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# Branching functions for winding subalgebras and tensor products

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## §0. Introduction.

0.1. One of the basic problems of representation theory is to find a decomposition of an irreducible representation of a group with respect to a subgroup. Namely, suppose that we have a representation $\pi$ of a group $G$ in a vector space $V$ and suppose that with respect to a subgroup $S$ this representation decomposes into a direct sum of irreducible representations:

$$\pi = \oplus_i \pi_i, \ V = \oplus_i V_i$$

Given an irreducible representation $\sigma$ of $S$ one denotes by $[\pi : \sigma]$ the number of representations of $S$ equivalent to $\sigma$ in this decomposition, and calls this number a *branching coefficient*. An important problem is to compute the branching coefficients.

A special case of this problem is the decomposition of a tensor product. In this case $G = S \times S$, $S$ is the diagonal subgroup of $G$, $V = V' \otimes V''$, where $(V', \pi')$ and $(V'', \pi'')$ are some irreducible representations of $S$, and the problem is to compute the numbers $[\pi' \otimes \pi'' : \sigma]$.

0.2. In the present paper we study branching coefficients for positive energy representations of affine algebras. Let us recall the basic definitions in the "non-twisted" case ($r = 1$). See [8] for details.

Let $\bar{\mathfrak{g}}$ be a complex simple finite-dimensional Lie algebra of rank $\ell$, and let $\phi(.,.)$ be its Killing form. Fix a triangular decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- + \bar{\mathfrak{h}} + \bar{\mathfrak{n}}_+$, where $\bar{\mathfrak{h}}$ is a Cartan subalgebra and $\bar{\mathfrak{n}}_\pm$ are maximal nilpotent subalgebras, and let $\theta^\vee \in \bar{\mathfrak{h}}$ be the coroot corresponding to the highest root. Let $h^\vee = \phi(\theta^\vee, \theta^\vee)$ be the *dual Coxeter number* and let $(x|y) = \phi(x,y)/2h^\vee$ be the *normalized invariant form* on $\bar{\mathfrak{g}}$. The *affine algebra* $\mathfrak{g}'$ *associated to* $\bar{\mathfrak{g}}$ (called also the affine algebra of type $X_\ell^{(1)}$, where $X_\ell$ is the type of $\bar{\mathfrak{g}}$) is constructed as follows. Let $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in $t$, and let us view the loop algebra $\bar{\mathfrak{g}}[t, t^{-1}] = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \bar{\mathfrak{g}}$ as an (infinite-dimensional) Lie algebra over $\mathbb{C}$. Then

$$\mathfrak{g}' = \bar{\mathfrak{g}}[t, t^{-1}] + \mathbb{C}K$$

is the unique non-trivial central extension of $\bar{\mathfrak{g}}[t, t^{-1}]$ by a 1-dimensional center $\mathbb{C}K$. Explicitly, it can be defined by the following commutation relations:

$$[x(m), y(n)] = [x, y](m+n) + m\delta_{m,-n}(x|y)K,$$

where $x(m) \in \bar{\mathfrak{g}}[t, t^{-1}]$ stands for $t^m \otimes x$ ($m \in \mathbb{Z}$, $x \in \bar{\mathfrak{g}}$). We identify $\bar{\mathfrak{g}}$ with the subalgebra $1 \otimes \bar{\mathfrak{g}}$ of $\mathfrak{g}'$, and let $\mathfrak{h}' = \bar{\mathfrak{h}} + \mathbb{C}K$ be the Cartan subalgebra of $\mathfrak{g}'$. Let also $\mathfrak{n}_\pm = \bar{\mathfrak{n}}_\pm + t^{\pm 1}\bar{\mathfrak{g}}[t^{\pm 1}]$. Then we have the *triangular decomposition* $\mathfrak{g}' = \mathfrak{n}_- + \mathfrak{h}' + \mathfrak{n}_+$.

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An arbitrary affine algebra is a direct sum of the Lie algebras of the form $\mathfrak{g}'$ and their twisted analogues $\mathfrak{g}'(\sigma, s)$ (see below).

It is often convenient to consider $\mathfrak{g}'$ as an ideal of codimension 1 in the Lie algebra $\mathfrak{g} = \mathfrak{g}' + \mathbf{C}d$, with commutation relations

$$[d, x(m)] = mx(m), \ [d, K] = 0,$$

and let $\mathfrak{h} = \mathfrak{h}' + \mathbf{C}d$ be its Cartan subalgebra. One extends the normalized bilinear form $(.|.)$ from $\overline{\mathfrak{h}}$ to $\mathfrak{h}$ by letting $(\overline{\mathfrak{h}}|\mathbf{C}K + \mathbf{C}d) = 0, \ (K|K) = (d|d) = 0, \ (K|d) = 1$.

A representation $\pi$ of the affine algebra $\mathfrak{g}'$ in a vector space $V$ is called a *positive energy* representation if a) $\pi(K) = kI, \ k \in \mathbf{C}$, and b) $\pi$ can be extended to the whole $\mathfrak{g}$ such that $-\pi(d)$ is diagonalizable and its eigenvalues are non–negative integers.

The number $k$ is called the *level* of $V$. The eigenspace decomposition

$$V = \oplus_{n \in \mathbf{Z}_+} V_{(n)}$$

with respect to $-\pi(d)$ is called the *energy decomposition*; if $v \in V_{(n)}$ we say that $v$ has energy $n$.

Since $[d, \overline{\mathfrak{g}}] = 0$, the energy decomposition is $\overline{\mathfrak{g}}$–invariant, and we denote by $\overline{\pi}$ the representation of $\overline{\mathfrak{g}}$ in $V_{(0)}$. It is easy to show that the map $\pi \longmapsto (k, \overline{\pi})$ establishes a bijective correspondence between the set of (equivalence classes of) positive energy irreducible representations $(\pi, V)$ such that $V_{(0)} \neq 0$ of $\mathfrak{g}'$ and the set of pairs $(k, \overline{\pi})$, where $k \in \mathbf{C}$ and $\overline{\pi}$ is an irreducible representation of $\overline{\mathfrak{g}}$ (considered up to equivalence). (Given $k \in \mathbf{C}$ and an irreducible representation $\overline{\pi}$ in $V_{(0)}$, we extend $\overline{\pi}$ to $\mathfrak{g}_+ := \overline{\mathfrak{g}}[t] + \mathbf{C}K + \mathbf{C}d$ by letting $\overline{\pi}(K) = kI_{V_{(0)}}, \ \overline{\pi}(d) = 0$ and $\overline{\pi}(x(n)) = 0$ for $n > 0$, and let $\pi$ be the irreducible quotient of the induced representation of $\mathfrak{g}$ in the space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_+)} V_{(0)}$. This gives us an irreducible positive energy representation $\pi$ of $\mathfrak{g}'$ corresponding to the pair $(k, \overline{\pi})$.)

0.3. We shall consider only the irreducible positive energy representations $(\pi, V)$ of $\mathfrak{g}'$ such that $(\overline{\pi}, V_{(0)})$ is an irreducible highest weight representation of $\overline{\mathfrak{g}}$. In other words, we shall assume that there exists a non–zero vector $v_{k,\overline{\lambda}} \in V_{(0)}$, where $\overline{\lambda} \in \overline{\mathfrak{h}}$, such that

$$\pi(\overline{\mathfrak{n}}_+)v_{k,\overline{\lambda}} = 0, \ \pi(h)v_{k,\overline{\lambda}} = \overline{\lambda}(h)v_{k,\overline{\lambda}} \text{ for } h \in \overline{\mathfrak{h}}.$$

These representations are parameterized by the pairs $(k, \overline{\lambda}), \ k \in \mathbf{R}, \ \overline{\lambda} \in \overline{\mathfrak{h}}^*$. It is more convenient to represent the pair $(k, \overline{\lambda})$ by an element $\lambda \in \mathfrak{h}'^*$ such that $\lambda|_{\overline{\mathfrak{h}}} = \overline{\lambda}$ and $\lambda(K) = k$. The corresponding representation $\pi_\lambda$ of $\mathfrak{g}'$ is denoted by $L(\lambda)$ and is called the *irreducible highest weight representation* of $\mathfrak{g}'$ with *highest weight* $\lambda$.

The vector $v_\lambda := v_{k,\overline{\lambda}}$ is called the highest weight vector; it is the unique up to a non–zero constant factor vector in $L(\lambda)$ satisfying equations

$$\pi_\lambda(\mathfrak{n}_+)v_\lambda = 0, \ \pi_\lambda(h)v_\lambda = \lambda(h)v_\lambda \text{ for } h \in \mathfrak{h}'.$$

Especially important are the representations $L(\lambda)$ of $\mathfrak{g}'$ which can be lifted to a (projective) representation of the corresponding loop group. These are called *integrable* highest weight representation. They are parameterized by the set

$$P_+ = \{\lambda \in \mathfrak{h}'^* | k := \lambda(K) \in \mathbf{Z}_+, \ \overline{\lambda} \in \overline{P}_+, \ (\overline{\lambda}|\theta^\vee) \leq k\}.$$

Here $\overline{P}_+ \subset \overline{\mathfrak{h}}^*$ is the set of highest weights of finite–dimensional irreducible representations of $\overline{\mathfrak{g}}$.

The basic tool for study of integrable $L(\lambda)$ is the so called Weyl–Kac character formula [6] for the function $\mathrm{ch}_\lambda$ on $Y := \{v \in \mathfrak{h} | \mathrm{Re}(v|K) > 0\}$ defined by

$$\mathrm{ch}_\lambda(v) = \mathrm{tr}_{L(\lambda)} e^v.$$

Recently a similar character formula has been established for a larger class of the $L(\lambda)$, those with an "admissible" highest weight [12]. Their "normalized" characters are Jacobi modular forms and, conjecturally, these representations are characterized by this property. (These are also the only ones for which the Kazhdan–Lusztig polynomials are trivial.) In the present paper we consider the best studied (see [13]), principal admissible highest weight representations. Their levels may be arbitrary rational numbers $k$ (called principal admissible) such that $k + h^\vee \geq h^\vee/u$, where $u \in \mathbb{N}$ is the denominator of $k$. In the case when $k \in \mathbb{Z}_+$ all principal admissible representations are integrable (but all representations of fractional level are not).

0.4. A natural class of subalgebras of the Lie algebra $\mathfrak{g}'$ (and similarly of an arbitrary affine algebra) to consider is the following. Let $\dot{\overline{\mathfrak{g}}}$ be a reductive subalgebra of $\overline{\mathfrak{g}}$, let $\sigma$ be an automorphism of $\dot{\overline{g}}$ and let $s \in \mathbb{N}$ be such that $\sigma^s = 1$. Define an automorphism $\tilde{\sigma}$ of the subalgebra $\dot{\overline{\mathfrak{g}}}[t, t^{-1}] + \mathbb{C}K$ of $\mathfrak{g}'$ by letting

$$\tilde{\sigma}(x(n)) = (\exp \frac{2\pi i n}{s})\sigma(x)(n), \ \sigma(K) = K,$$

and denote by $\dot{\mathfrak{g}}'(\sigma, s)$ the fixed point set of $\tilde{\sigma}$.

0.5. We shall consider only the representations $L(\lambda)$ of $\mathfrak{g}'$ which are completely reducible with respect to $\dot{\mathfrak{g}}'(\sigma, s)$. (This is always the case when $\lambda \in P_+$.) The branching coefficients of such $\dot{\mathfrak{g}}'(\sigma, s)$ in a representation $L(\lambda)$ of $\mathfrak{g}'$, i.e. the numbers $[L(\lambda) : \dot{L}(\mu)]$, where $\dot{L}(\mu)$ is an irreducible highest weight representation of $\dot{\mathfrak{g}}(\sigma, s)$, are almost always zero or infinity. To get around this, let $\dot{\mathfrak{g}}(\sigma, s) = \dot{\mathfrak{n}}_- + \dot{\mathfrak{h}} + \dot{\mathfrak{n}}_+$ be the induced triangular decomposition of $\dot{\mathfrak{g}}(\sigma, s)$, i.e. $\dot{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \cap \dot{\mathfrak{g}}(\sigma, s)$ and $\dot{\mathfrak{h}}' = \mathfrak{h}' \cap \dot{\mathfrak{g}}(\sigma, s)$, and let

$$[\lambda : \mu]_n = \dim\{v \in L(\lambda)_{(n)} | \pi_\lambda(\dot{\mathfrak{n}}_+)v = 0, \ \pi_\lambda(h)v = \mu(h)v \text{ for all } h \in \dot{\mathfrak{h}}'\}.$$

These numbers are always finite, and we can consider the power series

$$b_\mu^\lambda = q^{m_{\lambda,\mu}} \sum_{n \in \mathbb{Z}_+} [\lambda : \mu]_n q^n.$$

The number $m_{\lambda,\mu}$ is a rational number called the *modular anomaly* which is defined as follows. Let $q = e^{2\pi i \tau}$; then the above series converges to a holomorphic function in $\tau$ for $\mathrm{Im}\, \tau > 0$. According to [10, Proposition 4.36], provided that $\lambda \in P_+$, there exists a unique number $m_{\lambda,\mu}$ such that the function $b_\mu^\lambda(\tau)$ is a modular function, i.e. is fixed under the action of a principal congruence subgroup $\Gamma(N)$, some $N$. (Explicit formulas for the $m_{\lambda,\mu}$ may be found in [11] and in the paper; of course, these numbers depend also of the $\mathfrak{g}'$ and the subalgebra.)

