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Montecatini lectures on invariant theory

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In these notes I will discuss two approaches to the study of the orbits, invariants, etc, of a linear reductive group G operating on a finite dimensional vector space V. The two techniques are the "quiver method" and the "slice method", which are discussed in Chapters I and II respectively.

Undoubtedly, the slice method, based on Luna's slice theorem [1], is one of the most powerful methods in geometric invariant theory. Even in the case of binary forms the slice method gives results which were out of reach of mathematatucs of 19^{th} century (cf. [15]). For example, I show that for the action of $SL_2(\mathbb{C})$ on the space of binary forms of odd degree $d > 3$ the minimal number of generators of the algebra of invariant polynomials is greater than $p(d-2)$, where $p(n)$ is the classical partition function.

On the other hand, the quiver method can be applied to a (very special) class of representations for which the slice method often fails.

Most of the results of Chapter I are contained in [4] and [5]; on the most part I just give simpler versions of the proofs. Chapter II contains some new results (as, it seems, the one mentioned above).

I am mostly greatful to the organizers of the summer school in Montecatini Terme (Italy) for inviting me to give these lectures, especially to F. Gherardelli who convinced me to write the notes. My thanks go to J. Dixmier for sharing his knowledge and enthusiasm about invariant theory of binary forms, and to H. Kraft and R. Stanley for several important observations.

Chapter I. Representations of quivers.

The whole range of problems of linear algebra can be formulated in a uniform way in the context of representations of quivers introduced by Gabriel [2]. In this chapter I discuss the links of this with invariant theory and theory of generalized root systems ([3], [4]).

§1.1. Given a connected graph Γ with n vertices $\{1,\dots,n\}$ we introduce the associated root system $\Delta(\Gamma)$ as a subset in $\mathbb{Z}^n$ as follows. Let b_{ij} denote the number of edges connecting vertices i and j, if $i \neq j$, and twice the number of loops at i if $i=j$. Let $\alpha_i = (\delta_{i1},\dots,\delta_{in})$, $i=1,\dots,n$, be the standard basis of $\mathbb{Z}^n$. Introduce a bilinear form $(\ ,\ )$ on $\mathbb{Z}^n$ by:

$$(\alpha_i,\alpha_j) = \delta_{ij} - \tfrac{1}{2}b_{ij} \quad (i,j=1,\dots,n).$$

Denote by $Q(\alpha)$ the associated quadratic form. It is clear that this is a $\mathbb{Z}$-valued form. The element α_i is called a fundamental root if there is no edges-loops at the vertex i. Denote by Π the set of fundamental roots. For a fundamental root α define the fundamental reflection $r_\alpha \in \operatorname{Aut} \mathbb{Z}^n$ by

$$r_\alpha(\lambda) = \lambda - 2(\lambda,\alpha)\alpha \quad \text{for } \lambda \in \mathbb{Z}^n.$$

This is a reflection since $(\alpha,\alpha) = 1$ and hence $r_\alpha(\alpha) = -\alpha$, and also $r_\alpha(\lambda) = \lambda$ if $(\lambda,\alpha) = 0$. In particular, $(r_\alpha(\lambda), r_\alpha(\lambda)) = (\lambda,\lambda)$. The group $W(\Gamma) \subset \operatorname{Aut} \mathbb{Z}^n$ generated by all fundamental reflections is called the Weyl group of the graph Γ (for example $W = \{1\}$ if there is an edge-loop at any vertex of Γ). Note that the bilinear form $(\ ,\ )$ is $W(\Gamma)$-invariant. Define the set of real roots $\Delta^{re}(\Gamma)$ by:

$$\Delta^{re}(\Gamma) = \bigcup_{w\in W} w(\Pi).$$

For an element $\alpha = \sum_i k_i\alpha_i \in \mathbb{Z}^n$ we call the height of α (write: $\operatorname{ht}\alpha$) the number $\sum_i k_i$; we call the support of α (write: $\operatorname{supp}\alpha$) the subgraph of Γ consisting of those vertices i for which $k_i \neq 0$ and all the edges joining these vertices. Define the fundamental set $M \subset \mathbb{Z}^n$ by:

$M = \{\alpha \in \mathbb{Z}^n_+\setminus\{0\} \mid (\alpha,\alpha_i) \leq 0$ for all $\alpha_i \in \Pi$, and $\operatorname{supp}\alpha$ is connected$\}$ (Note that $(\alpha,\alpha_i) \leq 0$ if $\alpha_i \notin \Pi$, automatically).

Here and further on, $\mathbb{Z}_+ = \{0,1,2,\dots\}$.

Define the set of imaginary roots $\Delta^{im}(\Gamma)$ by:

$$\Delta^{im}(\Gamma) = \bigcup_{w\in W} w(M\cup -M).$$

Then the root system $\Delta(\Gamma)$ is defined as

$$\Delta(\Gamma) = \Delta^{re}(\Gamma) \cup \Delta^{im}(\Gamma).$$

An element $\alpha \in \Delta(\Gamma) \cap \mathbb{Z}_+^n$ is called a positive root. Denote by $\Delta_+(\Gamma)$ (resp. $\Delta_+^{re}(\Gamma)$ or $\Delta_+^{im}(\Gamma)$) the set of all positive (resp. positive real or positive imaginary) roots. When it does not cause a confusion we will write W, Δ, etc. instead of $W(\Gamma)$, $\Delta(\Gamma)$, etc.

It is obvious that $(\alpha,\alpha) = 1$ if $\alpha \in \Delta^{re}$. On the other hand, $(\alpha,\alpha) \leq 0$ if $\alpha \in \Delta^{im}$. (Indeed, one can assume that $\alpha = \Sigma k_i\alpha_i \in M$; but then $(\alpha,\alpha) = \sum_i k_i(\alpha,\alpha_i) \leq 0$.) Hence

$$\Delta^{re} \cap \Delta^{im} = \emptyset.$$

Furthermore, one has:

$$\Delta = \Delta_+ \bigsqcup -\Delta_+.$$

This statement is less obvious but will follow from the representation theory of quivers.

We shall need two more easy facts:

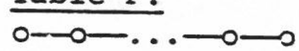
$$\Delta_+^{im} = \bigcup_{w\in W} w(M) = \{\alpha \in \Delta_+ | W(\alpha) \subset \Delta_+\};$$

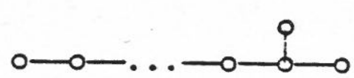
$\Delta_+^{re} = \{\alpha \in \Delta_+\setminus\{\alpha_1,\dots,\alpha_n\} |$ there exist $\alpha_1,\dots,\alpha_s \in \Pi$ such that $r_{\alpha_i}r_{\alpha_{i+1}}\dots r_{\alpha_s}(\alpha) \in \Delta_+$ for $1 \leq i \leq s$ and $r_{\alpha_1}\dots r_{\alpha_s}(\alpha)\in\Pi\}\cup\Pi$.

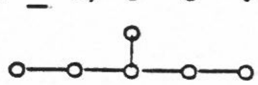
The proof of these facts can be found in [3].

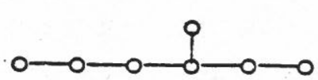
§1.2. According as the bilinear form (,) is positive definite, positive semidefinite or indefinite the (connected) graph Γ is called a graph of finite, tame and wild type respectively. The complete lists of finite and tame graphs are given in Tables F and T.

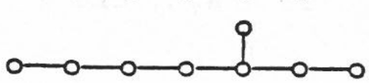
Table F.

$A_n (n \geq 1)$ 

$D_n (n \geq 4)$ 

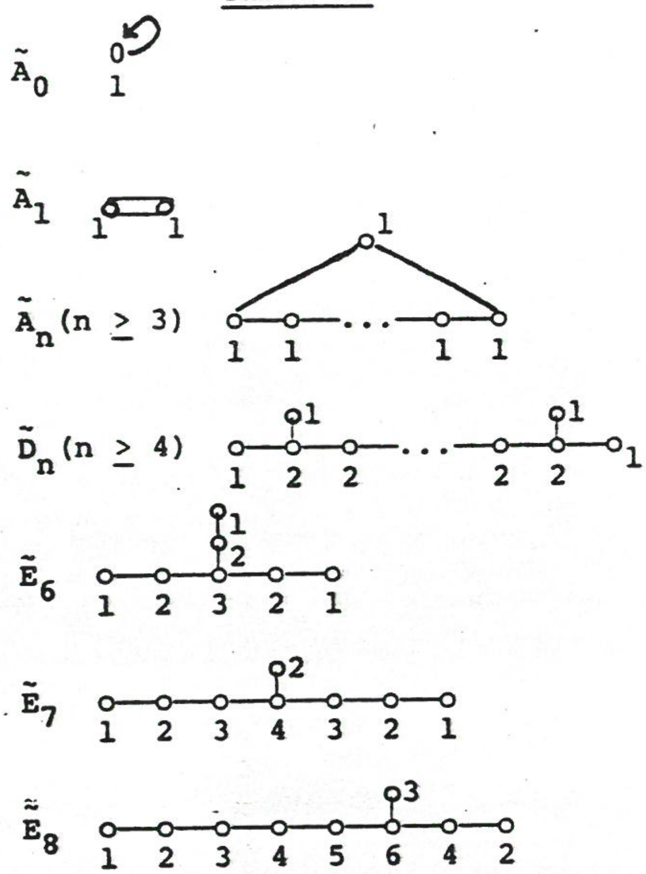
E_6 

E_7 

E_8 

The subscript in the notation of a graph in Table F equals to the number of vertices.

Table T.



The subscript in the notation of a graph in Table T plus 1 equals to the number of vertices. The kernel of the bilinear form (,) is $\mathbb{Z}\delta$, where $\delta = \sum_i a_i\alpha_i$, a_i being the lables by the vertices. It is easy to show that the converse is also true (see e.g. [3], p.61):

Proposition. If there exists $\delta \in \mathbb{Z}_+^n$ such that $(\delta,\alpha_i) = 0$ for all i and $\delta \neq 0$ then Γ is of tame type.

Here are some characterisations of graphs of finite, tame and wild types:

Γ is finite $\Longleftrightarrow$ $|W(\Gamma)| < \infty \Longleftrightarrow |\Delta(\Gamma)| < \infty \Longleftrightarrow \Delta^{im}(\Gamma) = \emptyset$,

Γ is tame $\Longleftrightarrow$ $\Delta^{im}(\Gamma)$ lies on a line;

Γ is wild $\Longleftrightarrow$ there exists $\alpha \in \Delta_+(\Gamma)$ such that $(\alpha,\alpha_i) < 0$ for all i and supp $\alpha = \Gamma$.

A graph of wild type is called hyperbolic if every one of its proper connected subgraph is of finite or tame type. In the case of a finite, affine or hyperbolic graph, there is a simple description of the root system $\Delta(\Gamma)$.

Proposition. *If a graph Γ is of finite, affine or hyperbolic type, then*

$$\Delta(\Gamma) = \{\alpha \in \mathbb{Z}^n \setminus \{0\} \mid (\alpha,\alpha) \leq 1\}.$$

In particular, if Γ is of finite type, then

$$\Delta(\Gamma) = \{\alpha \in \mathbb{Z}^n \mid (\alpha,\alpha) = 1\},$$

and if Γ is of affine type, then

$$\Delta^{re}(\Gamma) = \{\alpha \in \mathbb{Z}^n \mid (\alpha,\alpha) = 1\}; \Delta^{im}(\Gamma) = (\mathbb{Z}\setminus\{0\})\delta.$$

Proof. Let $\alpha \in \mathbb{Z}^n\setminus\{0\}$ be such that $(\alpha,\alpha) \leq 1$. We have to show that $\alpha \in \Delta(\Gamma)$. Note that $\operatorname{supp} \alpha$ is connected as in the contrary case $\alpha = \beta + \gamma$, where $\operatorname{supp} \beta$ and $\operatorname{supp} \gamma$ are unions of subgraphs of finite type and $(\beta,\gamma) = 0$, but then $(\alpha,\alpha) \geq 2$. Next, either α or $-\alpha \in \mathbb{Z}^n_+$. Indeed, in the contrary case, $\alpha = \beta-\gamma$, where $\beta,\gamma \in \mathbb{Z}^n_+$, $\operatorname{supp} \beta \cap \operatorname{supp} \gamma = \emptyset$, $\operatorname{supp} \beta$ is a union of subgraphs of finite type and $\operatorname{supp} \gamma$ is either a union of subgraphs of finite type or is a subgraph of affine type. But $(\alpha,\alpha) = (\beta,\beta) + (\gamma,\gamma)-2(\beta,\gamma) \leq 1$ and $(\beta,\gamma) \leq 0$. Hence the only possiblility is that $(\beta,\beta) = 1$, $(\gamma,\gamma) = 0$ and $(\beta,\gamma) = 0$. But then $\operatorname{supp} \gamma$ is a subgraph of affine type and (β,γ) must be < 0, a contradiction.

