The n-Component KP Hierarchy and Representation Theory

V.G. Kac[1] *and J.W. van de Leur*[2]

[1]Department of Mathematics, MIT, Cambridge, MA 02139, USA
[2]Department of Mathematics, University of Utrecht, Utrecht, The Netherlands

§0. Introduction.

0.1. The remarkable link between the soliton theory and the group GL_∞ was discovered in the early 1980s by Sato [S] and developed, making use of the spinor formalism, by Date, Jimbo, Kashiwara and Miwa [DJKM1,2,3], [JM]. The basic object that they considered is the KP hierarchy of partial differential equations, which they study through a sequence of equivalent formulations that we describe below. The first formulation is a deformation (or Lax) equation of a formal pseudo-differential operator $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \dots$, introduced in [S] and [W1]:

(0.1.1)
$$\frac{\partial L}{\partial x_n} = [B_n, L], \ n = 1, 2, \dots .$$

Here u_i are unknown functions in the indeterminates $x_1, x_2, \dots$, and $B_n = (L^n)_+$ stands for the differential part of L^n. The second formulation is given by the following zero curvature (or Zakharov-Shabat) equations:

(0.1.2)
$$\frac{\partial B_m}{\partial x_n} - \frac{\partial B_n}{\partial x_m} = [B_n, B_m], \ m, n = 1, 2, \dots .$$

These equations are compatibility conditions for the following linear system

(0.1.3)
$$Lw(x,z) = zw(x,z), \ \frac{\partial}{\partial x_n} w(x,z) = B_n w(x,z), \ n = 1, 2, \dots$$

on the wave function

(0.1.4)
$$w(x,z) = (1 + w_1(x)z^{-1} + w_2(x)z^{-2} + \dots)e^{x_1 z + x_2 z^2 + \dots}.$$

Provided that (0.1.2) holds, the system (0.1.3) has a unique solution of the form (0.1.4) up to multiplication by an element from $1 + z^{-1}\mathbb{C}[[z^{-1}]]$. Introduce the wave operator

(0.1.5)
$$P = 1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \dots ,$$

so that $w(x,z) = Pe^{x_1 z + x_2 z^2 + \dots}$. Then the existence of a solution of (0.1.3) is equivalent to the existence of a solution of the form (0.1.5) of the following Sato equation, which is the third formulation of the KP hierarchy [S], [W1]:

(0.1.6)
$$\frac{\partial P}{\partial x_k} = -(P \circ \partial \circ P^{-1})_- \circ P, \ k = 1, 2, \dots ,$$

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where the formal pseudo-differential operators P and L are related by

(0.1.7) $$L = P \circ \partial \circ P^{-1}.$$

Let $P^* = 1 + (-\partial)^{-1} \circ w_1 + (-\partial)^{-2} \circ w_2 + \dots$ be the formal adjoint of P and let

$$w^*(x,z) = (P^*)^{-1} e^{-x_1 z - x_2 z^2 - \dots}$$

be the adjoint wave function. Then the fourth formulation of the KP hierarchy is the following bilinear identity

(0.1.8) $$\mathrm{Res}_{z=0} w(x,z) w^*(x',z) dz = 0 \text{ for any } x \text{ and } x'.$$

Next, this bilinear identity can be rewritten in terms of Hirota bilinear operators defined for an arbitary polynomial Q as follows:

(0.1.9) $$Q(D) f(x) \cdot g(x) \overset{\mathrm{def}}{=} Q(\frac{\partial}{\partial y})(f(x+y) g(x-y))|_{y=0}.$$

Towards this end, introduce the famous τ-function $\tau(x)$ by the formulas:

(0.1.10) $$w(x,z) = \Gamma^+(z)\tau/\tau, \; w^*(x,z) = \Gamma^-(z)\tau/\tau.$$

Here $\Gamma^\pm(z)$ are the vertex operators defined by

(0.1.11) $$\Gamma^\pm(z) = e^{\pm(x_1 z + x_2 z^2 + \dots)} e^{\mp(z^{-1}\tilde{\partial}/\partial x_1 + z^{-2}\tilde{\partial}/\partial x_2 + \dots)},$$

where $\frac{\tilde{\partial}}{\partial x_j}$ stands for $\frac{1}{j}\frac{\partial}{\partial x_j}$. The τ-function exists and is uniquely determined by the wave function up to a constant factor. Substituting the τ-function in the bilinear identity (0.1.8) we obtain the fifth formulation of the KP hierarchy as the following system of Hirota bilinear equations:

(0.1.12) $$\sum_{j=0}^{\infty} S_j(-2y) S_{j+1}(\tilde{D}) e^{\sum_{r=1}^{\infty} y_r D_r} \tau \cdot \tau = 0.$$

Here $y = (y_1, y_2, \dots)$ are arbitrary parameters and the elementary Schur polynomials S_j are defined by the generating series

(0.1.13) $$\sum_{j \in \mathbb{Z}} S_j(x) z^j = \exp \sum_{k=1}^{\infty} x_k z^k.$$

The τ-function formulation of the KP hierarchy allows one to construct easily its N-soliton solutions. For that introduce the vertex operator [DJKM2,3]:

(0.1.14) $$\Gamma(z_1, z_2) =: \Gamma^+(z_1)\Gamma^-(z_2):$$

(where the sign of normal ordering : : means that partial derivatives are always moved to the right), and show using the bilinear identity (0.1.8) that if τ is a solution of (0.1.12), then $(1 + a\Gamma(z_1, z_2))\tau$, where $a, z_1, z_2 \in \mathbb{C}^\times$, is a solution as well. Since $\tau = 1$ is a solution, the function

(0.1.15) $$f_N \equiv (1 + a_1 \Gamma(z_1^{(1)}, z_2^{(1)})) \dots (1 + a_N \Gamma(z_1^{(N)}, z_2^{(N)})) \cdot 1$$

is a solution of (0.1.12) too. This is the τ-function of the N-soliton solution.

The first application of the KP hierarchy, as well as its name, comes from the fact that the simplest non-trivial Zakharov-Shabat equation, namely (0.1.2) with $m = 2$ and $n = 3$, is equivalent to the Kadomtsev-Petviashvili equation if we let $x_1 = x,\ x_2 = y,\ x_3 = t,\ u = 2u_1$:

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right). \tag{0.1.16}$$

Recall also that the celebrated KdV and Boussinesq equations are simple reductions of (0.1.16). Since the functions u and τ are related by

$$u = 2\frac{\partial^2}{\partial x^2}\log\tau, \tag{0.1.17}$$

the functions $2\frac{\partial^2}{\partial x^2}\log f_N$ are solutions of (0.1.16), called the N-soliton solutions.

0.2. The connection of the KP hierarchy to the representation theory of the group GL_∞ is achieved via the spinor formalism. Consider the Clifford algebra $C\ell$ on generators ψ_j^+ and ψ_j^- ($j \in \frac{1}{2}+\mathbb{Z}$) and the following defining relations (i.e. $\psi_i^\pm$ are free charged fermions):

$$\psi_i^+\psi_j^- + \psi_j^-\psi_i^+ = \delta_{i,-j},\ \psi_i^\pm\psi_j^\pm + \psi_j^\pm\psi_i^\pm = 0. \tag{0.2.1}$$

The algebra $C\ell$ has a unique irreducible representation in a vector space F (resp. F^*) which is a left (resp. right) module admitting a non-zero vector $|0\rangle$ (resp. $\langle 0|$) satisfying

$$\psi_j^\pm|0\rangle = 0 \text{ (resp. } \langle 0|\psi_{-j}^\pm = 0) \text{ for } j > 0. \tag{0.2.2}$$

These representations are dual to each other with respect to the pairing

$$\langle\langle 0|a,\ b|0\rangle\rangle = \langle 0|ab|0\rangle$$

normalized by the condition $\langle 0|1|0\rangle = 1$.

The Lie algebra $g\ell_\infty$ embeds in $C\ell$ by letting

$$r(E_{ij}) = \psi_{-i}^+\psi_j^-. \tag{0.2.3}$$

Exponentiating gives a representation R of the group GL_∞ on F and F^*. Let for $n \in \mathbb{Z}$:

$$\alpha_n = \sum_{j\in\frac{1}{2}+\mathbb{Z}} \psi_{-j}^+\psi_{j+n}^- \text{ for } n \neq 0,\ \alpha_0 = \sum_{j>0}\psi_{-j}^+\psi_j^- - \sum_{j<0}\psi_j^-\psi_{-j}^+. \tag{0.2.4}$$

and consider the following operator on F:

$$H(x) = \sum_{n=1}^{\infty} x_n\alpha_n. \tag{0.2.5}$$

For a positive integer m let

$$\langle\pm m| = \langle 0|\psi_{\frac{1}{2}}^\pm \ldots \psi_{m-\frac{1}{2}}^\pm \in F^* \text{ and } |\pm m\rangle = \psi_{-m+\frac{1}{2}}^\pm \ldots \psi_{-\frac{1}{2}}^\pm|0\rangle \in F.$$

Then the Fock space is realized on the vector space of polynomials $B = \mathbb{C}[x_1, x_2, \ldots; Q, Q^{-1}]$ via the isomorphism $\sigma:\ F \xrightarrow{\sim} B$ defined by

$$\sigma(a|0\rangle) = \sum_{m\in\mathbb{Z}} \langle m|e^{H(x)}a|0\rangle Q^m. \tag{0.2.6}$$

This remarkable isomorphism is called the boson-fermion correspondence and goes back to the work of Skyrme [Sk] and many other physicists; this beautiful form of it is an important part of the work of Date, Jimbo, Kashiwara and Miwa [DJKM2,3], [JM].

Using that

$$[\alpha_m, \alpha_n] = m\delta_{m,-n}, \tag{0.2.7}$$

(i.e that the α_n are free bosons), it is not difficult to show that the isomorphism σ is characterized by the following two properties [KP2]:

$$\sigma(|m\rangle) = Q^m, \ \sigma\alpha_n\sigma^{-1} = \frac{\partial}{\partial x_n} \text{ and } \sigma\alpha_{-n}\sigma^{-1} = nx_n \text{ if } n > 0. \tag{0.2.8}$$

Using (0.2.8), it is easy to recover the following well-known properties of the boson-fermion correspondence [DJKM2,3], [KP2]. Introduce the fermionic fields

$$\psi^{\pm}(z) = \sum_{j\in\frac{1}{2}+\mathbf{Z}} \psi_j^{\pm} z^{-j-1/2}.$$

Then one has:

$$\sigma\psi^{\pm}(z)\sigma^{-1} = Q^{\pm 1}z^{\pm\alpha_0}\Gamma^{\pm}(z), \tag{0.2.9}$$

$$\sigma\Big(\sum_{i,j\in\frac{1}{2}+\mathbf{Z}} r(E_{ij})z_1^{i-\frac{1}{2}}z_2^{-j-\frac{1}{2}}\Big)\sigma^{-1} = \frac{1}{z_1 - z_2}\Gamma(z_1, z_2). \tag{0.2.10}$$

Hence $\Gamma(z_1, z_2)$ lies in a "completion" of the Lie algebra $g\ell_\infty$ acting on B via the boson-fermion correspondence. Therefore, the group GL_∞ and its "completion" act on B and Date, Jimbo, Kashiwara and Miwa show that all elements of the orbit $\mathcal{O} = GL_\infty \cdot 1$ and its completions satisfy the bilinear identity (0.1.12). Since $\Gamma(z_1, z_2)^2 = 0$ and $\Gamma(z_1, z_2)$ lies in a completion of $g\ell_\infty$, we see that $\exp a\Gamma(z_1, z_2) = 1 + a\Gamma(z_1, z_2)$ leaves a completion of the orbit $\mathcal{O}$ invariant, which explains why (0.1.15) are solutions of the KP hierarchy.

Since the orbit $GL_\infty|0\rangle$ (which is the image of $\mathcal{O}$ in the fermionic picture) can be naturally identified with the cone over a Grassmannian, we arrive at the remarkable discovery of Sato that solutions of the KP hierarchy are parameterized by an infinite-dimensional Grassmannian [S].

0.3. It was subsequently pointed out in [KP2] and [KR] that the bilinear equation (0.1.8) (in the bosonic picture) corresponds to the following remarkably simple equation on the τ-function in the fermionic picture:

$$\sum_{k\in\frac{1}{2}+\mathbf{Z}} \psi_k^+\tau \otimes \psi_{-k}^-\tau = 0. \tag{0.3.1}$$

This is the fermionic formulation of the KP hierarchy. Since (0.3.1) is equivalent to

$$\mathrm{Res}_{z=0}\psi^+(z)\tau \otimes \psi^-(z)\tau = 0, \tag{0.3.2}$$

it is clear from (0.1.10 and 11) that equations (0.1.8) and (0.3.2) are equivalent. Since $\tau = |0\rangle$ obviously satisfies (0.3.1) and $R\otimes R(GL_\infty)$ commutes with the operator $\sum_k \psi_k^+ \otimes \psi_{-k}^-$, we see why any element of $R(GL_\infty)|0\rangle$ satisfies (0.3.1). Thus, the most natural approach to the KP hierarchy is to start with the fermonic formulation (0.3.1), go over to the bilinear identity (0.1.8) and then to all other formulations (see [KP2], [KR], [K]). This approach was generalized in [KW].

0.4. Our basic idea is to start once again with the fermonic formulation of KP, but then use the n-component boson-fermion correspondence, also considered by Date, Jimbo, Kashiwara and Miwa [DJKM1,2], [JM]. This leads to a bilinear equation on a matrix wave function, which in turn leads to a deformation equation for a matrix formal pseudo-differential operator, to matrix Sato equations and to matrix Zakharov-Shabat type equations.

The corresponding linear problem has been already formulated in Sato's paper [S] and Date, Jimbo, Kashiwara and Miwa [DJKM1] have written the corresponding bilinear equation for the wave function, but the connection between these formulations remained somewhat obscure.

It is the aim of the present paper to give all formulations of the n-component KP hierarchy and clarify connections between them. The generalization to the n-component KP is important because it contains many of the most popular systems of soliton equations, like the Davey-Stewartson system (for $n = 2$), the 2-dimensional Toda lattice (for $n = 2$), the n-wave system (for $n \geq 3$). It also allows us to construct natural generalizations of the Davey-Stewartson and Toda lattice systems. Of course, the inclusion of all these systems in the n-component KP hierarchy allows us to construct their solutions by making use of vertex operators.

Hirota's direct method [H] requires some guesswork to introduce a new function (the τ-function) for which the equations in question take a bilinear form. The inclusion of the equations in the n-component KP hierarchy provides a systematic way of construction of the τ-functions, the corresponding bilinear equations and a large family of their solutions.

The difficulty of the τ-function approach lies in the fact that the hierarchy contains too many Hirota bilinear equations. To deal with this difficulty we introduce the notion of an energy of a Hirota bilinear equation. We observe that the most interesting equations are those of lowest energy. For example, in the $n = 1$ case the lowest energy ($= 4$) non-trivial equation is the classical KP equation in the Hirota bilinear form, in the $n = 2$ case the lowest energy ($= 2$) equations form the 2-dimensional Toda chain and the energy 2 and 3 equations form the Davey-Stewartson system in the bilinear form, and in the $n \geq 3$ case the lowest energy ($= 2$) bilinear equations form the n-wave system in the bilinear form.

There is a new phenomenon in the n-component case, which does not occur in the 1-component case: the τ-function and the wave function is a collection of functions $\{\tau_\alpha\}$ and $\{W_\alpha\}$ parameterized by the elements of the root lattice M of type A_{n-1}. The set $\text{supp}\ \tau = \{\alpha \in M | \tau_\alpha \neq 0\}$ is called the support of the τ-function τ. We show that supp τ is a convex polyhedron whose edges are parallel to roots; in particular, supp τ is connected, which allows us to relate the behaviour of the n-component KP hierarchy at different points of the lattice M. It is interesting to note that the "matching conditions" which relate the functions W_α and W_β, $\alpha, \beta \in M$, involve elements from the subgroup of translations of the Weyl group [K, Chapter 6] of the loop group $GL(\mathbb{C}[z, z^{-1}])$ and are intimately related to the Bruhat decomposition of this loop group (see [PK]). We are planning to study this in a future publication.

The behaviour of solutions obtained via vertex operators in the n-component case is much more complicated than for the ordinary KP hiearchy. In particular, they are not necessarily multisoliton solutions (i.e. a collection of waves that preserve their form after interaction). For that reason we call them the multisolitary solutions. Some of the multisolitary solutions turn out to be the so called dromion solutions, that have become very popular recently [BLMP], [FS], [HH], [HMM]. This solutions decay exponentially in all directions (and they are not soliton solutions; in particular, they exist only for $n > 1$). It is a very interesting problem for which values of parameters the multisolitary solutions are soliton or dromion solutions.

Note also that the Krichever method for construction of the quasiperiodic solutions of the KP hierarchy as developed in [SW] and [Sh] applies to the n-component KP.

As shown in [S], [DJKM2], the m-th reduction of the KP hiearchy, i.e. the requirement that L^m is a differential operator, leads to the classical formulation of the celebrated KdV hierarchy for $m = 2$, Boussinesq for $m = 3$ and all the Gelfand-Dickey hierarchies for $m > 3$. The totality of τ-functions for the m-th reduced KP hierarchy turns out to be the orbit of the vacuum under the loop group of SL_m.

We define in a similar way the m-th reduction of the n-component KP and show that the totality of τ-functions is the orbit of the vacuum vector under the loop group of SL_{mn}. Even the case $m = 1$ turns out to be extremely interesting (it is trivial if $n = 1$), as it gives the $1 + 1$ n-wave system for $n \geq 3$ and the decoupled non-linear Schrödinger (or AKNS) system for $n = 2$. We note that the 1-reduced n-component KP, which we call the n-component NLS hierarchy, admits a natural generalization to the case of an arbitrary simple Lie group G (the n-component NLS corresponding to GL_n). These hierarchies which might be called the GNLS hierarchies, contain the systems studied by many authors [Di], [W1 and 2], [KW],... .

0.5. The paper is set out as follows. In §1 we explain the construction of the semi-infinite wedge representation F of the group GL_∞ and write down the equation of the GL_∞-orbit $\mathcal{O}$ of the vacuum $|0\rangle$ (Proposition 1.3). This equation is called the KP hierarchy in the fermionic picture. As usual, the Plücker map makes $\mathcal{O}$ a $\mathbb{C}^\times$-bundle over an infinite-dimensional Grassmannian. We describe the "support" of $\tau \in \mathcal{O}$ (Proposition 1.4).

In §2 we introduce the n-component bosonisation and write down the fermionic fields in terms of bosonic ones via vertex operators (Theorem 2.1). This allows us to transport the KP hierarchy from the fermionic picture to the bosonic one (2.3.3) and write down the n-component KP hierarchy as a system of Hirota bilinear equations (2.3.7). We describe the support of a τ-function in the bosonic picture (Proposition 2.3). At the end of the section we list all Hirota bilinear equations of lowest energy (2.4.3–9).

We start §3 with an exposition of the theory of matrix formal pseudo-differential operators, and prove the crucial Lemma 3.2. This allows us to reformulate the n-component KP hierarchy (2.3.3) in terms of formal pseudo-differential operators (see (3.3.4 and 12)). Using the crucial lemma we show that the bilinear equation (2.3.3) is equivalent to the Sato equation (3.4.2) and matching conditions (3.3.16) on the wave operators $P^+(\alpha)$. We show that Sato equation is the compatibility condition of Sato's linear problem (3.5.5) on the wave function (Proposition 3.5), and that compatibility of Sato equation implies the equivalent Lax and Zakharov-Shabat equations (Lemma 3.6). We prove that compatibility conditions completely determine the wave operators $P^+(\alpha)$ once one of them is given (Proposition 3.3). At the end of the section we write down explicitly the first Sato and Lax equations and relations between them.

In §4 we show that many well-known $2+1$ soliton equations are the simplest equations of the n-component KP hierarchy, and deduce from §3 expressions for their τ-functions and the corresponding Hirota bilinear equations.

Using vertex operators we write down in §5 the N-solitary solutions (5.1.11) of the n-component KP and hence of all its relatives. We discuss briefly the relation of this general solution to the known solutions to the relatives.