The functions $b_\mu^\lambda(\tau)$, clearly, completely describe the decomposition of $L(\lambda)$ with respect to the subalgebra in question. They are called *branching functions.* If $[L(\lambda):\dot{L}(\mu)] < \infty$, then, of course, $[L(\lambda):\dot{L}(\mu)] = \lim_{\tau\downarrow 0} b_\mu^\lambda(\tau)$. In general, we study the asymptotics of $b_\mu^\lambda(\tau)$ as $\tau \downarrow 0$ instead. Namely, since the $b_\mu^\lambda(\tau)$ are modular functions, we have as $\tau \downarrow 0$, provided that $b_\mu^\lambda \neq 0$:

$$b_\mu^\lambda(\tau) \sim a(\lambda,\mu) e^{\frac{\pi i}{12\tau} g(\lambda,\mu)},$$

where $a(\lambda,\mu) > 0$ and $g(\lambda,\mu) \geq 0$ are real numbers called the *asymptotic dimension* and *growth* of the branching function. Here and in the rest of the paper $f(\tau) \sim g(\tau)$ means that $\lim_{\tau\downarrow 0} f(\tau)/g(\tau) = 1$.

In all known examples the growth depends only on the algebra $\mathfrak{g}'$, the subalgebra and the level $k$ of $\lambda$, but this is an open problem, which we will refer to as the basic conjecture, even in the case of integrable $L(\lambda)$ and the subalgebra $\dot{\mathfrak{g}}(1,1)$ (cf. [11]). Incidentally, the knowledge of the above asymptotics allows one to compute the asymptotics of the branching coefficients $[\lambda:\mu]_n$ as $n \to \infty$ by making use of a Tauberian theorem (see [10] and [11]).

0.6. An important version of a special case of branching functions are *string functions* $c_\mu^\lambda$, which correspond to $\dot{\overline{\mathfrak{g}}} = \overline{\mathfrak{h}}$, $\sigma = 1$, $s = 1$, i.e. which describe the multiplicities of weights of $L(\lambda)$. Namely, given $\lambda, \mu \in \mathfrak{h}'^*$ of level $k$, we define as usual, the weight space $L(\lambda)_\mu = \{v \in L(\lambda) | h(v) = \mu(h)v \text{ for all } h \in \mathfrak{h}'\}$, and let

$$c_\mu^\lambda = q^{m_{\lambda,\mu}} \sum_{n\in\mathbb{Z}_+} (\dim L(\lambda)_\mu \cap L(\lambda)_{(n)}) q^n,$$

where we let $m_\lambda = |\overline{\lambda} + \overline{\rho}|^2/2(k+h^\vee) - |\overline{\rho}|^2/2h^\vee$ to be the modular anomaly of $\lambda$ (see below) and $m_{\lambda,\mu} = m_\lambda - |\overline{\mu}|^2/2k$ is the modular anomaly in this case (as usual, $\overline{\rho}$ is the half-sum of positive roots for $\overline{\mathfrak{g}}$). This is a modular form of weight $-\ell/2$, and it is related to the corresponding branching function by the equation

$$c_\mu^\lambda(\tau) = G(\tau)^{-1} b_\mu^\lambda(\tau),$$

where $G(\tau)$ is a modular form of weight $\ell/2$ given in §2.2.

String functions for integrable $L(\lambda)$ were studied in great detail in [9], [10], [11] (see also [8, Chapters 12 and 13]). The key result of this work is an explicit transformation formula for the normalized characters

$$\chi_\lambda := q^{m_\lambda} \mathrm{ch}_\lambda, \ \lambda \in P_+,$$

under the action of the involution $S: \tau \longmapsto -1/\tau$ (defined in §4.3). Here we identify $q$ with the function $e^{-K}(v) = e^{-(K|v)}$ on $\mathfrak{h}$. (This result was extended in [13, Theorem 3.6] to the case of principal admissible $L(\lambda)$; see formula (4.3.1) of the present paper.) One deduces from this result an explicit transformation formula for string functions under the involution $S$ [10, Theorem A]. Since the transformation formula together with the polar parts of the $q$-expansions completely determine modular forms, this allows one to compute the string functions explicitly in many interesting cases. Furthermore, it turns out that the asymptotics of the string functions $c^\lambda_\mu$ is independent of $\mu$, which allows one to find this asymptotics explicitly ([10, Proposition 4.21] or [8, Chapter 13],) proving thereby the validity of the basic conjecture in this case. (Incidentally, for admissible $\lambda$ the string functions $c^\lambda_\mu(\tau)$ fail to be modular forms [14].)

0.7. General branching functions in the case of the subalgebras of the form $\dot{\mathfrak{g}}'(1,1) = \dot{\overline{\mathfrak{g}}}[t,t^{-1}] \oplus \mathbb{C}K$ of $\mathfrak{g}'$ and integrable $L(\lambda)$ were studied in detail in [11]. Again from the tranformation formula for the normalized characters one deduces the transformation law for these branching functions under the involution $S$ [11, Theorem A]. One derives from this definitive results on asymptotics (which prove the validity of the basic conjecture) only in the case of tensor products [11, (2.7.15) and §3.4]. For general branching functions we derive the basic conjecture from the conjectural positivity of certain matrix elements of the transformations $S$ in the basis of branching functions [11, p. 188].

0.8. In the present paper we consider the subalgebras

$$\mathfrak{g}'_{[u]} := \mathfrak{g}'(1,u) = \overline{\mathfrak{g}}[t^u, t^{-u}] \oplus \mathbb{C}K$$

of $\mathfrak{g}'$, called the *winding* subalgebras. (This is the simplest case different from $\dot{\mathfrak{g}}'(1,1)$.) The first basic result of the paper is Theorem 2.1 which gives an explicit expression of the branching functions $b^\Lambda_\lambda$ for winding subalgebras in integrable highest weight representations $L(\Lambda)$ in terms of string functions. The special case of this theorem, when the level of $L(\Lambda)$ is 1, is Theorem 2.2, which gives a solution to Frenkel's conjecture [4]. Theorem 2.1 leads also to Conjecture 2.2 on the asymptotics of the branching functions for winding subalgebras. We were able to prove only that it holds for all sufficiently large $u$.

0.9. Next we compare branching functions $b^\Lambda_\lambda$ for winding subalgebras with branching functions $b^{\Lambda\otimes\mu}_\lambda$ for tensor products $L(\Lambda) \otimes L(\mu)$. It is clear from [13, Corollary 4.1 and Theorem 3.6] that the branching functions $b^{\Lambda\otimes\mu}_\lambda$ are modular functions provided that $\Lambda$ is integrable and $\mu$ is admissible. This is the case studied in the present paper. As in the case of winding subalgebras, we find an explicit expression of the $b^{\Lambda\otimes\mu}_\lambda$ in terms of string functions (Theorem 3.1). Comparing Theorems 2.1 and 3.1, we see that a branching function $b^\Lambda_\lambda$ for a winding subalgebra $\mathfrak{g}'_{[u]}$ coincides with a branching function for tensor product $L(\Lambda) \otimes L(\mu)$, where $\mu = ((u^{-1} - 1)h^\vee, 0)$, provided that $u$ is relatively prime to $h^\vee$ and $r^\vee$ (Proposition 3.2). This (still mysterious) coincidence indicates a remarkable interplay between the integrable and admissible representations. As in the case of winding subalgebras, Theorem 3.1 leads to Conjecture 3.1 on asymptotics of the $b^{\Lambda\otimes\mu}_\lambda$, generalizing the known result in the integrable case. After this paper was completed we received preprint [1] where Theorem 3.1 is derived in the integrable case using a free field resolution.

0.10. The remarkable feature of the theory of integrable, and, more generally, principal admissible highest weight representations is the $SL_2(\mathbb{Z})$-invariance of the $\mathbb{C}$-span of normalized characters (in the twisted case, $SL_2(\mathbb{Z})$ should be replaced by a slightly smaller subgroup $\Gamma$; see Proposition 4.3), hence the $SL_2(\mathbb{Z})$-invariance of the $\mathbb{C}$-span of branching functions $b^{\Lambda\otimes\mu}_\lambda$, where $\Lambda$ (resp. $\mu$) runs over all integrable (resp. principal admissible) weights of fixed level. (A similar result holds for arbitrary subalgebras of the form $\dot{\mathfrak{g}}'(1,1)$, see [11]. This follows from the $S$-invariance of normalized characters.) Due to the above coincidence, this is the case also for the branching functions for winding subalgebras $\mathfrak{g}'_{[u]}$

provided that $u$ is relatively prime to $h^\vee$ and $r^\vee$. However, in the general case we have only the $\Gamma_0(u)$–invariance (see §4.3). This is a general feature of the subalgebras $\dot{\mathfrak{g}}'(\sigma, s)$.

0.11. The last, Section 4, contains some preparatory material for our forthcoming paper with E. Frenkel [3]. It deals with functions $\varphi_{\lambda,\mu}$, which are branching functions for tensor products of the level 1 integrable representations with arbitrary principal admissible representations (see (4.1.1) and Theorem 4.1). These functions previously appeared in this context in [12, Proposition 3].

It turns out that the functions $\varphi_{\lambda,\mu}$ can be obtained by a simple limiting procedure from the characters of the principal admissible representations (Proposition 4.2). As will be explained in [3] this procedure naturally appears in the quantization of the Drinfeld–Sokolov reduction developed in [2]. As a result, one obtains that the functions $\varphi_{\lambda,\mu}$ are characters of the so called extended conformal algebras, which are higher rank generalizations of the Virasoro algebra (cf. [15]). In the particular case of $\overline{\mathfrak{g}} = s\ell_2(\mathbf{C})$, this procedure is equivalent to taking the residue of the admissible characters. As was shown previously in [16], this reproduces the Virasoro characters. It is worth mentioning that the limiting procedure gives a non–zero result only for "non–degenerate" principal admissible weights; in particular, the integrable characters always give zero.

The main result of this last section is Theorem 4.4 which give a transformation formula for the $\varphi_{\lambda,\mu}$ under the action of $S$, obtained from the limiting procedure (which is simpler than that obtained from tensor products). This formula will be applied in [3] to calculate the fusion rules for the extended conformal algebras.

Thus Theorem 4.1b means the coincidence of two theories of extended conformal algebras at least on the character level in the simply laced or twisted case. (In the case of $B_\ell^{(1)}$ they are different, as can be seen by comparing Theorem 2.2′ and Proposition 4.2.)

Note that though the set of principal admissible representations of given fractional level carries quite a few features of a conformal field theory (like modular invariance, the unique vacuum, the involution), it can't be a conformal field theory since, for example, its fusion rules computed by Verlinde's formula [17] may be negative. This makes it quite remarkable that a "reduction" of this theory indeed produces a conformal field theory.

0.12. We would like to thank E. Getzler and M. Hopkins who pointed out that the study of branching functions for winding subalgebras may be important for the theory of cohomological operations in the elliptic cohomology, which stimulated our research. We thank E. Frenkel for his patient explanations of his (joint with Feigin) work [2] and for collaboration in Section 4. We thank D. Jerison for consultations on asymptotics.

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## §1. Notation and preliminaries.

1.1. Let $I = \{0, 1, \dots, \ell\}$, $\ell \geq 1$. Recall that an *affine matrix* is a square matrix $A = (a_{ij})_{i,j \in I}$ such that $a_{ii} = 2$, $-a_{ij} \in \mathbf{Z}_+$ for $i \neq j$, $a_{ij} = 0$ implies $a_{ji} = 0$, and there exists a unique sequence $(a_0, \dots, a_\ell)$ of positive relatively prime integers, called the *null-vector* of $A$, such that $(a_0, \dots, a_\ell)({}^tA) = 0$.

Two affine matrices are called *equivalent* if they are obtained from each other by a reordering of $I$. A complete list of affine matrices up to equivalence is given in Tables Aff r of [8, Chapter 4]. Affine matrices in these tables are denoted by their type $X_N^{(r)}$, where $X_N$ is the type of the "underlying" simple finite–dimensional Lie algebra and $r$ is the so–called *tier number*. The number $h = \sum_{i \in I} a_i$ is called the *Coxeter number* of $A$.

Note that the transposed matrix ${}^tA$ is again affine. The corresponding null–vector is denoted by $(a_0^\vee, \dots, a_\ell^\vee)$, the tier number by $r^\vee$ and the Coxeter number by $h^\vee$. The latter is called the *dual Coxeter number* of $A$.

Let $\varepsilon_j = a_j^\vee a_j^{-1}$; then the matrix $(\varepsilon_i a_{ij})_{i,j \in I}$ is symmetric.