So, $\operatorname{supp} \alpha$ is connected and we can assume that $\alpha \in \mathbb{Z}^n_+$. We can assume that $W(\alpha) \cap \Pi = \emptyset$, otherwise there is nothing to prove. But then, clearly, $W(\alpha) \subset \mathbb{Z}^n_+$. Taking in $W(\alpha)$ an element of minimal height, we can assume that $(\alpha,\alpha_i) \leq 0$ for $\alpha_i \in \Pi$. Since, in addition, $\operatorname{supp} \alpha$ is connected, we deduce that α lies in the fundamental set. □

Using the proposition, one can describe $\Delta^{re}_+(\Gamma)$ for a tame graph $\Gamma = \tilde{A}_n, \tilde{D}_n$ or $\tilde{E}_n$ via the subset $\mathring{\Delta}_+ \subset \Delta^{re}_+(\Gamma)$ of positive roots of the subgraph A_n, D_n or E_n respectively as follows:

$$\Delta^{re}_+(\Gamma) = \{\alpha + n\delta \mid \pm\alpha \in \mathring{\Delta}_+, n \geq 1\} \cup \mathring{\Delta}_+.$$

One can show, that conversely, if $\Delta(\Gamma) = \{\alpha \in \mathbb{Z}^n \mid (\alpha,\alpha) \leq 1\}$, then Γ is of finite, affine or hyperbolic type.

Remark. The graphs of finite type are the so called simply laced Dynkin diagrams. They correspond to simple finite-dimensional Lie algebras with equal root length. The other graphs correspond to certain infinite-dimensional Lie algebras, the so called Kac-Moody algebras.

§1.3. Examples.

a) Denote by S_m the graph with one vertex and m edges loops. The associated quadratic form on $\mathbb{Z}$ is $Q(k\alpha) = (1-m)k^2$. $\Delta(S_0) = \{\pm\alpha\}$ and $\Delta(S_m) = (\mathbb{Z}\setminus\{0\})\alpha$ if $m > 0$. $W(S_0) = \{r_\alpha, 1\}$ and $W(S_m) = \{1\}$ if $m > 0$. S_m is of finite, tame or hyperbolic type iff $m = 0$, $m=1$ or $m > 1$ respectively.

b) Denote by P_m the graph with two vertices and m edges connecting these vertices. The associated quadratic form on $\mathbb{Z}^2$ is $Q(k_1\alpha_1+k_2\alpha_2) = k_1^2 - mk_1k_2 + k_2^2$. The set of positive roots is as follows:

$\Delta_+(P_1) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$;

$\Delta_+(P_2) = \{k\alpha_1 + (k-1)\alpha_2, (k-1)\alpha_1 + k\alpha_2, k\alpha_1 + k\alpha_2;\ k \geq 1\}$

the set $\{k\alpha_1 + k\alpha_2; k \geq 1\}$ being $\Delta_+^{im}(\Gamma)$;

$m \geq 3$: $\Delta_+(P_m) = \{k_1\alpha_1 + k_2\alpha_2 | k_1^2 - m\, k_1k_2 + k_2^2 \leq 1,\ k_1 \geq 0, k_2 \geq 0, k_1+k_2 > 0\}$;

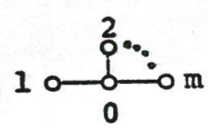
more explicitly,

$\Delta_+^{re}(P_m) = \{c_j\alpha_1 + c_{j+1}\alpha_2, c_{j+1}\alpha_1 + c_j\alpha_2,\ j \in \mathbb{Z}_+\}$, where $c_j (j \in \mathbb{Z}_+)$ are defined by the recurrent formula:

$$c_{j+2} = mc_{j+1} - c_j\ ,\ c_0 = 0,\ c_1 = 1.$$

$W(P_1)$ is the dihedral group of order 6 and $W(P_m)$ is the infinite dihedral group if $m \geq 2$. P_m is of finite, tame or hyperbolic type iff $m = 1$, $m = 2$ or $m \geq 3$ respectively.

c) Denote by V_m the graph:



Note that $V_1 = P_1$. The associated quadratic form on $\mathbb{Z}^{m+1}$ is:

$$Q(k_0\alpha_0+\ldots+k_m\alpha_m) = \sum_{i=0}^{m} k_i^2 - k_0 \sum_{i=1}^{m} k_i.$$

$\Delta(V_2) = \{\alpha_0, \alpha_1, \alpha_2, \alpha_0 + \alpha_1, \alpha_0 + \alpha_2, \alpha_0 + \alpha_1 + \alpha_2\}$.

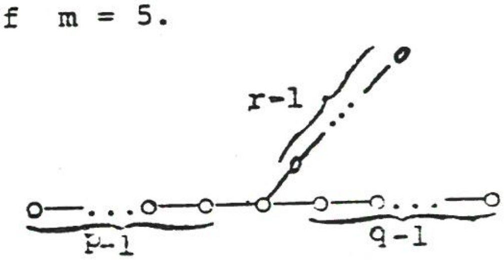
$\Delta(V_3) = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_0 + \alpha_1, \alpha_0 + \alpha_2, \alpha_0 + \alpha_3\ , \alpha_0 + \alpha_1 + \alpha_2, \alpha_0 + \alpha_1 + \alpha_3, \alpha_0 + \alpha_2 + \alpha_3, \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3\}$.

$\Delta^{im}(V_4) = (\mathbb{Z}\setminus\{0\})\delta$, where $\delta = 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$;

$\Delta^{re}(V_4) = \{n\delta \pm \alpha_i,\ i = 0,1,\ldots,4;\ n\delta \pm (\alpha_0 + \alpha_{i_1}+\ldots+\alpha_{i_s})$, where $1 \leq i_1 < i_2 <\ldots< i_s\ ,\ 1 \leq s \leq 4;\ n \in \mathbb{Z}\}$.

V_m is of finite, tame or wild type iff $m \leq 4$, $m = 4$ or $m \geq 5$ respectively; V_m is hyperbolic iff $m = 5$.

d) Denote by $T_{p,q,r}$ the graph:



set $c = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. Then $T_{p,q,r}$ is of finite, tame or wild type iff $c > 1$, $c = 1$ or $c < 1$ respectively. The only hyperbolic graphs among them are $T_{7,3,2}$, $T_{5,4,2}$ and $T_{4,3,3}$.

§1.4. There are at least two more equivalent difinitions of the set of positive roots $\Delta_+(\Gamma)$:

a) $\Delta_+(\Gamma)$ is a subset of $\mathbb{Z}^n_+\setminus\{0\}$ such that:

(i) $\alpha_1,\ldots,\alpha_n \in \Delta_+(\Gamma)$; $2\alpha_i \notin \Delta_+(\Gamma)$ for $\alpha_i \in \Pi$;

(ii) if $\alpha_i \notin \Pi$, $\alpha \in \Delta_+(\Gamma)$, then $\alpha + \alpha_i \in \Delta_+(\Gamma)$;

(iii) if $\alpha \in \Delta_+(\Gamma)$, then supp α is connected;

(iv) if $\alpha \in \Delta_+(\Gamma)$, $\alpha_i \in \Pi$ and $\alpha \neq \alpha_i$, then $[\alpha, r_{\alpha_i}(\alpha)] \cap \mathbb{Z}^n \subset \Delta_+(\Gamma)$.

b) Assume that Γ has no edges-loops. Extend the action of the group $W(\Gamma)$ to the lattice $\mathbb{Z}^n \oplus \mathbb{Z}\rho$ by: $r_{\alpha_i}(\rho) = \rho - \alpha_i$. Then one can show that $s(w) := \rho - w(\rho) \in \mathbb{Z}^n_+\setminus\{0\}$ for all $w \in W$. For $w \in W$ set $\varepsilon(w) = \det(w)$; it is clear that $\varepsilon(w) = \pm 1$. Introduce the notation: $x^\alpha = x_1^{k_1}\ldots x_n^{k_n}$, where $\alpha = \sum_i k_i\alpha_i$, and take the product decomposition of the following sum:

$$\sum_{w\in W} (\det w)x^{s(w)} = \prod_{\alpha\in\mathbb{Z}^n_+} (1 - x^\alpha)^{m_\alpha}.$$

Then one can show that $m_\alpha \in \mathbb{Z}_+$, and that

$$\Delta_+(\Gamma) = \{\alpha \in \mathbb{Z}^n_+ | m_\alpha > 0\}.$$

The positive integer m_α is called the <u>multiplicity</u> of the root α. Note that $m_\alpha = m_{w(\alpha)}$ for $w \in W$.

I do not know how to extend this definition to the case when Γ has edges-loops.

<u>Examples</u>. The multiplicity of a real root is 1. The multiplicity of an imaginary root of a tame graph $\tilde{A}_n$, $\tilde{D}_n$ or $\tilde{E}_n$ is n. This gives the multiplicity of any root α such that $(\alpha,\alpha) = 0$ since any such root is W-equivalent to a unique imaginary root β such that supp β is a tame graph. One knows that if $(\alpha,\alpha) < 0$, then mult $k\alpha$ growth exponentially as $k \to \infty$.

§1.5. Now we turn to the representation theory of quivers. If every edge of a graph Γ is equipped by an arrow, we say that Γ is equipped by an <u>orientation</u>, say Ω; an oriented graph (Γ,Ω) is called a <u>quiver</u>.

Fix a base field $\mathbb{F}$. A <u>representation</u> of a quiver (Γ,Ω) is a collection of finite dimensional vector spaces V_j, $j = 1,\ldots,n$, and linear maps $\phi_{ij}: V_i \longrightarrow V_j$ for every arrow $i \longrightarrow j$ of the quiver (Γ,Ω), everything defined over $\mathbb{F}$. The element $\alpha = \sum_i (\dim V_i)\alpha_i \in \mathbb{Z}_+^n$ is called the <u>dimension</u> of the representation. Morphisms and direct sums of representations of (Γ,Ω) are defined in an obvious way (the dimension of a direct sum is equal to the sum of dimensions). A representation is called <u>indecomposable</u> (resp. <u>absolutely indecomposable</u>) if it is not zero and cannot be decomposed into a direct sum of non-zero representations defined over $\mathbb{F}$ (resp. $\bar{\mathbb{F}}$, the algebraic closure of $\mathbb{F}$).

The main problem of the theory is to classify all representations of a quiver up to isomorphism. One knows that the decomposition of a representation into indecomposable ones is unique. So, for classification purposes it is sufficient to classify indecomposable representations.

Note that there exists a unique up to isomorphism representation of dimension $\alpha_i (i = 1,\ldots,n)$ and it is absolutely indecomposable, namely: $V_i = \mathbb{F}$, $V_j = 0$ for $j \neq i$ and all the maps are zero.

§1.6. Examples.

a) The graph S_m has a unique orientation. The problem of classification of the representations of this quiver is equivalent to the classification of m-tuples of $k \times k$-matrices up to a simultaneous conjugation by a non-degenerate matrix. For $m = 1$ this problem is "tame" and was solved by Weierstrass and Jordan (the so called Jordan normal form). For $m \geq 2$ the problem remains open and provides a typical example of a "wild" problem.

b) Put on P_m the orientation Ω for which all arrows point into the same direction. The corresponding problem is to classify all m-types of linear maps from one vector space into another. For $m = 1$ this is a trivial "finite" problem. For $m = 2$ this is a "tame" problem, which was solved by Kronecker. For $m \geq 3$ the problem becomes "wild".

c) Put on the graph V_m the orientation Ω for which all arrows point to the vertex 0. The corresponding problem is essentially equivalent to the problem of classification of m-tuples of subspaces in a vector space V up to an automorphism of V. For $m \leq 3$ the problem is "finite". For $m = 4$ the problem is "tame" and was solved by Nazarova and Gelfand-Ponomarev, for $m \geq 5$ the problem becomes "wild".

