mathbf{F}} \mathbf{F}[G]$ with the given values on $(e_i, 0)$, etc. Using Lemmas 2.1d and 2.2b (ii and iii), there exists a Q-gradation $\mathbf{F}[G] = \bigoplus R_\alpha$, where $R_\alpha = \{f \in \mathbf{F}[G] \mid f(h^{-1} g h) = h(\alpha) f(g)$ for all $h \in \tilde{H}\}$.

Thus, $(\mathbf{F}[G], \pi)$ is an integrable $\mathfrak{g}' \times \mathfrak{g}'$-module and $(\mathbf{F}[G], \pi_{reg})$ is the associated $G \times G$-module. Using Lemma 2.2b (iii), $U_- \times U_+$ acts locally-finitely on $\mathbf{F}[G]$. Hence, $\mathfrak{n}_- \times \mathfrak{n}_+$ acts locally-finitely, and therefore, using the Q-gradation, locally-nilpotently. Using the complete reducibility theorem [14, Proposition 2.9]), we deduce that the $G \times G$-module $\mathbf{F}[G]$ is isomorphic to a direct sum of modules of the form $L^*(\lambda) \otimes L(\mu)$ ($\lambda, \mu \in P_+$).

Now, regarding $v^* \otimes v$ as an operator on $L(\lambda)$, we have: $f_{v^*,v}(g) = tr(v^* \otimes v)\pi_\lambda(g)$, so that ϕ is a well-defined $G \times G$-module homomorphism. ϕ is injective since the $L^*(\lambda) \otimes L(\lambda)$ are irreducible and inequivalent $G \times G$-modules, and $\phi(v^*_\lambda \otimes v_\lambda) = \theta_\lambda \neq 0$. On the other hand, let ψ: $L^*(\lambda) \otimes L(\mu) \longrightarrow \mathbf{F}[G]$ be a $G \times G$-module homomorphism. Considering the action of $B_- \times U_+$ on $\psi(v^*_\lambda \otimes v_\mu)$ and using Lemma 2.1d, we obtain that $\psi(v^*_\lambda \otimes v_\mu) \in \mathbf{F}\theta_\lambda$. Hence, ϕ is surjective. Q.E.D.

2D) Later we will need the following corollary of the proof of Theorem 1.

Corollary 2.1. *Let $f_1, f_2 \in \mathbf{F}[G]$, $f_1 \neq 0$, and suppose that for each $k \in \mathbf{Z}_+$ and $\overline{\beta} \in (\Delta^{re})^k$ there exists a polynomial function $q_{\overline{\beta}}: \mathbf{F}^k \longrightarrow \mathbf{F}$ such that $q_{\overline{\beta}}(f_1 \circ x_{\overline{\beta}}) = f_2 \circ x_{\overline{\beta}}$. Then $f_1^{-1}f_2 \in \operatorname{Fract}\mathbf{F}[G]$ lies in $\mathbf{F}[G]$.*

Proof is that of Theorem 1, replacing $\mathbf{F}[G]$ by the subalgebra of $\operatorname{Fract}\mathbf{F}[G]$ consisting of all $f_1^{-1}f_2$, where f_1, f_2 satisfy the hypothesis of the corollary. Q.E.D.

For a subgroup P of G, let $\mathbf{F}[G]^P = \{f \in \mathbf{F}[G] \mid f(gp) = f(g) \text{ for all } g \in G \text{ and } p \in P\}$. This is a subalgebra of $\mathbf{F}[G]$, and G acts on it by left multiplication: $(g \cdot f)(g') = f(g^{-1}g')$. For $\Lambda \in P_+$, put:

$$S_\Lambda = \{f \in \mathbf{F}[G] \mid f(gb) = \theta_\Lambda(b)f(g) \text{ for all } g \in G \text{ and } b \in B_+\}.$$

This is a G-submodule of $\mathbf{F}[G]^{U_+}$.

Corollary 2.2. a) $\mathbf{F}[G]^{U_+} = \bigoplus_{\Lambda \in P_+} S_\Lambda$.

b) *The map $L^*(\Lambda) \to S_\Lambda$ defined by $v \mapsto f_{v,v_\Lambda}$ is a G-module isomorphism.*

c) $S_\Lambda S_M = S_{\Lambda+M}$.

Proof is immediate by Theorem 1 and the following facts: $L(\Lambda)^{U_+} = \mathbf{F}v_\Lambda$; $\theta_\Lambda\theta_M = \theta_{\Lambda+M}$; multiplication is G-equivariant. Q.E.D.

Remark 2.1. a) The algebra $\mathbf{F}[G]^{U_+}$ can be constructed without reference to the group G. Indeed, for $\Lambda, M \in P_+$ we have the Cartan product $L^*(\Lambda) \otimes L^*(M) \xrightarrow{\phi} L^*(\Lambda + M)$, defined by the properties that ϕ is a $\mathfrak{g}$-module homomorphism and $\phi(v_\Lambda^* \otimes v_M^*) = v_{\Lambda+M}^*$. Under the Cartan product, the space $\bigoplus_{\Lambda \in P_+} L^*(\Lambda)$ becomes an algebra, isomorphic to $\mathbf{F}[G]^{U_+}$ by Corollary 2.2.

b) Corollary 2.2 can be viewed as a Borel-Weil-type theorem. (A special case of this is considered in [13]). It should not be difficult, using the method of [5], to extend it to a Borel-Weil-Bott-type theorem.

Corollary 2.3. *Let $\Lambda \in P_+ \setminus \{0\}$ and let $B_\Lambda = \{b \in B_+ \mid \theta_\Lambda(b) = 1\}$. Then, provided that $\mathbf{F}$ is algebraically closed, one has:*

$$\mathbf{F}[G]^{B_\Lambda} \cong \bigoplus_{n \geq 0} L^*(n\Lambda),$$

where $L^*(n\Lambda)L^*(m\Lambda) = L^*((n+m)\Lambda)$ under the Cartan product.

Example 2.1. a) If G is of finite type, then $\mathbf{F}[G] = \mathbf{F}[G]_{w.r.}$ is the coordinate ring of the finite-dimensional affine variety G.

b) Let G be of affine type as in Example 1.1b. Then the only strongly regular functions f such that $f(cg) = f(g)$ for all $c \in \mathbf{F}^* \subset C$ and $g \in G$ are constants (by Theorem 1). On the other hand, given a rational N-dimensional representation π of $\underline{G}$, let $\pi(g) = (\sum_k a_{ij}^k(g)z^k)_{i,j=1}^N$ for $g \in \underline{G}(\mathbf{F}[z, z^{-1}])$; then the pullback of each function $g \mapsto a_{ij}^k(g)$ is a weakly regular function on G.

2E) We introduce the Zariski topology on G defined by strongly regular functions, i.e., a closed subset is the set of zeros of an ideal of $\mathbf{F}[G]$.

Note that the stabilizer or normalizer of a finite-dimensional subspace of $\bigoplus_{\Lambda \in P_+} L(\Lambda)$ or $\bigoplus_{\Lambda \in P_+} L^*(\Lambda)$ is a closed subgroup of G. It follows that $U_\pm$ and $B_\pm$ are closed subgroups and hence $H = B_+ \cap B_-$ is a closed subgroup. Similarly, the $G_iU_\pm$ are closed subgroups and hence the subgroups $G_i = (G_iU_+) \cap (G_iU_-)$ are closed. It is easy to show that $\varphi_i: SL_2(\mathbf{F}) \longrightarrow G_i$ is a Zariski homeomorphism. One can also show that H is homeomorphic to $(\mathbf{F}^*)^I$ and $U_+ \cap (wU_-w^{-1})$ to $\mathbf{F}^{l(w)}$.