We keep the ordering of $I$ given in [8, Chapter 4]. For this ordering we have in particular: $a_0^\vee = 1$ and $a_0 = 1$ (resp. $a_0 = 2$) if $A \neq A_{2\ell}^{(2)}$ (resp. $A = A_{2\ell}^{(2)}$).

In what follows we fix an affine matrix $A$ of type $X_N^{(r)}$.

1.2. Let $\mathfrak{h}'$ be the $\ell+1$–dimensional vector space over $\mathbf{C}$ with a basis $\Pi = \{\alpha_0, \dots, \alpha_\ell\}$ and a symmetric bilinear form defined in this basis by the following formula [8, Chapter 6]

$$(1.2.1) \qquad (\alpha_i | \alpha_j) = \varepsilon_i a_{ij}, \ i, j \in I.$$

Then $\Pi$ is called a *root basis* and $(.|.)$ is called the *normalized invariant form* (for the matrix $A$). The basis $\Pi^\vee = \{\alpha_0^\vee, \dots, \alpha_\ell^\vee\}$ defined by $a_i \alpha_i = a_i^\vee \alpha_i^\vee$ is called a *coroot basis.* Note that the kernel of the normalized invariant form consists of all multiples of the element

$$(1.2.2) \qquad K = \sum_{i \in I} a_i \alpha_i = \sum_{i \in I} a_i^\vee \alpha_i^\vee,$$

called the *canonical central element.*

Given a subset $L$ of $\mathfrak{h}'$, we let $\mathbf{Z}L$ (resp. $\mathbf{Z}_+ L$, $\mathbf{C}L$, etc.) denote the set of all linear combinations of elements from $L$ with coefficients from $\mathbf{Z}$ (resp. $\mathbf{Z}_+$, $\mathbf{C}$, etc.). We also let

$$L^* = \{\alpha \in \mathbf{C}L | (\alpha | L) \subset \mathbf{Z}\}.$$

The lattices $Q = \mathbf{Z}\Pi$ and $Q^\vee = \mathbf{Z}\Pi^\vee$ are called the *root* and *coroot* lattices respectively. We also let $Q_+ = \mathbf{Z}_+\Pi$, $Q_+^\vee = \mathbf{Z}_+\Pi^\vee$.

We let $\overline{I} = \{1, \dots, \ell\} \subset I$, $\overline{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$, $\overline{\Pi}^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}, \overline{Q} = \mathbf{Z}\overline{\Pi}$, $\overline{Q}^\vee = \mathbf{Z}\overline{\Pi}^\vee, \overline{\mathfrak{h}} = \mathbf{C}\overline{\Pi}$. One has [8, Chapter 6]:

$$\overline{Q} \supset \overline{Q}^\vee \text{ if } r = 1; \ \overline{Q} \subset \overline{Q}^\vee \text{ if } r > 1.$$

Define the following important lattices $M$ and $\tilde{M}$ by

$$M = \overline{Q} \text{ if } r^\vee = 1; \ M = \overline{Q}^\vee \text{ if } r^\vee > 1.$$
$$\tilde{M} = \overline{Q}^* \text{ if } r = 1; \ \tilde{M} = \overline{Q}^{\vee *} \text{ if } r > 1.$$

Since $(\overline{Q}|\overline{Q}^\vee) \subset \mathbf{Z}$, we have $M \subset \tilde{M} \subset M^*$. We also have:

(1.2.3) $$|M^*/\tilde{M}| \text{ (resp. } |\tilde{M}/M|) \text{ are the same for } A \text{ and } {}^tA,$$

(1.2.4) $$|M^*/\tilde{M}| = |\overline{Q}^\vee/M| \text{ if } r > 1.$$

1.3. Let $r_i \in \operatorname{Aut}\mathfrak{h}'$ be the *fundamental reflections*, i.e.

$$r_i(v) = v - (\alpha_i|v)\alpha_i^\vee,$$

and let $W = \langle r_i | i \in I \rangle$ be the Weyl group. Note that $Q$ and $Q^\vee$ are $W$-invariant.

Let $\Delta^{re} = W(\Pi)$, and $\Delta^{\vee re} = W(\Pi^\vee)$ be the sets of *real roots* and *real coroots* respectively. Let $\Delta_+^{re} = \Delta^{re} \cap Q_+$, $\Delta_+^{\vee re} = \Delta^{\vee re} \cap Q_+^\vee$. We also let $\overline{\Delta} = \Delta^{re} \cap \overline{Q}$, $\overline{\Delta}^\vee = \Delta^{\vee re} \cap \overline{Q}^\vee$, $\overline{\Delta}_+ = \Delta_+^{re} \cap \overline{Q}$, etc.

If $\alpha = w(\alpha_i) \in \Delta^{re}$, then $\alpha^\vee = w(\alpha_i^\vee) \in \Delta^{\vee re}$ is well-defined [8, Chapter 3] and $\alpha^\vee = 2\alpha/(\alpha|\alpha)$; letting $r_\alpha = wr_iw^{-1}$, we have:

(1.3.1) $$r_\alpha(v) = v - (\alpha|v)\alpha^\vee, \ v \in \mathfrak{h}'.$$

We have the following homomorphism $\alpha \longmapsto t_\alpha$ of $\mathfrak{h}'$ into $\operatorname{Aut}\mathfrak{h}'$ with kernel $\mathbf{C}K$ [8, Chapter 6]:

(1.3.2) $$t_\alpha(v) = v + (v|K)\alpha - ((v|\alpha) + \tfrac{1}{2}(\alpha|\alpha)(v|K))K.$$

For a subset $L$ of $\mathfrak{h}'$ we let $t_L = \{t_\alpha | \alpha \in L\}$.

We have [8, Chapter 6]:

(1.3.3) $$W = t_M \rtimes \overline{W},$$

where $\overline{W} = \langle r_i | i \in \overline{I} \rangle$ is a finite subgroup of $W$ (we shall write: $w = t_\alpha \overline{w}$).

The sets of real roots and coroots are invariant with respect to the group $\tilde{W} := t_{\tilde{M}} \rtimes \overline{W}$ containing $W$. Let $\tilde{W}_+ = \{w \in \tilde{W} | w(\Pi) = \Pi\}$, and let

$$J = \{j \in I | a_j = 1 \text{ (resp. } a_j^\vee = 1) \text{ if } r = 1 \text{ (resp. } r > 1)\}.$$

Then [10, §4.8] the group $\tilde{W}_+$ is isomorphic to $\tilde{M}/M$, acts simply transitively on $\{\alpha_i \in \Pi | i \in J\}$, and one has:

(1.3.4) $$\tilde{W} = \tilde{W}_+ \ltimes W.$$

The group $\tilde{W}_+$ can be described more explicitly as follows. Let $\overline{W}_+$ be the subgroup of $\overline{W}$ consisting of the elements preserving the subset $\{\overline{\alpha}_0 := a_0\alpha_0 - K, \overline{\alpha}_1 := \alpha_1, \ldots, \overline{\alpha}_\ell := \alpha_\ell\}$

of $\overline{Q}$. This group acts simply transitively on the set $\{\overline{\alpha}_j | j \in J\}$. Denote by $\overline{w}_j$ the element of $\overline{W}_+$ such that $\overline{w}_j(\overline{\alpha}_0) = \overline{\alpha}_j$. Then

$$w_j := t_{\Lambda_j - \Lambda_0} \overline{w}_j$$

is the element of $\tilde{W}_+$ such that $w_j(\alpha_0) = \alpha_j$.

Note also the following useful formulas:

$$|\tilde{M}/M| = |J|, \tag{1.3.5}$$

$$|M^*/M| = |J| \text{ if } r = r^\vee = 1 \text{ or } a_0 = 2. \tag{1.3.6}$$

1.4. Define the *fundamental weights* (resp. *fundamental coweights*) $\Lambda_i \in \mathfrak{h}'^*$ (resp. $\Lambda_i^\vee \in \mathfrak{h}'^*$), $i \in I$, by:

$$< \Lambda_i, \alpha_j^\vee > \text{(resp. } < \Lambda_i^\vee, \alpha_j >) = \delta_{ij},\ j \in I.$$

Let $P = \sum_{i \in I} \mathbf{Z}\Lambda_i$, $P^\vee = \sum_{i \in I} \mathbf{Z}\Lambda_i^\vee$ be the sets of integral weights and coweights and $P_+ = \sum_{i \in I} \mathbf{Z}_+\Lambda_i$, $P_+^\vee = \sum_{i \in I} \mathbf{Z}_+\Lambda_i^\vee$ the sets of dominant integral weights. Let

$$\rho = \sum_{i \in I} \Lambda_i,\ \rho^\vee = \sum_{i \in I} \Lambda_i^\vee.$$

Note that

$$< \rho^\vee, K >= h,\ < \rho, K >= h^\vee. \tag{1.4.1}$$

The number $k =< \lambda, K >$ is called the *level* of $\lambda \in \mathfrak{h}'^*$; the set of all $\lambda$ of level $k$ is denoted by $\mathfrak{h}'^{*k}$. The following map $\tau_k$ sends $\mathfrak{h}'^*$ to $\mathfrak{h}'^{*k}$ (since $< \Lambda_0, K >= 1$):

$$\tau_k(\lambda) = \lambda - (< \lambda, K > -k)\Lambda_0. \tag{1.4.2}$$

Fix $u \in \mathbf{N}$ relatively prime to $a_0$. It is easy to see from [8, Chapter 6] that the element $\dot{\alpha}_0 := a_0^{-1}(u-1)K + \alpha_0$ lies in $\Delta_+^{re}$ and the element $\dot{\alpha}_0^\vee := (u-1)K + \alpha_0^\vee$ lies in $\Delta_+^{\vee re}$. We let

$$\begin{aligned}
\Pi_{[u]} &= \{\dot{\alpha}_0, \dot{\alpha}_i := \alpha_i\ (i \in \overline{I})\}, \\
\Pi_{[u]}^\vee &= \{\dot{\alpha}_0^\vee, \dot{\alpha}_i^\vee := \alpha_i^\vee\ (i \in \overline{I})\}, \\
W_{[u]} &=< r_{\dot{\alpha}_i} | i \in I > .
\end{aligned}$$

Then $\Pi_{[u]}$ and $\Pi_{[u]}^\vee$ are root and coroot bases with the same normalized bilinear form as for $\Pi$ and $\Pi^\vee$. We shall indicate the objects associated to $\Pi_{[u]}$ by an overdot. For example, the canonical central element for $\Pi_{[u]}$ is

$$\dot{K} := \sum_{i \in I} a_i \dot{\alpha}_i^\vee = uK, \tag{1.4.3}$$

the fundamental weights are

$$\dot{\Lambda}_i = \Lambda_i + (u^{-1} - 1) a_i^\vee \Lambda_0, \tag{1.4.4}$$

hence

$$\dot{\rho} := \sum_{i \in I} \dot{\Lambda}_i = \rho + (u^{-1} - 1) h^\vee \Lambda_0, \ |\dot{\rho}|^2 = |\rho|^2, \tag{1.4.5}$$

$$\dot{t}_\alpha = t_{u\alpha}, \tag{1.4.6}$$

$$\dot{W} = W_{[u]} = t_{uM} \rtimes \overline{W}. \tag{1.4.7}$$

The following lemma, the proof of which is straightforward, will be useful in the sequel.

LEMMA 1.4. *If $y \in \tilde{W}$, $\gamma \in \overline{\mathfrak{h}}^*$ and $w \in yW_{[u]}y^{-1}$, so that $y = t_\beta \overline{y}$ for some $\beta \in \tilde{M}$ and $y \in \overline{W}$, and $w = yt_{u\alpha}\overline{w}y^{-1}$ for some $\alpha \in M$ and $\overline{w} \in \overline{W}$, then*

$$wyt_\gamma y^{-1} w^{-1} = t_{\overline{w}\ \overline{y}(\gamma)}. \qquad \square$$

1.5. Let $k \in \mathbf{Q}$ and let $u \in \mathbf{N}$ be the denominator of $k$ (i.e. $ku \in \mathbf{Z}$ and $(ku, u) = 1$). We say that $k$ is *principal admissible* if

$$u(k + h^\vee) \geq h^\vee \ \text{ and } \ (u, r^\vee) = 1. \tag{1.5.1}$$

Given $k \in \mathbf{Z}_+$ we let $P_+^k = P_+ \cap \mathfrak{h}'^{*k}$. Given a principal admissible $k$, we let $P_+^k = \tau_k(P^{u(k+h^\vee)-h^\vee})$. (Note that when $k \in \mathbf{Z}_+$, the two definitions of $P_+^k$ coincide.)

Given $\lambda \in \mathfrak{h}'^*$, we let

$$R^\lambda = \{\alpha \in \Delta^{\vee re} | < \lambda + \rho, \alpha > \in \mathbf{Z}\}, \ R_+^\lambda = R^\lambda \cap \Delta_+^{\vee re},$$

and denote by $S^\lambda$ the set of $\alpha \in R_+^\lambda$ which do not decompose into a sum of several elements from $R_+^\lambda$. Let $W^\lambda =< r_\alpha | \alpha \in R_+^\lambda >$; one can show that $W^\lambda =< r_\alpha | \alpha \in S^\lambda >$.