Now I shall give precise definitions. A quiver is called <u>finite</u> if it has only a finite number of indecomposable representations (up to isomorphism). Following Nazarova [10], we call a quiver (Γ,Ω) <u>wild</u> if there is an imbedding of the category of representations of the quiver S_2 into the category of representations of (Γ,Ω); a quiver which is not finite or wild is called <u>tame</u>.

Gabriel [2] proved that the quiver (Γ,Ω) is finite iff Γ is finite (i.e., appears in Table F); this will follow from our general theorems. Nazarova [10] proved that (Γ,Ω) is tame iff Γ is tame (i.e., appears in Table T).

Let me show on examples how to prove that (Γ,Ω) is wild.

For the quiver from c) take a vector space V, put $V_1 = V_2 = V$, and take the l'st map $V_1 \longrightarrow V_2$ to be an isomorphism. Then the category of representations of S_{m-1} is naturally imbedded in the category in question. So (P_m,Ω) is wild if $m \geq 3$.

For the quiver from c) take $V_0 = V \oplus V$, $V_1=V_2=\ldots=V_m=V$. Let $A_1,A_2,\ldots,A_{m-3}: V \longrightarrow V$ be some linear operators. Define the maps $\phi_i: V \longrightarrow V \oplus V$ $(i=1,\ldots,m)$ by:

$\phi_i(x) = x \oplus A(x)$ for $i = 1,\ldots,m-3$;

$\phi_{m-2}(x) = x \oplus 0$, $\phi_{m-1}(x) = 0 \oplus x$, $\phi_m(x) = x \oplus x$.

It is easy to see that this is an imbedding of the category of representation of S_{m-3} in the category in question. So the quiver (V_m,Ω) is wild if $m \geq 5$.

§1.7. One of the main technical tools of the representation theory of quivers are the so called reflection functors. Given an orientation Ω of a graph Γ and a vertex k, define a new orientation $\tilde{r}_k(\Omega)$ of Γ by reversing the direction of arrows along all the edges containing the vertex k. A vertex k of a quiver (Γ,Ω) is called a <u>sink</u> (resp. <u>source</u>) if for all edges for which k is a vertex, the arrows point to the vertex k (resp. to the other vertex). Note that if there is a loop at k, then k is neither a sink nor a source.

<u>Proposition</u> [1]. <u>Let</u> (Γ,Ω) <u>be a quiver and</u> k <u>a sink (resp. source)</u>. <u>Then there exists a functor</u> R_k^+ <u>(resp.</u> R_k^-<u>)</u> <u>from the category of representations of the quiver</u> (Γ,Ω) <u>to the category of representation of the quiver</u> $(\Gamma,\tilde{r}_k(\Omega))$ <u>such that</u>:

a) $R_k^{(\pm)}(U \oplus U') = R_k^{(\pm)}(U) \oplus R_k^{(\pm)}(U')$;

b) <u>If</u> U <u>is a representation of dimension</u> α_k, <u>then</u> $R_k^{(\pm)}(U) = 0$;

c) If U is an indecomposable representation of (Γ,Ω) and $\dim U \neq \alpha_k$, then

$$R_k^- R_k^+(U) \simeq U \quad (\text{resp. } R_k^+ R_k^-(U) \simeq U),$$
$$\text{and } \dim R_k^{\pm}(U) = r_{\alpha_k}(\dim U).$$

Corollary. Under the assumptions of c) of the proposition, $R_k^{\pm}(U)$ is an indecomposable representation of $(\Gamma, \tilde{r}_k(\Omega))$, and $\operatorname{End} U$ and $\operatorname{End} R_k^{\pm}(U)$ are canonically isomorphic.

We shall explain the construction of the reflection functors R_k^+ and R_k^- in the next section in a more general situation.

§1.8. Now we establish a link between representation theory of quivers and invariant theory.

Fix $\alpha = \sum_i k_i\alpha_i \in \mathbb{Z}_+^n$. Then the set of all, up to isomorphism, representations of dimension α of the quiver (Γ,Ω) is in 1-1 correspondence with the orbits of the group

$$G^\alpha(\mathbb{F}) := GL_{k_1}(\mathbb{F})\times\ldots\times GL_{k_n}(\mathbb{F})$$

operating in a natural way on the vector space

$$M^\alpha(\Gamma,\Omega) := \bigoplus_{i\to j} \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^{k_i}, \mathbb{F}^{k_j})$$

(here the summation is taken over all arrows of the quiver (Γ,Ω)).

Note that the subgroup $C = \{(t,\ldots,t), t \in \mathbb{F}^*\}$ operates trivially and that

$$(=) \qquad \dim G^\alpha - \dim M^\alpha(S,\Omega) = (\alpha,\alpha).$$

Furthermore, note that $U \in M^\alpha(\Gamma,\Omega)$ is an indecomposable representation of the quiver (Γ,Ω) iff $\operatorname{End} U$ contains no nontrivial projectors (recall that P is a projector if $P^2 = P$). U is an absolutely indecomposable representation iff $\operatorname{End} U$ contains no nontrivial semisimple elements, i.e., iff the stabilizer $(G^\alpha/C)_U$ of $U \in M^\alpha(S,\Omega)$ is a unipotent group.

Another observation: the group G_U^α is connected since it is the set of invertible elements in the ring $\operatorname{End} U$.

Now I shall explain, what are the reflection functors. Let G be a group and π_1, π_2 some representations of G on vector spaces V_1 and V_2, $\dim V_1 = m \geq k$. Then the group $G \times GL_k$ acts naturally on the space

$$M^+ = \operatorname{Hom}(V_1, \mathbb{F}^k) \oplus V_2.$$

Set $M_0^+ = \{\phi \oplus v \in M^+ \mid \phi \in \operatorname{Hom}(V_1,\mathbb{F}^k),\ v \in V_2,\ \operatorname{rank}\phi = k\}$. Furthermore, the group $GL_{m-k} \times G$ acts naturally on $M^- = \mathrm{Hom}(\mathbb{F}^{m-k}, V_1) \oplus V_2$. We set $M_0^- = \{\phi \oplus v \in M^- \mid \phi \in \mathrm{Hom}(\mathbb{F}^{m-k}, V_1), v \in V_2, \text{rank } \phi = m-k\}$. We define a map R^+ from the set of orbits on M_0^+ to the set of orbits on M_0^- as follows. If $\phi \oplus v$ lies on an orbit $\sigma \subset M_0^+$, choosing an isomorphism $\mathbb{F}^{m-k} \longrightarrow \mathrm{Ker}\ \phi$, we get a map $r^+(\phi)\colon \mathbb{F}^{m-k} \longrightarrow V_1$; denote by $R^+(\sigma)$ the orbit of $r^+(\phi) \oplus v \in M_0^-$. It is easy to see that R^+ is a well-defined map. Similarly we define the "dual" map R^- from the set of orbits on M_0^- to the set of orbits on M_0^+. One easily checks that R^-R^+ (resp. R^+R^-) is an identity map on the set of orbits in M_0^+ (resp. M_0^-).

Many people have discovered independently from each other this type of construction. For example, Sato and Kimura call it the "castling transform".

In order to get the reflection functor R_j^+ we apply the above construction to the group $G = \prod_{i \neq j} GL_{k_i}$, $k = k_j$, $V_1 = \bigoplus_{s \to j} \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{k_s}, \mathbb{F}^{k_j})$, $V_2 = \bigoplus_{\substack{s \to i \\ i \neq j}} \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{k_s}, \mathbb{F}^{k_j})$.

§1.9. We need a general remark about actions of a connected algebraic group G. Let G act on an irreducible algebraic variety X over field $\mathbb{F}$. Then by a theorem of Rosenlicht, there exists a dense open subset $X_0 \subset X$, an algebraic variety Z and a surjective morphism $X_0 \longrightarrow Z$, everything defined over $\mathbb{F}$, whose fibers are G-orbits. Z is called a geometric quotient of X_0.

Now, given an action of G on a constructible set X we can decompose X into a union of irreducible subsets and take a (finite) set of G-invariant algebraic subvarieties $Y_1, \ldots, Y_s \subset X$ such that $\dim X \backslash (Y_1 \cup \ldots \cup Y_s) < \dim X$ and each Y_i has a geometric quotient Z_i. Next, we apply the same procedure to $X \backslash \{Y_1 \cup \ldots \cup Y_s\}$, etc. After at most $\dim X$ steps we obtain (absolutely) irreducible varieties $Z_1, Z_2, \ldots$. We set $\mu(G,X) = \max_i \dim Z_i$. It is clear that this number is well defined. We say that the set of orbits of G on X depends on $\mu(G,X)$ parameters.

Denote by $M^\alpha_{ind}(\Gamma,\Omega)$ the set of all absolutely indecomposable representations from $M^\alpha(\Gamma,\Omega)$. This is a G^α-invariant set, which is constructible and defined over the prime field. Indeed, there exists a finite number of projectors $P_1, \ldots, P_s$ such that $M^\alpha_{ind}(\Gamma,\Omega) = M^\alpha(\Gamma,\Omega) \backslash (\bigcup_i G^\alpha(M^\alpha(\Gamma,\Omega)^{P_i}))$. Applying the above construction we obtain that the set of absolutely indecomposable representations (considered up to isomorphism) is parametrized by a finite union of algebraic varieties $Z_1, \ldots Z_2, \ldots$, defined over the prime field. We denote for short:

$$\mu_\alpha(\Gamma,\Omega) = \mu(G^\alpha, M^\alpha_{ind}(\Gamma,\Omega)).$$

§1.10 Now we can state the main theorem.

Theorem. Suppose that the base field $\mathbb{F}$ is algebraically closed. Let (Γ,Ω) be a quiver. Then

a) There exists an indecomposable representation of dimension $\alpha \in \mathbb{Z}_+^n\setminus\{0\}$ iff $\alpha \in \Delta_+(\Gamma)$.

b) There exists a unique indecomposable representation of dimension α iff $\alpha \in \Delta_+^{re}(\Gamma)$.

c) If $\alpha \in \Delta_+^{im}(\Gamma)$, then $\mu_\alpha(\Gamma,\Omega) = 1-(\alpha,\alpha) > 0$.

The proof of the theorem is based on two lemmas. We defer their proof to the next sections.

Lemma 1. Suppose that α lies in the fundamental set M and that, moreover, $(\alpha,\alpha_i) < 0$ for some i. Then

a) The set $M_0^\alpha(\Gamma,\Omega)$ of representations in $M^\alpha(\Gamma,\Omega)$ with a trivial endomorphism ring is a dense open G^α-invariant subset. In particular, $\mu(G^\alpha, M_0^\alpha(\Gamma,\Omega)) = 1-(\alpha,\alpha)$.

b) $\mu(G^\alpha, M_{ind}^\alpha(\Gamma,\Omega)\setminus M_0^\alpha(\Gamma,\Omega)) < 1-(\alpha,\alpha)$.

Lemma 2. The number of indecomposable representation of dimension α (if it is finite) and $\mu_\alpha(\Gamma,\Omega)$ are independent of the orientation Ω.

Proof of the theorem. Note that using the reflection functors, $\mu_{r_i(\alpha)}(\Gamma,\Omega) = \mu_\alpha(\Gamma,\Omega)$ if $\alpha \neq \alpha_i$ and i is a sink or a source of the quiver (Γ,Ω) (the same is true for the number of indecomposable representations). But using Lemma 2, we can always make the vertex i a sink provided that there is no loops at i. Hence the above statement always holds if there is no loops at i.

If $\alpha \in \Delta_+^{im}(\Gamma)$, by the above remarks we can assume that $\alpha \in M$. If $(\alpha,\alpha_i) = 0$ for all i, then supp α is a tame graph and $\alpha = k\delta$ (see §1.2), and case by case analysis in [10] gives the result. Now the part c) of the theorem follows from Lemma 1.