In §6 we discuss the m-reductions of the n-component KP hierarchy. They reduce the $2+1$ soliton equations to $1+1$ soliton equations. We show that at the group theoretic level it corresponds to a reduction from GL_∞ (or rather a completion of it) to the subgroup $SL_{mn}(\mathbb{C}[t,t^{-1}])$ (Proposition 6.1). We discuss in more detail the 1-reduced n-component KP, which is a generalization of the NLS system and which admits further generalization to any simple Lie group.

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§1. The semi-infinite wedge representation of the group GL_∞ and the KP hierarchy in the fermionic picture.

1.1. Consider the infinite complex matrix group

$GL_\infty = \{A = (a_{ij})_{i,j\in\mathbb{Z}+\frac12} | A$ is invertible and all but a finite number of $a_{ij} - \delta_{ij}$ are $0\}$
and its Lie algebra

$$gl_\infty = \{a = (a_{ij})_{i,j\in\mathbb{Z}+\frac12} |\ \text{all but a finite number of } a_{ij} \text{ are } 0\}$$

with bracket $[a,b] = ab - ba$. The Lie algebra gl_∞ has a basis consisting of matrices E_{ij}, $i,j \in \mathbb{Z}+\frac12$, where E_{ij} is the matrix with a 1 on the (i,j)-th entry and zeros elsewhere.

Let $\mathbb{C}^\infty = \bigoplus_{j\in\mathbb{Z}+\frac12} \mathbb{C}v_j$ be an infinite dimensional complex vector space with fixed basis $\{v_j\}_{j\in\mathbb{Z}+\frac12}$. Both the group GL_∞ and its Lie algebra gl_∞ act linearly on $\mathbb{C}^\infty$ via the usual formula:

$$E_{ij}(v_k) = \delta_{jk} v_i.$$

The well-known semi-infinite wedge representation is constructed as follows [KP2]. The semi-infinite wedge space $F = \Lambda^{\frac{1}{2}\infty}\mathbb{C}^\infty$ is the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \ldots$, where $i_1 > i_2 > i_3 > \ldots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. We can now define representations R of GL_∞ and r of gl_∞ on F by

$$R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots, \tag{1.1.1}$$

$$r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = \sum_k v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{k-1}} \wedge av_{i_k} \wedge v_{i_{k+1}} \wedge \cdots. \tag{1.1.2}$$

These equations are related by the usual formula:

$$\exp(r(a)) = R(\exp a) \text{ for } a \in gl_\infty.$$

1.2. The representation r of gl_∞ can be described in terms of a Clifford algebra. Define the wedging and contracting operators ψ_j^+ and ψ_j^- $(j \in \mathbb{Z} + \frac{1}{2})$ on F by

$$\psi_j^+(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \begin{cases} 0 & \text{if } j = i_s \text{for some } s \\ (-1)^s v_{i_1} \wedge v_{i_2} \cdots \wedge v_{i_s} \wedge v_{-j} \wedge v_{i_{s+1}} \wedge \cdots & \text{if } i_s > -j > i_{s+1} \end{cases}$$

$$\psi_j^-(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s \\ (-1)^{s+1} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots & \text{if } j = i_s. \end{cases}$$

These operators satisfy the following relations $(i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)$:

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j}, \tag{1.2.1}$$

hence they generate a Clifford algebra, which we denote by $C\ell$.
Introduce the following elements of F $(m \in \mathbb{Z})$:

$$|m\rangle = v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m-\frac{5}{2}} \wedge \cdots.$$

It is clear that F is an irreducible $C\ell$-module such that

$$\psi_j^\pm|0\rangle = 0 \text{ for } j > 0. \tag{1.2.2}$$

It is straightforward that the representation r is given by the following formula:

$$r(E_{ij}) = \psi_{-i}^+ \psi_j^-. \tag{1.2.3}$$

Define the *charge decomposition*

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)} \tag{1.2.4}$$

by letting

$$\text{charge}(v_{i_1} \wedge v_{i_2} \wedge \ldots) = m \text{ if } i_k + k = \tfrac{1}{2} + m \text{ for } k >> 0. \tag{1.2.5}$$

Note that

$$\text{charge}(|m\rangle) = m \text{ and charge}(\psi_j^\pm) = \pm 1. \tag{1.2.6}$$

It is clear that the charge decomposition is invariant with respect to $r(g\ell_\infty)$ (and hence with respect to $R(GL_\infty)$). Moreover, it is easy to see that each $F^{(m)}$ is irreducible with respect to $g\ell_\infty$ (and GL_∞). Note that $|m\rangle$ is its highest weight vector, i.e.

$$r(E_{ij})|m\rangle = 0 \text{ for } i < j,$$
$$r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i < m).$$

1.3. The main object of our study is the GL_∞-orbit

$$\mathcal{O} = R(GL_\infty)|0\rangle \subset F^{(0)}$$

of the vacuum vector $|0\rangle$.

Proposition 1.3 ([KP2]). *A non-zero element τ of $F^{(0)}$ lies in $\mathcal{O}$ if and only if the following equation holds in $F \otimes F$:*

$$\sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_k^+ \tau \otimes \psi_{-k}^- \tau = 0. \tag{1.3.1}$$

Proof. It is clear that $\sum_k \psi_k^+|0\rangle \otimes \psi_{-k}^-|0\rangle = 0$ and it is easy to see that the operator $\sum_k \psi_k^+ \otimes \psi_{-k}^- \in \text{End}(F \otimes F)$ commutes with $R(g) \otimes R(g)$ for any $g \in GL_\infty$. It follows that $R(g)|0\rangle$ satisfies (1.3.1). For the proof of the converse statement (which is not important for our purposes) see [KP2] or [KR]. □

Equation (1.3.1) is called the *KP hierarchy in the fermionic picture.*

Note that any non-zero element τ from the orbit $\mathcal{O}$ is of the form:

$$\tau = u_{-\frac{1}{2}} \wedge u_{-\frac{3}{2}} \wedge u_{-\frac{5}{2}} \wedge \dots, \text{where } u_j \in \mathbb{C}^\infty \text{ and } u_{-k} = v_{-k} \text{ for } k \gg 0. \tag{1.3.2}$$

This allows us to construct a canonical map $\varphi : \mathcal{O} \to \text{Gr}$ by $\varphi(\tau) = \sum_i \mathbb{C}u_{-i} \subset \mathbb{C}^\infty$, where Gr consists of the subspaces of $\mathbb{C}^\infty$ containing $\sum_{j=k}^\infty \mathbb{C}v_{-j-1/2}$ for $k \gg 0$ as a subspace of codimension k. It is clear that the map φ is surjective with fibers $\mathbb{C}^\times$.

1.4. Consider the free $\mathbb{Z}$-module $\tilde{L}$ with the basis $\{\delta_j\}_{j \in \frac{1}{2}+\mathbb{Z}}$, let $\tilde{\Delta}$ (resp. $\tilde{\Delta}_0$) $= \{\delta_i - \delta_j | i, j \in \frac{1}{2} + \mathbb{Z} \text{ (resp. } i, -j \in \frac{1}{2} + \mathbb{Z}_+), i \neq j\}$, and let $\tilde{M} \subset \tilde{L}$ (resp. $\tilde{M}_0 \subset \tilde{L}$) be the $\mathbb{Z}$-span of $\tilde{\Delta}$ (resp. $\tilde{\Delta}_0$). We define the weight of a semi-infinite monomial by

$$\text{weight}(\psi_{i_1}^+ \dots \psi_{i_s}^+ \psi_{j_1}^- \dots \psi_{j_t}^- |0\rangle) = \delta_{-i_1} + \dots + \delta_{-i_s} - \delta_{j_1} - \dots - \delta_{j_t}. \tag{1.4.1}$$

Note that weights of semi-infinite monomials from $F^{(0)}$ lie in $\tilde{M}_0$. Given $\tau \in F$ we denote by $f\text{supp}\,\tau$, and call it the *fermionic support of* τ, the set of weights of semi-infinite monomials that occur in τ with a non-zero coefficient.

Proposition 1.4. *If $\tau \in \mathcal{O}$, then $f\text{supp}\ \tau$ is the intersection of $\tilde{M}_0$ with a convex polyhedron with vertices in $\tilde{M}_0$ and edges in $\tilde{\Delta}_0$.*

Proof. According to the general result [PK, Lemma 4], the edges of the convex hull of $f\text{supp}\ \tau$ must be parallel to the elements of $\tilde{\Delta}_0$. But if the difference of weights of two semi-infinite monomials is a multiple of $\delta_i - \delta_j$, then it is clearly equal to $\pm(\delta_i - \delta_j)$. Hence edges of the convex hull of $f\text{supp}\ \tau$ are elements of $\tilde{\Delta}_0$, and the proposition follows. □

§2. The n-component bosonization and the KP hierarchy in the bosonic picture.

2.1. Using a bosonization one can rewrite (1.3.1) as a system of partial differential equations. There are however many different bosonizations. In this paper we focus on the n-component bosonizations, where $n = 1, 2, \dots$.

For that purpose we relabel the basis vectors v_i and with them the corresponding fermionic operators (the wedging and contracting operators). This relabeling can be done in many different ways, see e.g. [TV], the simplest one is the following.

Fix $n \in \mathbb{N}$ and define for $j \in \mathbb{Z},\ 1 \leq j \leq n,\ k \in \mathbb{Z} + \frac{1}{2}$:

$$v_k^{(j)} = v_{nk-\frac{1}{2}(n-2j+1)},$$

and correspondingly:

$$\psi_k^{\pm(j)} = \psi^{\pm}_{nk\pm\frac{1}{2}(n-2j+1)}.$$

Notice that with this relabeling we have:

$$\psi_k^{\pm(j)}|0\rangle = 0 \text{ for } k > 0.$$

The charge decomposition (1.2.5) can be further decomposed into a sum of *partial charges* which are denoted by charge_j, $j = 1, \dots, n$, defined for a semi-infinite monomial $v \equiv v_{i_1} \wedge v_{i_2} \wedge \dots$ of weight $\sum_i a_i \delta_i$ by

$$\text{(2.1.1)} \qquad \text{charge}_j(v) = \sum_{k\in\mathbf{Z}} a_{kn+j-1/2},$$

which is equivalent to

$$\text{charge}_j\ \psi_k^{\pm(i)} = \pm\delta_{ij},\ \ \text{charge}_j\ |0\rangle = 0.$$

Another important decomposition is the *energy decomposition* defined by

$$\text{(2.1.2)} \qquad \text{energy } |0\rangle = 0,\ \ \text{energy } \psi_k^{\pm(j)} = -k.$$

Note that energy is a non-negative number which can be calculated by

$$\text{(2.1.3)} \qquad \text{energy}(v) = \sum_{k\in\frac{1}{2}+\mathbf{Z}} a_k([k/n] + \tfrac{1}{2}).$$

Introduce the fermionic fields ($z \in \mathbb{C}^\times$):

$$\text{(2.1.4)} \qquad \psi^{\pm(j)}(z) \stackrel{\text{def}}{=} \sum_{k\in\mathbf{Z}+\frac{1}{2}} \psi_k^{\pm(j)} z^{-k-\frac{1}{2}}.$$

Next we introduce bosonic fields ($1 \leq i,j \leq n$):

$$\text{(2.1.5)} \qquad \alpha^{(ij)}(z) \equiv \sum_{k\in\mathbf{Z}} \alpha_k^{(ij)} z^{-k-1} \stackrel{def}{=} :\psi^{+(i)}(z)\psi^{-(j)}(z):,$$

where : : stands for the *normal ordered product* defined in the usual way ($\lambda, \mu = +$ or $-$):

$$\text{(2.1.6)} \qquad :\psi_k^{\lambda(i)}\psi_\ell^{\mu(j)}: = \begin{cases} \psi_k^{\lambda(i)}\psi_\ell^{\mu(j)} & \text{if } \ell > 0 \\ -\psi_\ell^{\mu(j)}\psi_k^{\lambda(i)} & \text{if } \ell < 0. \end{cases}$$

One checks (using e.g. the Wick formula) that the operators $\alpha_k^{(ij)}$ satisfy the commutation relations of the affine algebra $gl_n(\mathbb{C})^\wedge$ with central charge 1, i.e.:

$$\text{(2.1.7)} \qquad [\alpha_p^{(ij)}, \alpha_q^{(k\ell)}] = \delta_{jk}\alpha_{p+q}^{(i\ell)} - \delta_{i\ell}\alpha_{p+q}^{(kj)} + p\delta_{i\ell}\delta_{jk}\delta_{p,-q},$$

and that

(2.1.8) $$\alpha_k^{(ij)}|m\rangle = 0 \text{ if } k > 0 \text{ or } k = 0 \text{ and } i < j.$$

The operators $\alpha_k^{(i)} \equiv \alpha_k^{(ii)}$ satsify the canonical commutation relation of the associative oscillator algebra, which we denote by $\mathfrak{a}$:

(2.1.9) $$[\alpha_k^{(i)}, \alpha_\ell^{(j)}] = k\delta_{ij}\delta_{k,-\ell},$$

and one has

(2.1.10) $$\alpha_k^{(i)}|m\rangle = 0 \text{ for } k > 0.$$

It is easy to see that restricted to $g\ell_n(\mathbb{C})^\wedge$, $F^{(0)}$ is its basic highest weight representation (see [K, Chapter 12]). The $g\ell_n(\mathbb{C})^\wedge$-weight of a semi-infinite monomial v is as follows:

(2.1.11) $$\Lambda_0 + \sum_{j=1}^{n} \text{charge}_j(v)\delta_j - \text{ energy}(v)\tilde{\delta}.$$

Here Λ_0 is the highest weight of the basic representation, $\{\delta_j\}$ is the standard basis of the weight lattice of $g\ell_n(\mathbb{C})$ and $\tilde{\delta}$ is the primitive imaginary root ([K, Chapter 7]).

In order to express the fermionic fields $\psi^{\pm(i)}(z)$ in terms of the bosonic fields $\alpha^{(ii)}(z)$, we need some additional operators Q_i, $i = 1, \dots, n$, on F. These operators are uniquely defined by the following conditions:

(2.1.12) $$Q_i|0\rangle = \psi_{-\frac{1}{2}}^{+(i)}|0\rangle,\ Q_i\psi_k^{\pm(j)} = (-1)^{\delta_{ij}+1}\psi_{k\mp\delta_{ij}}^{\pm(j)}Q_i.$$

They satisfy the following commutation relations:

(2.1.13) $$Q_iQ_j = -Q_jQ_i \text{ if } i \neq j,\ [\alpha_k^{(i)}, Q_j] = \delta_{ij}\delta_{k0}Q_j.$$

Theorem 2.1. *([DJKM1], [JM])*

(2.1.14) $$\psi^{\pm(i)}(z) = Q_i^{\pm 1} z^{\pm\alpha_0^{(i)}} \exp(\mp\sum_{k<0}\frac{1}{k}\alpha_k^{(i)}z^{-k})\exp(\mp\sum_{k>0}\frac{1}{k}\alpha_k^{(i)}z^{-k}).$$

Proof. See [TV].

The operators on the right-hand side of (2.1.14) are called vertex operators. They made their first appearance in string theory (cf. [FK]).

We shall use below the following notation

(2.1.15) $$|k_1, \dots, k_n\rangle = Q_1^{k_1}\dots Q_n^{k_n}|0\rangle.$$

Remark 2.1. One easily checks the following relations:

$$[\alpha_k^{(i)}, \psi_m^{\pm(j)}] = \pm\delta_{ij}\psi_{k+m}^{\pm(j)}.$$

They imply formula (2.1.14) for $\psi^{\pm(i)}(z)$ except for the first two factors, which require some additional analysis.

2.2. We can describe now the n-component boson-fermion correspondence. Let $\mathbb{C}[x]$ be the space of polynomials in indeterminates $x = \{x_k^{(i)}\}$, $k = 1, 2, \dots$, $i = 1, 2, \dots, n$. Let L be a lattice with a basis $\delta_1, \dots, \delta_n$ over $\mathbb{Z}$ and the symmetric bilinear form $(\delta_i|\delta_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Let

$$\varepsilon_{ij} = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \le j. \end{cases} \tag{2.2.1}$$

Define a bimultiplicative function $\varepsilon : L \times L \longrightarrow \{\pm 1\}$ by letting

$$\varepsilon(\delta_i, \delta_j) = \varepsilon_{ij}. \tag{2.2.2}$$

Let $\delta = \delta_1 + \ldots + \delta_n$, $M = \{\gamma \in L | (\delta|\gamma) = 0\}$, $\Delta = \{\alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, n,\ i \ne j\}$. Of course M is the root lattice of $s\ell_n(\mathbb{C})$, the set Δ being the root system.

Consider the vector space $\mathbb{C}[L]$ with basis e^γ, $\gamma \in L$, and the following twisted group algebra product:

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}. \tag{2.2.3}$$

Let $B = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[L]$ be the tensor product of algebras. Then the n-component boson-fermion correspondence is the vector space isomorphism

$$\sigma : F \xrightarrow{\sim} B, \tag{2.2.4}$$

given by

$$\sigma(\alpha_{-m_1}^{(i_1)} \ldots \alpha_{-m_s}^{(i_s)} |k_1, \ldots, k_n)) = m_1 \ldots m_s x_{m_1}^{(i_1)} \ldots x_{m_s}^{(i_s)} \otimes e^{k_1\delta_1 + \ldots + k_n\delta_n}. \tag{2.2.5}$$

The transported charge and energy then will be as follows:

$$\text{charge}(p(x) \otimes e^\gamma) = (\delta|\gamma), \ \text{charge}_j(p(x) \otimes e^\gamma) = (\delta_j|\gamma), \tag{2.2.6}$$

$$\text{energy}(x_{m_1}^{(i_1)} \ldots x_{m_s}^{(i_s)} \otimes e^\gamma) = m_1 + \ldots + m_s + \tfrac{1}{2}(\gamma|\gamma). \tag{2.2.7}$$

We denote the transported charge decomposition by

$$B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}.$$

The transported action of the operators $\alpha_m^{(i)}$ and Q_j looks as follows:

$$\begin{cases} \sigma\alpha_{-m}^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) = m x_m^{(j)} p(x) \otimes e^\gamma, \text{ if } m > 0, \\ \sigma\alpha_m^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) = \frac{\partial p(x)}{\partial x_m} \otimes e^\gamma, \text{ if } m > 0, \\ \sigma\alpha_0^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) = (\delta_j|\gamma) p(x) \otimes e^\gamma, \\ \sigma Q_j \sigma^{-1}(p(x) \otimes e^\gamma) = \varepsilon(\delta_j, \gamma) p(x) \otimes e^{\gamma+\delta_j}. \end{cases} \tag{2.2.8}$$

2.3. Using the isomorphism σ we can reformulate the KP hierarchy (1.3.1) in the bosonic picture as a hierarchy of Hirota bilinear equations.