We call $\lambda \in \mathfrak{h}'^*$ an *admissible weight* if it satisfies the following two properties:

$$- < \lambda + \rho, \alpha > \notin \mathbf{Z}_+ \text{ for all } \alpha \in \Delta_+^{\vee re}, \tag{1.5.2}$$

$$\mathbf{Q}R^{\lambda} = \mathbf{Q}\Pi^{\vee}. \tag{1.5.3}$$

A weight $\lambda$ is called ***principal admissible*** if in addition to (1.5.2 and 3) we have:

$$\text{the matrix } (2(\alpha|\beta)/(\beta|\beta))_{\alpha,\beta\in S^{\lambda}} \text{ is equivalent to } A. \tag{1.5.4}$$

Note that all dominant integral weights are principal admissible.

We recall now the description of the set of all principal admissible weights. Given $y \in \tilde{W}$, denote by $P_{u,y}$ the set of all admissible $\lambda$ such that $S^{\lambda} = y(\Pi^{\vee}_{[u]})$; let $P^k_{u,y} = P_{u,y} \cap \mathfrak{h}'^{*k}$. Denote by $P^k(A)$ the set of all principal admissible weights of level $k$. Finally, recall the shifted action of $\tilde{W}$:

$$w.\lambda = w(\lambda + \rho) - \rho.$$

PROPOSITION 1.5. *[13, Theorem 2.1 and Proposition 2.1].*
*(a)* $P^k_{u,y} \neq \emptyset$ *if and only if*

$$k \textit{ is principal admissible and } u \textit{ is the denominator of } k, \tag{1.5.5}$$

$$y(\Pi^{\vee}_{[u]}) \subset \Delta^{\vee re}_{+}. \tag{1.5.6}$$

*(b) If* $(k, u, y)$ *and* $(k, u, y')$ *are two triples satisfying (1.5.5 and 6), then the following statements are equivalent:*
*(i)* $P^k_{u,y} \cap P^k_{u,y'} \neq \emptyset$,
*(ii)* $P^k_{u,y} = P^k_{u,y'}$,
*(iii)* $y(\Pi^{\vee}_{[u]}) = y'(\Pi^{\vee}_{[u]})$,.
*(iv) there exists* $\sigma = t_{\alpha}\overline{\sigma} \in \tilde{W}_{+}$ *such that* $y' = y t_{u\alpha}\overline{\sigma}$.

*(c) If (1.5.5 and 6) hold, then*

$$P^k_{u,y} = y.P^k_{+} = \{y.(\lambda - (u-1)(k+h^{\vee})\Lambda_0) | \lambda \in P^{u(k+h^{\vee})-h^{\vee}}_{+}\}.$$

*(d)* $P^k(A) \neq \emptyset$ *if and only if* $k$ *is principal admissible.*
*(e)* $P^k(A) = \cup_y P^k_{u,y}$, *where* $u$ *is the denominator of* $k$ *and* $y$ *satisfies (1.5.6).* □

PROOF: We shall prove that (iii) is equivalent to (iv). The rest of the statements are proved in [13]. Indeed, (iii) is equivalent to $y^{-1}y' \in \dot{\tilde{W}}_{+}$. But $t_{\alpha}\overline{\sigma} \in \tilde{W}_{+}$ if and only if $t_{u\alpha}\overline{\sigma} \in \dot{\tilde{W}}_{+}$. □

Remark 1.5. (a) If $k \in \mathbf{Z}_{+}$, then $P^k(A) = P^k_{+}$.

(b) Admissible weights are classified completely in [13]. In the case $A = A_\ell^{(1)}$ this is precisely the set $P(A)$ of all principal admissible weights. For all other affine matrices there are admissible weights which are not principal admissible, and their levels need not be principal admissible.

For $\lambda \in P_{u,y}^k$, let $\lambda^0 = \tau_k^{-1}(y^{-1}.\lambda)$, i.e.

$$\lambda = y.(\lambda^0 - (u-1)(k+h^\vee)\Lambda_0). \tag{1.5.7}$$

The map $\lambda \longmapsto \lambda^0$ defines a bijective correspondence between $P_{u,y}^k$ and $P_+^{u(k+h^\vee)-h^\vee}$.

Note that

$$W^\lambda = yW_{[u]}y^{-1} \text{ for } \lambda \in P_{u,y}^k. \tag{1.5.8}$$

The set $P^k(A)$ admits an important involution $\lambda \longmapsto {}^t\lambda$ defined as follows. Let $\lambda \in P_{u,y}^k$, $y = t_\beta \overline{y}$, $\beta \in \tilde{M}$, $\overline{y} \in \overline{W}$. First, note that $\overline{W}^\lambda := W^\lambda \cap \overline{W}$ is the Weyl group for the finite root system $\overline{\Delta}^\lambda := \overline{\Delta}^\vee \cap R^\lambda$. Denote by $\overline{w}^\lambda$ the longest element in $\overline{W}^\lambda$ (so that $\overline{w}^0$ is the longest element in $\overline{W}$). Recall that we have: $\overline{w}^\lambda\overline{\Delta}_+^\lambda = -\overline{\Delta}_+^\lambda$ and $(\overline{w}^\lambda)^2 = 1$. Define $w^\lambda \in \mathrm{Aut}\,\mathfrak{h}$ by

$$w^\lambda(v) = -\overline{w}^\lambda(v) \text{ if } v \in \overline{\mathfrak{h}},\ w^\lambda(K) = K,\ w^\lambda(d) = d. \tag{1.5.9}$$

Note that $w^\lambda \Delta_+^{\vee re} = \Delta_+^{\vee re}$. If $\Lambda \in P_+^m$ ($m \in \mathbf{Z}_+$) is a dominant integral weight, we let

$${}^t\Lambda = w^0(\Lambda) (= -\overline{w}^0(\Lambda) + 2m\Lambda_0 \in P_+^m). \tag{1.5.10}$$

Let

$${}^t\overline{y} = \overline{y}\overline{w}^\lambda\overline{w}^0,\ {}^ty = t_{-\beta}\,{}^t\overline{y}. \tag{1.5.11}$$

Finally, we let

$${}^t\lambda = {}^ty.({}^t(\lambda^0) - (u-1)(k+h^\vee)\Lambda_0). \tag{1.5.12}$$

It is easy to check that this is an involution that maps $P_{u,y}^k$ onto $P_{u,{}^ty}^k$, and that more invariantly this involution can be defined by

$${}^t\lambda = w^\lambda(\lambda+\rho) - \rho. \tag{1.5.13}$$

We clearly have

$$R^{{}^t\lambda} = w^\lambda(R^\lambda). \tag{1.5.14}$$

The following lemma will be used later.

LEMMA 1.5. *Let $y = t_\beta \overline{y} \in \tilde{W}$, where $\beta \in \tilde{M}$, $y \in \overline{W}$, and let $u \in \mathbb{N}$ be such that*

$$y(\Pi^\vee_{[u]}) \subset \Delta^{\vee re}_{+} \backslash \overline{\Delta}^\vee_{+}. \tag{1.5.15}$$

*Define $\omega_{u,y}$ by*

(1.5.16) $\omega_{u,y} + \rho (resp.\ + \rho^\vee) = u\Lambda_0 - \overline{y}^{-1}(\beta)$ *if* $r = r^\vee = 1$ *or* $r > 1$ *(resp. if* $r = 1,\ r^\vee > 1$*).*

*Then $\omega_{u,y} \in P_+^{u-h^\vee}$ (resp. $\in P_+^{\vee u-h}$).*

PROOF: We shall consider the case $r = r^\vee = 1$ or $r > 1$. The proof in the case $r = 1$, $r^\vee > 1$ is similar. It is clear that $\omega_{u,y} \in P^{u-h^\vee}$, so one has to show that $(\omega_{u,y}|\alpha_i^\vee) \geq 0$. For $i > 0$ we have: $0 < y(\dot{\alpha}_i^\vee) = \overline{y}\alpha_i^\vee - (\overline{y}\alpha_i^\vee|\beta)K$, hence $(\alpha_i^\vee|\overline{y}^{-1}\beta) < 0$ and $(\omega_{u,y}|\alpha_i^\vee) \geq 0$. Finally, $0 < y(\dot{\alpha}_0^\vee) = (u - 1 - (\alpha_0^\vee|\overline{y}^{-1}\beta))K + \overline{y}\alpha_0^\vee$ implies that $(u\Lambda_0 - \overline{y}^{-1}\beta|\alpha_0^\vee) \geq 0$. In the case of equality, $y(\dot{\alpha}_0^\vee) = \overline{y}(\alpha_0^\vee) - K \in \overline{\Delta}^\vee_{+}$, which contradicts (1.5.15). □

1.6. Let $\mathfrak{g}' = \mathfrak{g}(A)$ be the affine Kac–Moody algebra associated to an affine matrix $A$ of type $X_N^{(r)}$ and let $\mathfrak{h}$ be its Cartan subalgebra. Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra. Then $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ is the space introduced in §1.2. The bilinear form $(.|.)$ extends from $\mathfrak{h}'$ to the whole $\mathfrak{g}(A)$ to a non–degenerate bilinear invariant symmetric form $(.|.)$ (not in a unique way). We pick an element $d \in \mathfrak{h}$, called the *scaling element*, such that

$$(d|\alpha_j) = \delta_{0j},\ j \in I,\ (d|d) = 0.$$

Then $\mathfrak{h} = \mathfrak{h}' + \mathbb{C}d$. The space $\mathfrak{h}'^*$ will be identified with a subspace of $\mathfrak{h}^*$ via extending a linear function $\lambda$ on $\mathfrak{h}'$ to $\mathfrak{h}$ by $<\lambda, d> = 0$. Since the restriction of the bilinear form $(.|.)$ to $\mathfrak{h}$ is non–degenerate, it induces an isomorphism $\mathfrak{h} \tilde{\rightarrow} \mathfrak{h}^*$, and we shall identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via this isomorphism. Note that $d$ is then identified with $a_0\Lambda_0$. The action of the groups $W$ and $\tilde{W}$ extends from $\mathfrak{h}'$ to $\mathfrak{h}$ by (1.3.1 and 2), and the bilinear form $(.|.)$ is $\tilde{W}$–invariant.

Introduce the following domains in $\mathfrak{h}$:

$$Y = \{v \in \mathfrak{h} |\ \mathrm{Re}(K|v) > 0\},$$
$$D = \{v \in \mathfrak{h} |\ \mathrm{Re}(\alpha_i|v) > 0,\ i \in I\}.$$

Note that $D \subset Y$ and recall that $\overline{D}$, the closure of $D$ in $Y$ in metric topology, is a fundamental domain for $W$ in $Y$ (see [8, Chapter 3]).

Given $\lambda \in \mathfrak{h}$, we denote by $e^\lambda$ the function on $\mathfrak{h}$ defined by $e^\lambda(v) = e^{(\lambda|v)}$. The function $e^{-K}$ will usually be denoted by $q$.

Define a holomorphic function $R$ on $Y$ by

$$R = e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}\ \alpha}.$$

Here $\Delta_+ = \Delta_+^{re} \cup \mathbb{N}K$, mult $\alpha = 1$ if $\alpha \in \Delta_+^{re}$, mult $rjK = \ell$, mult $jK = (N - \ell)/(r - 1)$ if $r > 1$ and $j \not\equiv 0 \bmod r$ (see [8, Chapters 6 and 8] for details).

Given $\Lambda \in \mathfrak{h}^*$, let $L(\Lambda)$ denote the irreducible highest weight module over $\mathfrak{g}(A)$ with highest weight $\Lambda$ (see the Introduction). With respect to $d$ we have the *energy decomposition:*

$$L(\Lambda) = \oplus_{j \in \mathbf{Z}_+} L(\Lambda)_{(\Lambda(d)-j)},$$

where $L(\Lambda)_{(a)}$ is the $a$–eigenspace of $d$ in $L(\Lambda)$. We shall say that elements from $L(\Lambda)_{(\Lambda(d)-j)}$ have *energy* $j$. Let

$$\mathrm{ch}_\Lambda v = \mathrm{tr}_{L(\Lambda)} e^v := \sum_{\lambda \in \mathfrak{h}^*} \mathrm{mult}_\Lambda(\lambda) e^{(\lambda|v)}, \ v \in \mathfrak{h},$$

be the *character* of $L(\Lambda)$. By [8, Lemma 10.6b and (11.10.1)], this series converges in $D$ to a holomorphic function. The following result is a special case of [12, Theorem 1]:

PROPOSITION 1.6. *Let $\Lambda$ be a principal admissible weight. Then in the domain $D$, $ch_\Lambda$ is given by the following formula:*

$$ch_\Lambda = \sum_{w \in W^\Lambda} \varepsilon(w) e^{w(\Lambda+\rho)}/R. \ \square \tag{1.6.1}$$

Of course, a special case of (1.6.1) when $\Lambda = 0$ is the Macdonald identity

$$\sum_{w \in W} \varepsilon(w) e^{w(\rho)} = R. \tag{1.6.2}$$

Remark 1.6. Both the numerator and the denominator in (1.6.1) converge to holomorphic functions in $Y$ and the denominator $R$ does not vanish in $D$.