For $\overline{\beta} \in (\Delta^{re})^k$, let $\mathbf{F}[G]_{w.r.}^{\overline{\beta}} = \{f \in \mathbf{F}[G]_{w.r.} \mid f \text{ vanishes on } U_{\overline{\beta}}\}$. Taking the $\mathbf{F}[G]_{w.r.}^{\overline{\beta}}$ for a basis of neighborhoods of 0 makes $\mathbf{F}[G]_{w.r.}$ into a Hausdorff complete topological ring.

Remark 2.2. We have the canonical inclusion $G \to \operatorname{Specm}\mathbf{F}[G]$ (= set of all closed ideals of codimension 1). Let G be of infinite type (i.e., $\dim \mathfrak{g} = \infty$). Then $\mathfrak{m} := \phi(\bigoplus_{\Lambda \in P_+ \setminus \{0\}} L^*(\Lambda) \otimes L(\Lambda))$ is a closed ideal of codimension 1 in $\mathbf{F}[G]$ (this follows from the well-known fact that $(Q_+|_{\mathfrak{h}'}) \cap (P_+|_{\mathfrak{h}'}) = \{0\}$ in the infinite type case). Since $\mathfrak{m}$ is $G \times G$-invariant, we deduce that $\mathfrak{m} \in (\operatorname{Specm}\mathbf{F}[G]) \setminus G$.

§3. The Varieties $\mathcal{V}_\Lambda$

3A) Given a decomposition of a vector space V into a direct sum of finite-dimensional subspaces, $V = \bigoplus_\alpha V_\alpha$, we denote by $\mathbf{F}[V]$ the symmetric algebra over $\bigoplus_\alpha V_\alpha^* \subset V^*$. We call elements of $\mathbf{F}[V]$ *strongly regular functions* on V. The algebra $\mathbf{F}[V]$ is a polynomial algebra on a basis of $\bigoplus V_\alpha^*$ (it may be viewed as the coordinate ring of $\prod_\alpha V_\alpha$). It is a subalgebra of the algebra $\mathbf{F}[V]_r$ of *regular functions*, i.e., $\mathbf{F}$-valued functions on V whose restriction to any finite-dimensional subspace is a polynomial function. Taking the ideals of finite-dimensional subspaces of V for a basis of neighborhoods of zero makes $\mathbf{F}[V]_r$ into a complete topological ring.

We introduce the Zariski topology on V defined by strongly regular functions. For a closed subset $\mathcal{V}$ of V (resp. the zero set $\mathcal{V}$ of an ideal of $\mathbf{F}[V]_r$), we denote by $\mathbf{F}[\mathcal{V}]$ (resp. $\mathbf{F}[\mathcal{V}]_r$) the restriction of $\mathbf{F}[V]$ (resp. $\mathbf{F}[V]_r$) to $\mathcal{V}$.

These definitions will be applied in this section to the vector spaces $L(\Lambda)$ and $L^*(\Lambda)$ with the weight space decompositions and $\mathfrak{g}$ with the root space decomposition. Here $\mathbf{F}[L(\Lambda)] = \operatorname{Sym} L^*(\Lambda)$ and $\mathbf{F}[L^*(\Lambda)] = \operatorname{Sym} L(\Lambda)$.[2]

Remark 3.1. It is easy to see that the canonical map $V \to \operatorname{Specm} \mathbf{F}[V]$ is a bijection (cf. Remark 2.2); more generally, we have a bijection $\mathcal{V} \to \operatorname{Specm} \mathbf{F}[\mathcal{V}]$ for every closed subset $\mathcal{V}$ of V.

3B) For each $\alpha \in \Delta \cup \{0\}$, choose dual bases $\{e_\alpha^{(i)}\}$ of $\mathfrak{g}_\alpha$ and $\{f_\alpha^{(i)}\}$ of $\mathfrak{g}_{-\alpha}$. Let $\Lambda \in P_+$. Denote by $\mathcal{V}_\Lambda$ the set of all $v \in L(\Lambda)$ which satisfy the following equation in $L(\Lambda) \otimes L(\Lambda)$:

$$(\Lambda \mid \Lambda) v \otimes v = \sum_{\alpha \in \Delta \cup \{0\}} \sum_i f_\alpha^{(i)}(v) \otimes e_\alpha^{(i)}(v). \tag{3.1}$$

Note that the sum on the right-hand side is finite. (3.1) is equivalent to a (possibly infinite) system of equations of the form $P = 0$, where $P \in \operatorname{Sym}^2 L^*(\Lambda)$. We call these polynomials P (and their analogues in $\operatorname{Sym}^2 L(\Lambda)$) *Plücker polynomials.*

We have shown in [18] that:

$$\mathcal{V}_\Lambda = G(\mathbf{F} v_\Lambda). \tag{3.2}$$

---

[2] Here and further on, $\operatorname{Sym} V = \bigoplus_{k \geq 0} \operatorname{Sym}^k V$ denotes the symmetric algebra over a vector space V.

By the complete reducibility theorem ([12], [14]), we have: $\operatorname{Sym}^k L(\Lambda) = L(k\Lambda) \oplus J_k$, $\operatorname{Sym}^k L^*(\Lambda) = L^*(k\Lambda) \oplus J_k^*$, where $L(k\Lambda)$ (resp. $L^*(k\Lambda)$) is the $U(\mathfrak{g})$-submodule generated by v_Λ^k (resp. $(v_\Lambda^*)^k$) and J_k (resp. J_k^*) is the (unique) complementary submodule. Set $J = \bigoplus_{k \geq 2} J_k$, $J^* = \bigoplus_{k \geq 2} J_k^*$. Note that the restriction map $\phi\colon \mathbf{F}[L(\Lambda)] \longrightarrow \mathbf{F}[\mathcal{V}_\Lambda]$ is a G-module homomorphism by (3.2); it is surjective by definition. Note also that: $(v_\Lambda^*)^k(tv_\Lambda) = t^k$ and $J_k^*(tv_\Lambda) = 0$. Hence, J^* is the ideal of $G(\mathbf{F}v_\Lambda)$ in $\mathbf{F}[L(\Lambda)]$, so that by (3.2) and Remark 2.1a we have:

Lemma 3.1. *$\mathbf{F}[\mathcal{V}_\Lambda] \cong \mathbf{F}[L(\Lambda)]/J^*$ is isomorphic to $\bigoplus_{k \geq 0} L^*(k\Lambda)$ with the Cartan product.*

Theorem 2. a) *The ideals J and J^* are generated by the Plücker polynomials.*

b) *The algebra $\mathbf{F}[G]^{U_+}$ is the associative commutative $\mathbf{F}$-algebra with unity on generators $\bigoplus_i L^*(\Lambda_i)$ with defining relations*

$$(\Lambda_i \mid \Lambda_j) uv = \sum_{\alpha \in \Delta \cup \{0\}} \sum_s f_\alpha^{(s)}(u)\, e_\alpha^{(s)}(v),$$

where $i, j \in I$, $u \in L^*(\Lambda_i)$, $v \in L^*(\Lambda_j)$.

The proof of Theorem 2 uses the Casimir operator introduced in [11] (cf. [14]):

$$\Omega = 2\nu^{-1}(\rho) + \sum_s f_0^{(s)} e_0^{(s)} + 2 \sum_{\alpha \in \Delta_+} \sum_s f_\alpha^{(s)} e_\alpha^{(s)},$$

where $\rho = \sum_i \Lambda_i$. Recall that Ω acts on $L(\Lambda)$ as a scalar $c_\Lambda := (\Lambda + 2\rho \mid \Lambda)$ ([11], [12]).