We start by observing that (1.3.1) can be rewritten as follows:

$$\text{Res}_{z=0}\, dz (\sum_{j=1}^{n} \psi^{+(j)}(z)\tau \otimes \psi^{-(j)}(z)\tau) = 0, \ \tau \in F^{(0)}. \tag{2.3.1}$$

Here and further $\text{Res}_{z=0}\, dz \sum_j f_j z^j$ (where f_j are independent of z) stands for f_{-1}. Notice that for $\tau \in F^{(0)}$, $\sigma(\tau) = \sum_{\gamma \in M} \tau_\gamma(x) e^\gamma$. Here and further we write $\tau_\gamma(x)e^\gamma$ for $\tau_\gamma \otimes e^\gamma$. Using Theorem 2.1, equation (2.3.1) turns under $\sigma \otimes \sigma : F \otimes F \xrightarrow{\sim} \mathbb{C}[x', x''] \otimes (\mathbb{C}[L'] \otimes \mathbb{C}[L''])$ into the following equation:

$$\text{(2.3.2)}\quad \begin{aligned}&\mathrm{Res}_{z=0}\, dz(\sum_{j=1}^{n}\sum_{\alpha,\beta\in M}\varepsilon(\delta_j,\alpha-\beta)z^{(\delta_j|\alpha-\beta)}\\&\times\exp(\sum_{k=1}^{\infty}(x_k^{(j)'}-x_k^{(j)''})z^k)\exp(-\sum_{k=1}^{\infty}(\frac{\partial}{\partial x_k^{(j)'}}-\frac{\partial}{\partial x_k^{(j)''}})\frac{z^{-k}}{k})\\&\tau_\alpha(x')(e^{\alpha+\delta_j})'\tau_\beta(x'')(e^{\beta-\delta_j})'')=0.\end{aligned}$$

Hence for all $\alpha,\beta\in L$ such that $(\alpha|\delta)=-(\beta|\delta)=1$ we have:

$$\text{(2.3.3)}\quad \begin{aligned}&\mathrm{Res}_{z=0}(dz\sum_{j=1}^{n}\varepsilon(\delta_j,\alpha-\beta)z^{(\delta_j|\alpha-\beta-2\delta_j)}\\&\times\exp(\sum_{k=1}^{\infty}(x_k^{(j)'}-x_k^{(j)''})z^k)\exp(-\sum_{k=1}^{\infty}(\frac{\partial}{\partial x_k^{(j)'}}-\frac{\partial}{\partial x_k^{(j)''}})\frac{z^{-k}}{k})\\&\tau_{\alpha-\delta_j}(x')\tau_{\beta+\delta_j}(x''))=0.\end{aligned}$$

Now making the change of variables

$$x_k^{(j)}=\tfrac{1}{2}(x_k^{(j)'}+x_k^{(j)''}),\quad y_k^{(j)}=\tfrac{1}{2}(x_k^{(j)'}-x_n^{(j)''}),$$

(2.3.3) becomes

$$\text{(2.3.4)}\quad \begin{aligned}&\mathrm{Res}_{z=0}(dz\sum_{j=1}^{n}\varepsilon(\delta_j,\alpha-\beta)z^{(\delta_j|\alpha-\beta-2\delta_j)}\\&\times\exp(\sum_{k=1}^{\infty}2y_k^{(j)}z^k)\exp(-\sum_{k=1}^{\infty}\frac{\partial}{\partial y_k^{(j)}}\frac{z^{-k}}{k})\tau_{\alpha-\delta_j}(x+y)\tau_{\beta+\delta_j}(x-y))=0.\end{aligned}$$

We can rewrite (2.3.4) using the elementary Schur polynomials defined by (0.1.13):

$$\text{(2.3.5)}\quad \sum_{j=1}^{n}\varepsilon(\delta_j,\alpha-\beta)\sum_{k=0}^{\infty}S_k(2y^{(j)})S_{k-1+(\delta_j|\alpha-\beta)}(-\frac{\tilde\partial}{\partial y^{(j)}})\tau_{\alpha-\delta_j}(x+y)\tau_{\beta+\delta_j}(x-y)=0.$$

Here and further we use the notation

$$\frac{\tilde\partial}{\partial y}=(\frac{\partial}{\partial y_1},\frac{1}{2}\frac{\partial}{\partial y_2},\frac{1}{3}\frac{\partial}{\partial y_3},\ldots)$$

Using Taylor's formula we can rewrite (2.3.5) once more:

$$\text{(2.3.6)}\quad \begin{aligned}&\sum_{j=1}^{n}\varepsilon(\delta_j,\alpha-\beta)\sum_{k=0}^{\infty}S_k(2y^{(j)})S_{k-1+(\delta_j|\alpha-\beta)}(-\frac{\tilde\partial}{\partial u^{(j)}})\\&\times e^{\sum_{j=1}^{n}\sum_{r=1}^{\infty}y_r^{(j)}\frac{\partial}{\partial u_r^{(j)}}}\tau_{\alpha-\delta_j}(x+u)\tau_{\beta+\delta_j}(x-u)|_{u=0}=0.\end{aligned}$$

This last equation can be written as the following generating series of Hirota bilinear equations:

$$\text{(2.3.7)}\quad \begin{aligned}&\sum_{j=1}^{n}\varepsilon(\delta_j,\alpha-\beta)\sum_{k=0}^{\infty}S_k(2y^{(j)})S_{k-1+(\delta_j|\alpha-\beta)}(-\widetilde{D^{(j)}})\\&\times e^{\sum_{j=1}^{n}\sum_{r=1}^{\infty}y_r^{(j)}D_r^{(j)}}\tau_{\alpha-\delta_j}\cdot\tau_{\beta+\delta_j}=0\end{aligned}$$

for all $\alpha, \beta \in L$ such that $(\alpha|\delta) = -(\beta|\delta) = 1$. Hirota's dot notation used here and further is explained in Introduction (see (0.1.9)).

Equation (2.3.7) is known (see [DJKM1,2], [JM]) as the n-component KP hierarchy of Hirota bilinear equations. This equation still describes the group orbit: $\sigma(\mathcal{O}) = \sigma R \sigma^{-1}(GL_\infty) \cdot 1$.

Remark 2.3. Equation (2.3.7) is invariant under the transformations $\alpha \longmapsto \alpha + \gamma$, $\beta \longmapsto \beta + \gamma$, where $\gamma \in M$. Transformations of this type are called Schlessinger transformations.

Let $\tau = \sum_{\gamma \in L} \tau_\gamma(x) e^\gamma \in B$; the set supp $\tau \stackrel{\text{def}}{=} \{\gamma \in L | \ \tau_\gamma \neq 0\}$ is called the *support* of τ.

Proposition 2.3. *Let $\tau \in \mathbb{C}[[x]] \otimes \mathbb{C}[M]$ be a solution to the KP hierarchy (2.3.4). Then supp τ is the intersection of M with a convex polyhedron with vertices in M and edges parallel to elements of Δ.*

Proof. Consider the linear map $\overline{\sigma} : \tilde{L} \to L$ defined by $\overline{\sigma}(\delta_j) = \delta_{(j+1/2) \bmod n}$, where $a \bmod n$ stands for the element of the set $\{1, \dots, n\}$ congruent to $a \bmod n$. Then it is easy to see that for $\tau \in F$ we have:

$$\text{supp } \sigma(\tau) = \overline{\sigma}(f\text{supp } \tau).$$

Now Proposition 2.3 follows from Proposition 1.4. $\square$

2.4. The indeterminates $y_k^{(j)}$ in (2.3.7) are free parameters, hence the coefficient of a monomial $y_{k_1}^{(j_1)} \dots y_{k_s}^{(j_s)}$ ($k_i \in \mathbb{N}$, $k_1 \le k_2 \le \dots$, $j_i \in \{1, \dots, n\}$) in equation (2.3.7) gives us a Hirota bilinear equation of the form

$$\sum_{i=1}^{n} \sum_{\alpha,\beta} Q_{k;\alpha,\beta}^{(j)}(D) \tau_{\alpha-\delta_i} \cdot \tau_{\beta+\delta_i} = 0, \tag{2.4.1}$$

where $Q_{k,\alpha,\beta}^{(j)}$ are polynomials in the $D_r^{(i)}$, $k = (k_1, \dots, k_s)$, $j = (j_1, \dots, j_s)$ and $\alpha, \beta \in L$ are such that $(\alpha|\delta) = -(\beta|\delta) = 1$. Each of these equations is a PDE in the indeterminates $x_k^{(j)}$ on functions τ_γ, $\gamma \in M$.

Recall that an expression $Q(D)\tau_\alpha \cdot \tau_\beta$ is identically zero if and only if $\alpha = \beta$ and $Q(D) = -Q(-D)$. The corresponding Hirota bilinear equation is then called *trivial* and can be disregarded.

Let us point out now that the energy decomposition (2.2.7) induces the following energy decomposition on the space of Hirota bilinear equations:

$$\text{energy}(Q_{k;\alpha,\beta}^{(j)}(D)\tau_{\alpha-\delta_i} \cdot \tau_{\beta+\delta_i}) = k_1 + \dots + k_s + \tfrac{1}{2}((\alpha|\alpha) + (\beta|\beta)) \tag{2.4.2}$$

It is clear that the energy of a nontrivial Hirota bilinear equation is at least 2.

Below we list the Hirota bilinear equations of lowest energy for each n.

$n = 1$. In this case we may drop the superscript in $D_k^{(1)}$ and the subscript in τ_α (which is 0). Each monomial $y_{k_1} \dots y_{k_s}$ gives a Hirota bilinear equation of the form

$$Q_k(D)\tau \cdot \tau = 0$$

of energy $k_1 + \dots + k_s + 1$. An easy calculation shows that the lowest energy of a non-trivial equation is 4, and that there is a unique non-trivial equation of energy 4, the classical KP equation in the Hirota bilinear form:

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 0. \tag{2.4.3}$$

$n \geq 2$. There is an equation of energy 2 for each unordered pair of distinct indices i and k (recall that $\alpha_{ik} = \delta_i - \delta_k$ are roots):

$$D_1^{(i)} D_1^{(k)} \tau_0 \cdot \tau_0 = 2\tau_{\alpha_{ik}} \tau_{\alpha_{ki}}. \tag{2.4.4}$$

Furthermore, for each ordered pair of distinct indices i and j there are three equations of energy 3:

$$(D_2^{(i)} + D_1^{(i)2})\tau_0 \cdot \tau_{\alpha_{ij}} = 0, \tag{2.4.5}$$

$$(D_2^{(j)} + D_1^{(j)2})\tau_{\alpha_{ij}} \cdot \tau_0 = 0, \tag{2.4.6}$$

$$D_1^{(i)} D_2^{(j)} \tau_0 \cdot \tau_0 + 2D_1^{(j)} \tau_{\alpha_{ij}} \cdot \tau_{\alpha_{ji}} = 0. \tag{2.4.7}$$

$n \geq 3$. There is an equation of energy 2 and an equation of energy 3 for each ordered triple of distinct indices i, j, k:

$$D_1^{(k)} \tau_0 \cdot \tau_{\alpha_{ij}} = \varepsilon_{ik}\varepsilon_{kj}\varepsilon_{ij}\tau_{\alpha_{ik}}\tau_{\alpha_{kj}}, \tag{2.4.8}$$

$$D_2^{(k)} \tau_0 \cdot \tau_{\alpha_{ij}} = \varepsilon_{ij}\varepsilon_{kj}\varepsilon_{ik} D_1^{(k)} \tau_{\alpha_{ik}} \cdot \tau_{\alpha_{kj}}. \tag{2.4.9}$$

(Note that (2.4.6) is a special case of (2.4.9) where $k = j$.)

$n \geq 4$. There is an algebraic equation of energy 2 for each ordered quadruple of distinct indices i, j, k, ℓ:

$$\varepsilon_{ij}\varepsilon_{k\ell}\tau_0\tau_{\alpha_{ik}+\alpha_{j\ell}} + \varepsilon_{i\ell}\varepsilon_{jk}\tau_{\alpha_{ik}}\tau_{\alpha_{j\ell}} + \varepsilon_{ik}\varepsilon_{j\ell}\tau_{\alpha_{i\ell}}\tau_{\alpha_{jk}} = 0. \tag{2.4.10}$$

Equations (2.4.4–10), together with an algebraic equation of energy 3 for each ordered sixtuple of distinct indices similar to (2.4.10), form a complete list of non-trivial Hirota bilinear equations of energy ≤ 3 of the n-component KP hierarchy.

§3. The algebra of formal pseudo-differential operators and the n-component KP hierarchy as a dynamical system.

3.0. The KP hierarchy and its n-component generalizations admit several formulations. The one given in the previous section obtained by the field theoretical approach is the τ-function formulation given by Date, Jimbo, Kashiwara and Miwa [DJKM1]. Another well-known formulation, introduced by Sato [S], is given in the language of formal pseudo-differential operators. We will show that this formulation follows from the τ-function formulation given by equation (2.3.3).

3.1. We shall work over the algebra $\mathcal{A}$ of formal power series over $\mathbb{C}$ in indeterminates $x = (x_k^{(j)})$, where $k = 1, 2, \ldots$ and $j = 1, \ldots, n$. The indeterminates $x_1^{(1)}, \ldots, x_1^{(n)}$ will be viewed as variables and $x_k^{(j)}$ with $k \geq 2$ as parameters. Let

$$\partial = \frac{\partial}{\partial x_1^{(1)}} + \ldots + \frac{\partial}{\partial x_1^{(n)}}.$$

A formal $n \times n$ matrix pseudo-differential operator is an expression of the form

$$P(x, \partial) = \sum_{j \leq N} P_j(x)\partial^j, \tag{3.1.1}$$

where P_j are $n \times n$ matrices over $\mathcal{A}$. The largest N such that $P_N \neq 0$ is called the *order* of $P(x, \partial)$ (write *ord* $P(x, \partial) = N$). Let Ψ denote the vector space over $\mathbb{C}$ of all expressions (3.1.1). We have a linear isomorphism $s: \Psi \longrightarrow \mathrm{Mat}_n(\mathcal{A}((z)))$ given by $s(P(x, \partial)) = P(x, z)$. The matrix series $P(x, z)$ in indeterminates x and z is called the *symbol* of $P(x, \partial)$.

Now we may define a product $\circ$ on Ψ making it an associative algebra:

$$s(P \circ Q) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n s(P)}{\partial z^n} \partial^n s(Q).$$

We shall often drop the multiplication sign $\circ$ when no ambiguity may arise. Letting $\Psi(m) = \{P \in \Psi | ord\, \Psi \leq m\}$, we get a $\mathbb{Z}$-filtration of the algebra Ψ:

$$\cdots \Psi(m+1) \supset \Psi(m) \supset \Psi(m-1) \supset \cdots \tag{3.1.2}$$

One defines the differential part of $P(x,\partial)$ by $P_+(x,\partial) = \sum_{j=0}^N P_j(x)\partial^j$, and let $P_- = P - P_+$. We have the corresponding vector space decomposition:

$$\Psi = \Psi_- \oplus \Psi_+. \tag{3.1.3}$$

One defines a linear map $* : \Psi \to \Psi$ by the following formula:

$$(\sum_j P_j\partial^j)^* = \sum_j (-\partial)^j \circ {}^t P_j. \tag{3.1.4}$$

Here and further ${}^tP$ stands for the transpose of the matrix P. Note that $*$ is an anti-involution of the algebra Ψ. In terms of symbols the anti-involution $*$ can be written in the following closed form:

$$P^*(x,z) = (\exp \partial \frac{\partial}{\partial z})\ {}^tP(x,-z). \tag{3.1.5}$$

It is clear that the anti-involution $*$ preserves the filtration (3.1.2) and the decomposition (3.1.3).

3.2. Introduce the following notation

$$z \cdot x^{(j)} = \sum_{k=1}^{\infty} x_k^{(j)} z^k, \quad e^{z\cdot x} = diag(e^{z\cdot x^{(1)}}, \dots , e^{z\cdot x^{(n)}}).$$

The algebra Ψ acts on the space U_+ (resp. U_-) of formal oscillating matrix functions of the form

$$\sum_{j\leq N} P_j z^j e^{z\cdot x} \quad (\text{resp. } \sum_{j\leq N} P_j z^j e^{-z\cdot x}), \text{ where } P_j \in \text{Mat}_n(\mathcal{A}),$$

in the obvious way:

$$P(x)\partial^j e^{\pm z\cdot x} = P(x)(\pm z)^j e^{\pm z\cdot x}.$$

We can now prove the following fundamental lemma.

Lemma 3.2. *If $P, Q \in \Psi$ are such that*

$$Res_{z=0}(P(x,\partial)e^{z\cdot x})\ {}^t(Q(x',\partial')e^{-z\cdot x'})dz = 0, \tag{3.2.1}$$

then $(P \circ Q^*)_- = 0$.

Proof. Equation (3.2.1) is equivalent to

$$Res_{z=0}P(x,z)e^{z(x-x')}\ {}^tQ(x',-z)dz = 0. \tag{3.2.2}$$

The (i,m)-th entry of the matrix equation (3.2.2) is

$$Res_{z=0} \sum_{i=1}^{n} P_{ij}(x,z)Q_{mj}(x',-z)e^{z\cdot(x^{(j)}-x'^{(j)})}dz = 0.$$

Letting $y_k^{(j)} = x_k^{(j)} - x_k'^{(j)}$, this equation can be rewritten by applying Taylor's formula to Q:

(3.2.3) $$Res_{z=0}\sum_{j=1}^{n}P_{ij}(x,z)\exp\sum_{\ell=1}^{n}\sum_{k=1}^{\infty}y_k^{(\ell)}(\delta_{\ell j}z^k-\frac{\partial}{\partial x_k^{(\ell)}})Q_{mj}(x,-z)dz=0.$$

Letting $y_k^{(\ell)}=0$ for $k>1$ and $y_1^{(\ell)}=y$ for all ℓ, we obtain from (3.2.3):

(3.2.4) $$Res_{z=0}P(x,z)\sum_{k\geq 0}\frac{(-1)^k}{k!}\partial^k({}^tQ)(x,-z)y^k e^{yz}dz=0.$$

Notice that $y^k e^{yz}=(e^{yz})^{(k)}$. Here and further we write $\varphi^{(k)}$ for $\frac{\partial^k\varphi}{\partial z^k}$. Using integration by parts with respect to z, (3.2.4) becomes:

(3.2.5) $$Res_{z=0}\sum_{k\geq 0}\frac{1}{k!}(P(x,z)\partial^k({}^tQ)(x,-z))^{(k)}e^{yz}dz=0.$$

Using Leibnitz formula, the left-hand side of (3.2.5) is equal to

$$\begin{aligned}&Res_{z=0}\sum_{k\geq 0}\sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{\ell!(k-\ell)!}P^{(\ell)}(x,z)\left(\partial^k({}^tQ)\right)^{(k-\ell)}(x,-z)e^{yz}dz\\&=Res_{z=0}\sum_{\ell\geq 0}\frac{1}{\ell!}P^{(\ell)}(x,z)\partial^\ell\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\partial^k({}^tQ^{(k)})(x,-z)\right)e^{yz}dz\\&=Res_{z=0}\sum_{\ell\geq 0}\frac{1}{\ell!}P^{(\ell)}(x,z)\left(\frac{\partial^\ell Q^*(x,z)}{\partial x^\ell}\right)e^{yz}dz=Res_{z=0}(P\circ Q^*)(x,z)e^{yz}dz.\end{aligned}$$

So we obtain that

(3.2.6) $$Res_{z=0}(P\circ Q^*)(x,z)e^{yz}dz=0.$$

Now write $(P\circ Q^*)(x,z)=\sum_j A_j(x)z^j$ and $e^{yz}=\sum_{\ell=0}^{\infty}\frac{(zy)^\ell}{\ell!}$. Then from (3.2.6) we deduce:

$$0=Res_{z=0}\sum_j\sum_{\ell=0}^{\infty}A_j(x)\frac{y^\ell}{\ell!}z^{\ell+j}dz=\sum_{\ell=0}^{\infty}A_{-\ell-1}(x)\frac{y^\ell}{\ell!}.$$

Hence $A_j(x)=0$ for $j<0$, i.e. $(P\circ Q^*)_-=0$. □

3.3. We proceed now to rewrite the formulation (2.3.3) of the n-component KP hierarchy in terms of formal pseudo-differential operators.