Similarly, part b) of the theorem follows from the (trivial) fact that there exists a unique up to isomorphism representation whose dimension is equal to a fundamental root.

To prove c) take $\alpha \in \mathbb{Z}_+^n\setminus W(\Pi)$, $\alpha \neq 0$, and suppose that there exists an indecomposable representation of dimension α. Then, as before, there exists an indecomposable representation of dimension $r_\gamma(\alpha)$ for $\gamma \in \Pi$; in particular, $r_\gamma(\alpha) \in \mathbb{Z}_+^n$. Also supp α is connected. Hence $W(\alpha) \subset \mathbb{Z}_+^n\setminus W(\Pi)$. Taking $\beta \in W(\alpha)$ of minimal height, we have:

$(\beta,\alpha_i) \leq 0$ for all $\alpha_i \in \Pi$ and supp β is connected. Hence $\beta \in M$ and $\alpha \in \Delta_+^{im}(\Gamma)$. □

Remark. One can show (see e.g. [4]) that a generic representation of dimension $k\delta$ of a tame quiver decomposes into a direct sum of k representations of dimension δ.

§1.11. In this section we prove Lemma 1. Let first $\alpha = \Sigma k_i \alpha_i$ be an arbitrary non-zero element from $\mathbb{Z}_+^n$. Let $\alpha = \beta_1+\dots+\beta_s$, where $\beta_1 \geq \beta_2 \geq \dots$ (i.e., each coordinate $\geq$) be a decomposition of α into a sum of non-zero elements from $\mathbb{Z}_+^n$; let $\beta_k = \sum_i m_i^{(k)} \alpha_i$. Taking distinct elements $\lambda_1,\lambda_2,\dots \in \mathbb{F}^*$ defines a conjugacy class of semi-simple elements in G^α consisting of the elements $g = (g_1,\dots,g_n)$ such that λ_j is an eigenvalue of g_i with multiplicity $m_i^{(j)}$ for all $i = 1,\dots,n$. Denote by $S_{\beta_1,\dots,\beta_s} \subset G^\alpha$ the union of all such conjugacy classes. Then an easy computation shows that the dimensions of the centralizer G_g^α of $g \in G^\alpha$ and of the fixed point set $M^\alpha(\Gamma,\Omega)^g$ of g in $M^\alpha(\Gamma,\Omega)$ are independent of the choice of $g \in S_{\beta_1,\dots,\beta_s}$ and, moreover, we have:

$$(!) \qquad \dim G_g^\alpha - \dim M^\alpha(\Gamma,\Omega)^g = \sum_i (\beta_i,\beta_i).$$

It follows from the theory of sheets in GL_k[6] that $S_{\beta_1,\dots,\beta_s}$ is a locally closed irreducible subvariety in G^α; denote by $\hat{S}_{\beta_1,\dots,\beta_s}$ the union of orbits of the same dimension in the Zarisky closure of $S_{\beta_1,\dots,\beta_s}$. Then, as we saw, $S_{\beta_1,\dots,\beta_s} \subset \hat{S}_{\beta_1,\dots,\beta_s}$. Futhermore, it follows from the theory of sheets in GL_k[6] that $\hat{S}_{\beta_1,\dots,\beta_s}$ contains a unique unipotent conjugacy class u, which corresponds to the conjugate partition of α. A similar (but slightly more delicate computation, which can be found in [4]) shows that the above properties hold for $g = u$. By a deformation argument it follows that these properties hold for arbitrary $g \in G^\alpha$ (this also can be checked by a direct computation, cf §1.13). So, we have proved the following

Lemma. <u>For</u> $g \in \hat{S}_{\beta_1,\dots,\beta_s}$, <u>dimensions of</u> G_g^α <u>and</u> $M^\alpha(\Gamma,\Omega)^g$ <u>are independent of</u> g <u>and formula</u> (!) <u>holds</u>.

Note that $\hat{S}_{\beta_1,\dots,\beta_s}$ is a sheet in G^α, i.e., an irreducible component of the union of the orbits of G^α of the same dimension, so that G^α is a disjoint union of the sets $\hat{S}_{\beta_1,\dots,\beta_s}$ [6]. Note also that the trivial sheet $\hat{S}_\alpha$ coincides with C.

We need one more lemma. Its proof is based on the following identity:

$$(\#) \qquad \sum_{i,j=1}^{n} a_{ij} m_i (k_j - m_j) = \sum_{j=1}^{n} m_j (k_j - m_j) k_j^{-1} \left(\sum_{i=1}^{n} a_{ij} k_j \right) + \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij} \left(\frac{m_i}{k_i} - \frac{m_j}{k_j} \right)^2 k_i k_j,$$

provided that $a_{ij} = a_{ji}$ and $k_j \neq 0$ for all $i,j = 1,\ldots,n$. This can be checked directly.

<u>Lemma</u>. <u>Let</u> $\alpha \in M$. <u>Then</u>

$$(!!) \qquad \dim G^{\alpha} - \dim G^{\alpha}_{g} \leq \dim M^{\alpha}(\Gamma,\Omega) - \dim M^{\alpha}(\Gamma,\Omega)^{g}.$$

<u>The equality holds only in the following situation:</u>

(N) $(\alpha,\alpha_i) = 0$ <u>for all</u> $i \in \operatorname{supp}\alpha$ <u>or</u> $g \in C$.

<u>Proof</u>. Using formula (=) from §1.8 and formula (!), we have only to show that $(\alpha,\alpha) \leq \sum_i (\beta_i,\beta_i)$ and the equality holds only in the situation (N). This is equivalent to: $\sum_i (\alpha-\beta_i,\beta_i) \leq 0$ and the equality holds only on the situation (N). We can assume that $\operatorname{supp}\alpha = \Gamma$. Applying identity (#) we deduce:

$$(\alpha-\beta_t,\beta_t) = \sum_j m_j^{(t)} (k_j - m_j^{(t)}) k_j^{-1} (\alpha,\alpha_i) + \tfrac{1}{2} \sum_{i,j} (\alpha_i,\alpha_j) \left(\frac{m_i^{(t)}}{k_i} - \frac{m_j^{(t)}}{k_j} \right)^2 k_i k_j .$$

Since $(\alpha_i,\alpha_j) \leq 0$ for $i \neq j$ and $(\alpha,\alpha_i) \leq 0$, we deduce that both summands of the right-hand side are ≤ 0. This proves the inequality in question. In the case of equality, both summands are zero. Since the second summand is zero and Γ is connected, we deduce that α and β_t are proportional. Since $\alpha \neq \beta_t$, and the first summand is zero, we deduce that $(\alpha,\alpha_i) = 0$ for all i. □

Now we can easily complete the proof of Lemma 1. Indeed, if $g \in G^{\alpha}\setminus C$, then, by inequality (!!) we have:

$$\dim M^{\alpha}(\Gamma,\Omega) > \dim M^{\alpha}(\Gamma,\Omega)^{g} + (\dim G^{\alpha} - \dim G^{\alpha}_{g}).$$

It follows that $\dim M^{\alpha}(\Gamma,\Omega) > \dim(G^{\alpha}(M^{\alpha}(\Gamma,\Omega)^{g}))$ and therefore there exists a dense open set $M(g)$ in $M^{\alpha}(\Gamma,\Omega)$ such that the intersection of the conjugacy class of g with G^{α}_{U} is trivial for any $U \in M(g)$. Since there exists only a finite number of conjugacy classes of projectors in $\bigoplus_i gl_{k_i}(\mathbb{F})$, we deduce that there is a dense open set M'

in $\mathcal{M}^\alpha(\Gamma,\Omega)$ such that $(G^\alpha/C)_U$ is a unipotent group for any $U \in \mathcal{M}'$. Since there is only a finite number of unipotent classes in G^α_u we deduce that there is a dense open set in $\mathcal{M}^\alpha(\Gamma,\Omega)$ which consists of representations with a trivial endomorphism ring. This proves Lemma 1a).

To prove b) note that $\mu(\mathcal{M}^\alpha_{ind}(\Gamma,\Omega)\setminus\mathcal{M}^\alpha_0(\Gamma,\Omega)) \leq \max_u(\dim \mathcal{M}^\alpha(\Gamma,\Omega))^u - \dim(G^\alpha/C)_u$ where u ranges over a set of representatives of all non-trivial unipotent classes of G^α. But the right-hand side is (by (!!)) $< \dim \mathcal{M}^\alpha(\Gamma,\Omega) - \dim G^\alpha + 1$, which is equal to $1-(\alpha,\alpha)$. □

§1.12. Unfortunately, I do not know a direct proof of Lemma 2. The only known proof requires a reduction mod p argument and counting over a finite field. In this section we recall the necessary facts.

Let X be an absolutely irreducible N-dimensional algebraic variety over a finite field $\mathbb{F}_q$ of $q = p^s$ elements (p is a prime number). Then the number of points in X over the field $\mathbb{F}_{q^t}$ is equal to $q^{Nt} + \lambda_2^t + \ldots + \lambda_k^t - \nu_1^t - \ldots - \nu_s^t$, where $\lambda_i, \nu_j \in \mathbb{C}$ are independent of t and $|\lambda_i|, |\nu_j| < q^N$. This result is due to Grothendieck. In particular, knowing the number of points in X over all finite fields $\mathbb{F}_{q^t}$ we can compute the dimension of X.

Let now X be an absolutely irreducible N-dimensional algebraic variety over $\mathbb{Q}$. Then X can be represented as a union of open affine subvarieties, each of which is given by a system of polynomial equations over $\mathbb{Z}$, the transition functions being polynomials over $\mathbb{Z}$. Now we can reduce this modulo a prime p. Then for all but a finite number of primes we get an absolutely irreducible variety $X^{(p)}$ over $\mathbb{F}_p$ of dimension N.

This reduces the proof of Lemma 2 to the case when $\mathbb{F}$ is a field of prime characteristic p.

§1.13. In order to count the number of orbits of $G^\alpha(\mathbb{F}_q)$ on $\mathcal{M}^\alpha(\Gamma,\Omega)(\mathbb{F}_q)$ we employ the Burnside lemma: for the action of a finite group G on a finite set Y the number of orbits is:

$$|Y/G| = \frac{1}{|G|}\sum_{g\in G}|Y^g|.$$

(Here Y^g denote the fixed point set of g on Y and $|Z|$ denotes the cardinality of Z). Denoting by C_g the congugacy class of $g \in G$ and using $|C_g| = |G|/|G_g|$ we can rewrite this formula:

$$|Y/G| = \sum_g |Y^g|/|G_g|,$$

where the summation is taken over a set of representatives of conjugacy classes in G.

Now we need a Jordan canonical form for the elements from $GL_k(\mathbb{F}_q)$ (this information can be found, e.g., in [9], Chapter IV).

Denote by Φ the set of all irreducible polynomials in t over $\mathbb{F}_q$ with leading coefficient 1, excluding the polynomial t. Such a polynomial of degree d has the form

$$P(t) = \prod_{i=0}^{d-1} (t - \alpha^{q^i}), \quad \text{where} \quad \alpha \in \mathbb{F}_q^*, \alpha^{q^d} = \alpha.$$

It follows that the number of polynomials from Φ of degree d is equal to

$$q - 1 \ \text{if} \ d = 1 \ \text{and} \ d^{-1} \sum_{j|d} \mu(j) q^{d/j} \ \text{if} \ q > 1,$$

where μ denotes the classical Möbius function.

Let Par denote the set of all partitions, i.e., non-increasing finite sequences of non-negative integers: $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$. We denote by λ' the conjugate partition and by $m_i(\lambda)$ the multiplicity of i in λ; we denote: $|\lambda| = \sum_i \lambda_i$, $\langle\lambda,\mu\rangle = \sum_i \lambda_i \mu_i$.

Conjugacy classes C_ν in $GL_k(\mathbb{F}_q)$ are parametrizes by maps ν: $\Phi \longrightarrow \text{Par}$ such that $\sum_{P \in \Phi} (\deg P) |\nu(P)| = k$ as follows.