Lemma 3.2. *Let $\Lambda, \Lambda' \in \mathfrak{h}^*$ satisfy $\Lambda(h_i), \Lambda'(h_i) \in \mathbf{Z}_+$ for all $i \in I$. If $\Lambda > \Lambda'$, then $c_\Lambda - c_{\Lambda'} > 0$.*

Proof. $c_\Lambda - c_{\Lambda'} = (\Lambda + \Lambda' + 2\rho | \Lambda - \Lambda') > 0$. Q.E.D.

Proof of Theorem 2. To prove a), it suffices to consider J. Using Lemma 3.2 and the complete reducibility theorem applied to $\operatorname{Sym}^k L(\Lambda)$ we have:

$$J_k = (\Omega - c_{k\Lambda}) \operatorname{Sym}^k L(\Lambda). \tag{3.3}$$

It follows from (3.1) and (3.3) that J_2 is the space of Plücker polynomials. Furthermore, we have by an easy calculation in $\mathrm{Sym}^k L(\Lambda)$, $k \geq 2$:

$$(\Omega - c_{k\Lambda})v^k = \frac{1}{2}k(k-1)((\Omega - c_{2\Lambda})v^2)v^{k-2},$$

which shows that J_2 generates the ideal J.

The proof of b) is similar, using the identity

$$\Omega(xyz) = \Omega(xy)z + \Omega(yz)x + \Omega(zx)y - \Omega(x)yz - \Omega(y)zx - \Omega(z)xy.$$

Q.E.D.

Example 3.1. a) Let $G = SL_n(\mathbf{F})$; then $L(\Lambda_k)$ is the G-module $\Lambda^k\mathbf{F}^n$ and $\mathcal{V}_{\Lambda_k}$ is the set of all decomposable k-vectors, so that $\mathbf{P}\mathcal{V}_{\Lambda_k}$ is the Grassmann variety of k-dimensional subspaces of $\mathbf{F}^n$. The ideal of $\mathcal{V}_{\Lambda_k}$ is generated by the classical Plücker relations; this result is due to Plücker. Theorem 2 in the finite-dimensional case is due to Kostant. Our proof is essentially the same as Kostant's (presented in [16]).

b) Let G be a group of affine type and $L(\Lambda_0)$ its basic representation (see [12], [15]). Then the Plücker relations are equivalent to the hierarchy of Hirota bilinear equations studied in [4], the simplest case $A_1^{(1)}$ being equivalent to the celebrated KdV hierarchy. Theorem 2a shows that the ideal of equations satisfied by all polynomial solutions of these hierarchies is generated by Hirota bilinear equations.

c) Let K be a connected compact Lie group of type $X_n (= A_n, B_n, \ldots, E_8)$. Let σ be an automorphism of K of finite order m. Let k be the minimal positive integer such that σ^k is an inner automorphism of K and let K_0 be the fixed point set of σ. Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ be the unit circle; denote by $\Omega_\sigma(K)$ the space of all σ-equivariant polynomial loops on K, i.e., polynomial maps $g\colon S^1 \longrightarrow K$ such that $\sigma(g(z)) = g(z \exp \frac{2\pi i}{m})$. Then K_0 operates by right multiplication on $\Omega_\sigma(K)$ and we may consider the space $\Omega_\sigma(K)/K_0$. Let A be the generalized Cartan matrix of type $X_n^{(k)}$ [12] and G the associated group. Then we have a homeomorphism of topological spaces $\Omega_\sigma(K)/K_0 \cong \mathbf{P}\mathcal{V}_\Lambda$ for a suitable Λ (cf. [18]). Note that $\Omega_1(K)/K$ is the space of based loops; in this case A is of type $X_n^{(1)}$ and we have: $\Omega_1(K)/K \cong \mathbf{P}\mathcal{V}_{\Lambda_0}$. This allows one to compute the homology of certain loop spaces (cf. [2] and Theorem 4e in §4).

3C) In this subsection we use some elements of the theory of Coxeter groups, which can be found e.g. in [3]. Let $\Lambda \in P_+$; consider the orbit

$W(\Lambda)$. Recall the definition of the *Bruhat order* $\succeq$ on the set $W(\Lambda)$ [18]. This is the partial order generated by: $r_\alpha(\lambda) \succeq \lambda$ if $\alpha \in \Delta^{re}$ and $r_\alpha(\lambda) \geq \lambda$.

If $\Lambda \in P_{++}$, we may identify W with the set $W(\Lambda)$ by $w \Leftrightarrow w(\Lambda)$. We write $w' \leq w$ if $w'(\Lambda) \succeq w(\Lambda)$. It is easy to see that this definition is independent of the choice of $\Lambda \in P_{++}$ and coincides with the usual definition of the Bruhat order on W as the partial order generated by:

$$r_{i_1} \ldots r_{i_{s-1}} r_{i_{s+1}} \ldots r_{i_k} < w \quad (1 \leq s \leq k),$$

where $w = r_{i_1} \ldots r_{i_k}$ is a reduced expression; or, equivalently, generated by: $w_1 w_2 < w_1 r_i w_2$ if $w_1(\alpha_i) > 0$ and $w_2^{-1}(\alpha_i) > 0$.

Note that $w(\Lambda_i) < \Lambda_i$ if and only if w contains r_i in one (and hence every) reduced expression. Therefore, we have:

$$w(\Lambda_i) < \Lambda_i \quad \text{iff} \quad r_i \leq w. \tag{3.4}$$

The following lemma summarizes some of the results of [18, Theorem 1 and Corollaries 2 and 5].

Lemma 3.3. a) *For $\Lambda \in P_+$ and $\lambda \in W(\Lambda)$ let*

$$\mathcal{V}_\Lambda(\lambda)_\pm = U_\pm(L(\Lambda)_\lambda \setminus \{0\}).$$

Then[3]

(i) $\mathcal{V}_\Lambda \setminus \{0\} = \coprod_{\lambda \in W(\Lambda)} \mathcal{V}_\Lambda(\lambda)_\pm$;

(ii) $\overline{\mathcal{V}_\Lambda(\lambda)_+} \setminus \{0\} = \coprod_{\mu \succeq \lambda} \mathcal{V}_\Lambda(\mu)_+$;

(iii) $\overline{\mathcal{V}_\Lambda(\lambda)_-} \setminus \{0\} = \coprod_{\lambda \succeq \mu} \mathcal{V}_\Lambda(\mu)_-$.

b) *Given a subset X of I, let $W_X = \langle r_i \mid i \in X \rangle \subset W$ and $P_X = B_+ W_X B_+ \subset G$. Then:*

(i) $G = \coprod_{w \in W/W_X} B_+ w P_X$ (*Bruhat decomposition*);

[3] Here and further on, $\overline{M}$ denotes the Zariski closure of M unless otherwise specified.

(ii) $G = \coprod_{w \in W/W_X} B_- w P_X$ (*Birkhoff decomposition*);
(iii) $G = \bigcup_{w \in W} w B_- B_+$.