Let $1\leq a,b\leq n$ and recall formula (2.3.3) where α is replaced by $\alpha+\delta_a$ and β by $\beta-\delta_b$:

(3.3.1) $$\begin{aligned}&\mathrm{Res}_{z=0}(dz\sum_{j=1}^{n}\varepsilon(\delta_j,\alpha+\delta_a-\beta+\delta_b)z^{(\delta_j|\alpha+\delta_a-\beta+\delta_b-2\delta_j)}\\&\times\exp(\sum_{k=1}^{\infty}(x_k^{(j)'}-x_k^{(j)''})z^k)\exp(-\sum_{k=1}^{\infty}(\frac{\partial}{\partial x_k^{(j)'}}-\frac{\partial}{\partial x_k^{(j)''}})\frac{z^{-k}}{k})\\&\tau_{\alpha+\alpha_{a_j}}(x')\tau_{\beta-\alpha_{b_j}}(x''))=0\quad(\alpha,\beta\in M).\end{aligned}$$

For each $\alpha\in$ supp τ we define the (matrix valued) functions

(3.3.2) $$V^{\pm}(\alpha,x,z)=(V_{ij}^{\pm}(\alpha,x,z))_{i,j=1}^{n}$$

as follows:

$$(3.3.3)\qquad \begin{aligned} V_{ij}^{\pm}(\alpha,x,z) &\stackrel{\text{def}}{=} \varepsilon(\delta_j,\alpha+\delta_i)z^{(\delta_j|\pm\alpha+\alpha_{ij})} \\ &\times \exp(\pm\sum_{k=1}^{\infty} x_k^{(j)}z^k)\exp(\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k})\tau_{\alpha\pm\alpha_{ij}}(x)/\tau_\alpha(x). \end{aligned}$$

It is easy to see that equation (3.3.1) is equivalent to the following bilinear identity:

$$(3.3.4)\qquad Res_{z=0}V^+(\alpha,x,z)\ {}^tV^-(\beta,x',z)dz = 0 \text{ for all } \alpha,\beta\in M.$$

Define $n\times n$ matrices $W^{\pm(m)}(\alpha,x)$ by the following generating series (cf. (3.3.3)):

$$(3.3.5)\qquad \sum_{m=0}^{\infty} W_{ij}^{\pm(m)}(\alpha,x)(\pm z)^{-m} = \varepsilon_{ji}z^{\delta_{ij}-1}(\exp\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k})\tau_{\alpha\pm\alpha_{ij}}(x))/\tau_\alpha(x).$$

Note that

$$(3.3.6)\qquad W^{\pm(0)}(\alpha,x) = I_n,$$

$$(3.3.7)\qquad W_{ij}^{\pm(1)}(\alpha,x) = \begin{cases} \varepsilon_{ji}\tau_{\alpha\pm\alpha_{ij}}/\tau_\alpha & \text{if } i\neq j \\ -\tau_\alpha^{-1}\frac{\partial\tau_\alpha}{\partial x_1^{(i)}} & \text{if } i=j, \end{cases}$$

$$(3.3.8)\qquad W_{ij}^{\pm(2)}(\alpha,x) = \begin{cases} \mp\varepsilon_{ji}\frac{\partial\tau_{\alpha\pm\alpha_{ij}}}{\partial x_1^{(j)}}/\tau_\alpha & \text{if } i\neq j, \\ (\mp\frac{1}{2}\frac{\partial\tau_\alpha}{\partial x_2^{(i)}}+\frac{1}{2}\frac{\partial^2\tau_\alpha}{\partial x_1^{(i)2}})/\tau_\alpha & \text{if } i=j. \end{cases}$$

We see from (3.3.3) that $V^{\pm}(\alpha,x,z)$ can be written in the following form:

$$(3.3.9)\qquad V^{\pm}(\alpha,x,z) = (\sum_{m=0}^{\infty} W^{\pm(m)}(\alpha,x)R^{\pm}(\alpha,\pm z)(\pm z)^{-m})e^{\pm z\cdot x},$$

where

$$(3.3.10)\qquad R^{\pm}(\alpha,z) = \sum_{i=1}^{n}\varepsilon(\delta_i,\alpha)E_{ii}(\pm z)^{\pm(\delta_i|\alpha)}.$$

Here and further E_{ij} stands for the $n\times n$ matrix whose (i,j) entry is 1 and all other entries are zero. Now it is clear that $V^{\pm}(\alpha,x,z)$ can be written in terms of formal pseudo-differential operators

$$(3.3.11)\qquad P^{\pm}(\alpha)\equiv P^{\pm}(\alpha,x,\partial) = I_n + \sum_{m=1}^{\infty} W^{\pm(m)}(\alpha,x)\partial^{-m} \text{ and } R^{\pm}(\alpha) = R^{\pm}(\alpha,\partial)$$

as follows:

$$(3.3.12)\qquad V^{\pm}(\alpha,x,z) = P^{\pm}(\alpha)R^{\pm}(\alpha)e^{\pm z\cdot x}.$$

Since obviously

$$(3.3.13)\qquad R^-(\alpha,\partial)^{-1} = R^+(\alpha,\partial)^*,$$

using Lemma 3.2 we deduce from the bilinear identity (3.3.4):

$$(3.3.14)\qquad (P^+(\alpha)R^+(\alpha-\beta)P^-(\beta)^*)_- = 0 \text{ for any } \alpha,\beta\in \text{supp}\ \tau.$$

Furthermore, (3.3.14) for $\alpha = \beta$ is equivalent to

$$P^-(\alpha) = (P^+(\alpha)^*)^{-1}, \tag{3.3.15}$$

since $R^{\pm}(0) = I_n$ and $P^{\pm}(\alpha) \in I_n + \Psi_-$. Equations (3.3.14 and 15) imply

$$(P^+(\alpha)R^+(\alpha-\beta)P^+(\beta)^{-1})_- = 0 \text{ for all } \alpha, \beta \in \text{supp } \tau. \tag{3.3.16}$$

In the rest of this paper we sometimes write $P(\alpha)$ instead of $P^+(\alpha)$.

Proposition 3.3. *Given $\beta \in$ supp τ, all the pseudo-differential operators $P(\alpha)$, $\alpha \in$ supp τ, are completely determined by $P(\beta)$ from equations (3.3.16).*

Proof. We have for $i \neq j$: $R(\alpha_{ij}) = A\partial + B + C\partial^{-1}$, where

$$A = \varepsilon_{ij}E_{ii}, \; B = \sum_{\substack{k=1\\k\neq i,j}}^{n} \varepsilon_{ik}\varepsilon_{jk}E_{kk}, \; C = \varepsilon_{ji}E_{jj}. \tag{3.3.17}$$

For $P = I_n + \sum_{j=1}^{\infty} W^{(j)}\partial^{-j}$ we have

$$P^{-1} = I_n - W^{(1)}\partial^{-1} + (W^{(1)2} - W^{(2)})\partial^{-2} + \cdots. \tag{3.3.18}$$

Let $\alpha, \beta \in M$ be such that $\alpha - \beta = \alpha_{ij}$. It follows from (3.3.18) and (3.3.16) that $P(\alpha)R(\alpha-\beta)P(\beta)^{-1} = (P(\alpha)R(\alpha-\beta)P(\beta)^{-1})_+ = A\partial + B + W^{(1)}(\alpha)A - AW^{(1)}(\beta)$, or equivalently:

$$P(\alpha)(A\partial + B + C\partial^{-1}) = (A\partial + B + W^{(1)}(\alpha)A - AW^{(1)}(\beta))P(\beta).$$

Equating coefficients of ∂^{-m}, $m \geq 1$, we obtain:

$$\begin{aligned}&W^{(m+1)}(\alpha)A + W^{(m)}(\alpha)B + W^{(m-1)}(\alpha)C = \\&= A(\partial W^{(m)}(\beta)W^{(m+1)}(\beta) - W^{(1)}(\beta)W^{(m)}(\beta)) + BW^{(m)}(\beta) + W^{(1)}(\alpha)AW^{(m)}(\beta).\end{aligned}$$

Substituting expressions (3.3.17) for A, B and C, we obtain an explicit form of matching conditions ($m \geq 1$):

$$\begin{aligned}&\varepsilon_{ij}W^{(m+1)}(\alpha)E_{ii} + \sum_{k\neq i,j} \varepsilon_{ik}\varepsilon_{jk}W^{(m)}(\alpha)E_{kk} + \varepsilon_{ji}W^{(m-1)}(\alpha)E_{jj}\\&= \varepsilon_{ij}E_{ii}(\partial W^{(m)}(\beta) + W^{(m+1)}(\beta) - W^{(1)}(\beta)W^{(m)}(\beta))\\&+ \sum_{k\neq i,j} \varepsilon_{ik}\varepsilon_{jk}E_{kk}W^{(m)}(\beta) + \varepsilon_{ij}W^{(1)}(\alpha)E_{ii}W^{(m)}(\beta).\end{aligned} \tag{3.3.19}$$

It follows from (3.3.19) that $W^{(m+1)}(\alpha)$ for $m \geq 1$ can be expressed in terms of the $W^{(s)}(\beta)$ with $s \leq m+1$ and $W^{(1)}(\alpha)$. Looking at the (k,ℓ)-entry of (3.3.19) for $k, \ell \neq i, j$, we see that $W^{(1)}(\alpha)$ can be expressed in terms of $W^{(1)}(\beta)$ and $W^{(1)}_{ki}(\alpha)$, where $k \neq i, j$. The (k,j)-entry of (3.3.19) for $m = 1$ gives: $W^{(1)}_{ki}(\alpha)W^{(1)}_{ij}(\beta) = \varepsilon_{ik}\varepsilon_{jk}W^{(1)}_{kj}(\beta)$, and since the (j,j)-entry of this equation is $W^{(1)}_{ji}(\alpha)W^{(1)}_{ij}(\beta) = -1$, we see that $W^{(1)}_{ij}(\beta)$ is invertible, hence $W^{(1)}_{ki}(\alpha)$ is expressed in terms of $W^{(1)}(\beta)$.

Due to Proposition 2.3 for any $\alpha, \beta \in$ supp τ there exist a sequence $\gamma_1, \ldots, \gamma_k$ such that $\alpha = \gamma_1$, $\beta = \gamma_k$ and $\gamma_i - \gamma_{i+1} \in \Delta$ for all $i = 1, \ldots, k-1$. The proposition now follows. □

Remark 3.3. The functions $P^+(\alpha, x, z)$ $(\alpha \in M)$ determine the τ-function $\sum_\alpha \tau_\alpha(x)e^\alpha$ up to a constant factor. Namely, we may recover $\tau_\alpha(x)$ from functions $P^+_{jj}(\alpha, x, z)$ as follows. We have from (3.3.5):

$$\log P_{jj}^{+}(\alpha,x,z) = \log \tau_\alpha(x_\ell^{(p)} - \frac{\delta_{jp}}{\ell z^\ell}) - \log \tau_\alpha(x_\ell^{(p)}).$$

Applying to both sides the operator $\frac{\partial}{\partial z} - \sum_{k\geq 1} z^{-k-1}\frac{\partial}{\partial x_k^{(j)}}$ (that kills the first summand on the right), we obtain:

$$(\frac{\partial}{\partial z} - \sum_{k\geq 1} z^{-k-1}\frac{\partial}{\partial x_k^{(j)}})\log P_{jj}^{+}(\alpha,x,z) = \sum_{k\geq 1} z^{-k-1}\frac{\partial}{\partial x_k^{(j)}}\log \tau_\alpha(x).$$

Hence

$$\frac{\partial}{\partial x_k^{(j)}}\log \tau_\alpha(x) = Res_{z=0}\ dz\ z^k(\frac{\partial}{\partial z} - \sum_{k\geq 1} z^{-k-1}\frac{\partial}{\partial x_k^{(j)}})\log P_{jj}^{+}(\alpha,x,z). \tag{3.3.20}$$

This determines $\tau_\alpha(x)$ up to a constant factor. It follows from (3.3.7) and Proposition 2.3 that these constant factors are the same for all α.

3.4. Introduce the following formal pseudo-differential operators $L(\alpha)$, $C^{(j)}(\alpha)$, $L^{(j)}(\alpha)$ and differential operators $B_m(\alpha)$ and $B_m^{(j)}(\alpha)$:

$$\begin{aligned}
L(\alpha) &\equiv L(\alpha,x,\partial) = P^+(\alpha)\circ\partial\circ P^+(\alpha)^{-1},\\
C^{(j)}(\alpha) &\equiv C^{(j)}(\alpha,x,\partial) = P^+(\alpha)E_{jj}P^+(\alpha)^{-1},\\
L^{(j)}(\alpha) &\equiv C^{(j)}(\alpha)L(\alpha) = P^+(\alpha)E_{jj}\circ\partial\circ P^+(\alpha)^{-1},\\
B_m(\alpha) &\equiv (L(\alpha)^m)_+ = (P^+(\alpha)\circ\partial^m\circ P^+(\alpha)^{-1})_+,\\
B_m^{(j)}(\alpha) &\equiv (L^{(j)}(\alpha)^m)_+ = (P^+(\alpha)E_{jj}\circ\partial^m\circ P^+(\alpha)^{-1})_+.
\end{aligned} \tag{3.4.1}$$

Using Lemma 3.2 we can now derive the Sato equations from equation (3.3.4):

Lemma 3.4. *Each formal pseudo-differential operator $P = P^+(\alpha)$ satisfies the Sato equations:*

$$\frac{\partial P}{\partial x_k^{(j)}} = -(PE_{jj}\circ\partial^k\circ P^{-1})_-\circ P. \tag{3.4.2}$$

Proof. Notice first that

$$\begin{aligned}
&(\frac{\partial}{\partial x_k^{(j)}} - B_k^{(j)}(\alpha))V^+(\alpha,x,z) = (\frac{\partial}{\partial x_k^{(j)}} - B_k^{(j)}(\alpha))P^+(\alpha)R^+(\alpha)e^{z\cdot x}\\
&= (\frac{\partial P^+(\alpha)}{\partial x_k^{(j)}}R^+(\alpha) + P^+(\alpha)R^+(\alpha)E_{jj}\partial^k - B_k^{(j)}(\alpha)P^+(\alpha)R^+(\alpha))e^{z\cdot x}\\
&= (\frac{\partial P^+(\alpha)}{\partial x_k^{(j)}} + P^+(\alpha)E_{jj}\partial^k - B_k^{(j)}(\alpha)P^+(\alpha))R^+(\alpha)e^{z\cdot x}\\
&= (\frac{\partial P^+(\alpha)}{\partial x_k^{(j)}} + L^{(j)}(\alpha)^k P^+(\alpha) - B_k^{(j)}(\alpha)P^+(\alpha))R^+(\alpha)e^{z\cdot x}\\
&= (\frac{\partial P^+(\alpha)}{\partial x_k^{(j)}} + (L^{(j)}(\alpha)^k)_- P^+(\alpha))R^+(\alpha)e^{z\cdot x}
\end{aligned}$$

Next apply $\frac{\partial}{\partial x_k^{(j)}} - B_k^{(j)}(\alpha)$ to the equation (3.3.4) for $\alpha = \beta$ to obtain:

$$Res_{z=0}\ dz\ (\frac{\partial P^+(\alpha)}{\partial x_k^{(j)}} + (L^{(j)}(\alpha)^k)_-)(P^+(\alpha)R^+(\alpha)e^{z\cdot x})\ {}^t(P^-(\alpha)R^-(\alpha)e^{-z\cdot x'}) = 0.$$

Now apply Lemma 3.2 and (3.3.15) to obtain:

$$((\frac{\partial P^+(\alpha)}{\partial x_k^{(j)}} + (L^{(j)}(\alpha)^k)_- P^+(\alpha))P^+(\alpha)^{-1})_- = 0$$

which proves the lemma. □

Proposition 3.4. *Consider the formal oscillating functions $V^+(\alpha,x,z)$ and $V^-(\alpha,x,z)$, $\alpha \in M$, of the form (3.3.12), where $R^{\pm}(\alpha,z)$ are given by (3.3.10) and $P^{\pm}(\alpha,x,\partial) \in I_n + \Psi_-$. Then the bilinear identity (3.3.4) for all $\alpha,\beta \in$ supp τ is equivalent to the Sato equation (3.4.2) for each $P = P^+(\alpha)$ and the matching condition (3.3.14) for all $\alpha,\beta \in$ supp τ.*

Proof. We have proved already that the bilinear identity (3.3.4) implies (3.4.2) and (3.3.14). To prove the converse, denote by $A(\alpha,\beta,x,x')$ the left–hand side of (3.3.4). The same argument as in the proof of Lemma 3.4 shows that:

$$\left(\frac{\partial}{\partial x_k^{(j)}} - B_k^{(j)}(\alpha)\right) A(\alpha,\beta,x,x') = 0, \tag{3.4.3}$$

$$A(\alpha,\beta,x,x') = 0,.\ \text{ if } x_k^{(i)} = x_k'^{(i)} \text{ for } k \geq 2, \tag{3.4.4}$$

where $B_k^{(j)}(\alpha)$ is defined by (3.4.1).

Denote by $A_1(\alpha,\beta)$ the expression for $A(\alpha,\beta,x,x')$ in which we set $x_k^{(j)} = x_k'^{(j)} = 0$ if $k \geq 2$ and $x_1^{(1)} = \ldots = x_1^{(n)} = x_1$, $x_1'^{(1)} = \ldots = x_1'^{(n)} = x_1'$. Expanding $A(\alpha,\beta,x,x')$ in a power series in $x_k^{(i)} - x_k'^{(i)}$ for $k \geq 2$ and $x_1^{(i)} - x_1^{(j)}$, $x_1'^{(i)} - x_1'^{(j)}$, we see from (3.4.3) and (3.4.4) that it remains to prove

$$A_1(\alpha,\beta) = 0. \tag{3.4.5}$$

But the same argument as in the proof of Lemma 3.2 shows that

$$A_1(\alpha,\beta) = \mathrm{Res}_{z=0} W^+(\alpha,x_1,\partial)R^+(\alpha-\beta,\partial)W^-(\beta,x_1,\partial)^*\ e^{yz}dz,$$

where $y = x_1 - x_1'$. Hence, as at the end of the proof of Lemma 3.2, (3.4.5) follows from (3.3.14). □

3.5. Fix $\alpha \in M$; we have introduced above a collection of formal pseudo-differential operators $L \equiv L(\alpha)$, $C^{(i)} \equiv C^{(i)}(\alpha)$ of the form:

$$\begin{aligned} L &= I_n\partial + \sum_{j=1}^{\infty} U^{(j)}(x)\partial^{-j}, \\ C^{(i)} &= E_{ii} + \sum_{j=1}^{\infty} C^{(i,j)}(x)\partial^{-j},\ i = 1,2,\cdots,n, \end{aligned} \tag{3.5.1}$$

subject to the conditions

$$(3.5.2)\qquad \sum_{i=1}^{n} C^{(i)} = I_n,\ C^{(i)}L = LC^{(i)},\ C^{(i)}C^{(j)} = \delta_{ij}C^{(i)}.$$

They satisfy the following set of equations for some $P \in I_n + \Psi_-$:

$$(3.5.3)\qquad \begin{cases} LP = P\partial \\ C^{(i)}P = PE_{ii} \\ \dfrac{\partial P}{\partial x_k^{(i)}} = -(L^{(i)k})_- P, \text{ where } L^{(i)} = C^{(i)}L. \end{cases}$$

Notice that the first equation of (3.5.3) follows from the last one, since $L = I_n\partial + \sum_i (L^{(i)})_-$.

Proposition 3.5. *The system of equations (3.5.3) has a solution $P \in I_n + \Psi_-$ if and only if we can find a formal oscillating function of the form*

$$(3.5.4)\qquad W(x,z) = (I_n + \sum_{j=1}^{\infty} W^{(j)}(x)z^{-j})e^{z\cdot x}$$

that satisfies the linear equations

$$(3.5.5)\qquad LW = zW,\ C^{(i)}W = WE_{ii},\ \frac{\partial W}{\partial x_k^{(i)}} = B_k^{(i)}W.$$

Proof (3.5.3) $\Rightarrow$ (3.5.5): Put $W = Pe^{z\cdot x}$. Then we have:

$$\begin{aligned} LW &= LPe^{z\cdot x} = P\partial e^{z\cdot x} = zPe^{z\cdot x} = zW; \\ C^{(i)}W &= C^{(i)}Pe^{z\cdot x} = PE_{ii}e^{z\cdot x} = Pe^{z\cdot x}E_{ii} = WE_{ii}; \\ \frac{\partial W}{\partial x_k^{(i)}} &= \frac{\partial P}{\partial x_k^{(i)}} + P\frac{\partial e^{z\cdot x}}{\partial x_k^{(i)}} = -(L^{(i)k})_- Pe^{z\cdot x} + z^k PE_{ii}e^{z\cdot x} \\ &= -(L^{(i)k})_- W + PE_{ii}\partial^k e^{z\cdot x} = -(L^{(i)k})_- W + C^{(i)}P\partial^k e^{z\cdot x} \\ &= -(L^{(i)k})_- W + C^{(i)}L^k Pe^{z\cdot x} = -(L^{(i)k})_- W + L^{(i)k}W = B_k^{(i)}W. \end{aligned}$$

(3.5.5) $\Rightarrow$ (3.5.3): Define $P \in \Psi$ by $W = Pe^{z\cdot x}$. If $LW = zW$, then $LPe^{z\cdot x} = zPe^{z\cdot x} = P\partial e^{z\cdot x}$, hence $LP = P\partial$.