Define the *modular anomaly* of $\Lambda \in \mathfrak{h}'^{*k}$ by

$$m_\Lambda = \frac{|\Lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee}, \tag{1.6.3}$$

and the *normalized character* by

$$\chi_\Lambda = e^{-m_\Lambda K}\, \mathrm{ch}_\Lambda. \tag{1.6.4}$$

Note that $\chi_\Lambda$ depends only on $\Lambda$ mod $\mathbf{C}K$. Another advantage of introducing the normalized character is its modular invariance properties which will be discussed later. Note that using (1.6.1) functions $\mathrm{ch}_\Lambda$ and $\chi_\Lambda$ can be extended to meromorphic functions in $Y$, which are analytic at least on the set of regular elements $Y_{\mathrm{reg}} = \{v \in Y | (\alpha|v) \neq 0 \text{ for all } \alpha \in \Delta_+^{re}\}$.

Introduce coordinates $(\tau, z, t)$ on $Y$ as follows:

$$Y = \{(\tau, z, t) := 2\pi i(-\tau\Lambda_0 + z + tK) | \tau, t \in \mathbf{C}, \mathrm{Im}\ \tau > 0,\ z \in \overline{\mathfrak{h}}\}.$$

Then

$$q := e^{-K} = e^{2\pi i \tau}.$$

For $\lambda \in P_+^u$, $u \in \mathbb{N}$, let

$$A_\lambda = q^{|\lambda|^2/2u} \sum_{w\in W} \varepsilon(w) e^{w(\lambda)}. \tag{1.6.5}$$

This is holomorphic function on $Y$ (which is identically zero if $\lambda$ is not regular).

Let $\Lambda \in P_{u,y}^k$, $y = t_\beta \overline{y}$. Formula (1.6.1) may be rewritten in terms of the functions $A_\lambda$ as follows [13, Theorem 3.5]:

$$\chi_\Lambda(\tau, z, t) = A_{\Lambda^0+\rho}(u\overline{y}^{-1} t_{-\beta/u}(\tau, z/u, t/u^2))/A_\rho(\tau, z, t). \tag{1.6.6}$$

Using this formula, it is straightforward to derive the behaviour of the characters under the involution $\Lambda \longmapsto {}^t\Lambda$:

$$\chi_{{}^t\Lambda}(\tau, z, t) = \varepsilon(\overline{w}^\Lambda)\varepsilon(\overline{w}^0)\chi_\Lambda(\tau, -z, t). \tag{1.6.7}$$

For $\mu \in \mathfrak{h}'^{*m}$, $m \neq 0$, we have the following simple but useful identity:

$$A_\lambda(m\tau, -\tau\overline{\mu}, \tau\frac{|\mu|^2}{2m}) = \sum_{w\in W} \varepsilon(w) q^{\frac{um}{2}|\frac{w(\lambda)}{u} - \frac{\mu}{m}|^2}. \tag{1.6.8}$$

Now (1.6.8) for $\mu = \rho$ and (1.6.2) give the following identity:

$$\begin{aligned} A_\lambda(h^\vee\tau, -\tau\overline{\rho}, \tau|\rho|^2/2h^\vee) &= \sum_{w\in W} \varepsilon(w) q^{|h^\vee w(\lambda) - u\rho|^2/2uh^\vee} \\ &= q^{|h^\vee\lambda - u\rho|^2/2uh^\vee} \prod_{\alpha\in\Delta_+} (1 - q^{(\lambda|\alpha)})^{\text{mult }\alpha}. \end{aligned} \tag{1.6.9}$$

Here and further, given $\lambda \in \mathfrak{h}^*$, we denote by $\overline{\lambda} \in \overline{\mathfrak{h}}$ the restriction of $\lambda$ to $\overline{\mathfrak{h}}$. Finally, [13, Lemma 3.1] gives us the following asymptotics as $\tau \downarrow 0$:

$$\begin{aligned} A_\lambda(m\tau, -\tau\overline{\mu}, \tau|\mu|^2/2m) \sim (um)^{-\ell/2}|M^*/M|^{-1/2} &\prod_{\alpha\in\overline{R}_+} 4\sin\frac{\pi(\overline{\lambda}|\alpha)}{u}\sin\frac{\pi(\overline{\mu}|\alpha)}{m} \\ &\times (-i\tau)^{-\ell/2}\exp -\frac{\pi i}{12\tau}\frac{h^\vee \dim \mathfrak{g}(X_N)}{rum}, \end{aligned} \tag{1.6.10}$$

where $\overline{R}_+$ is defined by

$$\overline{R}_+ = \overline{\Delta}_+^\vee \text{ if } r^\vee = 1, \text{ and } = \overline{\Delta}_+ \text{ if } r^\vee > 1. \tag{1.6.11}$$

## §2. Branching functions for winding subalgebras.

2.1. Let $u \in \mathbf{N}$ be relatively prime to $a_0$, and let $y \in \tilde{W}$ satisfy (1.5.6). Denote by $\mathfrak{g}_{[u,y]}$ the subalgebra of the affine algebra $\mathfrak{g}$ generated by $\mathfrak{h}$ and the root vectors attached to the roots from $\pm y(\Pi_{[u]})$, and let $\mathfrak{g}_{[u]} = \mathfrak{g}_{[u,1]}$. The subalgebras $\mathfrak{g}_{[u,y]}$ will be called *winding subalgebras.*

Example 2.1. Let $A$ be of type $X_\ell^{(1)}$. This is usually referredto as the non-twisted case. Let $\bar{\mathfrak{g}}$ be the simple finite-dimensional Lie algebra of type $X_\ell$. Then

$$\mathfrak{g} = \mathfrak{g}(A) = \mathbf{C}[t, t^{-1}] \otimes_{\mathbf{C}} \bar{\mathfrak{g}} + \mathbf{C}K + \mathbf{C}d$$

with well-known commutation relations (see the Inroduction), and one has

$$\mathfrak{g}_{[u]} = \mathbf{C}[t^u, t^{-u}] \otimes \bar{\mathfrak{g}} + \mathbf{C}K + \mathbf{C}d.$$

One of the main objectives of the paper is to describe the decomposition of an integrable $\mathfrak{g}$-module $L(\Lambda)$ with respect to the subalgebra $\mathfrak{g}_{[u,y]}$. It turns out that this can be obtained in terms of the so called string functions. Recall that, given $\lambda \in \mathfrak{h}^*$, the *string function* $c_\lambda^\Lambda$ of the $\mathfrak{g}$-module $L(\Lambda)$ of level $k$ is defined by:

$$c_\lambda^\Lambda = q^{m_\Lambda - \frac{|\lambda|^2}{2k}} \sum_{n \in \mathbf{Z}} \operatorname{mult}_\Lambda(\lambda - nK) q^n. \tag{2.1.1}$$

(Recall that $q = e^{-K}$.) This series converges to a holomorphic function in $Y$ [ , Chapter 11]. Note that $c_\lambda^\Lambda$ depends on $\lambda$ mod $\mathbf{C}K$, that $c_\lambda^\Lambda \neq 0$ implies $\Lambda - \lambda \in Q$, and that $c_{w\lambda}^\Lambda = c_\lambda^\Lambda$ for $w \in W$.

Let $k \in \mathbf{N}$ and let $\Lambda \in P_+^k$. Then by [8, Proposition 11.8], the $\mathfrak{g}$-module $L(\Lambda)$ decomposes as a $\mathfrak{g}_{[u,y]}$-module into a direct sum of integrable irreducible highest weight modules $\dot{L}(\lambda)$ of level $uk$ (by (1.4.3)), each appearing with a finite multiplicity. Denote by $[\Lambda : \lambda]$ the multiplicity of the occurrence of $\dot{L}(\lambda)$ in this decomposition. Introduce the *branching function* $b_\lambda^\Lambda = b_\lambda^\Lambda(\mathfrak{g}_{[u,y]})$ for the winding subalgebra, where $\Lambda \in P_+^k$, $\lambda \in y(P_+^{uk})$, by

$$b_\lambda^\Lambda = q^{m_\Lambda - u m_\lambda} \sum_{n \in \mathbf{Z}} [\Lambda : (\lambda - nK)] q^n. \tag{2.1.2}$$

This series converges to a holomorphic function in $Y$. Note that as before, $b_\lambda^\Lambda$ depends on $\lambda$ mod $\mathbf{C}K$. Note also

$$b_\lambda^\Lambda \neq 0 \text{ implies } \lambda - \Lambda \in (u-1)k\Lambda_0 + Q. \tag{2.1.3}$$

By the Weyl-Kac character formula (which is a special case of (1.6.1) when $\Lambda \in P_+$) we have for an integrable $\mathfrak{g}$-module $L(\Lambda)$:

$$\chi_\Lambda = A_{\Lambda+\rho} / A_\rho. \tag{2.1.4}$$

Similarly we have for an integrable $\mathfrak{g}_{[u,y]}$-module $\dot{L}(\lambda)$:

$$\dot{\chi}_\lambda = \dot{A}_{y(\dot{\lambda}+\dot{\rho})}/\dot{A}_{y(\dot{\rho})}, \tag{2.1.5}$$

where for $\mu \in P_+^s$, $\mu = \sum_i m_i \Lambda_i$, one lets $\dot{\mu} = \sum_i m_i \dot{\Lambda}_i$, and $\dot{\Lambda}_i$ and $\dot{\rho}$ are defined by (1.4.4 and 5), and where

$$\dot{A}_{y(\dot{\mu})} = e^{\frac{|\dot{\mu}|^2}{2s}\dot{K}} \sum_{w \in y\dot{W}y^{-1}} \varepsilon(w) e^{w(y\dot{\mu})}. \tag{2.1.6}$$

Note that

$$\dot{\mu} = \mu + (u-1)s\Lambda_0. \tag{2.1.7}$$

Finally, by the very definitions we have

$$\chi_\Lambda = \sum_{\lambda \in P_+^{uk}} b^\Lambda_{y(\lambda)} \dot{\chi}_\lambda. \tag{2.1.8}$$

We can prove now the main result of this section.

THEOREM 2.1. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$, let $u \in \mathbf{N}$, $(u, a_0) = 1$ and let $y \in \tilde{W}$ satisfy (1.5.6). Let $\Lambda \in P_+^k$ and let $\lambda \in P_+^{uk}$. Then*

$$b^\Lambda_{y(\lambda)}(\mathfrak{g}_{[u,y]}) = \sum_{w \in W} \varepsilon(w) q^{\frac{h^\vee(uk+h^\vee)}{2k}\left|\frac{w(\lambda+\rho)}{uk+h^\vee} - \frac{\rho}{h^\vee}\right|^2} c^\Lambda_{y(w(\lambda+\rho)-\rho-(u-1)k\Lambda_0)}. \tag{2.1.9}$$

PROOF: In view of (2.1.8) and (2.1.5), it suffices to show

$$\dot{A}_{y(\dot{\rho})}\chi_\Lambda = \sum_{\lambda \in P_+^{uk}} b^\Lambda_{y(\lambda)} \dot{A}_{y(\dot{\lambda}+\dot{\rho})}, \tag{2.1.10}$$

where $b^\Lambda_{y(\lambda)}$ is given by the right-hand side of (2.1.9).