Remark 3.2. a) If G is of finite type, then Lemma 3.3b (i) and (ii) give equivalent decompositions of G. This decomposition is due to Gauss, Gelfand-Naimark, Bruhat and Harish-Chandra.

b) Let G be a group of affine type from Example 1.1b, so that $G/\mathbf{F}^* = \underline{G}(\mathbf{F}[z,z^{-1}])$. Let $\underline{H}$ be a Cartan subgroup of $\underline{G}$. Then $W = W_X \ltimes T$ is the affine Weyl group of $\underline{G}$, where W_X is the Weyl group of $\underline{G}$ and T is the group of "translations", which is isomorphic to $\underline{H}(\mathbf{F}[z,z^{-1}])/\underline{H}(\mathbf{F})$. Furthermore, $P_X/\mathbf{F}^* = G(\mathbf{F}[z])$ and $B_-/\mathbf{F}^* \subset \underline{G}(\mathbf{F}[z^{-1}])$. Then Lemma 3.3b (ii) gives:

$$\underline{G}(\mathbf{F}[z,z^{-1}]) = \underline{G}(\mathbf{F}[z^{-1}])\underline{H}(\mathbf{F}[z,z^{-1}])\,\underline{G}(\mathbf{F}[z]),$$

a result usually attributed to Grothendieck [8]. A special case of this is the decomposition:

$$SL_n(\mathbf{F}[z,z^{-1}]) = \coprod_{\substack{k_1 \le \cdots \le k_n \\ \sum k_i = 0.}} SL_n(\mathbf{F}[z^{-1}])\,\mathrm{diag}\,(z^{k_1},\ldots,z^{k_n}) SL_n(\mathbf{F}[z]).$$

This is due to Dedekind-Weber and Birkhoff [1].

Lemma 3.4. *Let $\mathcal{T}$ be a G-biinvariant topology on G such that (i) Zariski-closed subsets are $\mathcal{T}$-closed, and (ii) G_i lies in the $\mathcal{T}$-closure of $U_{-\alpha_i} H_i U_{\alpha_i}$ for all $i \in I$. Then for all $w \in W$, we have:*

(a) $$\coprod_{w' \le w} B_+ w' B_+ \text{ is the } \mathcal{T}\text{-closure of } B_+ w B_+$$

(b) $$\coprod_{w \le w'} B_- w' B_+ \text{ is the } \mathcal{T}\text{-closure of } B_- w B_+.$$

Proof. Fix $\Lambda \in P_{++}$ and consider the map $\phi: G \longrightarrow \mathcal{V}_\Lambda$ defined by $g \mapsto g(v_\Lambda)$. The map ϕ is Zariski-continuous and [18]:

$$\phi^{-1}(U_\pm(L(\Lambda)_{w(\Lambda)})) = B_\pm w B_+.$$

Hence, by (i) and Lemma 3.3 a, the $\mathcal{T}$-closure of $B_\pm w B_+$ is contained in the union in question.

In order to prove the reverse inclusion in a), suppose that $w = w_1 r_i w_2$, where $w_1(\alpha_i) > 0$, $w_2^{-1}(\alpha_i) > 0$. Then we have:

$$\begin{aligned} B_+ w B_+ = B_+ w_1 r_i w_2 B_+ &= B_+ (w_1 r_i U_{-\alpha_i} r_i w_1^{-1}) w_1 r_i w_2 (w_2^{-1} U_{\alpha_i} w_2) B_+ \\ &= B_+ w_1 r_i U_{-\alpha_i} H_i U_{\alpha_i} w_2 B_+. \end{aligned}$$

Since $\mathcal{T}$ is biinvariant we get (here $\overline{M}$ denotes the $\mathcal{T}$-closure of M):

$$\overline{B_+ w B_+} \supset B_+ w_1 r_i \overline{(U_{-\alpha_i} H_i U_{\alpha_i})} w_2 B_+.$$

Since, by (ii), $N_i \subset \overline{U_{-\alpha_i} H_i U_{\alpha_i}}$, we deduce that $\overline{B_+ w B_+} \supset B_+ w_1 w_2 B_+$. Similarly, we have:

$$\begin{aligned} B_- w_1 w_2 B_+ &= B_- (w_1 U_{-\alpha_i} w_1^{-1}) w_1 w_2 (w_2^{-1} U_{\alpha_i} w_2) B_+ \\ &= B_- w_1 U_{-\alpha_i} H_i U_{\alpha_i} w_2 B_+ \end{aligned}$$

and hence

$$\overline{B_- w_1 w_2 B_+} \supset B_- w_1 \overline{U_{-\alpha_i} H_i U_{\alpha_i}} w_2 B_+ \supset B_- w_1 r_i w_2 B_+ = B_- w B_+,$$

proving the reverse inclusion in b). Q.E.D.

Remark 3.3. The Zariski topology on G satisfies the hypothesis of Lemma 3.4.

Let $\Gamma_i = \{g \in G \mid \theta_{\Lambda_i}(g) = 0\}$. We deduce from (3.4) and Remark 3.3:

Corollary 3.1. a) $\Gamma_i = \overline{B_- r_i B_+} (i \in I)$.

b) $U_- H U_+ = G \setminus \bigcup_i \Gamma_i$ *is open in G, and therefore* $G = \bigcup_{w \in W} w U_- H U_+$ *is a covering of G by open sets.*

Remark 3.4. One can show that $w' \le w$ if and only if

$$B_+ w B_+ \cap B_- w' B_+ \ne \emptyset.$$

§4. The Structure of the Algebra $\mathbf{F}[G]$

4A) Recall that the Lie algebra $\mathfrak{g}$ carries the *principal gradation* $\mathfrak{g} = \bigoplus_{j\in\mathbf{Z}} \mathfrak{g}_j$ defined by $\deg e_i = -\deg f_i = 1$, $\deg \mathfrak{h} = 0$; let $\mathfrak{g}_{(j)} = \bigoplus_{i\geq j} \mathfrak{g}_i$ be the associated filtration. Let $U_j (j \geq 1)$ be the descending central series of U_+: $U_1 = U_+$, $U_{j+1} = (U_+, U_j)$ for $j \geq 1$.

Lemma 4.1. *For any $u \in U_j (j \geq 1)$ there exists a unique element $\phi_j(u)$ of $\mathfrak{g}_j$ such that*

(i) $Ad(u)x - x \equiv [\phi_j(u), x] \bmod \mathfrak{g}_{(j+k+1)}$ *for all* $x \in \mathfrak{g}_{(k)}$, $k \in \mathbf{Z}$; *moreover, we have:*

(ii) $\phi_1(\exp te_i) = te_i \quad (i \in I)$, $\phi_1(\exp te_\alpha) = 0$ *if* $\alpha \in \Delta_+^{re} \setminus \Pi$;

(iii) $\phi_j(uu') = \phi_j(u) + \phi_j(u')$;

(iv) $\phi_{j+j'}((u,u')) = [\phi_j(u), \phi_{j'}(u')]$;

(v) $\phi_j \quad (j \geq 1)$ *is surjective.*

Proof. (i) for $k = 0$ gives uniqueness of $\phi_j(u)$. It is easy to check that (i) implies (ii), (iii), (iv). We construct $\phi_j(u)$ satisfying (i) by induction on $j \geq 1$ using (ii), (iii), (iv). Finally, (v) follows from (ii), (iii) and (iv). Q.E.D.