If $C^{(i)}W = WE_{ii}$, then $C^{(i)}Pe^{z\cdot x} = Pe^{z\cdot x}E_{ii} = PE_{ii}e^{z\cdot x}$, hence $C^{(i)}P = PE_{ii}$.

Finally, the last equation of (3.5.5) gives: $\frac{\partial}{\partial x_k^{(i)}}(Pe^{z\cdot x}) = -(L^{(i)k})_- Pe^{z\cdot x} + L^{(i)k}Pe^{z\cdot x}$.

Since we have already proved the first two equations of (3.5.3), we derive (as above): $L^{(i)k}Pe^{z\cdot x} = z^k Pe^{z\cdot x} = P\frac{\partial e^{z\cdot x}}{\partial x_k^{(i)}}$, hence: $\frac{\partial P}{\partial x_k^{(i)}}e^{x\cdot z} = -(L^{(i)k})_- Pe^{x\cdot z}$, which proves that P satisfies the Sato equations. □

Remarks 3.5. (a) It is easy to see that the collection of formal pseudo-differential operators $\{L, C^{(1)}, \dots, C^{(n)}\}$ of the form (3.5.1) and satisfying (3.5.2) can be simultaneously conjugated to the trivial collection $\{\partial, E_{11}, \dots, E_{nn}\}$ by some $P \in I_n + \Psi_-$. It follows that the solution of the form (3.5.4) to the linear problem (3.5.5) is unique up to multiplication on the right by a diagonal matrix of the form

$$(3.5.6)\qquad D(z) = \exp - \sum_{j=1}^{\infty} a_j z^{-j}/j,$$

where the a_j are diagonal matrices over $\mathbb{C}$ (indeed, this is the case for the trivial collection). The space of all solutions of (3.5.5) in formal oscillating functions is obtained from one of the form (3.5.4) by multiplying on the right by a diagonal matrix over $\mathbb{C}((z))$. For that reason the (matrix valued) functions

$$(3.5.7) \qquad W^+(\alpha, x, z) = P^+(\alpha)e^{x\cdot z}, \ \alpha \in \text{ supp } \tau,$$

are called the *wave functions* for τ. The formal pseudo-differential operator $P^+(\alpha)$ is called the wave operator. The functions $W^-(\alpha, x, z) = P^-(\alpha)e^{-x\cdot z}$ are called the adjoint wave functions and the operators $P^-(\alpha)$ (which are expressed via $P^+(\alpha)$ by (3.3.15)) are called the adjoint wave operators. Note that $V^+(\alpha, x, z)$ are solutions of (3.5.5) as well since they are obtained by multiplying $W^+(\alpha, x, z)$ on the right by $R^+(\alpha, z)$. (b) Multiplying the wave function $W^+(\alpha, x, z)$ on the right by $D(z)$ given by (3.5.6) corresponds to multiplying the corresponding τ-function by $\exp tr \sum_{k=1}^{\infty} a_k x_k$, where $x_k = \text{diag}\,(x_k^{(1)}, \dots, x_k^{(n)})$.

(c) The collection $\{L, C^{(1)}, \dots, C^{(n)}\}$ determines uniquely $P \in I_n + \Psi_-$ up to the multiplication of P on the right by a formal pseudo-differential operator with constant coefficients from $I_n + \Psi_-$.

3.6. In this section we shall rewrite the compatibility conditions of the system (3.5.3) (or equivalent compatibility conditions of the system (3.5.5)) in the form of Lax equations and Zakharov-Shabat equations.

Lemma 3.6. *If for every $\alpha \in M$ the formal pseudo-differential operators $L \equiv L(\alpha)$ and $C^{(j)} \equiv C^{(j)}(\alpha)$ of the form (3.5.1) satisfy conditions (3.5.2) and if the equations (3.5.3) have a solution $P \equiv P(\alpha) \in I_n + \Psi_-$, then the differential operators $B_k^{(j)} \equiv B_k^{(j)}(\alpha) = (L^{(j)}(\alpha)^k)_+$ satisfy one of the following equivalent conditions:*

$$(3.6.1) \qquad \begin{cases} \dfrac{\partial L}{\partial x_k^{(j)}} = [B_k^{(j)}, L] \\ \dfrac{\partial C^{(i)}}{\partial x_k^{(j)}} = [B_k^{(j)}, C^{(i)}] \end{cases}$$

$$(3.6.2) \qquad \frac{\partial L^{(i)}}{\partial x_k^{(j)}} = [B_k^{(j)}, L^{(i)}]$$

$$(3.6.3) \qquad \frac{\partial B_\ell^{(i)}}{\partial x_k^{(j)}} - \frac{\partial B_k^{(j)}}{\partial x_\ell^{(i)}} = [B_k^{(j)}, B_\ell^{(i)}].$$

Proof (cf. [Sh]). To derive the first equation of (3.6.1) we differentiate the equation $LP = P\partial$ by $x_k^{(j)}$:

$$\frac{\partial L}{\partial x_k^{(j)}}P + L\frac{\partial P}{\partial x_k^{(j)}} = \frac{\partial P}{\partial x_k^{(j)}}\partial,$$

and substitute Sato's equation (see (3.5.3)). Then one obtains:

$$\frac{\partial L}{\partial x_k^{(j)}}P = (B_k^{(j)}L - LB_k^{(j)})P$$

from which we derive the desired result. The second equation of (3.6.1) is proven analogously: differentiate $C^{(i)}P = PE_{ii}$, substitute Sato's equation and use the fact that $[L^{(j)k}, C^{(i)}] = 0$.

Next we prove the equivalence of (3.6.1), (3.6.2) and (3.6.3). The implication (3.6.1) $\Rightarrow$ (3.6.2) is trivial. To prove the implication (3.6.2) $\Rightarrow$ (3.6.1) note that $L = \sum_{j=1}^n L^{(j)}$ implies that the first equation of (3.6.1) follows immediately. As for the second one, we have:

$$\frac{\partial C^{(i)}}{\partial x_k^{(j)}} = (\frac{\partial L^{(i)}}{\partial x_k^{(j)}} - C^{(i)}\frac{\partial L}{\partial x_k^{(j)}})L^{-1}$$

$$= ([B_k^{(j)}, L^{(i)}] - C^{(i)}[B_k^{(j)}, L])L^{-1}$$
$$= ([B_k^{(j)}, C^{(i)}]L)L^{-1} = [B_k^{(j)}, C^{(i)}].$$

Next, we prove the implication (3.6.2) $\Rightarrow$ (3.6.3). Since both $\frac{\partial}{\partial x_k^{(j)}}$ and *ad* $B_k^{(j)}$ are derivations, (3.6.2) implies:

$$\frac{\partial L^{(i)\ell}}{\partial x_k^{(j)}} = [B_k^{(j)}, L^{(i)\ell}].$$

Hence:

$$\left(\frac{\partial B_\ell^{(i)}}{\partial x_k^{(j)}} - \frac{\partial B_k^{(j)}}{\partial x_\ell^{(i)}} - [B_k^{(j)}, B_\ell^{(i)}]\right) + \left(\frac{\partial (L^{(i)\ell})_-}{\partial x_k^{(j)}} - \frac{\partial (L^{(j)k})_-}{\partial x_\ell^{(i)}} + [(L^{(j)k})_-, (L^{(i)\ell})_-]\right)$$
$$= [B_k^{(j)}, L^{(i)\ell}] - [B_\ell^{(i)}, L^{(j)k}] - [B_k^{(j)}, B_\ell^{(i)}] + [(L^{(j)k})_-, (L^{(i)\ell})_-]$$
$$= [L^{(j)k}, L^{(i)\ell}] = 0.$$

Since $\Psi_- \cap \Psi_+ = \{0\}$, both terms on the left-hand side are zero proving (3.6.3).

Finally, we prove the implication (3.6.3) $\Rightarrow$ (3.6.2). We rewrite (3.6.3):

$$\frac{\partial L^{(i)\ell}}{\partial x_k^{(j)}} - [B_k^j, L^{(i)\ell}] = \frac{\partial (L^{(i)\ell})_-}{\partial x_k^{(j)}} + \frac{\partial B_k^{(j)}}{\partial x_\ell^{(i)}} - [B_k^{(j)}, (L^{(i)\ell})_-]$$

This right-hand side has order $k-1$, hence

$$\frac{\partial L^{(i)\ell}}{\partial x_k^{(j)}} - [B_k^{(j)}, L^{(i)\ell}] \in \Psi(k-1) \text{ for every } \ell > 0. \tag{3.6.4}$$

Now suppose that $\frac{\partial L^{(i)}}{\partial x_k^{(j)}} - [B_k^{(j)}, L^{(i)}] \neq 0$. Then:

$$\lim_{\ell \to \infty} ord(\frac{\partial L^{(i)\ell}}{\partial x_k^{(i)}} - [B_k^{(j)}, L^{(i)\ell}]) = \infty$$

which contradicts (3.6.4). $\square$

Equations (3.6.1) and (3.6.2) are called *Lax type* equations. Equations (3.6.3) are called the *Zakharov-Shabat type* equations. The latter are the compatibility conditions for the linear problem (3.5.5). Indeed, since $\frac{\partial}{\partial x_k^{(j)}} \frac{\partial}{\partial x_\ell^{(i)}} W = \frac{\partial}{\partial x_\ell^{(i)}} \frac{\partial}{\partial x_k^{(j)}} W$, one finds

$$0 = \frac{\partial}{\partial x_k^{(j)}}(B_\ell^{(i)} W) - \frac{\partial}{\partial x_\ell^{(i)}}(B_k^{(j)} W) = (\frac{\partial B_\ell^{(i)}}{\partial x_k^{(j)}} - \frac{\partial B_k^{(j)}}{\partial x_\ell^{(i)}} - [B_k^{(j)}, B_\ell^{(i)}])W.$$

Notice that as a byproduct of the proof of Proposition 3.6, we obtain complementary Zakharov-Shabat equations:

$$\frac{\partial (L^{(i)\ell})_-}{\partial x_k^{(j)}} - \frac{\partial (L^{(j)k})_-}{\partial x_\ell^{(i)}} = [(L^{(i)\ell})_-, (L^{(j)k})_-]. \tag{3.6.5}$$

Proposition 3.6. *Sato equations (3.4.2) on $P \in I_n + \Psi_-$ imply equations (3.6.3) on differential operators $B_k^{(i)} = (L^{(i)k})_+$.*

Proof is the same as that of the corresponding part of Lemma 3.6. $\square$

Remark 3.6. The above results may be summarized as follows. The n-component KP hierarchy (2.3.7) of Hirota bilinear equations on the τ-function is equivalent to the bilinear equation (3.3.4) on the wave function, which is related to the τ-function by formula (3.3.3) and Remark 3.3. The bilinear equation (3.3.4) for each $\alpha = \beta$ implies the Sato equation (3.4.2) on the formal pseudo-differential operator $P \equiv P(\alpha)$. Moreover, equation (3.4.2) on $P(\alpha)$ for each α together with the matching conditions (3.3.14) are equivalent to the bilinear identity (3.3.4). Also, the Sato equation (or rather (3.5.3)) is a compatibility condition for the linear problem (3.5.5) for the wave function. The Sato equation in turn implies the system of Lax type equations (3.6.2) (or equivalent systems (3.6.1) or (3.6.3), which is the most familiar form of the compatibility condition) on formal pseudo-differential operators $L^{(i)}$ (resp. L and $C^{(i)}$ satisfying constraints (3.5.2)). The latter formal pseudo-differential operators are expressed via the wave function by formulas (3.4.1), (3.3.9–12).

3.7. In this section we write down explicitly some of the Sato equations (3.4.2) on the matrix elements $W_{ij}^{(s)}$ of the coefficients $W^{(s)}(x)$ of the pseudo-differential operator

$$P = I_n + \sum_{m=1}^{\infty} W^{(m)}(x)\partial^{-m}.$$

We shall write W_{ij} for $W_{ij}^{(1)}$ to simplify notation. We have for $i \neq k$:

$$\frac{\partial W_{ij}}{\partial x_1^{(k)}} = W_{ik}W_{kj} - \delta_{jk}W_{ij}^{(2)}, \tag{3.7.1}$$

$$\frac{\partial W_{ij}^{(2)}}{\partial x_1^{(k)}} = W_{ik}W_{kj}^{(2)} - \delta_{jk}W_{ij}^{(3)}. \tag{3.7.2}$$

Next, calculating $\frac{\partial W_{ij}}{\partial x_2^{(k)}}$ from (3.4.2) and substituting (3.7.1) and (3.7.2) in these equations, we obtain:

$$\frac{\partial W_{ij}}{\partial x_2^{(k)}} = W_{ik}\frac{\partial W_{kj}}{\partial x_1^{(k)}} - \frac{\partial W_{ik}}{\partial x_1^{(k)}}W_{kj} \text{ if } k \neq i \text{ and } k \neq j, \tag{3.7.3}$$

$$\frac{\partial W_{ij}}{\partial x_2^{(j)}} = 2\frac{\partial W_{jj}}{\partial x_1^{(j)}}W_{ij} - \frac{\partial^2 W_{ij}}{\partial x_1^{(j)2}} \text{ if } i \neq j, \tag{3.7.4}$$

$$\frac{\partial W_{ij}}{\partial x_2^{(i)}} = -2\frac{\partial W_{ii}}{\partial x_1^{(i)}}W_{ij} + \frac{\partial^2 W_{ij}}{\partial x_1^{(i)2}} \text{ if } i \neq j, \tag{3.7.5}$$

$$\frac{\partial W_{ii}}{\partial x_2^{(i)}} = \frac{\partial^2 W_{ii}}{\partial x_1^{(i)2}} + 2\sum_{p\neq i} W_{ip}\frac{\partial W_{pi}}{\partial x_1^{(i)}} - 2W_{ii}\partial W_{ii} + 2\partial W_{ii}^{(2)}. \tag{3.7.6}$$

Remark 3.7. Substituting expressions for the $W_{ij} = W_{ij}^{(1)}(\alpha = 0, x)$ given by (3.3.7), the above equations turn into the Hirota bilinear equations found in §2.4 as follows:

(3.7.1) for $i = j$ $\Rightarrow$ (2.4.4)
(3.7.1) for $i \neq j$ $\Rightarrow$ (2.4.8)
(3.7.5) $\Rightarrow$ (2.4.5),
(3.7.4) $\Rightarrow$ (2.4.6),
(3.7.3) for $i = j$ $\Rightarrow$ (2.4.7) (with j replaced by k),
(3.7.3) for $i \neq j$ $\Rightarrow$ (2.4.9).

3.8. In this section we write down explicitly some of the Lax equations (3.6.1) of the n-component KP hierarchy and auxiliary conditions (3.5.2) for the formal pseudo-differential operators

(3.8.1) $$L = I_n\partial + \sum_{j=1}^{\infty} U^{(j)}(x)\partial^{-j} \text{ and } C^{(i)} = E_{ii} + \sum_{j=1}^{\infty} C^{(i,j)}(x)\partial^{-j} \ (i = 1, \dots , n).$$

For the convenience of the reader, recall that x stands for all indeterminates $x_i^{(k)}$, where $i = 1, 2, \dots$ and $k = 1, \dots , n$, that the auxiliary conditions are

(3.8.2) $$\sum_{i=1}^{n} C^{(i)} = I_n, \ C^{(i)}C^{(j)} = \delta_{ij}C^{(i)}, \ C^{(i)}L = LC^{(i)},$$

and that the Lax equations of the n-component KP hierarchy are

(3.8.3a)$_i$ $$\frac{\partial L}{\partial x_i^{(k)}} = [B_i^{(k)}, L],$$

(3.8.3b)$_i$ $$\frac{\partial C^{(\ell)}}{\partial x_i^{(k)}} = [B_i^{(k)}, C^{(\ell)}],$$

where $B_i^{(k)} = (C^{(k)}L^i)_+$. For example, we have:

(3.8.4) $$B_1^{(k)} = E_{kk}\partial + C^{(k,1)}, \ B_2^{(k)} = E_{kk}\partial^2 + C^{(k,1)}\partial + 2E_{kk}U^{(1)} + C^{(k,2)}.$$

Denote by $C_{ij}^{(k,\ell)}$ and $U_{ij}^{(k)}$ the (i, j)-th entries of the $n \times n$ matrices $C^{(k,\ell)}$ and $U^{(k)}$ respectively. Then the ∂^{-1} term of the second equation (3.8.2) gives:

(3.8.5) $$C_{ij}^{(k,1)} = 0 \text{ if } i \neq k \text{ and } j \neq k, \text{ or } i = j = k,$$

(3.8.6) $$C_{kj}^{(k,1)} = -C_{kj}^{(j,1)}.$$

Hence the matrices $C^{(j,1)}$ are expressed in terms of the functions

$$A_{ij} := C_{ij}^{(j,1)} \ \text{(note that } A_{ii} = 0).$$

The ∂^{-2} term of the second equation (3.8.2) allowes one to express most of the $C_{ij}^{(k,2)}$ in terms of the A_{ij}:

(3.8.7) $$C_{ij}^{(k,2)} = -A_{ik}A_{kj} \text{ if } i \neq k \text{ and } j \neq k,$$

(3.8.8) $$C_{k,k}^{(k,2)} = \sum_{p=1}^{n} A_{kp}A_{pk}.$$

Furthermore, the ∂^{-1} term of the Lax equation (3.8.3b)$_1$ gives:

(3.8.9) $$\frac{\partial A_{ij}}{\partial x_1^{(k)}} = A_{ik}A_{kj} \text{ for distinct } i, j, k,$$

(3.8.10) $$C_{ij}^{(j,2)} = -\frac{\partial A_{ij}}{\partial x_1^{(j)}} \text{ for } i \neq j,$$

(3.8.11) $$C_{ij}^{(i,2)} = \sum_{\substack{p=1 \\ p\neq i}}^{n} \frac{\partial A_{ij}}{\partial x_1^{(p)}} \text{ for } i \neq j.$$

The ∂^{-2} term of that equation gives for $i \neq j$ (recall that $\partial = \frac{\partial}{\partial x_1^{(1)}} + \ldots + \frac{\partial}{\partial x_1^{(n)}}$.):

$$C_{ij}^{(i,3)} = -\frac{\partial C_{ij}^{(i,2)}}{\partial x_1^{(j)}} + A_{ij} C_{jj}^{(i,2)} - \sum_{p=1}^{n} (A_{ip} \partial A_{pj} + C_{ip}^{(i,2)} A_{pj}), \tag{3.8.12}$$

$$C_{ij}^{(j,3)} = -\sum_{\substack{p=1 \\ p\neq i}}^{n} \frac{\partial C_{ij}^{(j,2)}}{\partial x_1^{(p)}} - A_{ij} C_{ii}^{(j,2)} + \sum_{p=1}^{n} A_{ip} C_{pj}^{(j,2)}. \tag{3.8.13}$$

Substituting (3.8.7, 8 and 11) (resp. (3.8.7, 8 and 10)) in (3.8.12) (resp. in (3.8.13)) we obtain for $i \neq j$:

$$C_{ij}^{(i,3)} = -(\partial - \frac{\partial}{\partial x_1^{(i)}})^2 A_{ij} - 2\sum_{\substack{p=1 \\ p\neq i}}^{n} A_{ip} A_{pi} A_{ij}, \tag{3.8.14}$$

$$C_{ij}^{(j,3)} = \frac{\partial^2 A_{ij}}{\partial x_1^{(j)2}} + 2\sum_{\substack{p=1 \\ p\neq j}}^{n} A_{ij} A_{jp} A_{pj}. \tag{3.8.15}$$

Furthermore, the ∂^0 and ∂^{-1} terms of the Lax equation $(3.8.3\text{a})_1$ give respectively for $i \neq j$:

$$U_{ij}^{(1)} = -\partial A_{ij}, \tag{3.8.16}$$

$$\frac{\partial U_{ii}^{(1)}}{\partial x_1^{(j)}} = -\partial(A_{ij} A_{ji}). \tag{3.8.17}$$