To each $f = t^d - \sum_{i=1}^{d} a_i t^{i-1}$ we associate the "companion matrix"

$$J(f) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_d \end{bmatrix},$$

and for each integer $m \geq 1$ let

$$J_m(f) = \begin{bmatrix} J(f) & 1 & 0 & \dots & 0 \\ 0 & J(f) & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & J(f) \end{bmatrix}$$

with m diagonal blocks $J(f)$. Then the Jordan canonical form for elements of the conjugacy class C_ν is the diagonal sum of matrices $J_{\nu(f)_i}(f)$ for all $i \geq 1$ and $f \in \Phi$.

The order of the centralizer of each $g \in C_\nu$ is

$$a_\nu(q) = q^{\sum_{P\in\Phi} (\deg P) \langle \nu(P)', \nu(P)' \rangle} \prod_{P\in\Phi} b_{\nu(P)}(q^{-\deg P})$$

where for $\lambda \in \mathrm{Par}$, $b_\lambda(q) = \prod_{i\geq 1} (1-q^{-1})(1-q^{-2})\dots(1-q^{-m_i(\lambda)})$

Finally, if $g \in C_\nu$ and $h \in C_\gamma \subset GL_m$, then (see e.g. [4]):

$$\dim(\mathbb{C}^k \otimes \mathbb{C}^m)^{g\otimes h} = \sum_{P\in\Phi} (\deg P) \langle \nu(P)', \gamma(P)' \rangle.$$

Let now (Γ, Ω) be an oriented graph, and $\alpha = \Sigma k_i \alpha_i \in \mathbb{Z}_+^n$. The conjugacy classes of G^α are parametrized by the maps $\Phi \longrightarrow \mathrm{Par}^n$. An element $\lambda \in \mathrm{Par}^n$ is an n-tuple of partitions $\lambda^{(i)} = \{\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \dots\}$; set $\lambda_j = (\lambda_j^{(1)}, \dots, \lambda_j^{(n)}) \in \mathbb{Z}_+^n$. For $\lambda, \mu \in \mathrm{Par}^n$ we define $(\lambda, \mu) = \sum_j (\lambda_j, \lambda_j)$, where the bilinear form $(\, , \,)$ on $\mathbb{Z}^n$ is the one associated to Γ. This pairing depends on the graph Γ but is independent of Ω.

Using the Burnside formula we easily deduce the following formula for the number of orbits $d_\alpha(q)$ of $G^\alpha(\mathbb{F}_q)$ on $M^\alpha(\Gamma, \Omega)(\mathbb{F}_q)$:

$$d_\alpha(q) = \sum_\nu \frac{q^{-\sum_{P\in\Phi} (\deg P) (\nu', \nu')_q}}{\prod_{k=1}^{n} \prod_{P\in\Phi} b_{\nu(P)(k)}(q^{\deg P})},$$

where ν ranges over all maps $\nu\colon \Phi \longrightarrow \mathrm{Par}^n$ such that $\sum_{P\in\Phi} (\deg P) |\nu(P)^{(i)}| = k_i$. This formula (derived jointly with R. Stanley) is quite intractible. However, the following two important corollaries of this formula are clear:

The number $d_\alpha(q)$ of isomorphism classes of representations of the quiver (Γ, Ω) over $\mathbb{F}_q$ is independent of the orientation Ω, and is a polynomial in q with rational coefficients.

(It is immediate that $d_\alpha(q)$ is a rational function in q over $\mathbb{Q}$, but since $d_\alpha(q) \in \mathbb{Z}$ for all $q = p^s$, p prime, $s \in \mathbb{Z}_+$, it follows, that, in fact, $d_\alpha(q)$ is a polynomial.)

We deduce by induction on $\mathrm{ht}\,\alpha$ the following

Lemma. The number of isomorphism classes of indecomposable representations of dimension α of the quiver (Γ, Ω) over the field $\mathbb{F}_q$ is a polynomial in q with rational coefficients, independent of the orientation Ω.

§1.14. It remains to pass from indecomposable representations to absolutely indecomposable ones. For that we need the following general result, the proof of which can be found e.g., in [13] (see also [3]).

Proposition. Let G be a connected algebraic group operating transitively on an algebraic variety X over a finite field $\mathbb{F}_q$. Suppose that the stablizer G_x of $x \in X$ is connected. Then the set $X(\mathbb{F}_q)$ of points defined over $\mathbb{F}_q$ is non-empty and $G(\mathbb{F}_q)$ operates transitively on it.

Since all the stabilizers of the action of G^α on $M^\alpha(\Gamma,\Omega)$ are connected, by the proposition, counting the points over $\mathbb{F}_q$ of the geometric quotients $Z_1, Z_2, \ldots$ is the same as counting the orbits of $G^\alpha(\mathbb{F}_q)$ in $M^\alpha_{ind}(\Gamma,\Omega)(\mathbb{F}_q)$.

In order to count the number of absolutely indecomposable representations over $\mathbb{F}_q$ we need the following lemma, which follows easily from the proposition (see [3],p.90).

Lemma. a) A representation U of (Γ,Ω), defined over a finite field $\mathbb{F}$, has a unique minimal field of definition $\mathbb{F}'$. If $\sigma \in \mathrm{Gal}(\mathbb{F}':\mathbb{F}_p)$ and U is isomorphic to U^σ, then $\sigma = 1$.

b) Let $U \in M^\alpha(\Gamma,\Omega)$ be an absolutely indecomposable representation of (Γ,Ω) with a finite minimal field of definition $\mathbb{F}'$. Let $\mathbb{F}_q \subset \mathbb{F}'$ and set $G = \mathrm{Gal}(\mathbb{F}':\mathbb{F}_q)$. Set $\tilde{U} = \bigoplus_{\sigma\in G} U^\sigma$. Then

(i) $\tilde{U} \in M^{n\alpha}(\Gamma,\Omega)$ is indecomposable over $\mathbb{F}_q$ and $\mathbb{F}_q$ is the minimal field of definition for $\tilde{U}$;

(ii) two such representations $\tilde{U}$ and $\tilde{V}$ are isomorphic over $\mathbb{F}_q$ iff U is isomorphic over $\mathbb{F}'$ to a G-conjugate of V;

(iii) every indecomposable representation for which $\mathbb{F}_q$ is the minimal field of definition can be obtained in the way described above.

Now we can easily finish the proof of Lemma 2 (and of the theorem). Denote by $m(\Gamma,\alpha;q)$ (resp. $m'(\Gamma,\alpha,q)$) the set of absolutely indecomposable (resp. indecomposable) representations over $\mathbb{F}_q$ of dimension α of the quiver (Γ,Ω). Then we deduce from the lemma that for an indivisible $\alpha \in \mathbb{Z}^n_+$ one has:

$$(\alpha) \qquad m'(\Gamma,r\alpha;q) = \sum_{d|r} \frac{1}{d} \sum_{k|d} \mu(k)\, m(\Gamma,\tfrac{r}{d}\alpha;q^{\frac{d}{k}}),$$

where μ is the classical Möbius function. From this one expresses

$m(\Gamma, r\alpha; q)$ via $m'(\Gamma, d\alpha; q^s)$ where $d \mid r$. Hence $m(\Gamma, r\alpha; q)$ is independent of Ω.

□

§1.15. Note that we have also the following

<u>Proposition</u>. $m(\Gamma,\alpha;q) = q^{\mu_\alpha} + a_1 q^{\mu_\alpha - 1} + \ldots + a_{\mu_\alpha}$, <u>where</u> $\mu_\alpha = 1-(\alpha,\alpha)$ <u>and</u> $a_1, a_2, \ldots$ <u>are integers, independent of</u> Ω <u>and</u> q; <u>moreover</u>, $m(\Gamma, w(\alpha); q) = m(\Gamma,\alpha;q)$ <u>for any</u> $w \in W$.

<u>Proof</u>. It follows from the remarks in §1.12 and the main theorem that

$$m(\Gamma,\alpha;q^t) = q^{\mu_\alpha t} + \lambda_2^t + \ldots + \lambda_k^t - \nu_1^t - \ldots - \nu_s^t, \text{ where } |\lambda_i|, |\nu_j| < q^{\mu_\alpha}.$$

On the other hand, by §1.13, $m'(\Gamma,\alpha;q)$ and hence $m(\Gamma,\alpha;q)$ is a polynomial in q with rational coefficients. Since $m(\Gamma,\alpha;q) \in \mathbb{Z}$ for all $q = p^s$, it follows that the coefficients are integers. The rest of the statements were proved in the previous sections. □

<u>Conjecture</u> 1. a_{μ_α} = mult α (provided that Γ has no edges-loops).

<u>Conjecture</u> 2. $a_i \geq 0$ for all i.

I have no idea what is the meaning of the rest of a_i's.

<u>Examples</u>. a) If $\alpha \in \Delta_+^{re}$, then $m(\Gamma,\alpha;q) = 1$.

b) If Γ is a tame quiver with $n + 1$ vertices, and $\alpha \in \Delta_+^{im}$, then $m(\Gamma,\alpha;q) = q + n$.

c) Let (Γ,Ω) be the quiver V_k from §1.6 and let $\beta_k = 2\alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k \in \Delta_+(V_k)$. Then one can show (using Peterson's reccurent formula) that the multiplicity of β_k satisfies the following reccurent relation:

$$(k-1)(\text{mult } \beta_k) = k(\text{mult } \beta_{k-1}) + 2^{k-2}(k-2); \text{ mult } \beta_3 = 1.$$

From this we deduce: mult $\beta_k = 2^{k-1} - k$.

On the other hand one has (as D. Peterson pointed out):

$$m(V_k, \beta_k) = (q+1)^{k-3} + 3m(V_{k-1}, \beta_{k-1}) - 2m(V_{k-2}, \beta_{k-2}),$$

which gives:

$$m(V_k,\beta_k) = q^{k-3} + \binom{k}{1} q^{k-4} + \left(\binom{k}{2} + \binom{k}{0}\right) q^{k-5} + \left(\binom{k}{3} + \binom{k}{1}\right) q^{k-6} + \\ + \left(\binom{k}{4} + \binom{k}{2} + \binom{k}{0}\right) q^{k-7} + \ldots$$

and $m(V_k, \beta_k; 0) = 2^{k-1} - k$.

All the examples agree with the conjectures!

<u>Remark</u>. The constant terms of the polynomials $m(\Gamma,\alpha;q)$ and $m'(\Gamma,\alpha;q)$ are equal. Indeed, by formula (α) we have:

$$m'(\Gamma,r\alpha;0) = \sum_{d|r} \frac{1}{d} \sum_{k|d} \mu(k) m(\Gamma,\frac{r}{d}\alpha;0) = \sum_{d|r} \frac{1}{d} m(\Gamma,\frac{r}{d}\alpha;0) (\sum_{k|d} \mu(k)) =$$

$$= m(\Gamma,r\alpha;0), \quad \text{since} \quad \sum_{k|d} \mu(k) = 0 \quad \text{unless} \quad d = 1.$$

Conjecture 2 naturally suggests one more

<u>Conjecture 3</u>. The set of isomorphism classes of indecomposable representations of a quiver admits a cellular decomposition by locally closed subvarieties isomorphic to affine spaces, a_i's being the number of cells of dimension $\mu_\alpha - i$.

It follows from the proofs that the minimal field of definition of the (unique) representation of (Γ,Ω) of dimension $\alpha \in \Delta_+^{re}$ is $\mathbb{F}_p$ if char $\mathbb{F} = p$. Ironically enough, I do not know how to prove

<u>Conjecture 4</u>. If char $\mathbb{F} = 0$, the representation of (Γ,Ω) of dimension $\alpha \in \Delta_+^{re}$ is defined over $\mathbb{Q}$.

It would be interesting to give an explicit construction of this representation.

More general is the following

<u>Conjecture 5.</u> The main theorem holds over an arbitrary field $\mathbb{F}$.

Note that it is clear that if $\alpha \notin (\mathbb{Z}_+\Delta_+^{re}) \cup \Delta_+^{im}$, then there is no indecomposable representations of dimension α over $\mathbb{F}$. It would follow from Conjecture 4 that this is the case also for $\alpha \notin \Delta_+$.