Recall that by definition of string functions we have [8, Chapter 12]:

$$\chi_\Lambda = \sum_{\substack{\xi \in \Lambda + Q + \mathbf{C}K \\ \bmod (kM + \mathbf{C}K)}} \sum_{\gamma \in M} q^{\frac{|\xi|^2}{2k}} e^{t_\gamma(\xi)} c^\Lambda_\xi. \tag{2.1.11}$$

The left-hand side of (2.1.10) is equal to

$$e^{-\frac{|\dot{\rho}|^2}{2h^\vee}\dot{K}} \sum_{w \in y\dot{W}y^{-1}} \varepsilon(w) e^{w(y\dot{\rho})} \chi_\Lambda. \tag{2.1.12}$$

We can write $w = yt_{u\alpha}\overline{w}y^{-1}$, where $\alpha \in M$, $\overline{w} \in \overline{W}$. Letting $\gamma = \overline{yw}\gamma'$, $\xi = w^{-1}y^{-1}\xi'$ and using Lemma 1.4 and (1.4.3–6), we can rewrite (2.1.12) as follows

$$
\begin{aligned}
&q^{\frac{u|\rho|^2}{2h^\vee}} \sum_{w \in y\dot{W}y^{-1}} \sum_{\substack{\xi' \in (wy)^{-1}\Lambda + Q + \mathbf{C}K \\ \bmod (kM + \mathbf{C}K)}} \sum_{\gamma \in M} \varepsilon(w) q^{\frac{|\xi'|^2}{2k}} e^{wy(\dot{\rho} + t_\gamma \xi')} c^\Lambda_{wy\xi'} \\
&= q^{\frac{u|\rho|^2}{2h^\vee}} \sum_{w \in y\dot{W}y^{-1}} \sum_{\substack{\xi \in y^{-1}\Lambda + Q + \mathbf{C}K \\ \bmod (kM + \mathbf{C}K)}} \sum_{\gamma \in M} \varepsilon(w) q^{\frac{|\xi + k\gamma|^2}{2k}} e^{wy(\dot{\rho} + \xi + k\gamma)} c^\Lambda_{y\xi}.
\end{aligned}
\tag{2.1.13}
$$

We may assume that $y(\dot{\rho} + \xi + k\gamma)$ is regular with respect to $y\dot{W}y^{-1}$. Then there exists a unique element $\tilde{\nu}$ from the set $y(P_+^{uk})$ of dominant integral weights of level $uk$ for the basis $y\Pi^\vee_{[u]}y^{-1}$ and a unique $\sigma \in y\dot{W}y^{-1}$ such that

$$
y(\dot{\rho} + \xi + k\gamma) = \sigma(\tilde{\nu} + y(\dot{\rho})) + aK, \text{ where } a \in \mathbf{C}. \tag{2.1.14}
$$

Plugging (2.1.14) in (2.1.13) we obtain:

$$
\sum_{w \in y\dot{W}y^{-1}} \sum_{\sigma \in y\dot{W}y^{-1}} \sum_{\tilde{\nu} \in y(P_+^{uk})} \varepsilon(w) e^{w\sigma(\tilde{\nu} + y\dot{\rho})} q^b c^\Lambda_{\sigma(\tilde{\nu} + y\dot{\rho}) - y\dot{\rho}} \tag{2.1.15}
$$

where

$$
\begin{aligned}
b &= -a + \frac{|\sigma(\tilde{\nu} + y\dot{\rho}) - y\dot{\rho} + aK|^2}{2k} + \frac{u|\rho|^2}{2h^\vee} \\
&= \frac{|\sigma(\tilde{\nu} + y\dot{\rho}) - y\dot{\rho}|^2}{2k} + \frac{u|\rho|^2}{2h^\vee} \\
&= \frac{h^\vee(uk + h^\vee)}{2k} \left| \frac{\sigma(\tilde{\nu} + y\dot{\rho})}{uk + h^\vee} - \frac{y(\dot{\rho})}{h^\vee} \right|^2 + \frac{u|\tilde{\nu} + y\dot{\rho}|^2}{2(uk + h^\vee)}.
\end{aligned}
$$

Plugging this expression of $b$ in (2.1.15) and putting $w' = w\sigma$, $\dot{\nu} = y^{-1}\tilde{\nu}$, we obtain:

$$
\sum_{\dot{\nu} \in P_+^{uk}} \sum_{\sigma \in y\dot{W}y^{-1}} \varepsilon(\sigma) \dot{A}_{y(\dot{\nu} + \dot{\rho})} q^{\frac{h^\vee(uk + h^\vee)}{2k} \left| \frac{\sigma y(\dot{\nu} + \dot{\rho})}{uk + h^\vee} - \frac{y(\dot{\rho})}{h^\vee} \right|^2} c^\Lambda_{\sigma y(\dot{\nu} + \dot{\rho}) - y\dot{\rho}}. \tag{2.1.16}
$$

Using (2.1.7) completes the proof. $\square$

Since $c^\Lambda_\lambda = c^\Lambda_{w(\lambda)}$ for $w \in W$, we obtain, in view of (1.3.4), the following corollary of (2.1.9):

COROLLARY 2.1. *Decompose $y \in \tilde{W}$ according to (1.3.4): $y = \tilde{y}w$, $\tilde{y} \in \tilde{W}_+$, $w \in W$. Then*

$$b_\lambda^\Lambda(\mathfrak{g}_{[u,y]}) = b_\lambda^{\tilde{y}^{-1}(\Lambda)}(\mathfrak{g}_{[u]}). \quad \square$$

This corollary allows one to reduce the study of branching functions for the subalgebra $\mathfrak{g}_{[u,y]}$ to that for the subalgebra $\mathfrak{g}_{[u]}$.

Since $b_\lambda^\Lambda(\mathfrak{g}_{[1]}) = \delta_{\Lambda,\lambda}$, we obtain another corollary of (2.1.9). For $\Lambda, \lambda \in P_+^k$ one has:

$$\sum_{w \in W} \varepsilon(w) q^{\frac{h^\vee(k+h^\vee)}{2k}\left|\frac{w(\lambda+\rho)}{k+h^\vee} - \frac{\rho}{h^\vee}\right|^2} c_{w(\lambda+\rho)-\rho}^\Lambda = \delta_{\Lambda,\lambda}. \tag{2.1.17}$$

2.2. In this subsection we consider the special case $k = 1$ of Theorem 2.1. Recall the following

LEMMA 2.2. *([10, 4.6], [8, Chapter 13]). Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$ where $X_N$ is of type $A_N$, $D_N$ or $E_N$, and let $\Lambda, M \in P_+^1 + \mathbb{C}K$. Then either $\Lambda - M \notin Q$ and $c_M^\Lambda = 0$, or $\Lambda - M \in Q$ and*

$$c_M^\Lambda = q^{m_\Lambda - |\Lambda|^2/2} \prod_{n \in \mathbb{N}} (1 - q^n)^{-\mathrm{mult}\ nK}. \quad \square$$

Using Lemma 2.2 and (1.6.9), formula (2.1.9) can be rewritten in the case $\Lambda \in P_+^1$ as follows:

THEOREM 2.2. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$, where $X_N$ is of type $A_N$, $D_N$ or $E_N$ (i.e. either $r = r^\vee = 1$ or $r > 1$), and let $\Lambda \in P_+^1$. Let $u \in \mathbb{N}$, let $\lambda \in P_+^u$, and consider the subalgebra $\mathfrak{g}_{[u]}$ of $\mathfrak{g}$. Then either $\Lambda - (\lambda - (u-1)\Lambda_0) \notin Q$ and $[\Lambda : (\lambda - nK)] = 0$ for all $n$, or $\Lambda - (\lambda - (u-1)\Lambda_0) \in Q$ and*

$$\sum_{n \in \mathbb{Z}} [\Lambda : (\lambda - nK)] q^n = \frac{q^{(|\lambda|^2 - |\Lambda|^2)/2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda+\rho|\alpha)})^{\mathrm{mult}\ \alpha}}{\prod_{n \in \mathbb{N}} (1 - q^n)^{\mathrm{mult}\ nK}}. \quad \square \tag{2.2.1}$$

This result was proved in the case of $A_\ell^{(1)}$ in [4] by a quite complicated method and was conjectured there for the simply laced case (except that the power of $q$ is missing there).

COROLLARY 2.2. *Let $\mathfrak{g}$ be as in Theorem 2.2 and let $u \in \mathbb{N}$.*

*(a) If $\Lambda \in P_+^1$ and $\lambda \in P_+^u$ are such that $\lambda - \Lambda \in (u-1)\Lambda_0 + Q$ then the minimal $n$ for which $[\Lambda : (\lambda - nK)] \neq 0$ is equal to $\frac{1}{2}(|\lambda|^2 - |\Lambda|^2)$, and $[\Lambda : (\lambda - \frac{1}{2}(|\lambda|^2 - |\Lambda|^2)K)] = 1$.*

*(b) In the $\mathfrak{g}$–module $\oplus_{\Lambda \in P_+^1} L(\Lambda)$ viewed as a $\mathfrak{g}'_{[u]}$–module all integrable highest weight modules of level $u$ occur, the minimal energy of occurring of the highest weight vector of $L(\lambda)$, $\lambda \in P_+^u$, being equal to $\frac{1}{2}(|\overline{\lambda}|^2 - |\overline{\Lambda}|^2)$.* $\square$

According to Corollary 2.2b, given $\mathfrak{g}$ as in Theorem 2.2 and $u \in \mathbb{N}$, for each $\lambda \in P_+^u$ there exists a canonically defined 1–dimensional subspace in the space $V = \oplus_{\Lambda \in P_+^1} L(\Lambda)$, which we denote by $V_\lambda^{[u]}$, whose non-zero vectors are all highest weight vectors of weight $\lambda$ for $\mathfrak{g}'_{[u]}$ of lowest energy. (Comparing this with [8, Exercise 12.17], we see that $V_\lambda^{[u]}\backslash\{0\}$ is also the set in $V$ of highest weight vectors of weight $\overline{\lambda}$ for $\overset{\circ}{\mathfrak{g}}$ of lowest energy.) Since all weights of $L(\Lambda)$, $\Lambda \in P_+^1$, are of the form $t_\gamma(\Lambda) - nK$, $\gamma \in \overline{Q}$, $n \in \mathbf{Z}_+$, we conclude that

$$V_\lambda^{[u]} = L(\Lambda)_{t_\gamma(\Lambda)}, \text{ where } \lambda - \Lambda = (u-1)\Lambda_0 + \gamma \bmod \mathbb{C}K. \tag{2.2.2}$$

Let $N' = N$ for $A = X_N^{(r)}$ except for $A = A_{2\ell-1}^{(2)}$ and $D_{\ell+1}^{(2)}$, when we let $N' = \ell + 1$ and $2\ell - 1$ respectively. Introduce the following function,

$$G(\tau) = q^{N'/24a_0} \prod_{n\geq 1} (1-q^n)^{\text{mult } nK}.$$

This function, along with the related values of $|\overline{Q}^\vee/M|$, is given by the following table:

| $Type X_N^{(r)}$ | $G(\tau)$ | $\lvert\overline{Q}^\vee/M\rvert$ |
|---|---|---|
| $X_\ell^{(1)}$ or $A_{2\ell}^{(2)}$ | $\eta(\tau)^\ell$ | 1 |
| $A_{2\ell-1}^{(2)}$ | $\eta(\tau)^{\ell-1}\eta(2\tau)$ | 2 |
| $D_{\ell+1}^{(2)}$ | $\eta(\tau)\eta(2\tau)^{\ell-1}$ | $2^{\ell-1}$ |
| $E_6^{(2)}$ | $\eta(\tau)^2\eta(2\tau)^2$ | 4 |
| $D_4^{(3)}$ | $\eta(\tau)\eta(3\tau)$ | 3 |

Then formula (2.2.1) can be rewritten as follows:

$$b_\lambda^\Lambda = \frac{q^{|h^\vee\lambda - u\rho|^2/2(u+h^\vee)h^\vee} \prod_{\alpha\in\Delta_+}(1-q^{(\lambda+\rho|\alpha)})^{\text{mult }\alpha}}{G(\tau)}. \tag{2.2.3}$$

Remark 2.2. Formulas (1.2.3), (1.2.4), (1.3.5) and (1.3.6) together with the above table allow one to compute the values of $|M^*/M|$, $|\tilde{M}/M|$, etc.

Later we shall need the following asymptotics as $\tau \downarrow 0$:

$$G(\tau)^{-1} \sim |\overline{Q}^\vee/M|^{1/2}(-i\tau)^{\ell/2} \exp\frac{\pi i N}{12r\tau}, \tag{2.2.4}$$

which can be deduced either by using the above table and the asymptotics of $\eta(\tau)^{-1}$, or by using the asymptotics of the string function $c_{\Lambda_0}^{\Lambda_0}$ [8, Chapter 13].

There is one more case when branching functions for winding subalgebras have very simple expressions: $\mathfrak{g}$ is of type $B_\ell^{(1)}$ and $k = 1$. In this case all (up to equivalence and symmetry of the Dynkin diagram) non-zero string functions are given by the following formulas [10, 4.6]:

$$c_{\Lambda_0}^{\Lambda_0} + c_{\Lambda_1}^{\Lambda_0} = q^{m_{\Lambda_0}} \prod_{n\in\mathbf{N}} (1-q^n)^{-\ell}(1+q^{n-1/2}), \tag{2.2.5}$$

$$c_{\Lambda_\ell}^{\Lambda_\ell} = q^{m_{\Lambda_\ell}-|\Lambda_\ell|^2/2} \prod_{n\in\mathbf{N}} (1-q^n)^{-\ell}(1+q^n). \tag{2.2.6}$$

THEOREM 2.2. *′ Let $\mathfrak{g}$ be an affine algebra of type $B_\ell^{(1)}$ and let $\Lambda \in P_+^1 = \{\Lambda_0, \Lambda_1, \Lambda_\ell\}$. Let $u \in \mathbf{N}$, $\lambda \in P_+^u$ and consider the subalgebra $\mathfrak{g}_{[u]}$ of $\mathfrak{g}$. Then either all branching coefficients $[\Lambda : (\lambda - nK)]$ are zero, or $\lambda \in u\Lambda_0 + Q$ (resp. $\in \Lambda_\ell + (u-1)\Lambda_0 + Q$) and we have respectively:*

$$\begin{aligned} &\sum_{n\in\mathbf{Z}} [\Lambda_0 :\ (\lambda - nK)]q^n + \sum_{n\in\mathbf{Z}} [\Lambda_1 :\ (\lambda - nK)]q^{n+1/2} \\ &= q^{|\lambda|^2/2} \prod_{\alpha\in\Delta_+} (1-q^{(\lambda+\rho|\alpha)})^{\text{mult } \alpha} \prod_{n\in\mathbf{N}} (1-q^n)^{-\ell}(1+q^{n-1/2}), \end{aligned} \tag{2.2.7}$$

(2.2.8)

$$\begin{aligned} &\sum_{n\in\mathbf{Z}} [\Lambda_\ell :\ (\lambda - nK)]q^n \\ &= q^{(|\lambda|^2-|\Lambda_\ell|^2)/2} \prod_{\alpha\in\Delta_+} (1-q^{(\lambda+\rho|\alpha)})^{\text{mult } \alpha} \prod_{n\in\mathbf{N}} (1-q^n)^{-\ell}(1+q^n). \end{aligned}$$

PROOF: Formula (2.2.8) follows from (2.1.9), (1.6.9), (2.2.6) and the fact that any non–zero string function of the form $c_\lambda^{\Lambda_\ell}$ is equal to $c_{\Lambda_\ell}^{\Lambda_\ell}$.