Corollary 4.1. *Let $h \in \mathfrak{h}$ be such that $\alpha(h) \neq 0$ for all $\alpha \in \Delta$. Then for all $k \geq 1$, $Ad(U_+)$ acts transitively on $(h + \mathfrak{n}_+) \bmod \mathfrak{g}_{(k)}$.*

4B) Given two sets B_1 and B_2, we have the canonical inclusion

$$\mathbf{F}^{B_1} \otimes \mathbf{F}^{B_2} \to \mathbf{F}^{B_1 \times B_2}$$

given by $(\phi_1 \otimes \phi_2)(b_1, b_2) = \phi_1(b_2)\phi_2(b_2)$. Let P be a group, $\mathbf{F}[P]$ an algebra of $\mathbf{F}$-valued functions on P containing $\mathbf{F}$. We say that $\mathbf{F}[P]$ is *naturally a Hopf algebra* if for the multiplication map μ: $P \times P \longrightarrow P$, we have $\mu^*(\mathbf{F}[P]) \subset \mathbf{F}[P] \otimes \mathbf{F}[P]$, and for the inversion map i: $P \longrightarrow P$, we have $i^*(\mathbf{F}[P]) = \mathbf{F}[P]$.

For any subgroup U of U_+ or U_- considered in the sequel, we denote by $\mathbf{F}[U]$ the restriction of $\mathbf{F}[G]$ to U.

Lemma 4.2. $\mathbf{F}[U_+]$ *and* $\mathbf{F}[U_-]$ *are naturally Hopf algebras.*

Proof. We prove the lemma for U_+. By Theorem 1, every $f \in \mathbf{F}[U_+]$ is a linear combination of functions $f_{v^*,v}$ where $v \in L(\Lambda)$, $v^* \in L^*(\Lambda) (\Lambda \in P_+)$. Since U_+ acts locally unipotently on $L(\Lambda)$, and $\pi_\Lambda(u^{-1}) = \exp(-\log \pi_\Lambda(u))$, the lemma is clear. Q.E.D.

Remark 4.1. $\mathbf{F}[G]$ is naturally a Hopf algebra if and only if $\dim \mathfrak{g} < \infty$. If $\dim \mathfrak{g} = \infty$, then $\mu^*(\mathbf{F}[G]) \not\subset \mathbf{F}[G] \otimes \mathbf{F}[G]$ and $i^*(\mathbf{F}[G]) \not\subset \mathbf{F}[G]$. Note that $i^*(\mathbf{F}[G]) = \omega^*(\mathbf{F}[G])$.

4C) **Lemma 4.3.** a)*Let $h \in \mathfrak{h}$ be such that $\alpha(h) \neq 0$ for all $\alpha \in \Delta$. Then the map ψ: $U_+ \longrightarrow \mathfrak{n}_+$ defined by $\psi(u) = Ad(u)h - h$ induces an isomorphism $\psi^* : \mathbf{F}[\mathfrak{n}_+] \xrightarrow{\sim} \mathbf{F}[U_+]$.*

b) *Fix $\Lambda \in P_{++}$; $\mathbf{F}[U_+]$ is a polynomial algebra on generators $f_{v_\Lambda^*, x_i(v_\Lambda)}$ (restricted to U_+), where $\{x_i\}$ is a basis of $\mathfrak{n}_-$.*

Proof. b) follows from a) and the formula:

$$\langle a(v_\Lambda), v_\Lambda^* \rangle = (a \mid \nu^{-1}(\Lambda)) \quad \text{for } a \in \mathfrak{g}.$$

Indeed, by this formula, $(Ad(u)\nu^{-1}(\Lambda) \mid a) = f_{v_\Lambda^*, a(v_\Lambda)}(u^{-1})$, and we apply Lemma 4.2.

To prove a), fix $\lambda \in P_{++}$; by [18, Lemma 5b], the map ϕ: $\mathfrak{n}_+ \longrightarrow L^*(\lambda)$ defined by $\phi(n) = n(v_\lambda^*)$ is injective. Hence, we may identify $\mathfrak{n}_+$ with its image in $L^*(\lambda)$ and $\mathbf{F}[\mathfrak{n}_+]$ with the restriction of $\mathbf{F}[L^*(\lambda)]$ to $\phi(\mathfrak{n}_+)$. Take $v \in L(\lambda)$. By Lemma 4.2, we may write: $u^{-1}(v) = \sum_i f_i(u) v_i$ (finite sum) for $u \in U_+$, where $f_i \in \mathbf{F}[U_+]$ and $v_i \in L(\lambda)$. Hence, the function $u \mapsto \langle v, (Ad(u)h)v_\lambda^* \rangle = -\sum_i f_i(u) \langle u(h(v_i)), v_\lambda^* \rangle$ lies in $\mathbf{F}[U_+]$, showing that $\psi^*(\mathbf{F}[\mathfrak{n}_+]) \subset \mathbf{F}[U_+]$. ψ^* is injective by Corollary 4.1.

To show that ψ^* is surjective, choose a basis $e_\alpha^{(j)}$ of $\mathfrak{g}_\alpha$ for each $\alpha \in \Delta_+$, such that $e_{\alpha_i}^{(1)} = e_i$. Then we have: $Ad(u)h = h + \sum_{\alpha\in\Delta_+} \sum_j \varphi_\alpha^{(j)}(u) e_\alpha^{(j)}$, where $\varphi_\alpha^{(j)} \in B := \psi^*(\mathbf{F}[\mathfrak{n}_+])$. Choose $h' \in \mathfrak{h}$ such that $\alpha(h') \neq 0$ for all non-zero $\alpha \in Q_+$. Then from $[Ad(u)h, Ad(u)h'] = 0$ we deduce that $Ad(u)h' = h' + \sum_{\alpha\in\Delta_+} \sum_j \tilde{\varphi}_\alpha^{(j)}(u) e_\alpha^{(j)}$, where $\tilde{\varphi}_\alpha^{(j)} \in B$ by induction on $ht\,\alpha$. Using this, the equation $[Ad(u)h', Ad(u)f_i] = -\alpha_i(h') Ad(u) f_i$ gives:

$$Ad(u)f_i = f_i - \alpha_i(h')^{-1} \tilde{\varphi}_{\alpha_i}^{(1)}(u) h_i + \sum_{\alpha\in\Delta_+} \sum_j \varphi_{\alpha,i}^{(j)}(u) e_\alpha^{(j)},$$

where $\varphi_{\alpha,i}^{(j)} \in B$, again by induction on $ht\,\alpha$.

Now, functions of the form $f_{v^*,v}$, where $v = f_{i_1}\dots f_{i_k}(v_\mu)$, $\mu \in P_+$, $i_1, \dots, i_k \in I$ and $v^* \in L^*(\mu)$, generate $\mathbf{F}[U_+]$. But

$$f_{v^*,v}(u) = \langle (Ad(u)f_{i_1})\dots(Ad(u)f_{i_k})v_\mu, v^*\rangle$$

so that $f_{v^*,v} \in B$ since the $\varphi_{\alpha,i}^{(j)} \in B$. Q.E.D.

Remark 4.2. The map $\psi: U_+ \longrightarrow \mathfrak{n}_+$ is injective; however, ψ is surjective only if $\dim \mathfrak{g} < \infty$.

4D) Put $S = \{\theta_\lambda | \lambda \in P_+\} \subset \mathbf{F}[G]$. This is a multiplicative set since $\theta_\lambda\theta_\mu = \theta_{\lambda+\mu}$. We put $\theta_i = \theta_{\Lambda_i}$ for short. Denote by $\mathbf{F}[H]$ the algebra of functions on H generated by S and S^{-1}. We have: $\mathbf{F}[H] \cong \mathbf{F}[\theta_i, \theta_i^{-1};\ \ i \in I]$, the coordinate ring of $(\mathbf{F}^*)^I$.