Finally, the ∂^{-1} term of the Lax equation $(3.8.3\text{b})_2$ gives

$$\frac{\partial A_{ij}}{\partial x_2^{(j)}} = -2A_{ij} U_{jj}^{(1)} - C_{ij}^{(j,3)} \text{ for } i \neq j, \tag{3.8.18}$$

$$\frac{\partial A_{ij}}{\partial x_2^{(i)}} = \partial^2 A_{ij} - 2\partial C_{ij}^{(i,2)} - C_{ij}^{(i,3)} + 2U_{ii}^{(1)} A_{ij} \text{ for } i \neq j, \tag{3.8.19}$$

$$\frac{\partial A_{ij}}{\partial x_2^{(k)}} = A_{ik} \frac{\partial A_{kj}}{\partial x_1^{(k)}} - A_{kj} \frac{\partial A_{ik}}{\partial x_1^{(k)}} \text{ for } i \neq k \text{ and } j \neq k. \tag{3.8.20}$$

3.9. Finally, we write down explicitly expressions for $U^{(1)}$ and $C^{(i,1)}$ in terms of τ-functions. Recall that

$$P = I_n + \sum_{j=1}^{\infty} W^{(j)}(x) \partial^{-j},$$

$$L = P\partial P^{-1} = I_n \partial + \sum_{j=1}^{\infty} U^{(j)} \partial^{-j},$$

$$C^{(i)} = P E_{ii} P^{-1} = E_{ii} + \sum_{j=1}^{\infty} C^{(i,j)} \partial^{-j}.$$

Using (3.3.18) we have:

$$U^{(1)} = -\partial W^{(1)}, \tag{3.9.1}$$

(3.9.2) $$U^{(2)} = W^{(1)}\partial W^{(1)} - \partial W^{(2)},$$

(3.9.3) $$C^{(i,1)} = [W^{(1)}, E_{ii}],$$

(3.9.4) $$C^{(i,2)} = [W^{(2)}, E_{ii}] + [E_{ii}, W^{(1)}]W^{(1)}.$$

Using (3.3.7) we obtain from (3.9.1) and (3.9.3) respectively:

(3.9.5) $$U_{ij}^{(1)} = \begin{cases} -\varepsilon_{ji}\partial(\tau_{\alpha+\alpha_{ij}}/\tau_\alpha) & \text{if } i \neq j, \\ \partial(\frac{\partial \tau_\alpha}{\partial x_1^{(i)}}/\tau_\alpha) & \text{if } i = j. \end{cases}$$

(3.9.6) $$A_{ij} \equiv C_{ij}^{(j,1)} = \varepsilon_{ji}\tau_{\alpha+\alpha_{ij}}/\tau_\alpha.$$

(Recall that by (3.8.5 and 6) all the matrices $C^{(k,1)}$ can be expressed via the functions A_{ij}.) Using (3.3.7 and 8) and (3.9.2 and 4) one also may write down the matrices $U^{(2)}$ and $C^{(i,2)}$ in terms of τ-functions, but they are somewhat more complicated and we will not need them anyway.

§4. The n-wave interaction equations, the generalized Toda chain and the generalized Davey-Stewartson equations as subsystems of the n-component KP.

4.0. In this section we show that some well-known soliton equations, as well as their natural generalizations, are the simplest equations of the various formulations of the n-component KP hierarchy. To simplify notation, let

(4.0.1) $$t_i = x_2^{(i)},\ x_i = x_1^{(i)},\ \text{so that } \partial = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}.$$

4.1. Let $n \geq 3$. Then the n-component KP in the form of Sato equation contains the system (3.7.1) of $n(n-1)(n-2)$ equations on $n^2 - n$ functions W_{ij} $(i \neq j)$ in the indeterminates x_i (all other indeterminates being parameters):

(4.1.1) $$\frac{\partial W_{ij}}{\partial x_k} = W_{ik}W_{kj} \text{ for distinct } i, j, k.$$

The τ-function is given by the formula (3.3.7) for a fixed $\alpha \in M$:

(4.1.2) $$W_{ij} = \varepsilon_{ji}\ \tau_{\alpha+\alpha_{ij}}/\tau_\alpha.$$

Substituting this in (4.1.1) gives the Hirota bilinear equation (2.4.8):

(4.1.3) $$D_1^{(k)}\tau_\alpha \cdot \tau_{\alpha+\alpha_{ij}} = \varepsilon_{ik}\varepsilon_{kj}\varepsilon_{ij}\tau_{\alpha+\alpha_{ik}}\tau_{\alpha+\alpha_{kj}}$$

Note that due to (3.9.3), $W_{ij} = A_{ij}$ if $i \neq j$, hence (4.1.1) is satisfied by the A_{ij} as well.

One usually adds to (4.1.1) the equations

(4.1.4) $$\partial W_{ij} = 0,\ i \neq j.$$

We shall explain the group theoretical meaning of this constraint in §6.

Let now $a = \text{diag}(a_1, \ldots, a_n)$, $b = \text{diag}(b_1, \ldots, b_n)$ be arbitrary diagonal matrices over $\mathbb{C}$. We reduce the system (4.1.1) to the plane [D]:

(4.1.5) $$x_k = a_k x + b_k t.$$

A direct calculation shows that (4.1.1) reduces then to the following equation on the matrix $W = (W_{ij})$ (note that its diagonal entries don't occur):

$$[a, \frac{\partial W}{\partial t}] - [b, \frac{\partial W}{\partial x}] = [[a, W], [b, W]] + b\partial W a - a\partial W b. \tag{4.1.6}$$

Hence, imposing the constraint (4.1.4), we obtain the famous $1+1$ n-wave system (cf. [D], [NMPZ]):

$$[a, \frac{\partial W}{\partial t}] - [b, \frac{\partial W}{\partial x}] = [[a, W], [b, W]]. \tag{4.1.7}$$

Let now

$$x_k = a_k x + b_k t - y. \tag{4.1.8}$$

Then equation (4.1.6) gives

$$[a, \frac{\partial W}{\partial t}] - [b, \frac{\partial W}{\partial x}] - a\frac{\partial W}{\partial y}b + b\frac{\partial W}{\partial y}a = [[a, W], [b, W]]. \tag{4.1.9}$$

If we let

$$Q_{ij} = -(a_i - a_j)W_{ij}.$$

equation (4.1.9) turns into the following system, which is called in [AC, (5.4.30a,c)] the $2+1$ n-wave interaction equations ($i \neq j$):

$$\frac{\partial Q_{ij}}{\partial t} = a_{ij}\frac{\partial Q_{ij}}{\partial x} + b_{ij}\frac{\partial Q_{ij}}{\partial y} + \sum_k (a_{ik} - a_{kj})Q_{ik}Q_{kj}, \tag{4.1.10}$$

where

$$a_{ij} = (b_i - b_j)/(a_i - a_j), \;\; b_{ij} = b_i - a_i a_{ij}. \tag{4.1.11}$$

On the other hand, letting (we assume that $a_1 > \ldots > a_n$):

$$w_{ij} = W_{ij}/(a_i - a_j)^{1/2}, \tag{4.1.12}$$

the equation (4.1.6) gives for $i \neq j$:

$$\frac{\partial w_{ij}}{\partial t} - a_{ij}\frac{\partial w_{ij}}{\partial x} - b_{ij}\frac{\partial w_{ij}}{\partial y} = \sum_k \varepsilon_{ijk} w_{ik} w_{kj}, \tag{4.1.13}$$

where

$$\varepsilon_{ijk} = \frac{a_i b_k + a_k b_j + a_j b_i - a_k b_i - a_j b_k - a_i b_j}{((a_i - a_k)(a_k - a_j)(a_i - a_j))^{1/2}}. \tag{4.1.14}$$

Imposing the constraint $\overline{w}_{ij} = -w_{ji}$, we obtain from (4.1.13) the following Hamiltonian system (considered in [NMPZ, pp 175, 242] for $n = 3$ and called there the $2+1$ 3-wave system) ($i < j$):

$$\frac{\partial w_{ij}}{\partial t} - a_{ij}\frac{\partial w_{ij}}{\partial x} - b_{ij}\frac{\partial w_{ij}}{\partial y} = \frac{\partial H}{\partial \overline{w}_{ij}}, \tag{4.1.15}$$

where

$$H = \sum_{\substack{i,k,j \\ i<k<j}} \varepsilon_{ijk}(w_{ik}w_{kj}\overline{w}_{ij} + \overline{w}_{ik}\overline{w}_{kj}w_{ij}). \tag{4.1.16}$$

Finally, let $n = 3$ and let $u_1 = iw_{13}$, $u_2 = i\overline{w}_{13}$, $u_3 = iw_{12}$, $a_1 = -a_{23}$, $b_1 = -b_{23}$, $a_2 = -\overline{a}_{13}$, $b_2 = -\overline{b}_{13}$, $a_3 = -a_{12}$, $b_3 = -b_{13}$. Then, after imposing the constraint $\varepsilon_{132} = 1$, equations (4.1.15) turn into the well-known $2+1$ 3-wave interaction equations (see [AC, (5.4.27)]):

$$\frac{\partial u_j}{\partial t} + a_j \frac{\partial u_j}{\partial x} + b_j \frac{\partial u_j}{\partial y} = i\overline{u}_k \overline{u}_\ell, \tag{4.1.17}$$

where (j, k, ℓ) is an arbitrary cyclic permutation of $1, 2, 3$.

4.2. Let $n \geq 2$. Then the n-component KP in the form of Sato equations contains the following subsystem of the system of equations (3.7.1) for arbitrary $\alpha \in M$ on the functions $W_{ij}(\alpha)$ in the indeterminates x_i (all other indeterminates being parameters):

$$\frac{\partial W_{ii}(\alpha)}{\partial x_j} = W_{ij}(\alpha) W_{ji}(\alpha) \text{ if } i \neq j. \tag{4.2.1}$$

The τ-function is given by (3.3.7) ($\alpha \in M$):

$$W_{ij}(\alpha) = \begin{cases} \varepsilon_{ji} \tau_{\alpha + \alpha_{ij}} / \tau_\alpha & \text{if } i \neq j \\ -\frac{\partial}{\partial x_i} \log \tau_\alpha & \text{if } i = j. \end{cases} \tag{4.2.2}$$

Substituting this in (4.2.1) gives the Hirota bilinear equations (2.4.4):

$$D_i D_j \tau_\alpha \cdot \tau_\alpha = 2 \tau_{\alpha + \alpha_{ij}} \tau_{\alpha - \alpha_{ij}}. \tag{4.2.3}$$

In order to rewrite (4.2.1) in a more familiar form, let for $i \neq j$:

$$U_{ij}(\alpha) = \log \varepsilon_{ji} W_{ij}(\alpha) = \log(\tau_{\alpha + \alpha_{ij}} / \tau_\alpha). \tag{4.2.4}$$

Note that $\log(\tau_{\alpha+\alpha_{ij}}/\tau_\alpha) = -\log(\tau_{(\alpha+\alpha_{ij})-\alpha_{ij}}/\tau_{\alpha+\alpha_{ij}}$. Hence from (4.2.2) we obtain

$$U_{ij}(\alpha) = -U_{ji}(\alpha + \alpha_{ij}) \text{ if } i \neq j. \tag{4.2.5}$$

Furthermore, we have:

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} U_{ij}(\alpha) &= \frac{\partial^2}{\partial x_i \partial x_j} \log \tau_{\alpha + \alpha_{ij}} - \frac{\partial^2}{\partial x_i \partial x_j} \log \tau_\alpha \\ &= \frac{\partial W_{ii}(\alpha)}{\partial x_j} - \frac{\partial W_{ii}(\alpha + \alpha_{ij})}{\partial x_j} = W_{ij}(\alpha) W_{ji}(\alpha) - W_{ij}(\alpha + \alpha_{ij}) W_{ji}(\alpha + \alpha_{ij}) \\ &= -\frac{\tau_{\alpha+\alpha_{ij}}}{\tau_\alpha} \frac{\tau_{\alpha-\alpha_{ij}}}{\tau_\alpha} + \frac{\tau_{\alpha+2\alpha_{ij}}}{\tau_{\alpha+\alpha_{ij}}} \frac{\tau_\alpha}{\tau_{\alpha+\alpha_{ij}}} = e^{U_{ij}(\alpha+\alpha_{ij}) - U_{ij}(\alpha)} - e^{U_{ij}(\alpha) - U_{ij}(\alpha - \alpha_{ij})}. \end{aligned}$$

Thus the functions $U_{ij}(\alpha)$ ($i \neq j$) satisfy the following generalized Toda chain (with constraint (4.2.5)):

$$\frac{\partial^2 U_{ij}(\alpha)}{\partial x_i \partial x_j} = e^{U_{ij}(\alpha+\alpha_{ij}) - U_{ij}(\alpha)} - e^{U_{ij}(\alpha) - U_{ij}(\alpha - \alpha_{ij})}. \tag{4.2.6}$$

Note also that (4.1.1) for distinct i, j and k becomes:

$$\frac{\partial U_{ij}(\alpha)}{\partial x_k} = \varepsilon_{ik} \varepsilon_{kj} \varepsilon_{ji} e^{U_{ij}(\alpha) + U_{ik}(\alpha) + U_{kj}(\alpha)}. \tag{4.2.7}$$

One should be careful about the boundary conditions. Let $S =$ supp τ; recall that by

Proposition 2.4, S is a convex polyhedron with vertices in M and edges parallel to roots. It follows that (4.2.6) should be understood as follows:

(i) if $\alpha \notin S$, then $U_{ij}(\alpha) = 0$ and (4.2.6) is trivial,

(ii) if $\alpha \in S$, but $\alpha + \alpha_{ij} \notin S$, then (4.2.6) is trivial,

(iii) if $\alpha \in S$, but $\alpha - \alpha_{ij} \notin S$, then the second term on the right-hand side of (4.2.6) is removed,

(iv) if $\alpha \in S$, $\alpha + \alpha_{ij} \in S$, but $\alpha + 2\alpha_{ij} \notin S$, then the first term on the right-hand side of (4.2.6) is removed.

Let now $n = 2$, and let $u_n = U_{12}(n\alpha_{12})$. Then we get the usual Toda chain:

$$\frac{\partial^2 u_n}{\partial x_1 \partial x_2} = e^{u_{n+1}-u_n} - e^{u_n - u_{n-1}},\ n \in \mathbb{Z}. \tag{4.2.8}$$

It is a part of the Toda lattice hierarchy discussed in [UT].

4.3. Let $n \geq 2$. Then the n-component KP in the form of Sato equations contains the system of equations (3.7.4), (3.7.5), (3.7.3) and (3.7.1) for $j \neq k$ on n^2 functions W_{ij} in the indeterminates x_k and t_k $(k = 1, \dots, n)$ (all other indeterminates being parameters):

$$\frac{\partial W_{ij}}{\partial t_j} = -\frac{\partial^2 W_{ij}}{\partial x_j^2} + 2\frac{\partial W_{jj}}{\partial x_j} W_{ij} \text{ if } i \neq j, \tag{4.3.1}$$

$$\frac{\partial W_{ij}}{\partial t_i} = \frac{\partial^2 W_{ij}}{\partial x_i^2} - 2\frac{\partial W_{ii}}{\partial x_i} W_{ij} \text{ if } i \neq j, \tag{4.3.2}$$

$$\frac{\partial W_{ij}}{\partial t_k} = W_{ik}\frac{\partial W_{kj}}{\partial x_k} - \frac{\partial W_{ik}}{\partial x_k} W_{kj} \text{ if } i \neq k \text{ and } j \neq k, \tag{4.3.3}$$

$$\frac{\partial W_{ij}}{\partial x_k} = W_{ik} W_{kj} \text{ if } i \neq k \text{ and } j \neq k. \tag{4.3.4}$$

This is a system of $n^3 - n$ evolution equations (4.3.1–3) and $n(n-1)^2$ constraints (4.3.4) which we call the generalized Davey-Stewartson system.

Note that the τ-functions of this system are given by (3.3.7), where we may take $\alpha = 0$. The corresponding to (4.3.1)–(4.3.4) Hirota bilinear equations are (2.4.6); (2.4.5); (2.4.7) if $i = j$ and (2.4.9) if $i \neq j$; (2.4.4) if $i = j$ and (2.4.8) if $i \neq j$, respectively.

Now, note that letting

$$\varphi_{ij} = \frac{1}{2}\left(\frac{\partial W_{ii}}{\partial x_i} + \frac{\partial W_{jj}}{\partial x_j} + \frac{\partial W_{ii}}{\partial x_j} + \frac{\partial W_{jj}}{\partial x_i}\right) (= \varphi_{ji}),$$

and subtracting (4.3.2) from (4.3.1) we obtain using (4.3.4):

$$\frac{\partial W_{ij}}{\partial t_j} - \frac{\partial W_{ij}}{\partial t_i} = -\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2}\right) W_{ij} + 2W_{ij}(\varphi_{ij} - W_{ij}W_{ji}). \tag{4.3.5}$$

Also, from (4.3.4) we obtain

$$\frac{\partial^2 \varphi_{ij}}{\partial x_i \partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right)^2 (W_{ij}W_{ji}). \tag{4.3.6}$$

Let now $n = 2$; to simplify notation, let

$$q = W_{12},\ r = W_{21},\ \varphi = \varphi_{12} = \varphi_{21}.$$

Then, making the change of indeterminates

(4.3.7) $$s = -2i(t_1 + t_2),\ t = -2i(t_1 - t_2),\ x = x_1 + x_2,\ y = x_1 - x_2,$$

equations (4.3.5 and 6) turn into the decoupled Davey-Stewartson system:

(4.3.8) $$\begin{cases} i\frac{\partial q}{\partial t} = -\frac{1}{2}(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}) + q(\varphi - qr) \\ i\frac{\partial r}{\partial t} = \frac{1}{2}(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}) - r(\varphi - qr) \\ \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = 2\frac{\partial^2 (qr)}{\partial x^2}. \end{cases}$$

Due to (3.3.7), the corresponding τ-functions are given by the following formulas, where we let $\tau_n = \tau_{n\alpha_{12}}$:

(4.3.9) $$q = -\tau_1/\tau_0,\ r = \tau_{-1}/\tau_0,\ \varphi = -\frac{\partial^2}{\partial x^2}\log\tau_0,$$

the Hirota bilinear equations being (cf. [HH]):

(4.3.10) $$\begin{aligned} &(iD_t + \frac{1}{2}D_x^2 + \frac{1}{2}D_y^2)\tau_1 \cdot \tau_0 = 0 \\ &(-iD_t + \frac{1}{2}D_x^2 + \frac{1}{2}D_y^2)\tau_{-1} \cdot \tau_0 = 0 \\ &(D_x^2 - D_y^2)\tau_0 \cdot \tau_0 = 2\tau_1\tau_{-1}. \end{aligned}$$

Finally, imposing the constraint

(4.3.11) $$r = \kappa\bar{q},\ \text{where } \kappa = \pm 1,$$

we obtain the classical Davey-Stewartson system

(4.3.12) $$\begin{cases} i\frac{\partial q}{\partial t} + \frac{1}{2}(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}) = (\varphi - \kappa|q|^2)q \\ \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = 2\kappa\frac{\partial^2 |q|^2}{\partial x^2}. \end{cases}$$

Remark 4.3. It is interesting to compare the above results with that obtained via the Lax equations. To simplify notation, let $U_i = U_{ii}^{(1)}$. Substituting (3.8.15) (resp. (3.8.14)) in (3.8.18) (resp. (3.8.19)), we obtain for $i \neq j$:

(4.3.13) $$\frac{\partial A_{ij}}{\partial t_j} = -\frac{\partial^2 A_{ij}}{\partial x_j^2} - 2A_{ij}U_j - 2\sum_{k\neq j} A_{ij}A_{jk}A_{kj}$$

(4.3.14) $$\frac{\partial A_{ij}}{\partial t_i} = \frac{\partial^2 A_{ij}}{\partial x_i^2} + 2A_{ij}U_i + 2\sum_{k\neq i} A_{ij}A_{ik}A_{ki}$$

These equation together with (3.8.9, 17 and 20) give a slightly different version of the generalized DS system (recall that $A_{ij} = W_{ij}$ if $i \neq j$ and $U_i = -\partial W_{ii}$). For $n = 2$ we get again the classical DS system after the change of indeterminates (4.3.7) if we let $\varphi = -\frac{1}{2}(U_1 + U_2)$.