Adding up formulas (2.1.9) for $\Lambda = \Lambda_0$ and $\Lambda_1$, we have:

(2.2.9)

$$b_\lambda^{\Lambda_0} + b_\lambda^{\Lambda_1} = \sum_{w\in W} \varepsilon(w) q^{\frac{h^\vee(u+h^\vee)}{2}\left|\frac{w(\lambda+\rho)}{u+h^\vee} - \frac{\rho}{h^\vee}\right|^2} (c^{\Lambda_0}_{w(\lambda+\rho)-\rho-(u-1)\Lambda_0} + c^{\Lambda_1}_{w(\lambda+\rho)-\rho-(u-1)\Lambda_0}).$$

Since $w(\lambda+\rho)-\rho-(u-1)\Lambda_0 \in P^1$, it is $Q$–congruent to $\Lambda_0$, hence is $W$–conjugate to $\Lambda_0$ or $\Lambda_1$. Hence $c^{\Lambda_0}_{w(\lambda+\rho)-\rho-(u-1)\Lambda_0} + c^{\Lambda_1}_{w(\lambda+\rho)-\rho-(u-1)\Lambda_0} = c^{\Lambda_0}_{\Lambda_0} + c^{\Lambda_1}_{\Lambda_0}$ or $= c^{\Lambda_0}_{\Lambda_1} + c^{\Lambda_1}_{\Lambda_1}$, which are equal. Now we can apply to (2.2.9) formulas (1.6.9) and (2.2.5). □

2.3. In the section we discuss the asymptotics of the branching functions $b_\lambda^\Lambda(\mathfrak{g}_{[u]}; \tau)$ as $\tau \downarrow 0$.

Introduce the following notation (see (1.6.11)):

$$a(\Lambda) = (k+h^\vee)^{-\ell/2} |M^*/\overline{Q}^\vee|^{-\frac{1}{2}} \prod_{\alpha\in\overline{R}_+} 2\sin\frac{\pi(\Lambda+\rho|\alpha)}{k+h^\vee},\ \Lambda \in P_+^k; \tag{2.3.1}$$

$$c_k = \frac{k \dim \mathfrak{g}(X_N)}{k+h^\vee},\ k \in \mathbf{Q}. \tag{2.3.2}$$

We can state now the following

CONJECTURE 2.2. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$, let $k \in \mathbb{N}$, $\Lambda \in P_+^k$ and let $u \in \mathbb{N}$, $u > 1$, $\lambda \in P_+^{uk}$ be such that $\lambda - \Lambda \in (u-1)k\Lambda_0 + Q$. Then*

$$b_\lambda^\Lambda(\mathfrak{g}_{[u]};\tau) \sim |J|a(\Lambda)a(\lambda)\exp\frac{\pi i}{12\tau r}(c_k - u^{-1}c_{uk}). \tag{2.3.3}$$

Note that asymptotics (2.3.3) is obtained from (2.1.9) if one replaces thoughtlessly the string functions by their asymptotics (given in [10, §4.7] or [8, Chapter 13]):

$$c_\lambda^\Lambda(\tau) \sim |J||\overline{Q}^{\vee *}/M|^{-1}|M^*/M|^{1/2}a(\Lambda)(-i\tau/k)^{\ell/2}e^{\pi i c_k/12r\tau} \text{ if } \Lambda - \lambda \in Q, \tag{2.3.4}$$

and then uses (1.6.8 and 10). This procedure gives however a wrong result when $u = 1$. A more careful argument using the second term in the asymptotic expansion of the $c_\lambda^\Lambda$ shows that (2.3.3) holds for sufficiently large $u$. More precisely, let

$$b_k = \min_{\lambda \in P_+^k \setminus \{k\Lambda_j, j \in J\}} (2k(\lambda|\rho) - h^\vee|\lambda|^2)k^2,$$

and let, as usual, $b_k'$ denote the same quantity for the adjacent root system ([10,§1.5] or [13, §3]). (Note that $b_k = b_k'$ if $r = 1$ or $a_0 = 2$. Recall also that $b_k > 0$ [10, Proposition 4.14]. For example, $b_k = \ell(k-1)/k^2$ for $A_\ell^{(1)}$, $k > 1$.) Then (2.3.3) holds if

$$12a_0b_k'/(k+h^\vee) > \dim \mathfrak{g}(X_N)/(uk+h^\vee).$$

Note also that (2.3.3) holds for $k = 1$ if $r = r^\vee = 1$ or $r > 1$. This follows from Lemma 2.2 by making use of the "thoughtless" argument.

2.4. In this section we will assume (for simplicity) that $r = 1$, hence $\mathfrak{g}' = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathring{\mathfrak{g}} + \mathbb{C}K$ as described in Example 2.1. The well-known Sugawara construction extends any representation of $\mathfrak{g}$ in a vector space $V$ with $K = kI_V$ and with spectrum of $d$ bounded below, to the semidirect product Vir $\ltimes \mathfrak{g}'$ (see e.g. [11, §3.4] where this is described also in the twisted case). Recall that Vir is spanned by operators $L_n$ ($n \in \mathbb{Z}$) and $I$, which satisfy the usual Virasoro relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c_k,$$

where the *central charge* $c_k$ is given by (2.3.2), and that $[t^m \otimes x, L_n] = mt^{m+n} \otimes x$.

We fix $u \in \mathbb{N}$ and let

$$\tilde{L}_n = u^{-1}L_{un} + \delta_{n,0}(u-u^{-1})c_k/24.$$

These operators satisfy the Virasoro relations with central charge $uc_k$, and for $x(um) \in \mathfrak{g}'_{[u]}$ we have: $[x(um), \tilde{L}_n] = mx(u(m+n)) \in \mathfrak{g}'_{[u]}$. (Recall that $x(m)$ stands for $t^m \otimes x$.)

Let $\dot{L}_n$ ($n \in \mathbb{Z}$) be the operators given by the Sugawara construction for the $\mathfrak{g}'_{[u]}$-module $V$. They satisfy Virasoro relations with central charge $c_{uk}$. Hence the coset Virasoro operators

$$L_n^{[u]} := \tilde{L}_n - \dot{L}_n \tag{2.4.1}$$

satisfy the Virasoro relations with central charge

$$c_k^{[u]} := uc_k - c_{uk}. \tag{2.4.2}$$

This is a variation of the well-known coset construction [5].

The operators $L_n^{[u]}$ commute with $\mathfrak{g}'_{[u]}$, hence act on each subspace $\mathcal{U}(\Lambda, \lambda)$ of highest weight vectors with weight $\lambda \in P_+^{ku}$ for $\mathfrak{g}'_{[u]}$ in a $\mathfrak{g}$-module $L(\Lambda)$, $\Lambda \in P_+^k$. This representation of Vir in $\mathcal{U}(\Lambda, \lambda)$ is unitary with central charge $c_k^{[u]}$ and with the lowest eigenvalue of $L_0^{[u]}$ equal to

$$h_{\Lambda,\lambda}^{[u]} = u^{-1} m_\Lambda - \dot{m}_\lambda + c_k^{[u]}/24 + u^{-1} n, \tag{2.4.3}$$

where $n$ is the lowest energy of $d$ in $\mathcal{U}(\Lambda, \lambda)$.

Using the above facts it is easy to compute the branching functions in terms of Virasoro characters $\chi_{r,s}^{(m)}$ in the case when $c_k^{[u]} < 1$, using the method explained in [11] (we use notation of [11]).

PROPOSITION 2.4. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine algebra associated to a symmetric matrix $A$ and let $u > 1$. Let $c = c_k^{[u]} < 1$; then $k = 1$ and $u = 2$, and all these cases are listed below (in all formulas $b = b(\tau)$ and $\chi = \chi(2\tau)$):*

0) $A_1^{(1)}$, $c = \frac{1}{2}$:
$b_{2\Lambda_0}^{\Lambda_0} = b_{2\Lambda_1}^{\Lambda_0} = \chi_{2,2}^{(1)}$; $b_{\Lambda_0+\Lambda_1}^{\Lambda_1} = \chi_{1,1}^{(1)} + \chi_{2,1}^{(1)}$.

*1)* $E_8^{(1)}$, $c = \frac{1}{2}$:
$b_{2\Lambda_0}^{\Lambda_0} = \chi_{2,1}^{(1)}$; $b_{\Lambda_1}^{\Lambda_0} = \chi_{2,2}^{(1)}$; $b_{\Lambda_7}^{\Lambda_0} = \chi_{1,1}^{(1)}$.

*2)* $E_7^{(1)}$, $c = \frac{7}{10}$:
$b_{2\Lambda_0}^{\Lambda_0} = b_{2\Lambda_6}^{\Lambda_0} = \chi_{2,1}^{(2)}$; $b_{\Lambda_1}^{\Lambda_0} = b_{\Lambda_5}^{\Lambda_0} = \chi_{2,2}^{(2)}$; $b_{\Lambda_0+\Lambda_6}^{\Lambda_6} = \chi_{3,2}^{(2)} + \chi_{3,3}^{(2)}$; $b_{\Lambda_7}^{\Lambda_6} = \chi_{1,1}^{(2)} + \chi_{3,1}^{(2)}$.

*3)* $A_2^{(1)}$, $c = \frac{4}{5}$:
$b_{2\Lambda_0}^{\Lambda_0} = b_{2\Lambda_2}^{\Lambda_1} = b_{2\Lambda_1}^{\Lambda_2} = \chi_{4,2}^{(3)} + \chi_{4,4}^{(3)}$; $b_{\Lambda_1+\Lambda_2}^{\Lambda_0} = b_{\Lambda_0+\Lambda_1}^{\Lambda_1} = b_{\Lambda_0+\Lambda_2}^{\Lambda_2} = \chi_{2,2}^{(3)} + \chi_{3,2}^{(3)}$.

*4)* $E_6^{(1)}$, $c = \frac{6}{7}$:
$b_{2\Lambda_0}^{\Lambda_0} = b_{2\Lambda_5}^{\Lambda_1} = b_{2\Lambda_1}^{\Lambda_5} = \chi_{2,1}^{(4)} + \chi_{4,1}^{(4)}$, $b_{\Lambda_6}^{\Lambda_0} = b_{\Lambda_4}^{\Lambda_1} = b_{\Lambda_2}^{\Lambda_5} = \chi_{2,2}^{(4)} + \chi_{4,2}^{(4)}$,
$b_{\Lambda_1+\Lambda_5}^{\Lambda_0} = b_{\Lambda_0+\Lambda_1}^{\Lambda_1} = b_{\Lambda_0+\Lambda_5}^{\Lambda_5} = \chi_{4,3}^{(4)}$. □

Remark 2.4. Comparing (for $A$ of type $E_8^{(1)}$, $\Lambda = \Lambda_0$, $\lambda = \Lambda_1$) formula (2.2.1) with Proposition 2.4, we obtain the following curious identity:

$$\frac{\prod_{\alpha \in \Delta_+(E_8^{(1)})} (1 - q^{(\Lambda_1+\rho|\alpha)})^{\text{mult } \alpha}}{\prod_{n \in \mathbb{N}} (1 - q^n)^8} = \prod_{n \in \mathbb{N}} (1 + q^{2n}).$$

## §3. Comparison with branching functions for tensor products.

3.1. Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine algebra of type $X_N^{(r)}$. Let $m \in \mathbb{N}$, let $k \in \mathbb{Q}$ be principal admissible, and let $\Lambda \in P_+^m$, $\mu \in P^k(A)$ (see §1.6). Then by [13, Corollary 4.1] the module $L(\Lambda) \otimes L(\mu)$ decomposes with respect to $\mathfrak{g}$ into a direct sum of irreducible highest weight modules $L(\lambda)$, where $\lambda \in P^{m+k}(A) + \mathbb{C}K$, each appearing with finite multiplicity. Denote by $[\Lambda \otimes \mu : \lambda]$ this multiplicity. Define the branching function $b_\lambda^{\Lambda\otimes\mu}$ as follows:

$$b_\lambda^{\Lambda\otimes\mu} = q^{m_\Lambda + m_\mu - m_\lambda} \sum_{n\in\mathbb{Z}} [\Lambda \otimes \mu : (\lambda - nK)] q^n.$$

Note that $b_\lambda^{\Lambda\otimes\mu} = 0$ unless $\Lambda + \mu - \lambda \in Q$.