Lemma 4.4. *The map $\phi: U_- \times H \times U_+ \longrightarrow G$ defined by*

$$\phi(u_-, h, u_+) = u_- h u_+$$

induces an isomorphism ϕ^*: $S^{-1}\mathbf{F}[G] \xrightarrow{\sim} \mathbf{F}[U_-] \otimes \mathbf{F}[H] \otimes \mathbf{F}[U_+]$. In particular, (by Lemma 4.3b), $S^{-1}\mathbf{F}[G]$ is a unique factorization domain.

Proof. Using Theorem 1, one can easily check that

$$\phi^*(S^{-1}\mathbf{F}[G]) \subset \mathbf{F}[U_-] \otimes \mathbf{F}[H] \otimes \mathbf{F}[U_+];$$

ϕ^* is injective by Lemma 2.1d.

To prove surjectivity of ϕ^* we use the formulas:

$$\phi^*(\theta_\lambda) = 1 \otimes \theta_\lambda|_H \otimes 1,$$
$$\phi^*(\theta_\lambda^{-1} f_{v_\lambda^*,v}) = 1 \otimes 1 \otimes f_{v_\lambda^*,v}|_{U_+},$$
$$\phi^*(\theta_\lambda^{-1} f_{v^*,v_\lambda}) = f_{v^*,v_\lambda}|_{U_-} \otimes 1 \otimes 1,$$

and apply Lemma 4.3b. Q.E.D.

Corollary 4.2. *Let $\mathbf{F}$ be algebraically closed. If $\mathfrak{a}$ is a finitely generated ideal of $S^{-1}\mathbf{F}[G]$ and $f \in S^{-1}\mathbf{F}[G]$ vanishes on the zero set of $\mathfrak{a}$ in U_-HU_+, then $f \in \sqrt{\mathfrak{a}}$.*

Proof. Recall the map $\psi: U_+ \longrightarrow \mathfrak{n}_+$ defined in Lemma 4.3; similarly, we define the map $\psi_-: U_- \longrightarrow \mathfrak{n}_-$. Define a map $\sigma: U_-HU_+ \longrightarrow \mathfrak{n}_- \times H \times \mathfrak{n}_+$ by $\sigma(u_-hu_+) = (\psi_-(u_-), h, \psi(u_+))$. Then, by Lemmas 4.3 and 4.4, σ induces an isomorphism σ^*: $\mathbf{F}[\mathfrak{n}_-] \otimes \mathbf{F}[H] \otimes \mathbf{F}[\mathfrak{n}_+] \xrightarrow{\sim} S^{-1}\mathbf{F}[G]$. By Corollary 4.1, given $q_1, \dots, q_s \in \mathbf{F}[\mathfrak{n}_-] \otimes \mathbf{F}[H] \otimes \mathbf{F}[\mathfrak{n}_+]$ and $x \in \mathfrak{n}_- \times H \times \mathfrak{n}_+$, there exists $x' \in \sigma(U_-HU_+)$ such that $q_i(x) = q_i(x')(1 \le i \le s)$. Now we apply Hilbert's Nullstellensatz to $(\sigma^*)^{-1}f$ and $(\sigma^*)^{-1}\mathfrak{a}$. Q.E.D.

Lemma 4.5. a) *Let $w \in W$ and let $U_1 = U_+ \cap wU_-w^{-1}$, $U_2 = U_+ \cap wU_+w^{-1}$. Then the map $\psi: U_1 \times U_2 \longrightarrow U_+$ defined by $\psi(u_1, u_2) = u_1u_2$ induces an isomorphism ψ^*: $\mathbf{F}[U_+] \xrightarrow{\sim} \mathbf{F}[U_1] \otimes \mathbf{F}[U_2]$.*

b) *Moreover, let $\alpha \in \Pi$ be such that $w(\alpha) \in \Delta_+$ and let $U_3 = U_+ \cap (wr_\alpha)U_-(wr_\alpha)^{-1}$. Then the map $\phi: U_1 \times U_{w(\alpha)} \longrightarrow U_3$ defined by $\phi(u, u') = uu'$ induces an isomorphism ϕ^*: $\mathbf{F}[U_3] \xrightarrow{\sim} \mathbf{F}[U_1] \otimes \mathbf{F}[U_{w(\alpha)}]$.*

c) *Let $\beta \in \Delta^{re}$. Then $\mathbf{F}[U_\beta]$ is a polynomial algebra over $\mathbf{F}$ in one variable x, where $x(\exp te_\beta) = t$.*

Proof. By Lemma 4.2, $\psi^*(\mathbf{F}[U_+]) \subset \mathbf{F}[U_1] \otimes \mathbf{F}[U_2]$. ψ^* is injective since ψ is onto by [18, Corollary 5b]. To see that $\psi^*(\mathbf{F}[U_+]) \supset \mathbf{F}[U_1] \otimes \mathbf{F}[U_2]$, fix $\lambda \in P_{++}$ and choose $n \in wH$. Then, for $v \in L(\lambda)$ and $v^* \in L^*(\lambda)$, we have:

$$\psi^*(f_{v^*,n(v_\lambda)}) = f_{v^*,n(v_\lambda)}|_{U_1} \otimes 1,$$
$$\psi^*(f_{n(v_\lambda^*),v}) = 1 \otimes f_{n(v_\lambda^*),v}|_{U_2}.$$

But the $f_{v^*,n(v_\lambda)}|_{U_1}$ (resp. $f_{n(v_\lambda^*),v}|_{U_2}$) generate $\mathbf{F}[U_1]$ (resp. $\mathbf{F}[U_2]$), as seen by applying Lemmas 4.2 and 4.3b and conjugating by n. This proves a).

Since $U_{w(\alpha)} \subset U_2$ and $U_1U_{w(\alpha)} = U_3$, b) is clear from a) by restriction. c) for $\beta \in \Pi$ follows from the proof of a) in the case $w = r_\beta$; the general case then follows by conjugating by elements of N. Q.E.D.

4E) We proceed to prove the main result of this section:

Theorem 3. *The ring $\mathbf{F}[G]$ is a unique factorization domain (UFD).*

The proof is based on the following simple fact. (Its proof can be easily extracted from [17, p. 43].)

Lemma 4.6. *Let R be an integral domain and $p_1, \ldots, p_m$ prime elements of R (p is called prime if $p \neq 0$ and (p) is a prime ideal). Suppose that:*

(i) $\bigcap_{k=1}^{\infty}(p_i^k) = 0$ *for all i;*
(ii) $S^{-1}R$ *is a UFD, where S is the multiplicative system generated by* $p_1, \ldots, p_m$.

Then R is a UFD.

We apply this lemma to $R = \mathbf{F}[G]$ and $p_i = \theta_i (i \in I)$. Using Lemmas 2.1 a,b and 4.4, it suffices to show that the elements θ_i are prime.

For $f \in \mathbf{F}[G]$ and $n \in G$, we denote by ${}^n f$ the strongly regular function ${}^n f(g) = f(ng)$, $g \in G$. We will deduce that θ_i is prime from the following lemma.

Lemma 4.7. *For $i \in I$ and $n \in N$, ${}^n\theta_i$ is either a prime element or a unit in $S^{-1}\mathbf{F}[G]$.*

We may (and will) assume in the proof of Lemma 4.7 and the following deduction from it that θ_i is prime that $\mathbf{F}$ is algebraically closed.