§1.16. It is easy to see (see [3]) that the theorem, as well as Conjectures 4 and 5 would follow from the following

<u>Conjecture 6</u>. Let G be a linear algebraic group operating on the vector space V, all defined over $\mathbb{F}$, such that char $\mathbb{F} = 0$ or char $\mathbb{F} > \dim V$. Denote by V_0 (resp V_0^*) the sets of points with a unipotent stabilizer in V (resp. V^*). Then the number of orbits of G on V_0 is equal to that of G on V_0^* and $\mu(G,V_0) = \mu(G,V_0^*)$.

Example: $\mathbb{F} = \mathbb{R}$, $G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a > 0\}$, $V = \mathbb{R}^2$, action on V

(resp. V*) is the multiplication on a vector-column from the left (resp. vector row from the right). For the action on V the orbits of $\binom{1}{0}$ and $\binom{-1}{0}$ are the two (1-dimensional) orbits with a (1-dimensional) unipotent stabilizer. For the action of G on V* the orbits of (10) and (-10) are the two open orbits with a trivial stabilizer.

The following generalization of Conjecture 6 was suggested by Dixmier.

<u>Conjecture 7</u>. Let $S \subset G$ be a reductive subgroup of G. Denote by V_S (resp. V_S^*) the set of points $x \in V$ (resp. $\in V^*$) such that a Levi factor of G_x is a conjugate of S. Then the numbers of orbits of G in V_S and V_S^* are equal and $\mu(G,V_S) = \mu(G,V_S^*)$.

<u>Remark</u>. It is easy to deduce from Conjecture 6 the following statement: Fix a miximal torus $T \subset G$ and denote by V_T(resp. V_T^*) the set of $x \in V$ (resp. $\in V^*$) such that a maximal torus of G_x is a conjugate of T. Then the numbers of orbits of G in V_T and V_T^* are equal and $\mu(G,V_T) = \mu(G,V_T^*)$.

More general is the following

<u>Conjecture 8</u>. Let N be the unipotent radical of G; G/N acts on the sets of orbits V/N and V*/N. These two actions are equivalent.

§1.17. Examples. a) If (Γ,Ω) is a finite type quiver there is no imaginary roots and we recover Gabrial's theorem: $U \longmapsto \dim U$ gives a 1 - 1 correspondence between the set of isomorphism classes of indecomposable representations of a finite type quiver (Γ,Ω) and the set $\Delta_+(\Gamma)$.

We consider in more detail the finite type quiver V_3, which corresponds to the problem of classification of triples of subspaces U_1,U_2,U_3 in a given vector space U_0 up to an automorphism of U_0. There are 12 roots in $\Delta_+(\Gamma)$. Apart from the roots $\alpha_1,\alpha_2,\alpha_3$ which correspond to $U_0 = 0$ we have 9 nontrivial indecomposable triples. The corresponding dimensions ($\dim U_0$; $\dim U_1$, $\dim U_2$, $\dim U_3$) are

(1;0,0,0), (1;1,0,0), (1;0,1,0), (1;0,0,1), (1;1,1,0), (1;1,0,1),

(1;0,1,1), (1;,1,1,1) and (2;1,1,1).

Let $a_0,a_1,\ldots,a_8$ be the number of times these representations appear as indecomposable direct summands in a given representation $(U_0;U_1,U_2,U_3)$. Then we have:

$$\sum_{i=0}^{7} a_i + 2a_8 = \dim U_0, \quad \sum_{i=1}^{8} a_i + 2a_8 = \dim(V_1 + V_2 + V_3),$$

$$a_1 + a_4 + a_5 + a_7 + a_8 = \dim V_1, \quad a_2 + a_4 + a_6 + a_7 + a_8 = \dim V_2,$$

$$a_3 + a_5 + a_6 + a_7 + a_8 = \dim V_3, \quad a_4 + a_7 = \dim V_1 \cap V_2,$$

$$a_5 + a_7 = \dim V_1 \cap V_3, \quad a_6 + a_7 = \dim V_2 \cap V_3, \quad a_7 = \dim V_1 \cap V_2 \cap V_3.$$

It is clear from this system of equation that the nine discrete parameters $\dim U_i (i = 0,\dots,4)$, $\dim U_i \cap U_j$ $(i,j = 1,2,3,\ i = j)$, $\dim U_1 \quad U_2 \quad U_3$ and $\dim(U_1 + U_2 + U_3)$ determine the triple of subspaces U_1, U_2, U_3 is the vector space U_0 up to isomorphism.

b) The quiver P_2 from §1.6 corresponds to the problem of classification of pairs of linear maps $A,B: V_1 \longrightarrow V_2$, the problem solved by Kronecker. We assume that $\mathbb{F}$ is algebraically closed. Then the complete list of indecomposable pairs is in some bases $\{e_i\}$ and $\{f_i\}$ of V_1 and V_2 as follows $(k = 1,2,\dots)$:

$\dim V_1 = k$, $\dim V_2 = k + 1$:

$A(e_i) = f_i; B(e_i) = f_{i+1} (i = 1,\dots,k)$.

$\dim V_1 = k + 1$, $\dim V_2 = k$:

$A(e_i) = f_i (i = 1,\dots,k)$, $A(e_{k+1}) = 0$;

$B(e_i) = f_{i-1} (i = 2,\dots,k + 1)$, $B(e_1) = 0$.

$\dim V_1 = \dim V_2 = k$:

$A(e_i) - f_i (i = 1,\dots,k)$: $B(e_i) = \lambda f_i + f_{i+1} (i = 1,\dots,k - 1)$,

$B(e_k) = \lambda f_k$. Here $\lambda \in \mathbb{F}$ is arbitrary .

$A(e_i) = f_{i+1} (i = 1,\dots,k - 1)$, $A(e_k) = 0$;

$B(e_i) = f_i$.

§1.18. Since the problems of classifcation of all representations of an arbitrary quiver (Γ,Ω) seems to be too difficult, we shall try to understand a simpler question: what is the structure of a generic representation of given dimension α. It is easy to see that there exists a unique decomposition

$$\alpha = \beta_1+\ldots+\beta_s , \quad \text{where} \quad \beta_i \in \mathbb{Z}_+^n\setminus\{0\},$$

such that the set $M_0^\alpha(\Gamma,\Omega) := \{U \in M^\alpha(\Gamma,\Omega) \mid U \simeq \bigoplus_{i=1}^{s} U_i,\ \dim U_i = \beta_i$ and all U_i are indecomposable$\}$ is a dense open subset in $M^\alpha(\Gamma,\Omega)$. This is called the <u>canonical decomposition</u> of α. Further on we assume the base field $\mathbb{F}$ to be algebraically closed.

In order to study this decomposition we need the following definition. A representation $U \in M^\alpha(\Gamma,\Omega)$ is called a <u>Schur representation</u> if $\operatorname{End} U = \mathbb{F}$ (or, equivalently, $(G^\alpha/C)_U = 1$). An element $\alpha \in \mathbb{Z}_+^n\setminus\{0\}$ is called a <u>Schur root</u> for the quiver (Γ,Ω) if $M^\alpha(\Gamma,\Omega)$ contains a Schur representation. In this case the set of Schur representations form a dense open subset $M_0^\alpha(\Gamma,\Omega)$ in $M^\alpha(\Gamma,\Omega)$. Note that $\alpha = \alpha$ is the canonical decomposition of a Schur root. Conversely, if there exists a dense open subset in $M^\alpha(\Gamma,\Omega)$ consisting of indecomposable representations (i.e., $\alpha = \alpha$ is the canonical decomposition of α), then α is a Schur root. Indeed, otherwise,

$$\mu_\alpha(\Gamma,\Omega) > \dim M^\alpha(\Gamma,\Omega) - \dim G^\alpha + 1 = 1 - (\alpha,\alpha),$$

a contradiction with the statement c) of the main theorem.

The set of Schur roots is a subset in $\Delta_+(\Gamma)$ (by the main theorem); we denote it by $\Delta_+^{\mathrm{Schur}}(\Gamma,\Omega)$. As will be clear from examples, this set (as well as the canonical decomposition) depends on the orientation Ω of the quiver.

<u>Remark</u>. One can show that even a stronger result holds [4]: If a representation $U \in M^\alpha(\Gamma,\Omega)$ is stably indecomposable, i.e., all representations from a neighbourhood of U are indecomposable, then U is a Schur representation (the converse is obvious). The quiver S_1 with relation $A^2 = 0$ shows that this property fails for quiver with relations. It might be interesting to study the rings R which have the property that every its stably indecomposable representation has a trivial endomorphism ring.

<u>Example</u>. Consider in the 3-dimensional space V_0 a quadruple of subspaces V_1,V_2,V_3,V_4 of dimensions $2,2,1,1$ respectively. This quadruple is indecomposable iff $V_1 = V_2$, and $V_3 + V_4$ is a 2-dimensional

subspace different from V_1 and V_2 and $\dim V_1 \cap V_2 \cap (V_3 + V_4) = 1$ (all such quadruples are equivalent). However, the generic quadruple is decomposable and the canonical decomposition is as follows:

$$\alpha: = 3\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = (2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_0 + \alpha_1 + \alpha_2)$$

So, α is a (real) root but not a Schur root.

§1.19. Given a quiver (Γ,Ω), let r_{ij} denote the number of arrows with the initial vertex i and the final vertex j. We define the (in general non symmetric) bilinear form R (R in honour of Ringel) on $\mathbb{Z}^n$ by [11]:

$$R(\alpha_i,\alpha_j) = \delta_{ij} - r_{ij}.$$

Note that $(\alpha,\beta) = \frac{1}{2}(R(\alpha,\beta) + R(\beta,\alpha))$ is the associated symmetric bilinear form. The following proposition is crucial.

<u>Proposition [11]</u>. <u>Let</u> U <u>and</u> V <u>be representations of a quiver</u> (Γ,Ω) <u>of dimensions</u> α <u>and</u> β <u>respectively</u>. <u>Then</u>

$$\dim \mathrm{Hom}(U,V) - \dim \mathrm{Ext}(U,V) = R(\alpha,\beta).$$

Using formula (=) from §1.8 we deduce the following well-known formula for an arbitrary representation $U \in M^\alpha(\Gamma,\Omega)$:

$$(\equiv) \qquad \dim M^\alpha(\Gamma,\Omega) - \dim G^\alpha(U) = \dim \mathrm{Ext}(U,U).$$

We need another formula, which also can be derived from the proposition by a straight forward computation.

<u>Lemma</u>. <u>Let</u> $U_j \in M_0^{\beta_j}(\Gamma,\Omega)$ $(j = 1,\dots,s)$ <u>and</u> $\alpha = \sum_{j=1}^{s} \beta_j$. <u>Let</u> $S \in G^\alpha$ <u>be a semisimple element, such that</u> $\alpha = \sum_j \beta_j$ <u>is the corresponding partition of</u> α. <u>Then</u>

$$\dim M^\alpha(\Gamma,\Omega) - \dim G^\alpha\ (M^\alpha(\Gamma,\Omega)^S) = \sum_{i\neq j} \dim \mathrm{Ext}(U_i,U_j)$$

<u>Proof</u>. It is clear that the left-hand side of the formula is equal to:

$\dim M^\alpha(\Gamma,\Omega) - \dim G^\alpha(U) - \dim M^\alpha(\Gamma,\Omega)^S + \dim G^\alpha_S = \dim M^\alpha(\Gamma,\Omega) - \dim G^\alpha +$
$+ \dim G^\alpha_U + \dim G^\alpha_S - \dim M^\alpha(\Gamma,\Omega)^S = -(\alpha,\alpha) + \sum_j (\beta_j,\beta_j) + \dim G^\alpha_U$ by formula (=) from §1.8 and (!) from §1.11. Since $\dim G^\alpha_U = \dim \mathrm{Hom}(U,U)$ the lemma follows from the proposition. □

Corollary. Let $U_i \in M_0^{\beta_i}(\Gamma,\Omega)$ and $U = \bigoplus_i U_i$, $\alpha = \sum_i \beta_i$. Then $U \in M_0^\alpha(\Gamma,\Omega)$ iff $\mathrm{Ext}(U_i,U_j) = 0$ for all $i \neq j$. In particular, if all β_i are Schur roots, then $\alpha = \sum_i \beta_i$ is the canonical decomposition of α.