4.4. Finally, we explain what happens in the well-known case $n = 1$. In this case $C^{(1)} = 1$ and auxiliary conditions (3.8.2) are trivial. Lax equation (3.8.3b) is trivial as well, and Lax equation (3.8.3a) becomes

(4.4.1) $$\frac{\partial L}{\partial x_i} = [B_i, L],\ i = 1, 2, \dots,$$

where $L = \partial + \sum_{j=1}^{\infty} u_j(x)\partial^{-j}$, $\partial = \frac{\partial}{\partial x_1}$ and $B_i = (L^i)_+$. Thus, the KP hierarchy is a system of partial differential equations (4.4.1) on unknown functions $u_1, u_2, \dots$ in indeterminates $x_1, x_2, \dots$. By Lemma 3.6, (4.4.1) is equivalent to the following system of Zakharov-Shabat equations:

$$(4.4.2)_{k,\ell} \qquad \frac{\partial B_\ell}{\partial x_k} - \frac{\partial B_k}{\partial x_\ell} = [B_k, B_\ell].$$

By (3.8.4) we have:

$$(4.4.3) \qquad B_1 = \partial, \ B_2 = \partial^2 + 2u_1.$$

Furthermore, we have:

$$(4.4.4) \qquad B_3 = \partial^3 + 3u_1\partial + 3u_2 + 3\frac{\partial u_1}{\partial x_1}$$

Thus we see that equations $(4.4.2)_{k,1}$ are all trivial, the first non-trivial equation of (4.4.2) being

$$\frac{\partial B_2}{\partial x_3} - \frac{\partial B_3}{\partial x_2} = [B_2, B_3].$$

Substituting in it (4.4.3 and 4), the coefficients of ∂^0 and ∂^1 give respectively:

$$(4.4.5) \qquad 2\frac{\partial u_1}{\partial x_3} - 2\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - 6u_1\frac{\partial u_1}{\partial x_1} = 3\frac{\partial u_2}{\partial x_2} - 3\frac{\partial^2 u_2}{\partial x_1^2}$$

$$(4.4.6) \qquad 6\frac{\partial u_2}{\partial x_1} = 3\frac{\partial u_1}{\partial x_2} - \frac{\partial^2 u_1}{\partial x_1^2}.$$

Differentiating (4.4.5) by x_1 and substituting $\frac{\partial u_2}{\partial x_1}$ from (4.4.6) gives a PDE on $u = 2u_1$, where we let $x_1 = x,\ x_2 = y, x_3 = t$:

$$(4.4.7) \qquad \frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right).$$

This is the classical KP equation. Due to (3.9.5), the connection between u and the τ-function is given by the famous formula

$$(4.4.8) \qquad u = 2\frac{\partial^2}{\partial x^2}\log\tau.$$

Substituting u in (4.4.7) gives the Hirota bilinear equation (2.4.3).

§5. Soliton and dromion solutions.

5.1. We turn now to the construction of solutions of the n-component KP hierarchy. As in [DJKM3] we make use of the vertex operators (2.1.14). When transported via the n-component boson-fermion correspondence σ from F to $B = \mathbb{C}[x] \otimes \mathbb{C}[L]$, they take the following form:

$$(5.1.1) \qquad \psi^{\pm(i)}(z) = Q_i^{\pm 1} z^{\pm\alpha_0^{(i)}}(\exp \pm \sum_{k=1}^{\infty} z^k x_k^{(i)})(\exp \mp \sum_{k=1}^{\infty} \frac{z^{-k}}{k}\frac{\partial}{\partial x_k^{(i)}}).$$

Note that for $z, w \in \mathbb{C}^\times$ such that $|w| < |z|$ we have ($\lambda, \mu = +$ or $-$):

$$\psi^{\lambda(i)}(z)\psi^{\mu(j)}(w) = (z-w)^{\delta_{ij}\lambda\mu}Q_i^{\lambda 1}Q_j^{\mu 1}z^{\alpha_0^{(i)}}w^{\alpha_0^{(j)}}$$
(5.1.2)
$$\times \exp\sum_{k=1}^{\infty}(\lambda z^k x_k^{(i)} + \mu w^k x_k^{(j)}) \exp - \sum_{k=1}^{\infty}(\lambda\frac{z^{-k}}{k}\frac{\partial}{\partial x_k^{(i)}} + \mu\frac{w^{-k}}{k}\frac{\partial}{\partial x_k^{(j)}}).$$

We let for $0 < |w| < |z|$:
(5.1.3)

$$\Gamma_{ij}(z,w) \stackrel{\text{def}}{=} \psi^{+(i)}(z)\psi^{-(j)}(w) = (z-w)^{-\delta_{ij}}Q_iQ_j^{-1}z^{\alpha_0^{(i)}}w^{-\alpha_0^{(j)}}\exp\sum_{k=1}^{\infty}(z^k x_k^{(i)} - w^k x_k^{(j)})$$
$$\times \exp - \sum_{k=1}^{\infty}(\frac{z^{-k}}{k}\frac{\partial}{\partial x_k^{(i)}} - \frac{w^{-k}}{k}\frac{\partial}{\partial x_k^{(j)}}).$$

Using (5.1.2), we obtain for $|z_1| > |z_2| > \ldots > |z_{2N-1}| > |z_{2N}| > 0$:

$$\Gamma_{i_1i_2}(z_1,z_2)\ldots\Gamma_{i_{2N-1}i_{2N}}(z_{2N-1},z_{2N}) = \prod_{1\le k<\ell\le 2N}(z_k - z_\ell)^{(-1)^{k+\ell}\delta_{i_k i_\ell}}$$
(5.1.4)
$$\times Q_{i_1}Q_{i_2}^{-1}\ldots Q_{i_{2N-1}}Q_{i_{2N}}^{-1}\prod_{m=1}^{2N} z_m^{(-1)^{m-1}\alpha_0^{(i_m)}}\exp(-\sum_{m=1}^{2N}\sum_{k=1}^{\infty}(-1)^m z_m^k x_k^{(m)})$$
$$\times \exp(\sum_{m=1}^{2N}\sum_{k=1}^{\infty}(-1)^m\frac{z_m^{-k}}{k}\frac{\partial}{\partial x_k^{(m)}}).$$

We may analytically extend the right-hand side of (5.1.4) to the domain $\{z_i \neq 0,\ z_i \neq z_j \text{ if } i \neq j,\ i,j = 1,\ldots,2N\}$. Then we deduce from (5.1.4) for $N = 2$ that in this domain we have:

(5.1.5)
$$\Gamma_{i,i_2}(z_1,z_2)\Gamma_{i_3,i_4}(z_3,z_4) = \Gamma_{i_3i_4}(z_3,z_4)\Gamma_{i,i_2}(z_1,z_2),$$

(5.1.6)
$$\Gamma_{ij}(z_1,z_2)^2 \equiv \lim_{\substack{z_3\to z_1\\ z_4\to z_2}}\Gamma_{ij}(z_1,z_2)\Gamma_{ij}(z_3,z_4) = 0.$$

Remark 5.1. Let $A = (a_{ij})$ be a $n\times n$ matrix over $\mathbb{C}$ and let z_i, w_i $(i = 1,\ldots,n)$ be non-zero complex numbers such that $z_i \neq w_j$. Due to (1.2.3) the sum

(5.1.7)
$$\Gamma_A(z,w) = \sum_{i,j=1}^{n} a_{ij}\Gamma_{ij}(z_i,w_j)$$

lies in a completion of $r(g\ell_\infty)$.

By (5.1.5–6) we obtain:

(5.1.8)
$$\exp\Gamma_A(z,w) = \prod_{i,j=1}^{n}(1 + a_{ij}\Gamma_{ij}(z_i,w_j)).$$

Lemma 5.1. *(a) If τ is a solution of the n-component KP hierarchy (2.3.7) of Hirota bilinear equations, then* $(\exp\Gamma_A(z,w))\tau$ *is a solution as well for any complex $n\times n$ matrix A and any $z = (z_1,\ldots z_n)$, $w = (w_1,\ldots,w_n) \in \mathbb{C}^{\times n}$ such that $z_i \neq w_j$.*

(b) For any collection of complex $n\times n$-matrices $A_1,\ldots,A_N$ and any collection $z^{(1)},\ldots,z^{(N)},w^{(1)},\ldots,w^{(N)} \in \mathbb{C}^{\times n}$ with all coordinates distinct, the function

$$\exp\Gamma_{A_1}(z^{(1)},w^{(1)})\dots\exp\Gamma_{A_N}(z^{(N)},w^{(N)})\cdot 1 \tag{5.1.9}$$

is a solution of the n-component KP hierarchy (2.3.7).

Proof. (a) follows from Proposition 1.3 and Remark 5.1. (b) follows from (a) since the function $1=\sigma|0\rangle$ satisfies (1.3.1). □

We call (5.1.9) the *N-solitary τ-function* (of the n-component KP hierarchy).

In order to write down (5.1.9) in a more explicit form, introduce the lexicographic ordering on the set S of all triples $s=(p,i,j)$, where $p\in\{1,\dots,N\}$, $i,j\in\{1,\dots,n\}$ (i.e. $s_1<s_2$ if $p_1<p_2$, or $p_1=p_2$ and $i_1<i_2$ or $p_1=p_2$, $i_1=i_2$ and $j_1<j_2$). Given N $n\times n$ complex matrices $A_p=(a_{ij}^{(p)})$, we let $a_s=a_{ij}^{(p)}$ for $s=(p,i,j)\in S$; given in addition two sets of non-zero complex numbers z_s and w_s, all distinct, parametrized by $s\in S$, introduce the following constants

$$\begin{aligned} c(s_1,\dots,s_r) &= \prod_{k=1}^{r} a_{s_k}\prod_{\ell=k+1}^{r}\varepsilon_{i_k i_\ell}\varepsilon_{i_k j_\ell}\varepsilon_{j_k i_\ell}\varepsilon_{j_k j_\ell} \\ &\times \prod_{1\le k<\ell\le r}\frac{(z_{s_k}-z_{s_\ell})^{\delta_{i_k i_\ell}}(w_{s_k}-w_{s_\ell})^{\delta_{j_k j_\ell}}}{(z_{s_k}-w_{s_\ell})^{\delta_{i_k j_\ell}}(w_{s_k}-z_{s_\ell})^{\delta_{j_k i_\ell}}}. \end{aligned} \tag{5.1.10}$$

Then the N-solitary solution (5.1.9) can be written as follows

$$\begin{aligned} &1+\sum_{r=1}^{Nn^2}\sum_{(1,1,1)\le s_1<\dots<s_r\le(N,n,n)} c(s_1,\dots,s_r) \\ &\times(\exp\sum_{k=1}^{r}\sum_{m=1}^{\infty}(z_{s_k}^m x_m^{(i_k)}-w_{s_k}^m x_m^{(j_k)}))e^{\sum_{k=1}^r \alpha_{i_k j_k}}. \end{aligned} \tag{5.1.11}$$

5.2. Let $n=1$. Then the index set S is naturally identified with the set $\{1,\dots,N\}$, the two sets of complex numbers we denote by z_{2j-1} and z_{2j}, $j=1,\dots,N$, and we let $A_p=(z_{2p-1}-z_{2p})^{-1}a_p$, where a_p are some constants. Then (5.1.11) becomes the well-known formula (see [DJKM3]) for the τ-function of the N-soliton solution:

$$\begin{aligned} \tau^{(N)} &= 1+\sum_{r=1}^{N}\sum_{1\le j_1<\dots<j_r\le N}\prod_{k=1}^{r}a_{j_k}\prod_{1\le k<\ell\le 2r}(z_{j_k}-z_{j_\ell})^{(-1)^{k+\ell}} \\ &\times\exp\sum_{k=1}^{r}\sum_{m=1}^{\infty}(z_{j_{2k-1}}^m-z_{j_{2k}}^m)x_m. \end{aligned} \tag{5.2.1}$$

Letting $x_1=x$, $x_2=y$, $x_3=t$ and all other indeterminates constants $x_4=c_4,\dots$, we obtain, due to (4.4.8), the soliton solution of the classical KP equation (4.4.7):

$$u(t,x,y)=2\frac{\partial^2}{\partial x^2}\log\tau^{(N)}(x,y,t,c_4,c_5,\dots). \tag{5.2.2}$$

In particular, the τ-function of the 1-soliton solution is

$$\tau^{(1)}(x,y,t)=1+\frac{a}{z_1-z_2}\exp((z_1-z_2)x+(z_1^2-z_2^2)y+(z_1^3-z_2^3)t+\text{ const.}) \tag{5.2.3}$$

and we get the corresponding 1-soliton solution of the classical KP equation (4.4.7):

$$u(x,y,t)=\frac{(z_1-z_2)^2}{2}\cosh^{-2}(\frac{1}{2}((z_1-z_2)x+(z_1^2-z_2^2)y+(z_1^3-z_2^3)t)+\text{ const.}). \tag{5.2.4}$$

5.3. Let $n = 2$. Then any $\tau \in \mathbb{C}[x] \otimes \mathbb{C}[M]$ can be written in the form

$$\tau = \sum_{\ell \in \mathbf{Z}} \tau_\ell e^{\ell \alpha_{12}}, \text{ where } \tau_\ell \equiv \tau_{\ell\alpha_{12}}.$$

For a N-solitary solution $\tau^{(N)}$ given by (5.1.11) we then have

$$\tau_\ell^{(N)} = \delta_{\ell,0} + \sum_{r=1}^{4N} \sum_{(s_1,\dots,s_r)} c(s_1,\dots,s_r) \exp \sum_{k=1}^{r} \sum_{m=1}^{\infty} (z_{s_k}^m x_m^{(i_k)} - w_{s_k}^m x_m^{(j_k)}), \tag{5.3.1}$$

where $(s_1,\dots,s_r)$ run over the subset $(5.3.2)_2$ of S^r, where

$$(5.3.2)_n \qquad \begin{cases} (1,1,1) \le s_1 < s_2 \dots < s_r \le (N,n,n) \\ \#\{(i_k,j_k)|i_k > j_k\} - \#\{(i_k,j_k)|i_k < j_k\} = \ell. \end{cases}$$

Letting (cf. (4.3.9)):

$$(5.3.3) \qquad \begin{aligned} q &= -\frac{\tau_1(x,y,t,c,c_{3,\dots}^{(1)})}{\tau_0(x,y,t,c,c_3^{(1)},\dots)}, \quad r = \frac{\tau_{-1}(x,y,t,c,c_3^{(1)},\dots)}{\tau_0(x,y,t,c,c_3^{(1)},\dots)}, \\ \varphi &= -\frac{\partial^2}{\partial x^2}(\log \tau_0(x,y,t,c,c_3^{(1)},\dots)), \end{aligned}$$

where $x = x_1^{(1)} + x_2^{(1)}$, $y = x_1^{(1)} - x_2^{(1)}$, $t = -2i(x_2^{(1)} - x_2^{(2)})$, $c = -2i(x_2^{(1)} + x_2^{(2)})$ and all other indeterminates $x_k^{(j)}$ are arbitrary constants $c_k^{(j)}$, we obtain a N-solitary solution of the decoupled Davey-Stewartson system (4.3.8).

We turn now to the classical Davey-Stewartson system (4.3.12) for $\kappa = -1$. The constraint (4.3.11) gives

$$\tau_1/\tau_0 = \overline{\tau_{-1}/\tau_0}.$$

One way of satisfying this constraint is to let

$$(5.3.4) \qquad \begin{aligned} a_{ij}^{(p)} &= (-1)^{i+j}\overline{a}_{ji}^{(p)}, \quad z_{(p,i,j)} = -\overline{w}_{(p,j,i)}, \\ c_2 &= 0, \quad c_k^{(j)} \in i^{k+1}\mathbf{R}. \end{aligned}$$

We shall concentrate now on the case $N = 1$. It will be convenient to use the following notation:

$$\begin{aligned} &x_1 = x_1^{(1)}, \quad x_2 = x_1^{(2)}, \\ &z_{ij} = z_{(1,i,j)}, \quad a_i = a_{(1,i,i)} \in \mathbf{R}_+ \ (1 \le i,j \le 2), \quad a_3 = a_{(1,1,2)} \in \mathbb{C}, \\ &C(z,w) = \frac{z-w}{z+\overline{w}}, \quad D(z,w) = \frac{|a_3|^2}{(z+\overline{z})(w+\overline{w})}, \\ &A_j(z) = (z+\overline{z})(x_j - (-1)^j it\frac{z-\overline{z}}{4}) + \sum_{k=3}^{\infty}(z^k - (-\overline{z})^k)c_k^{(j)} \ (j = 1,2), \\ &A_3(z,w) = zx_1 + \overline{w}z_2 + it(\frac{z^2}{4} + \frac{\overline{w}^2}{4}) + \sum_{k=3}^{\infty}(z^k c_k^{(1)} - (-\overline{w})^k c_k^{(2)}). \end{aligned}$$

Then $q = -\tau_1/\tau_0$ and $\varphi = -\frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})^2 \log \tau_0$ is a solution of (4.3.12), where

$$\tau_1 = a_3 e^{A_3(z_{12},z_{21})}(1 + a_1 C(z_{12},z_{11})e^{A_1(z_{11})})(1 + a_2 C(\overline{z}_{21},\overline{z}_{22})e^{A_2(z_{22})}), \tag{5.3.5a}$$

$$\begin{aligned}(5.3.5b)\qquad \tau_0 &= (1+a_1e^{A_1(z_{11})})(1+a_2e^{A_2(z_{22})})\\ &+ D(z_{12},z_{21})e^{A_3(z_{12},z_{21})+\overline{A_3(z_{12},z_{21})}}(1+a_1|C(z_{12},z_{21})|^2e^{A_1(z_{11})})\\ &\times(1+a_2|C(z_{21},z_{22})|^2e^{A_2(z_{22})}).\end{aligned}$$

Consider now two special cases of (5.3.5a,b):

(D) $z_1 \equiv z_{11} = z_{12}$ and $z_2 \equiv z_{22} = z_{21}$,

(S) $a_i = 0 \quad (i = 1, 2)$,

and let $T = D$ or S. Then (5.3.5a and b) reduce to

$$(5.3.6a)\qquad \tau_1 = a_3e^{A_3(z_1,z_2)} \text{ in both cases,}$$

$$(5.3.6b)\qquad \tau_0^{(T)} = (1+\delta_{TD}a_1e^{A_1(z_1)})(1+\delta_{TD}a_2e^{A_2(z_2)}) + D(z_1,z_2)e^{A_1(z_1)+A_2(z_2)},$$

so that $q^{(T)} = -\tau_1/\tau_0^{(T)}$, $\varphi^{(T)} = -\frac{1}{2}(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2})^2\log\tau_0^{(T)}$ is a solution of (4.3.12).