Assume that Lemma 4.7 holds. Suppose that θ_i divides $f_1 f_2$, where $f_1, f_2 \in \mathbf{F}[G]$; we must show that θ_i divides one of f_1, f_2. By Corollary 3.1a, the set Γ_i of zeros of θ_i on G is the closure of $B_- r_i B_+ = r_i U_-^{\alpha_i} B_+$, where $U_-^{\alpha_i} = U_- \cap r_i U_- r_i^{-1}$ (cf. [18]). By Lemmas 4.4 and 4.5a, the restriction of $\mathbf{F}[G]$ to $U_-^{\alpha_i} B_+$ is an integral domain. Hence, one of the f_k, say f_1, vanishes on Γ_i. Lemma 4.7 and Corollary 4.2 now imply that $({}^n\theta_i)^{-1}({}^n f_1) \in S^{-1}\mathbf{F}[G]$ for all $n \in N$. Corollaries 3.1b and 2.1 now force $\theta_i^{-1} f_1 \in \mathbf{F}[G]$, proving Theorem 3. Q.E.D.

Proof of Lemma 4.7. We proceed by induction on $l(w)$, where $n \in wH$, $w \in W$. If $l(w) = 0$, i.e. $n \in H$, then ${}^n\theta_i \in \mathbf{F}^* S$ is a unit in $S^{-1}\mathbf{F}[G]$. Otherwise, choose $j \in I$ such that $l(r_j w) < l(w)$. Put $w' = r_j w$, choose $n_j \in r_j H$ and put $n' = n_j^{-1} n$. If $j \neq i$, then ${}^n\theta_i = \theta_i(n_j)({}^{n'}\theta_i) \in \mathbf{F}^*({}^{n'}\theta_i)$ is prime or a unit in $S^{-1}\mathbf{F}[G]$ by the inductive assumption. If $j = i$, put $U_0 = U_- \cap w^{-1} U_- w$, $U_1 = U_{-\alpha_i}$, $U_2 = U_- \cap (w'^{-1} U_+ w')$, and define the map

$$\psi \colon (U_0 \times U_1 \times U_2) \times H \times U_+ \longrightarrow G$$

by $\psi(u_0, u_1, u_2, h, u_+) = u_0(n'^{-1} u_1 n') u_2 h u_+$. Then Lemmas 4.2, 4.4 and 4.5 a,b show that ψ induces an isomorphism

$$\psi^* \colon S^{-1}\mathbf{F}[G] \longrightarrow \mathbf{F}[U_0] \otimes \mathbf{F}[U_1] \otimes \mathbf{F}[U_2] \otimes \mathbf{F}[H] \otimes \mathbf{F}[U_+].$$

Put $f = \theta_i^{-1}({}^n\theta_i)$, $f' = \theta_i^{-1}({}^{n'}\theta_i)$, $x = {}^{n_i}\theta_i|_{U_1}$. Then x generates the polynomial algebra $\mathbf{F}[U_1]$ by Lemma 4.5c and we compute, using $n_i u_1^{-1} n_i^{-1}(v_{\Lambda_i}^*) = v_{\Lambda_i}^* + x(u_1) n_i(v_{\Lambda_i}^*)$, that:

$$\psi^*(f') = 1 \otimes 1 \otimes f'|_{U_2} \otimes 1 \otimes 1. \tag{4.1}$$

$$\psi^*(f) = 1 \otimes x \otimes f'|_{U_2} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes f|_{U_2} \otimes 1 \otimes 1. \tag{4.2}$$

Suppose that ${}^n\theta_i$ is not prime or a unit in $S^{-1}\mathbf{F}[G]$, so that $\psi^*(f)$ is not prime or a unit. Since $S^{-1}\mathbf{F}[G]$ is a UFD, (4.1) and (4.2) show that $\psi^*(f)$ and $\psi^*(f')$ have a nontrivial common factor. Hence, by the inductive assumption, $\psi^*(f')$ is prime. Now, the set $P := ((w')^{-1} B_- r_i B_+) \cap (U_- H U_+)$ is non-empty by Corollary 4.2, since $(w')^{-1}\overline{B_- r_i B_+}$ is the set of zeros of ${}^{n'}\theta_i$ on G and $U_- H U_+$ is open (see Corollary 3.1). But ${}^n\theta_i$ vanishes nowhere on P, since $nP \subset U_- H U_+$. Hence, $\psi^*(f')$ does not divide $\psi^*(f)$. This contradiction completes the proof of the lemma and of Theorem 3. Q.E.D.

Remark 4.3. It is easy to see that $f \in \mathbf{F}[G]$ is divisible by θ_i if it vanishes on Γ_i, even if $\mathbf{F}$ is not algebraically closed.

Corollary 4.3. a) $\mathbf{F}[G]^{U_+}$ *is a UFD.*
b) $\mathbf{F}[G]$, $\mathbf{F}[G]^{U_+}$ *and* $\mathbf{F}[\mathcal{V}_\Lambda]$, $\Lambda \in P_+$, *are integrally closed.*
c) $\mathbf{F}[\mathcal{V}_\Lambda]$, $\Lambda \in P_+$, *is a UFD if and only if* $\Lambda = \Lambda_i$ *for some* $i \in I$ *or* $\Lambda = 0$.

Proof. The group U_+ acts by automorphisms locally unipotently on the UFD $\mathbf{F}[G]$ with unit group $\mathbf{F}^*$; a) follows. Since a UFD is integrally closed, and since the ring of invariants of a group acting by automorphisms on an integrally closed domain is integrally closed, b) follows from Theorem 3, using Corollary 2.3 and Lemma 3.1. c) is proved using the P_+-gradation $\mathbf{F}[G]^{U_+} \cong \bigoplus_\Lambda L^*(\Lambda)$ (see Corollary 2.2) and Lemma 3.1. Q.E.D.

Remark 4.4. a) The fact that the coordinate ring of a connected simply-connected simple algebraic group is a UFD is well-known. The earliest reference that we know is Voskresenskii [22] (see also [19]).
b) It is not difficult to see that the results of [21] can be extended to our setup.

Remark 4.5. Assume that $\mathbf{F}$ is algebraically closed. Let M be a subset of G. A function f on M is called *strongly regular* if for every $x \in M$ there exist a neighborhood U of x and functions $f_1, f_2 \in \mathbf{F}[G]$, such that f_2 does not vanish on $M \cap U$, for which $f = f_1/f_2$ on $M \cap U$. Denote by $\mathbf{F}[M]$ the ring of strongly regular functions on M. This definition coincides with the original one when $M = G$, $U_+ \cap wU_\pm w^{-1} (w \in W)$ or H.

Remark 4.6. It is clear that $\mathbf{F}[G]|_H$ is spanned by the characters of $H \simeq (\mathbf{F}^*)^I$ which appear as weights of the G-modules $L(\Lambda)(\Lambda \in P_+)$. But the union of the sets of weights of all $L(\Lambda)(\Lambda \in P_+)$, restricted to $\mathfrak{h}'$, coincides with $\sum_i \mathbf{Z}\Lambda_i|_{\mathfrak{h}'}$ if and only if G is of finite type. It follows that $\mathbf{F}[G]|_H = \mathbf{F}[H]$ if and only if G is of finite type. This phenomenon is related to Remark 2.2.