§1.20. We call an element $\alpha \in \mathbb{Z}_+^n\setminus\{0\}$ indecomposable if α cannot be decomposed into a sum $\alpha = \beta + \gamma$, where $\beta,\gamma \in \mathbb{Z}_+^n\setminus\{0\}$ and $R(\beta,\gamma) \geq 0$, $R(\gamma,\beta) \geq 0$. One deduces immediately from the remarks in §1.17 and the corollary from §1.18 the following facts:

Proposition. a) If α is an indecomposable element, then α is a Schur root.

b) Let $\alpha \in \mathbb{Z}_+^n\setminus\{0\}$ and $\alpha = \beta_1+\ldots+\beta_s$ be the canonical decomposition of α. Then all β_i are Schur roots and $R(\beta_i,\beta_j) \geq 0$ for all $i \neq j$.

Conjecture 9. If α is a Schur root, then α is indecomposable.

Conjecture 10. Provided that (Γ,Ω) has no oriented cycles, each $\alpha \in \mathbb{Z}^n\setminus\{0\}$ admits a unique decomposition $\alpha = \sum_j \beta_j$ such that β_j are indecomposable and $R(\beta_i,\beta_j) \geq 0$ for $i \neq j$. (See [4] for a version of this conjecture without assumptions on (Γ,Ω)).

If the conjectures 9 and 10 were true, we obtain that the decomposition of α given by conjecture 10 coincides with its canonical decomposition.

In [4] conjectures 9 and 10 are checked for finite and tame quivers, and for rank 2 quivers.

Example. If (Γ,Ω) is finite, then conjecture 9 holds since then any root is indecomposable. Indeed, if $\alpha = \beta + \gamma$, where $R(\beta,\gamma) \geq 0$ and $R(\gamma,\beta) \geq 0$, then $1 = (\alpha,\alpha) = (\beta,\beta) + (\gamma,\gamma) + R(\beta,\gamma) + R(\gamma,\beta) \geq 2$. Similarly, we show that if (Γ,Ω) is a tame quiver and α is a root such that its defect $R(\delta,\alpha) \neq 0$ then α is indecomposable.

Remark. If $(\alpha,\alpha_i) \leq 0$ for all i and $(\alpha,\alpha_i) < 0$ for some i, then α is indecomposable by the identity (#), and hence is a Schur root. This gives another proof of Lemma 1a).

§1.21. The following simple facts proved in [4] show that many questions about the action of G^α on $M^\alpha(\Gamma,\Omega)$ can be answered in terms of the canonical decomposition.

Proposition. Let $\alpha \in \mathbb{Z}_+^n\setminus\{0\}$ *and let* $\alpha = \beta_1+\dots+\beta_k$ *be the canonical decomposition of* α.

a) $\operatorname{tr\,deg} \mathbb{F}(M^\alpha(\Gamma,\Omega))^{G^\alpha} = \sum_{i=1}^{k} (1-(\beta_i,\beta_i))$.

b) $\operatorname{tr\,deg} \mathbb{F}(M^\alpha(\Gamma,\Omega))^{(G^\alpha,G^\alpha)} = \sum_{i=1}^{k} (1-(\beta_i,\beta_i)) + |\operatorname{supp}\alpha| - s - r$, *where* s *and* r *are the number of distinct real roots and the dimension of the* $\mathbb{Q}$*-span of all imaginary roots in the canonical decomposition of* α, *respectively*.

c) G^α *has a dense orbit in* $M^\alpha(\Gamma,\Omega)$ *iff all* β_i *are real, the principal stabilizer being reductive iff* $R(\beta_i,\beta_j) = 0$ *whenever* $\beta_i \neq \beta_j$.

d) *If* G^α *has a dense orbit* $\mathcal{O}$ *in* $M^\alpha(\Gamma,\Omega)$, *then we have for the categorical quotient:*

$$M^\alpha(\Gamma,\Omega)/(G^\alpha,G^\alpha) \simeq \mathbb{F}^{|\operatorname{supp}\alpha| - s},$$

where s *is the same as in* b) *(and also is the number of distinct indecomposable summands of a representation from* $\mathcal{O}$).

e) *The generic* (G^α,G^α)*-orbit in* $M^\alpha(\Gamma,\Omega)$ *is closed iff* $R(\beta_i,\beta_j) \neq 0$ *whenever* $\beta_i \neq \beta_j$.

Remarks. a) If (Γ,Ω) is a finite type quiver, then G^α always has a dense orbit in $M^\alpha(\Gamma,\Omega)$ (since it has a finite number of orbits or by part c) of the proposition), and hence formula from d) always holds. This has been found by Happel.

b) If (Γ,Ω) is a tame quiver then G^α has a dense orbit in $M^\alpha(\Gamma,\Omega)$ iff the defect $R(\delta,\alpha) \neq 0$ (by part c) of the proposition).

Chapter II. The slice method.

The slice method is based on Luna's slice theorem [7] and was for the first time applied in [5] for the classification of irreducible representations of connected simple linear groups for which the ring of invariants is a polynomial ring. In this chapter I discuss some examples of applications of this method, mainly to invariant theory of binary forms.

§2.1. Let G be a linear reductive group operating on a finite-dimensional vector space V, both defined over $\mathbb{C}$. For $p \in V$ let G_p denote the stabilizer of p and T_p the tangent space to the orbit $G(p)$ of p. Then T_p is G_p-invariant and we can consider the action of G_p on the vector space $S_p := V/T_p$. If the orbit $G(p)$ is closed, the action of G_p on the space S_p is called a slice representation. Note that G_p is a reductive group (since G/G_p is an affine variety and by Matsushima criterion G/H is affine iff H is a reductive subgroup); therefore, we can identify S_p with a G_p-invariant complementary to T_p subspace in V.

The slice method is based on the following principle:

Given a representation of a reductive group, every its slice representation is "better" than the representation itself.

§2.2. In order to make this principle more precise we have to introduce the so called categorical quotient. Let $\mathbb{C}[V]$ denote the ring of polynomials on V and $R = \mathbb{C}[V]^G$ the subring of G-invariant polynomials. Then it follows from the complete reducibility of the action of G on $\mathbb{C}[V]$ that there exists a linear map $\mathbb{C}[V] \longrightarrow \mathbb{C}[V]^G$, denoted by $f \longmapsto f^\natural$ with the following properties:

(i) if $U \subset \mathbb{C}[V]$ is G-invariant, then $U^\natural \subset U$;

(ii) if $f \in \mathbb{C}[V]^G$, $g \in \mathbb{C}[V]$, then $(fg)^\natural = fg^\natural$.

One immediately deduces the classical fact that the algebra $\mathbb{C}[V]^G$ is finitely generated. Indeed, let $I \subset \mathbb{C}[V]$ be the ideal generated by all homogeneous invariant polynomials of positive degree. By Hilbert's basis theorem, it is generated by a finite number of invariant polynomial, say $P_1, \ldots, P_N$. We prove by induction on the degree of a homogeneous polynomial $P \in \mathbb{C}[V]^G$ that P lies in the subalgebra generated by $P_1, \ldots, P_N$. We have

$$P = \sum_{i=1}^{N} Q_i P_i, \quad \text{where} \quad \deg Q_i < \deg P.$$

Applying to both sides the operator $\natural$ we get:

$$P = \sum_i Q_i^{\natural} P_i, \quad \text{where} \quad \deg Q_i^{\natural} = \deg Q_i < \deg P,$$

and applying the inductive assumption to $Q_i^{\natural}$ completes the proof.

Denote by V/G the affine variety for which $\mathbb{C}[V]^G$ is the coordinate ring. Then the inclusion $\mathbb{C}[V]^G \hookrightarrow \mathbb{C}[V]$ induces a map $\pi: V \longrightarrow V/G$ called the <u>quotient map</u>. The pair $\{\pi, V/G\}$ is called the <u>categorical quotient</u> because it satisfies the following characteristic properties:

(i) the fibers of π a G-invariant;

(ii) if $\pi': V \longrightarrow M$ is a morphism, such that M is in affine variety and the fibers of π' are G-invariant, then there exists a unique map $\psi: V/G \longrightarrow M$ such that $\pi' = \psi \circ \pi$.

Note that V/G is a (weighted) cone (i.e., it has a closed imbedding $V/G \hookrightarrow \mathbb{C}^m$, which is invariant under transformations $(c_1,\ldots,c_m) \longrightarrow (t^{s_1}c_1,\ldots,t^{s_m}c_m)$, $t \in \mathbb{C}$, $s_i > 0$).

Note that $\mathbb{C}[V]^G$ is a polynomial ring $\Leftrightarrow$ the vertex of V/G is a regular point $\Leftrightarrow$ V/G is smooth.

§2.3. Here we prove the following classical fact: V/G <u>parametrizes the closed orbits</u>, i.e., for each $x \in V/G$, the fiber $\pi^{-1}(x)$ contains a unique closed orbit.

This follows from the following two facts:

(i) if $M_1, M_2 \subset V$ are two closed disjoint G-invariant subvarieties, then there exists an invariant polynomial P which is identically 0 on M_1 and identically 1 on M_2;

(ii) the map π is surjective.

Then the closed orbit in a fiber is an orbit of minimal dimension in the fiber.

For (i), let P_i's and Q_j's be generators of defining ideals I_1 and I_2 for M_1 and M_2. Then by Hilbert's Nullstelensatz, $\{P_i, Q_j\}_{i,j}$ generate $\mathbb{C}[V]$ is an ideal, hence $\sum_i g_i P_i + \sum_j g_j' Q_j = 1$ for some g_i, g_j'. Denoting the first summand by f_1 and second by f_2, we have:

$$f_1 + f_2 = 1, \quad \text{where} \quad f_1 \in I_1, \quad f_2 \in I_2.$$

Applying $\natural$, we get:

$$f_1^\natural + f_2^\natural = 1, \quad \text{where} \quad f_i^\natural \in I_i.$$

Set $P = f_1^\natural$; clearly, $P|_{M_1} = 0$ since $P \in I_1$, so $P|_{M_2} = 1$.

If (ii) fails then there exists a maximal ideal $I \subseteq \mathbb{C}[V]^G$ which generates $\mathbb{C}[V]$ as an ideal; then we have: $1 = \sum_i f_i P_i$, where $f_i \in \mathbb{C}[V]$, and $P_i \in I$. Applying $\natural$, we get:

$$1 = \sum_i f_i^\natural P_i.$$

Hence $I = \mathbb{C}[V]^G$, a contradiction. □

§2.4. Let $p \in V$ be such that the orbit $G(p)$ is closed. We have: $V = T_p \oplus S_p$ (G_p-invariant decomposition). We have by restriction: $\pi: S_p \longrightarrow V/G$, and by the universality property, this can be pushed down to the morphism:

$$\pi_p: \; S_p/G_p \longrightarrow V/G.$$

It is clear that $\dim S_p/G_p = \dim V/G$. From Luna's slice theorem we deduce the following lemma [5].

<u>Lemma</u>. <u>The morphism</u> π_p <u>induces an isomorphism of completions of local rings of</u> $\pi(p) \in V/G$ <u>and the vertex of the cone</u> S_p/G_p

Note that this lemma says, in particular, that π_p is an analytic isomorphism on some neighbourhood of the vertex of S_p/G_p (in complex topology), i.e., this vertex has the same singularity as $\pi(p) \in V/G$. Moreover, it follows from §2.3 that we get all the singularities of V/G in this way.

§2.5. Here are some precise special cases of our general principle. Let $p \in V$ be such that the orbit $G(p)$ is closed. Set $R = \mathbb{C}[V]^G$ and $R_1 = \mathbb{C}[S_p]^{G_p}$.