In order to rewrite $q^{(T)}$ in a more familiar form, let ($j = 1, 2$ and $a_j(z_j+\overline{z}_j) > 0$):

$$\begin{aligned}&p_j^{(T)} = (a_j(z_j+\overline{z}_j))^{-1/2} \text{ if } T = D \text{ and } = 1 \text{ if } T = S,\\ &\mu_1 = \mu_{1R}+i\mu_{1I} = \tfrac{1}{2}\overline{z}_1,\ \mu_2 = \mu_{2R}+i\mu_{2I} = \tfrac{1}{2}z_2,\\ &m_j^{(T)} = 2\sqrt{2}\mu_{jR}p_j^{(T)}e^{-\sum_{k=3}^{\infty}(-1)^j((-1)^j2\mu_j)^kc_k^{(j)}},\\ &\xi_j = 2x_j+2\mu_{jI}t,\ \tilde{\xi}_j = \frac{1}{\mu_{jR}}\log\frac{|m_j^{(T)}|}{\sqrt{2\mu_{jR}}},\\ &\rho^{(T)} = -a_3p_1^{(T)}p_2^{(T)}.\end{aligned}$$

Then we obtain the following expression for $q^{(T)}$:

$$\frac{4\rho^{(T)}(\mu_{1R}\mu_{2R})^{1/2}\exp\{-(\mu_{1R}(\xi_1-\tilde{\xi}_1)+\mu_{2R}(\xi_2-\tilde{\xi}_2))+i(-(\mu_{1I}\xi_1+\mu_{2I}\xi_2)+(|\mu_1|^2+|\mu_2|^2)t+\arg m_1m_2)\}}{((\delta_{TD}+\exp(-2\mu_{1R}(\xi_1-\tilde{\xi}_1))(\delta_{TD}+\exp(-2\mu_{2R}(\xi_2-\tilde{\xi}_2))+|\rho^{(T)}|^2)}$$

The function $q^{(D)}$ is precisely the $(1,1)$-dromion solution of the Davey-Stewartson equations (4.3.12) with $\kappa = -1$ found in [FS] (provided that $\mu_{jR} \in \mathbb{R}_+$). On the other hand, if we let $\mu_{1I} = \mu_{2I} = 0$, then $q^{(T)}$ reduces to the 2-dimensional breather solution found in [BLMP]. Finally, $q^{(S)}$ is a 1-soliton solution.

Recall that the dromion solutions of the DS equation were originally discovered in [BLMP] and [FS] (see also [HH]). The dromion solutions of the DS equation were first studied from the point of view of the spinor formalism by [HMM].

5.4. Similarly, we obtain the following solutions of the 2-dimensional Toda chain (4.2.8):

$$(5.4.1)\qquad u_\ell = \begin{cases}\log(\tau_{\ell+1}^{(N)}/\tau_\ell^{(N)}) & \text{if } -N \le \ell \le N-1\\ 0 & \text{otherwise}\end{cases}$$

where the τ-functions $\tau_\ell^{(N)}$ are obtained from (5.3.1) by letting all indeterminates $x_m^{(j)}$ with $m > 1$ to be arbitrary constants:

$$(5.4.2)\qquad \tau_\ell^{(N)} = \delta_{\ell,0} + \sum_{r=1}^{4N} \sum_{(s_1,\dots,s_r)} c(s_1,\dots,s_r)c_{s_1}\dots c_{s_r} \exp \sum_{k=1}^{r}(x_{i_k}z_{s_k} - x_{j_k}w_{s_k}),$$

where $(s_1,\dots,s_r)$ runs over $(5.3.2)_2$ and c_s $(s \in S)$ are arbitrary constants.

5.5. Let now $n \geq 3$. Then we obtain solutions of the $2+1$ n-wave system (4.1.9) as follows. For $1 \leq i,j \leq n$ let

$$(5.5.1)\qquad \begin{aligned} \tau_{ij}^{(N)} = \delta_{ij} + \sum_{r=1}^{Nn^2} \sum_{(s_1,\dots,s_r)} c_{s_1}\dots c_{s_r} \\ \times \exp \sum_{k=1}^{r} (a_{i_k}x + b_{i_k}t - y)z_{s_k} - (a_{j_k}x + b_{j_k}t - y)w_{s_k}, \end{aligned}$$

where $(s_1,\dots,s_r)$ runs over $(5.3.2)_n$ and c_s $(s \in S)$ are arbitrary constants. Then $W_{ij} = \varepsilon_{ji}\tau_{ij}/\tau_0$ $(i \neq j)$ is a solution of (4.1.9), and $Q_{ij} = \varepsilon_{ij}(a_i - a_j)\tau_{ij}/\tau_0$ $(i \neq j)$ is a solution of (4.1.10).

§6. m-reductions of the n-component KP hierarchy.

6.1. Fix a positive integer m and let $\omega = \exp \frac{2\pi i}{m}$. Introduce the following mn^2 fields $(1 \leq i,j \leq n,\ 1 \leq k \leq m)$ [TV]:

$$(6.1.1)\qquad \alpha^{(ijk)}(z) \equiv \sum_{p\in\mathbb{Z}} \alpha_p^{(ijk)} z^{-p-1} =: \psi^{+(i)}(z)\psi^{-(j)}(\omega^k z) :,$$

where the normal ordering is defined by (2.1.6). Note that

$$(6.1.2)\qquad \alpha^{(ijm)}(z) = \alpha^{(ij)}(z),$$

where $\alpha^{(ij)}(z)$ are the bosonic fields, defined by (2.1.5), which generate the affine algebra $g\ell_n(\mathbb{C})^\wedge$ with central charge 1 (see (2.1.7)). It is easy to check that for arbitrary m, the fields $\alpha^{(ijk)}(z)$ generate the affine algebra $g\ell_{mn}(\mathbb{C})^\wedge$ with central charge 1. More precisely, all the operators $\alpha_p^{(ijk)}$ $(1 \leq i,j \leq n,\ 1 \leq k \leq m,\ p \in \mathbb{Z})$ together with 1 form a basis of $g\ell_{mn}(\mathbb{C})^\wedge$ in its representation in F with central charge 1, the charge decomposition being the decomposition into irreducibles. Hence, using (2.1.14), (2.2.5 and 8), we obtain the vertex operator realization of this representation of $g\ell_{mn}(\mathbb{C})^\wedge$ in the vector space B (see [TV] for details).

Now, restricted to the subalgebra $s\ell_{mn}(\mathbb{C})^\wedge$, the representation in $F^{(0)}$ is not irreducible any more, since $s\ell_{mn}(\mathbb{C})^\wedge$ commutes with all the operators

$$(6.1.3)\qquad \beta_k^{(n)} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{j=1}^{n} \alpha_k^{(j,j,m)}, \quad k \in m\mathbb{Z}.$$

In order to describe the irreducible part of the representation of $s\ell_{mn}(\mathbb{C})^\wedge$ in $B^{(0)}$ containing the vacuum 1, we choose the complementary generators of the oscillator algebra $\mathfrak{a}$ contained in $s\ell_{mn}(\mathbb{C})^\wedge$ $(k \in \mathbb{Z})$:

(6.1.4)

$$\beta_k^{(j)} = \begin{cases} \alpha_k^{(jjm)} & \text{if } k \notin m\mathbb{Z}, \\ \frac{1}{j(j+1)}(\alpha_k^{(11m)} + \dots + \alpha_k^{(jjm)} - j\alpha_k^{(j+1j+1m)}) & \text{if } k \in m\mathbb{Z} \text{ and } 1 \leq j < n, \end{cases}$$

so that the operators (6.1.3 and 4) also satisfy relations (2.1.9). Then the operators $1, \alpha_p^{(ijk)}$

for $i \neq j$, together with operators (6.1.4) form a basis of $s\ell_{mn}(\mathbb{C})^\wedge$. Hence, introducing the new indeterminates

$$(6.1.5) \qquad y_k^{(j)} = \begin{cases} x_k^{(j)} & \text{if } k \notin m\mathbb{N}, \\ \frac{1}{j(j+1)}(x_k^{(1)} + \ldots + x_k^{(j)} - jx_k^{(j+1)}) & \text{if } k \in m\mathbb{N} \text{ and } j < n, \\ \frac{1}{n}(x_k^{(1)} + \ldots + x_k^{(n)}) & \text{if } k \in m\mathbb{N} \text{ and } j = n, \end{cases}$$

we have: $\mathbb{C}[x] = \mathbb{C}[y]$ and

$$(6.1.6) \qquad \sigma(\beta_k^{(j)}) = \frac{\partial}{\partial y_k^{(j)}} \text{ and } \sigma(\beta_{-k}^{(j)}) = k y_k^{(j)} \text{ if } k > 0.$$

Now it is clear that the irreducible with respect to $s\ell_{mn}(\mathbb{C})^\wedge$ subspace of $B^{(0)}$ containing the vacuum 1 is the vector space

$$(6.1.7) \qquad B_{[m]}^{(0)} = \mathbb{C}[y_k^{(j)} | 1 \le j < n,\ k \in \mathbb{N},\ \text{or } j = n,\ k \in \mathbb{N}\backslash m\mathbb{Z}] \otimes \mathbb{C}[M].$$

The vertex operator realization of $s\ell_{mn}(\mathbb{C})^\wedge$ in the vector space $B_{[m]}^{(0)}$ is then obtained by expressing the fields $\alpha^{(ijk)}(z)$ for $i \neq j$ in terms of vertex operators (2.1.14), which are expressed via the operators (6.1.4), the operators $Q_i Q_j^{-1}$ and $\alpha_0^{(i)} - \alpha_0^{(j)}$ $(1 \le i < j \le n)$ (see [TV] for details).

The n-component KP hierarchy of Hirota bilinear equations on $\tau \in B^{(0)} = \mathbb{C}[y] \otimes \mathbb{C}[M]$ when restricted to $\tau \in B_{[m]}^{(0)}$ is called the *m-th reduced* KP *hierarchy*. It is obtained from the n-component KP hierarchy by making the change of variables (6.1.5) and putting zero all terms containing partial derivates by $y_m^{(n)}, y_{2m}^{(n)}, y_{3m}^{(n)}, \ldots$.

It is clear from the definitions and results of §3 that the condition on the n-component KP hierarchy to be m-th reduced implies the following equivalent conditions (cf. [DJKM3]):

$$(6.1.8) \qquad L(\alpha)^m \text{ is a differential operator,}$$

$$(6.1.9) \qquad \sum_{j=1}^{n} \frac{\partial W(\alpha)}{\partial x_m^{(j)}} = z^m W(\alpha),$$

$$(6.1.10) \qquad \sum_{j=1}^{n} \frac{\partial \tau}{\partial x_m^{(j)}} = \lambda \tau, \quad \text{for some } \lambda \in \mathbb{C}.$$

It follows from (6.1.8) that these conditions automatically imply that all of them hold if m is replaced by any multiple of m.

The totality of solutions of the m-th reduced KP hierarchy is given by the following

Proposition 6.1. *Let $\mathcal{O}_{[m]}$ be the orbit of 1 under the (projective) representation of the loop group $SL_{mn}(\mathbb{C}[t, t^{-1}])$ corresponding to the representation of $s\ell_{mn}(\mathbb{C})^\wedge$ in $B_{[m]}^{(0)}$. Then*

$$\mathcal{O}_{[m]} = \sigma(\mathcal{O}) \cap B_{[m]}^{(0)}.$$

In other words, the τ-functions of the m-th reduced KP hierarchy are precisely the τ-functions of the KP hierarchy in the variables $y_k^{(j)}$, which are independent of the variables $y_{m\ell}^{(n)}$, $\ell \in \mathbb{N}$.

Proof is the same as of a similar statement in [KP2]. □

Remark 6.1. The above representation of $s\ell_{mn}(\mathbb{C})^\wedge$ in $B_{[m]}^{(0)}$ is a vertex operator construction of the basic representation corresponding to the element of the Weyl group S_{mn} of $s\ell_{mn}(\mathbb{C})$ consisting of n cycles of length m (see [KP1] and [TV]). In particular, for $n = 1$ this is the principal realization [KKLW], and for $m = 1$ this is the homogeneous realization [FK]. The m-th reduced 1-component KP was studied in a great detail in [DJKM2] (see also [KP2]).

6.2. Let $n = 1$. Then the 2-reduced KP hierarchy becomes the celebrated KdV hierarchy on the differential operator $S \equiv (L^2)_+ = \partial^2 + u$, where $u = 2u_1$:

$$\frac{\partial}{\partial x_{2n+1}} S^{\frac{1}{2}} = [(S^{n+\frac{1}{2}})_+, S^{\frac{1}{2}}],\ n = 1, 2, \dots, \tag{6.2.1}$$

the first equation of the hierarchy being the classical Korteweg-deVries equation

$$4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}. \tag{6.2.2}$$

Of course, the 3-reduced KP is the Boussinesq hierarchy, and the general m-reduced KP are the Gelfand-Dickey hierarchies.

6.3. Let $n = 2$. The equations of the 1-reduced 2-component KP are independent of x, hence equation (4.3.8) becomes independent of x and φ becomes 0 (see (4.3.9)). Thus, equation (4.3.8) turns into the decoupled non-linear Schrödinger system (called also the AKNS system):

$$\begin{aligned} i\frac{\partial q}{\partial t} &= -\frac{1}{2}\frac{\partial^2 q}{\partial y^2} - q^2 r \\ i\frac{\partial r}{\partial t} &= \frac{1}{2}\frac{\partial^2 r}{\partial y^2} + qr^2. \end{aligned} \tag{6.3.1}$$

Thus (6.3.1) is a part of the 1-reduced 2-component KP. For that reason the 1-reduced 2-component KP is sometimes called the non-linear Schrödinger hierarchy. Of course, under the constraint (4.3.11), we get the non-linear Schrödinger equation

$$i\frac{\partial q}{\partial t} = -\frac{1}{2}\frac{\partial^2 q}{\partial y^2} - \kappa|q|^2 q. \tag{6.3.2}$$

Similarly, under the same reduction the 2-dimensional Toda chain (4.2.8) turns into the 1-dimensional Toda chain

$$\frac{\partial^2 u_n}{\partial x^2} = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n} \text{ (here } x = 2y_1^{(1)}\text{)}. \tag{6.3.3}$$

Thus, the 1-dimensional Toda chain is a part of the non-linear Schrödinger hierarchy. It was studied from the representation theoretical point of view in [TB].

6.4. Let $n \geq 3$. Since the constraint (4.1.4) is contained among the constraints of the 1-reduced n-component KP hierarchy, we see that the $1 + 1$ n-wave system (4.1.7) is a part of the 1-reduced n-component KP hierarchy. Note also that the 1-reduction of the n-component KP reduces the $2+1$ n-wave interaction system (4.1.10) into the $1+1$ system (4.1.7).

6.5. Since the non-linear Schrödinger system (6.3.1) is a part of the 1-reduced 2-component KP hierarchy, the 1-reduced n-component KP hierarchy will be called the n-component NLS. Let us give here its formulation since it is especially simple.

Given a $n \times n$ matrix $C(z) = \sum_j C_j z^j$, we let

$$C(z)_- = \sum_{j<0} C_j z^j,\ C(z)_+ = \sum_{j\geq 0} C_j z^j.$$

Also, given a diagonal complex matrix $a = \text{diag}\,(a_1, \dots, a_n)$ we let

$$x_k^a = \sum_{j=1}^n a_k x_k^{(j)}, \quad \frac{\partial}{\partial x_k^a} = \sum_{j=1}^n a_k \frac{\partial}{\partial x_k^{(j)}}.$$

Let $\mathfrak{h}$ denote the set of all traceless diagonal matrices over $\mathbb{C}$.

The n-component NLS hierarchy is the following system on matrix valued functions

$$P(\alpha) \equiv P(\alpha, x, z) = 1 + \sum_{j>0} W^{(j)}(\alpha, x) z^{-j}, \quad \alpha \in M,$$

where $x = \{x_k^{(a)} | a \in \mathfrak{h}, \ k = 1, 2, \dots\}$:

$$\frac{\partial P(\alpha)}{\partial x_k^{(a)}} = -(P(\alpha) a P(\alpha)^{-1} z^k)_- P(\alpha) \tag{6.5.1}$$

with additional matching conditions

$$(P(\alpha) R(\alpha - \beta, z) P(\beta)^{-1})_- = 0, \quad \alpha, \beta \in M, \tag{6.5.2}$$

where $R(\gamma, z) \equiv R^+(\gamma, z)$ is defined by (3.3.10).

This formulation implies the Lax form formulation if we consider $C^{(a)}(x,z) = P(\alpha) a P(\alpha)^{-1}$ for each $a \in \mathfrak{h}$ and fixed α. Consider a family of commuting matrix valued functions of the form

$$C^{(a)} \equiv C^{(a)}(x, z) = a + \sum_{j>0} C_j^{(a)}(x) z^{-j},$$

depending linearly on $a \in \mathfrak{h}$, and let $B_k^{(a)} = (C^{(a)} z^k)_+$. Then the Lax form of the n-component NLS is

$$\frac{\partial C^{(a)}}{\partial x_k^{(b)}} = [B_k^{(b)}, C^{(a)}], \quad a, b \in \mathfrak{h}, \ k = 1, 2, \dots. \tag{6.5.3}$$

The equivalent zero curvature form of the n-component NLS is

$$\frac{\partial B_\ell^{(a)}}{\partial x_k^{(b)}} - \frac{\partial B_k^{(b)}}{\partial x_\ell^{(a)}} = [B_k^{(b)}, B_\ell^{(a)}], \quad a, b \in \mathfrak{h}, \ k, \ell = 1, 2, \dots. \tag{6.5.4}$$

Since for the 1-reduced n-component KP one has: $L = \partial$, i.e. all $U^{(j)} = 0$, we see from Remark 4.3 that the n-component NLS in the form (6.5.3) contains the following system of equations on functions $A_{ij} \equiv (C_1^{E_{jj}})_{ij}$ $(i \neq j)$:

$$\begin{aligned}
\frac{\partial A_{ij}}{\partial t_j} &= -\frac{\partial^2 A_{ij}}{\partial x_j^2} - 2 \sum_{k \neq j} A_{ij} A_{jk} A_{kj}, \\
\frac{\partial A_{ij}}{\partial t_i} &= \frac{\partial^2 A_{ij}}{\partial x_i^2} + 2 \sum_{k \neq i} A_{ij} A_{ik} A_{ki}, \\
\frac{\partial A_{ij}}{\partial t_k} &= A_{ik} \frac{\partial A_{kj}}{\partial x_k} - A_{kj} \frac{\partial A_{ik}}{\partial x_k} \qquad \text{if } i \neq k, \ j \neq k, \\
\frac{\partial A_{ij}}{\partial x_k} &= A_{ik} A_{kj} \qquad \text{if } i \neq k, \ j \neq k, \\
\sum_k \frac{\partial A_{ij}}{\partial x_k} &= \sum_k \frac{\partial A_{ij}}{\partial t_k} = 0.
\end{aligned} \tag{6.5.5}$$

This reduces to (6.3.1) if $n = 2$.

Remark 6.5. Equations (6.5.1), (6.5.3) and (6.5.4) still make sense if we consider an arbitrary algebraic group G and a reductive commutative subalgebra $\mathfrak{h}$ of its Lie algebra $\mathfrak{g}$. The functions $P(\alpha)$ take values in $G(\mathcal{A}((z)))$ and the functions $C^{(a)}$ take values in $\mathfrak{g}(\mathcal{A}((z)))$. If G is a simply laced simple Lie group, the element $R(\gamma, z) \in G(\mathbb{C}[z, z^{-1}])$ in matching conditions (6.5.2) can be generalized as follows. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, normalize the Killing form on $\mathfrak{g}$ by the condition that $(\alpha|\alpha) = 2$ for any root α, and identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using this form. Let M (resp. L) $\subset \mathfrak{h}^* = \mathfrak{h}$ be the root (resp. weight) lattice and let $\varepsilon(\alpha, \beta) : M \times M \longrightarrow \{\pm 1\}$ be a bimultiplicative function such that $\varepsilon(\alpha, \alpha) = (-1)^{\frac{1}{2}(\alpha|\alpha)}$, $\alpha \in M$. Define $R(\alpha, z) \in H(\mathbb{C}[z, z^{-1}])$ for each α as follows:

$$R(\alpha, z) = c_\alpha z^\alpha, \tag{6.5.6}$$

where in any finite-dimensional representation V of G, $c_\alpha \in H$ and $z^\alpha \in H$ for $z \in \mathbb{C}^\times$ are defined by

$$c_\alpha(v) = \varepsilon(\beta, \alpha)v, \;\; z^\alpha(v) = z^{(\alpha|\beta)}v \text{ if } v \in V_\beta. \tag{6.5.7}$$

Note that this GNLS hierarchy is closely related to the Bruhat decomposition in the loop group $G(\mathbb{C}((z)))$.

6.6. It is clear that we get the τ-function of the m-th reduced n-component KP hierarchy if we let in (5.1.9)

$$w_s = \omega_s z_s, \;\; s \in S, \tag{6.6.1}$$

where ω_s are arbitrary m-th roots of 1. The totality of τ-functions is (a completion of) the orbit of $1 \in B^{(0)}$ under the group $SL_{mn}(\mathbb{C}[t, t^{-1}])$.

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