4F) The inversion map $i\colon G \longrightarrow G$ clearly induces an automorphism

$$i^*\colon \mathbf{F}[G]_{w.r.} \longrightarrow \mathbf{F}[G]_{w.r.}.$$

However, $i^*(\mathbf{F}[G]) \neq \mathbf{F}[G]$ if G is of infinite type (cf. Remarks 4.1 and 4.6). Let $\mathbf{F}[G]_r$ denote the closure in $\mathbf{F}[G]_{w.r.}$ of the subalgebra generated by $\mathbf{F}[G]$ and $i^*(\mathbf{F}[G])$. Elements of $\mathbf{F}[G]_r$ are called *regular.*

We now indicate how G may be viewed as an infinite-dimensional affine group in the sense of Shafarevich [20] with coordinate ring $\mathbf{F}[G]_r$. Let $\mathbf{A} = \bigoplus_i \mathbf{F}_i$ be a direct sum, possibly infinite, of copies of $\mathbf{F}$. Then we can apply the terminology of subsection 3A. Let $\mathcal{V} \subset \mathbf{A}$ be the zero set of an ideal of $\mathbf{F}[\mathbf{A}]_r$; closed subvarieties of intersections of $\mathcal{V}$ with finite-dimensional subspaces of $\mathbf{A}$ are called *finite subvarieties* of $\mathcal{V}$. Such a subset $\mathcal{V}$ of $\mathbf{A}$ with a group structure is called an *affine group of Shafarevich type* if the multiplication map $\mu\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and the inversion map $i\colon \mathcal{V} \longrightarrow \mathcal{V}$ have the following property: for every finite subvariety M of $\mathcal{V}$ there exists a finite subvariety N of $\mathcal{V}$ such that $\mu(M \times M) \subset N$, $i(M) \subset N$ and the induced maps $\mu\colon M \times M \longrightarrow N$, $i\colon M \longrightarrow N$ are morphisms.

Now we construct an injection

$$\phi\colon G \to \mathbf{A} := \Big(\bigoplus_{i\in I} L(\Lambda_i)\Big) \oplus \Big(\bigoplus_{i \in I} L^*(\Lambda_i)\Big).$$

Let $v = \sum_i v_{\Lambda_i}$, $v^* = \sum_i v^*_{\Lambda_i}$, and define $\phi(g) = g(v + v^*)$; ϕ is injective by [18, Corollary 3a]. Furthermore, $\phi(G) \subset \mathbf{A}$ is defined by the following system of equations: $x = \sum x_i + \sum x_i^*$, where $x_i \in L(\Lambda_i)$, $x_i^* \in L^*(\Lambda_i)$, lies in $\phi(G)$ if and only if

$$x_i \otimes x_j \in L(\Lambda_i + \Lambda_j) \subset L(\Lambda_i) \otimes L(\Lambda_j),$$
$$x_i^* \otimes x_j^* \in L^*(\Lambda_i + \Lambda_j) \subset L^*(\Lambda_i) \otimes L^*(\Lambda_j),$$

and $\langle x_i, x_i^* \rangle = 1$ for all $i, j \in I$. This follows easily from [18, Theorem 1b], using the idea of the proof of Theorem 2.

Furthermore, one can show that ϕ induces an isomorphism

$$\phi^*\colon \mathbf{F}[\phi(G)]_r \xrightarrow{\sim} \mathbf{F}[G]_r$$

and that $\phi(G)$ is an affine group of Shafarevich type with Lie algebra $\mathfrak{g}'$. One can show that G operates morphically on $L(\Lambda)\,(\Lambda \in P_+)$ and $\mathfrak{g}'$; in particular, the matrix coefficients of G on $L(\Lambda)$, $L^*(\Lambda)$ and $\mathfrak{g}'$ are regular.

4G) Let $\mathbf{F}$ be a non-discrete locally-compact topological field. We call a subset U of G *open* if $x_{\overline{\beta}}^{-1}(U) \subset \mathbf{F}^k$ is open for all $\overline{\beta} \in (\Delta^{re})^k$, $k \in \mathbf{Z}_+$. G is a Hausdorff σ-compact topological group (and hence is paracompact) in this topology. Similarly, we call a subset U of $L(\Lambda)$ *open* if $x^{-1}(U) \subset \mathbf{F}^k$ is open for all $x \in \mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^k, L(\Lambda)), k \in \mathbf{Z}_+$. The following results will be proved in a subsequent paper. (See [18] for definitions.)

Theorem 4. *Let $\Lambda \in P_+$ and let $X = \{i \in I | \Lambda(h_i) = 0\}$. Then:*
a) *The multiplication map $U_- \times H \times U_+ \to U_- H U_+$ is a homeomorphism and $U_- H U_+$ is open in G.*

b) *The canonical map $G \to G/P_X$ is a fibration, and the map $gP_X \mapsto g(\mathbf{F}^* v_\Lambda)$ of G/P_X onto $\mathbf{P}\mathcal{V}_\Lambda$ is a homeomorphism.*

c) *If $\mathbf{F} = \mathbf{C}$, then G is a connected simply-connected topological group.*

d) *If $\mathbf{F} = \mathbf{C}$, then $H_+ \times U_+$ is contractible and the multiplication map $K \times H_+ \times U_+ \to G$ is a homeomorphism.*

e) *If* $\mathbf{F} = \mathbf{C}$, *then* G/P_X *is a CW-complex with cells* B_+wP_X/P_X, *where* $w \in W/W_X$, *of dimension* $2d_X(w)$, *where* $d_X(w)$ *is the length of the shortest element of* wW_X.

4H) *Open problems.*

a) Is it true that the rings $\mathbf{F}[\overline{\mathcal{V}_\Lambda(\lambda)_\pm}]$ are integrally closed? (This would imply that the closures of finite Schubert cells are normal (see [18, Remark (iii)]), as is known for finite type groups [6].)

b) Compute $\operatorname{Specm}\mathbf{F}[G]$. (Recall that $\operatorname{Specm}\mathbf{F}[G]$ is larger than G if G is of infinite type, by Remark 2.2).

c) Is it true that the sum of two closed ideals of $\mathbf{F}[G]$ (or $\mathbf{F}[G]_{w.r.}$) is closed? In particular, is it true that every finitely-generated ideal of $\mathbf{F}[G]$ is closed?

d) Let $\mathbf{F}$ be algebraically closed. Is it true that every proper finitely-generated ideal of $\mathbf{F}[G]$ vanishes at some point of G? (It is obviously true for principal ideals.)

e) Is it true that $\mathbf{F}[G]_{w.r.} = \mathbf{F}[G]_r$?

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Examples of Surfaces of General Type with Vector Fields

William E. Lang

To I.R. Shafarevich

The purpose of this paper is to introduce some new surfaces of general type, called generalized Raynaud surfaces, and to prove that in many cases these surfaces possess global vector fields, contradicting a guess of Rudakov-Shafarevich [3].

In a lecture at M.I.T. in October 1981, H. Kurke announced that he and P. Russell had found surfaces of general type with vector fields. These surfaces were of the form Y^D, where Y is a ruled surface, and D is a p-closed vector field with divisorial singularities. While all details were not given, the calculations seemed rather involved. The structure of the resulting surface, however, was quite simple. Inspired by Kurke's talk (and by conversations with M. Artin), I tried to generalize the elementary construction of Raynaud surfaces in characteristic three studied in [1] to higher characteristic, and finished the construction given here in November 1981. These surfaces are also of the form Y^D, and I suspect that they are deformations of the Kurke-Russell examples; however, both the construction of the surfaces and the method used to prove that some of the surfaces have vector fields are quite different from those of Kurke and Russell, and I hope more transparent.

1. Construction of Generalized Raynaud Surfaces

Let p be a prime number, and let n and d be positive integers, such that if $p \neq 2$, and d is odd, n is also odd. Let k be an algebraically closed field of characteristic p.

Definition. A *generalized Tango curve* over k of type (p, n, d) is a triple $(C, \mathcal{L}, dt)$, where C is a smooth curve over k, $\mathcal{L}$ is a line bundle on C of degree d, and dt is a nowhere vanishing section of $K_C \otimes \mathcal{L}^{\otimes p(1-mp)}$ which is locally exact.