<u>Proposition</u>. a) <u>If</u> R <u>is a polynomial ring (resp. complete intersection), then</u> R_1 <u>is a polynomial ring (resp. complete intersection) too</u>.

b) <u>Let</u> m <u>(resp.</u> m_p<u>) denote the minimal number of homogeneous generators of</u> R <u>(resp.</u> R_1<u>). Then</u> $m \geq m_p$.

c) Set $A = \mathbb{C}[z_1,\dots,z_m]$, $A_1 = \mathbb{C}[z_1,\dots,z_{m_p}]$ and let

$$\dots \longrightarrow A^{r_2} \longrightarrow A^{r_1} \longrightarrow R \longrightarrow 0 \quad \text{and} \quad \dots \longrightarrow A^{r_{2p}} \longrightarrow A^{r_{1p}} \longrightarrow R_1 \longrightarrow 0$$

be the minimal free resolutions for R and R_1. Then

$$r_1 \geq r_{1p},\ r_2 \geq r_{2p},\ \dots .$$

Proof. follows from §2.4 and the following remarks. From the point of view of the properties we are interested in

(i) the behaviour of a local ring is the same as the one of its completion;

(ii) the behaviour of the local ring of the vertex of a cone is the same as the one of the coordinate ring of the cone;

(iii) the local ring of the vertex is the "worst" among all the local rings of a cone.

All these statements about local rings are quite simple and can be found e.g. in [14]. □

§2.6. One often can find slice representations for which G_p is a finite group. Then one can apply the Shepard-Todd-Chevalley theorem to check that R is not a polynomial ring. In order to check that R is not a complete intersection one can apply the following result [16].

Proposition. Let G be a finite linear group operating on a vector space V of dimension n. Suppose that $\mathbb{C}[V]^G$ has m generators and that the ideal of relations (i.e., the kernel of the surjection $\mathbb{C}[z_1,\dots,z_m] \longrightarrow \mathbb{C}[V]^G$) is generated by $m - n + s$ elements (note that $m \geq n$ and $s \geq 0$). Then G is generated by those σ such that $\operatorname{rank}(\sigma - I) \leq s + 2$. In particular, if V/G is a complete intersection, then $G = \langle \sigma \in G \mid \operatorname{rank}(\sigma - I) \leq 2 \rangle$.

Proof. Let F_g denote the fixed point set of $g \in G$. Denote by Z the union of all $F_g \subset V$ such that $\operatorname{codim}_V F_g \geq s + 3$. Then G acts on $X := V \setminus Z$ and $X/G = (V/G) \setminus (Z/G)$. Note that V/G is simply connected since, being a cone, it is contractible to the vertex. Furthermore, X/G is simply connected by the following fact, proved by Goresky and Macpherson. Let M be a closed affine subvariety in $\mathbb{C}^m$ of dimension n and suppose that the ideal of M is generated by $m - n + s$ elements. Then if M is simply connected, $M \setminus Y$ is simply connected for any sibvariety $Y \subset M$ of codimension $\geq s + 3$.

So, G acts on a connected variety X such taht X/G is simply connected. But then $G = \langle G_x | x \in X \rangle$. Indeed, let G_1 denote the right-hand side. Then G/G_1 acts on X/G_1 such that $g \neq e$ has no fixed points. Since $X/G = (X/G_1)/(G/G_1)$, we deduce that $G/G_1 = e$. This completes the proof. □

Note that the same proof gives the "only if" part of the Shepard-Todd-Chevalley theorem.

Remark. If $G \subset SL_2(\mathbb{C})$, then $\mathbb{C}^2/G$ is always a complete intersection (F. Klein). But for $G = \{\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}\}$, where ε is a cube root of 1, $\mathbb{C}^2/G$ is not a complete intersection. Indeed, $u_1 = x^3$, $u_2 = x^2y$, $u_3 = xy^2$, $u_4 = y^3$ is a minimal system of generating invariants, and a minimal system of relations is: $u_1u_4 = u_2u_3$, $u_1u_3 = u_2^2$, $u_2u_4 = u_3^2$. This is the simpliest counter-example to the converse statement of the proposition.

§2.7. In order to apply the slice method one should be able to check that an orbit $G(x)$ is closed. For this one can use the Hilbert-Mumford-Richardson criterion, which I will not discuss here, or the following

Proposition [8]. Let G be a reductive group operating on a vector space V, $p \in V$ and $H \subset G_p$ be a reductive subgroup. The normalizer N of H in G acts on the fixed point set L of H in V. The orbit $G(p)$ in V is closed iff the orbit $N(p)$ in L is closed.

Note also that $G(p)$ is closed iff $G^0(p)$ is closed, where G^0 is the connected component of the unity of G.

§2.8. Now we turn to an example of the action of the group $G = SL_2(\mathbb{C})$ on the space V of binary forms fo degree d by

$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} P(x,y) = P(\alpha x + \beta y, \gamma x + \delta y)$.

Fix the following basis of V_d:

$$v_0 = x^d,\ v_2 = x^{d-1}y, \ldots, v_{d-1} = xy^{d-1},\ v_d = y^d.$$

We consider separately the cases d odd and even.

a) d odd and ≥ 3. Set $p = x^{d-1}y + xy^{d-1}$ and $\varepsilon = \exp \frac{2\pi i}{d-2}$. Then $G_p = \{A_k = \begin{pmatrix}\varepsilon^k & 0\\ 0 & \varepsilon^{-k}\end{pmatrix}, k - 1, \ldots, d-2\}$ is a cyclic group of order $d - 2$. The fixed point set of G_p is $L = \mathbb{C}v_1 + \mathbb{C}v_{d-1}$; the connected component of the unity of the normalizer of G_p is $N^0 = \{\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}, t \in \mathbb{C}^*\}$. The orbit $N^0(p)$ is clearly closed. Hence by §2.7, the orbit

$G(p)$ is closed.

$\therefore$ The tangent space T_p to $G.p$ is

$$\left[\mathbb{C}x \frac{\partial}{\partial y} + \mathbb{C}y \frac{\partial}{\partial x} + \mathbb{C}(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y})\right](p) ,$$

hence the eigenvalues of A on T_p are : $\varepsilon^2, \varepsilon^{-2}$ and 1.

On the other hand, we have:

$$A_1(v_j) = \varepsilon^{(d-2j)} v_j .$$

Hence the eigenvalues of A_1 on $S_p = V/T_p$ are:

$$1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots, \varepsilon^{(d-3)} .$$

So, according to our principle, the representation of a cyclic group $H = \langle A_1 \rangle$ of order $d - 2$ on $\mathbb{C}^{d-2}$ acting by $A_1(e_j) = \varepsilon^{-j} e_j$ in some basis $e_1, \ldots, e_{d-2}$, is "better" than the action of $SL_2(\mathbb{C})$ on the space of binary forms of odd degree $d > 1$.

Let $f_1, \ldots, f_{d-2}$ be the basis dual to $e_1, \ldots, e_{d-2}$. Then the monomials $f_1^{k_1} \cdots f_{d-2}^{k_{d-2}}$ such that

$$(*) \quad \sum_j jk_j \equiv 0 \bmod (d - 2) ;$$

form a basis of the space of invariant polynomials for this action of H.

An integral solution of (*) is called positive if all $k_i \geq 0$ and not all of them $= 0$; a positive solution is called indecomposable if it is not a sum of two positive solutions. It is clear that the minimal number of generating invariant polynomials for the action of H on $\mathbb{C}^{d-2}$ is equal to the number of indecomposable positive solutions of (*). I do not know how to compute this number. However, (as observed by R. Stanley) it is clear that if a solution $(k_1, \ldots, k_{d-2})$ is positive and the left-hand side of (*) is equal to $d - 2$, then this is an indecomposable solution. Also it is clear that $(d - 2)(\delta_{1i}, \ldots, \delta_{d-2,i})$ is an indecomposable solution provided that i and $d-2$ are relatively prime and $i \neq 1$. This gives us the following estimate: (number of positive indecomposable solutions of (*)) $\geq p(d-2) + \phi(d-2) - 1$, where $p(k)$ is the classical partition function and $\phi(k)$ is the number of $1 \leq j \leq k$ relatively prime to k.

For the discussion of the number of relations we need the following defintiion. Let $R = \mathbb{C}[z_1, \ldots, z_m]/I$ be a finitely generated ring, where m is the minimal number of generators; let $n (\leq m)$ be the dimension of R. We say that R <u>requires at least</u> s <u>extra equations</u> if the minimal system of generators of the ideal I contains at least

$m - n + s$ elements.

It is clear by §2.6 that for the action of the cyclic group H the ring of invariants requires at least $d-5$ extra relations. Again, applying the principle, this gives an estimate for the number of relations between invariants of binary forms.

The obtained results are sumarized in this following

Proposition. *Let R be the ring of invariant polynomials for the action of $SL_2(\mathbb{C})$ on the space of binary forms of degree $d > 1$, d odd. Then the minimal number of generators of R is $\geq p(d-2)+q(d-2)-1$ and R requires at least $d - 5$ extra relations.*

A complete information about degrees of polynomials in a minimal system of homogeneous generators of R (it is easy to see that these are well-defined numbers) and the generating relations are known (for odd d) only for $d \leq 5$. Namely, for $d = 3$, R is generated by a homogeneous polynomial of degree 4; for $d = 5$, R is generated by homogeneous polynomials of degrees 4,8,12 and 18 and there is exactly one generating relation.

Note that for $d = 3$ and 5 our low bounds are exact. However, for $d \geq 7$ the low bounds given by the slice method are far from being exact. For instance taking $p = x^7 + y^7$ for $d = 7$ gives the best low bound, which is 17, for the minimal number of generating invariants; it is known, however, that this number lies between 28 and 33 [15].

b) d even and ≥ 4. We take $p = x^d + y^d$ and let $\varepsilon = \exp \frac{2\pi i}{d}$. Then $G_p = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$. The same argument as in a) shows that the orbit $G(p)$ is closed, and our principle gives similar low bounds. In particular, we get that R requires at least $\frac{3}{4}d - 5$ (resp. $\frac{1}{4}(3s + 2) - 5$) extra relations if $4|d$ (resp. $4|d + 2$).

One has a complete information about R (for even d) only for $d \leq 8$. Namely for $d = 2$, R is generated by one polynomial of degree 2; for $d = 4$, R is freely generated by polynomials for degree 2 and 3; for $d = 6$, R is generated by polynomials of degree 2,4,6, 10 and 15 and there is exactly one generating relation; for $d = 8$ R is generated by polynomials of degree 2,3,4,...,10 and requires two extra relations.

One can see that for $d = 4$, 6 and 8 our low bounds are exact.

§2.9. It follows from the results of §2.8 that for the action of $SL_2(\mathbb{C})$ on the space V_d of binary forms of degree d, the ring R of invariant polynomials is a complete intersection iff $d \leq 6$ (and is a polynomial ring iff $d \leq 4$).

Similarly, one can apply §2.7 to the classification of reductive linear groups for which the ring of invariants is a complete intersection. As an example, let us prove the following

Proposition. For the action of $SL_n(\mathbb{C})$ $(n > 1)$ on the space $S^d(\mathbb{C}^n)$ the ring of invariants R is a complete intersection iff either $d \leq 2$, or $n = 2$ and $d \leq 6$, or $n = d = 3$, or $n = 4$, $d = 3$. Moreover if R is a complete intersection but is not a polynomial ring, then $(n,d) = (2,5)$ or $(2,6)$ or $(4,3)$ and R is the coordinate ring of a hypersurface.

Proof. The case $(4,3)$ was worked out by Salmon about a hundred years ago [12]. He showed that R is generated by invariants of degree 8, 16,24,32,40 and 100 with one generating relation. It is well known that in the case $(3,3)$, R is a polynomial ring generated by invarinats of degree 4 and 6. The case $d \leq 2$ is obvious.

In order to show that in the remaining cases R is not a complete intersection take $p = \sum_{i=1}^{n} z_i^d \in S^d(\mathbb{C}^n)$. Then as in §2.8 we show that the orbit of p is closed. Using §§2.5 and 2.6 we deduce that R is not a complete intersection in all cases in question except $(3,4)$. In the last case one should take $p = z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1$. □

§2.10. Let $G = SL_n(\mathbb{C})$ and V be the direct sum of $m \geq n$ copies of the natural representation of SL_n on $\mathbb{C}^n$. Let $p = (v_1,\dots,v_m) \in V$, $p \neq 0$. Then the orbit $G(p)$ is closed iff $\operatorname{rank}(v_1 \dots v_m) = n$; in this case $G_p = \{e\}$. So all non-trivial slice representations are nice. However, if $m > n + 1$, the point $\pi(0)$ is (the only) singular point of V/G.

In other words, for these representations the slice method does not simplify the problem. In fact the slice principle works best of all for irreducible representations.

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