We have seen already in §2.1 that the branching functions for winding subalgebras can be expressed via string functions. We will show now that this holds for tensor products as well. Recall the map $\lambda \longmapsto \lambda^0$ defined by (1.5.7).

THEOREM 3.1. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$, let $m \in \mathbb{N}$ and $\Lambda \in P_+^m$, and let $k$ be a principal admissible rational number with denominator $u$ and $\mu =$ $y.(\mu^0 - (u-1)(k+h^\vee)\Lambda_0) \in P_{u,y}^k$. Then $b_\lambda^{\Lambda\otimes\mu} \neq 0$ implies that $\lambda \in P_{u,y}^{m+k}$ and in this case one has:*

$$(3.1.1) \qquad b_\lambda^{\Lambda\otimes\mu} = \sum_{w\in W} \varepsilon(w) q^{\frac{(k+h^\vee)(m+k+h^\vee)}{2m} \left| \frac{w(\lambda^0+\rho)}{m+k+h^\vee} - \frac{\mu^0+\rho}{k+h^\vee} \right|^2} c^\Lambda_{y(w(\lambda^0+\rho)-(\mu^0+\rho)-(u-1)m\Lambda_0)}.$$

PROOF: By the very definitions we have:

$$(3.1.2) \qquad \chi_\Lambda \chi_\mu = \sum_{\lambda\in P_{u,y}^{m+k}} b_\lambda^{\Lambda\otimes\mu} \chi_\lambda.$$

By (1.6.1 and 2) we have

$$(3.1.3) \qquad \chi_\mu = \hat{A}_{\mu+\rho}/A_\rho,$$

where

$$(3.1.4) \qquad \hat{A}_{\mu+\rho} = q^{\frac{|\mu^0+\rho|^2}{2(k+h^\vee)}} \sum_{w\in y\mathring{W}y^{-1}} \varepsilon(w) e^{w(\mu+\rho)}.$$

A similar formula holds for $\chi_\lambda$. Using (3.1.2 and 3), it suffices to show

$$(3.1.5) \qquad \chi_\Lambda \hat{A}_{\mu+\rho} = \sum_{\lambda\in P_{u,y}^{m+k}} b_\lambda^{\Lambda\otimes\mu} \hat{A}_{\lambda+\rho},$$

where $b_\lambda^{\Lambda\otimes\mu}$ is given by the right-hand side of (3.1.1). Using (2.1.11) and (3.1.4) we can rewrite the left-hand side of (3.1.5) as follows:

$$q^{\frac{|\mu^0+\rho|^2}{2(k+h^\vee)}} \sum_{\substack{\xi\in\Lambda+Q+\mathbf{C}K \\ \bmod (mM+\mathbf{C}K)}} \sum_{w\in y\dot{W}y^{-1}} \varepsilon(w)e^{w(\mu+\rho)} \sum_{\gamma\in M} q^{\frac{|\xi|^2}{2m}} e^{t_\gamma(\xi)} c_\xi^\Lambda. \tag{3.1.6}$$

Recall that $w = yt_{u\alpha}\overline{w}y^{-1}$; let $\gamma = \overline{y}\ \overline{w}\gamma'$ and $\xi = \overline{w}\ \overline{y}\xi'$. Then, using Lemma 1.4, we can rewrite (3.1.6) as follows:

$$q^{\frac{|\mu^0+\rho|^2}{2(k+h^\vee)}} \sum_{w\in y\dot{W}y^{-1}} \sum_{\substack{\xi'\in y^{-1}(\Lambda)+Q+\mathbf{C}K \\ \bmod (mM+\mathbf{C}K)}} \sum_{\gamma'\in M} \varepsilon(w)e^{w(\mu+\rho+y(\xi'+m\gamma'))} q^{\frac{|\xi'+m\gamma|^2}{2m}} c_{y(\xi')}^\Lambda. \tag{3.1.7}$$

We may assume that $\mu+\rho+y(\xi'+m\gamma')$ is regular with respect to $y\dot{W}y^{-1}$. Then there exists a unique dominant integral weight $\tilde{\nu}$ of level $u(m+k+h^\vee)-h^\vee$ for the simple coroot basis $y(\Pi_{[u]})$ and a unique $\sigma\in y\dot{W}y^{-1}$ such that

$$(\mu+\rho+y(\xi'+m\gamma'))-\sigma(\tilde{\nu}+y(\dot{\rho}))\in\mathbf{C}K, \tag{3.1.8}$$

where $\dot{\rho}$ is given by (1.4.5).

We can write $y^{-1}(\tilde{\nu})$ in the form:

$$y^{-1}(\tilde{\nu}) = \sum_i m_i\dot{\Lambda}_i,\ m_i\in\mathbf{Z}_+,\ \sum_i a_i^\vee m_i = u(m+k+h^\vee)-h^\vee,$$

where the $\dot{\Lambda}_i$ are defined by (1.4.4). Let $\nu+\rho=\tilde{\nu}+y(\dot{\rho})$. Then we have by (1.4.4 and 5):

$$\nu+\rho = y(\nu^0+\rho-(u-1)(m+k+h^\vee)\Lambda_0), \tag{3.1.9}$$

where $\nu^0=\Sigma m_i\Lambda_i$, hence $\nu\in P_{u,y}^{m+k}$.

Using (3.1.9) we can rewrite (3.1.8) as follows:

$$\mu+\rho+y(\xi'+m\gamma') = \sigma(\nu+\rho)+aK,\ a\in\mathbf{C}. \tag{3.1.10}$$

Since the string functions are invariant under translations by elements from $mM$, we see from (3.1.10), that

$$c_{y(\xi')}^\Lambda = c_{\sigma(\nu+\rho)-(\mu+\rho)}^\Lambda. \tag{3.1.11}$$

Plugging (3.1.10) and (3.1.11) in (3.1.7) we obtain

$$\sum_{\nu\in P_{u,y}^{m+k}} \sum_{\sigma\in y\dot{W}y^{-1}} \varepsilon(\sigma)\hat{A}_{\nu+\rho}q^b c_{\sigma(\nu+\rho)-(\mu+\rho)}^\Lambda, \tag{3.1.12}$$

where

$$\begin{aligned} b &= -a+\frac{|\sigma(\nu+\rho)-(\mu+\rho)+aK|^2}{2m}+\frac{|\mu^0+\rho|^2}{2(k+h^\vee)}-\frac{|\nu^0+\rho|^2}{2(m+k+h^\vee)} \\ &= \frac{|\sigma(\nu+\rho)-(\mu+\rho)|^2}{2m}+\frac{|\mu^0+\rho|^2}{2(k+h^\vee)}-\frac{|\nu^0+\rho|^2}{2(m+k+h^\vee)}. \end{aligned}$$

Furthermore, we have: $\sigma = yt_{u\alpha}\overline{\sigma}y^{-1}$, $\alpha\in M$, $\overline{\sigma}\in\overline{W}$. Hence, using (3.1.9), we obtain:

$$\sigma(\nu+\rho) = y(t_\alpha\overline{\sigma}(\nu^0+\rho)-(u-1)(m+k+h^\vee)\Lambda_0) \bmod \mathbf{C}K. \tag{3.1.13}$$

Formulas (3.1.12 and 13) prove (3.1.5) and the theorem. □

CONJECTURE 3.1. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$, let $m \in \mathbb{N}$ and $\Lambda \in P_+^m$, and let $\mu \in P_{u,y}^k$ and $\lambda \in P_{u,y}^{m+k}$ be such that $\Lambda + \mu - \lambda \in Q$. Then*

$$b_\lambda^{\Lambda\otimes\mu}(\tau) \sim |J|a(\Lambda)a(\lambda^0)a(\mu^0)\exp\frac{\pi i}{12\tau r}(c_m + c_k - c_{m+k}). \tag{3.1.14}$$

Note that asymptotics (3.1.14) is proved in the case $k \in \mathbb{N}$ [11, (2.7.15)]. As in the case of Conjecture 2.1, one can show that (3.1.14) holds for sufficiently large $u$, namely, when the following inequality is satisfied:

$$\frac{12a_0 b'_m}{m+h^\vee} > \frac{h^\vee \dim\mathfrak{g}(X_N)}{u^2(k+h^\vee)(m+k+h^\vee)}.$$

3.2. We shall compare now Theorems 2.1 and 3.1.

PROPOSITION 3.2. *Let $\mathfrak{g}, m$ and $\Lambda$ be as in Theorem 3.1. Let $u \in \mathbb{N}$ be such that $(u, h^\vee) = 1$ and $(u, r^\vee) = 1$. Let $k = (u^{-1}-1)h^\vee$ and let $y \in \tilde{W}$ satisfy (1.5.6). Then (by Proposition 1.5) $P_{u,y}^k = \{\mu := y.(k\Lambda_0)\}$, and for any $\lambda = y.(\lambda^0 - (u-1)(m+u^{-1}h^\vee)\Lambda_0) \in P_{u,y}^{m+k}$ (where $\lambda^0 \in P_+^{um}$) one has:*

$$b_\lambda^{\Lambda\otimes\mu} = b_{\lambda^0}^\Lambda(\mathfrak{g}_{[u,y]}). \tag{3.2.1}$$

PROOF: Just compare formulas (3.1.1) and (2.1.9). □

## §4. Functions $\varphi_{\lambda,\mu}$ and modular invariance.

4.1. Branching functions for winding subalgebras are intimately related to the functions $\varphi_{\lambda,\mu}$ defined as follows. As in Lemma 1.5, we shall distinguish cases 1) $r = r^\vee = 1$ (i.e. $A$ is symmetric) or $r > 1$, and 2) $r = 1$ but $r^\vee > 1$, putting the second case in parenthesis. Let $p$ and $p'$ be positive integers such that $p \geq h^\vee$ and $p' \geq h^\vee$ (resp. $p' \geq h$). For $\lambda \in P_+^{p-h^\vee}$ and $\mu \in P_+^{p'-h^\vee}$ (resp. $\mu \in P_+^{\vee p'-h}$) let

$$\varphi_{\lambda,\mu}(\tau) = \frac{1}{G(\tau)}\sum_{w\in W}\varepsilon(w)q^{\frac{pp'}{2}|\frac{w(\lambda+\rho)}{p} - \frac{\mu+\rho(\text{resp.}+\rho^\vee)}{p'}|^2}. \tag{4.1.1}$$

The connection of $\varphi_{\lambda,\mu}$ to branching functions is given by

THEOREM 4.1. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$ where either $r = r^\vee = 1$ or $r > 1$, let $\Lambda \in P_+^1$ and let $u \in \mathbb{N}$.*

*(a) If $\lambda \in P_+^u$ is such that $\lambda - \Lambda \in (u-1)\Lambda_0 + Q$, then*

$$b_\lambda^\Lambda(\mathfrak{g}_{[u]}) = \varphi_{\lambda,0}.$$

*(b) If $k \in \mathbb{Q}$ is a principal admissible rational number with the denominator $u$ and $\mu \in P_{u,y}^k$, $\lambda \in P_{u,y}^{k+1}$ are such that $y(\lambda^0 - \mu^0 - (u-1)\Lambda_0) - \Lambda \in Q$, then*

$$b_\lambda^{\Lambda\otimes\mu} = \varphi_{\lambda^0,\mu^0}.$$

PROOF: a) (resp. b)) follows from Theorem 2.1 (resp. Theorem 3.1) and Lemma 2.2. □

Theorem 4.1b was obtained in [12, Proposition 3].

PROPOSITION 4.1. *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$ and let $\lambda \in P_+^{p-h^\vee}$, $\mu \in P_+^{p'-h^\vee}$ (resp. $\in P_+^{\vee p'-h}$). One has the following asymptotics as $\tau \downarrow 0$:*

$$\varphi_{\lambda,\mu}(\tau) \sim (pp')^{-\ell/2}|M^*/\overline{Q}^\vee|^{-1/2} \prod_{\alpha\in\overline{R}_+} 4\sin\frac{\pi(\overline{\lambda}+\overline{\rho}|\alpha)}{p}\sin\frac{\pi(\overline{\mu}+\overline{\rho}(resp. +\overline{\rho}^\vee|\alpha)}{p'}$$

$$\times \exp\frac{\pi i N}{12r\tau}(1-\frac{h^\vee(h^\vee(resp.\ h)+1)}{pp'}).$$

PROOF: This follows from (1.6.8), (1.6.10) and (2.2.4), using

$$\dim \mathfrak{g}(X_N) = N(h^\vee(\text{resp. } h)+1). \quad \square \tag{4.1.2}$$

Remark 4.1. Note that the group Aut $\Pi^\vee \subset$ Aut $\mathfrak{h}'$ leaves the sets $P_+^m + \mathbb{C}K$ invariant, leaves $(.|.)$ invariant, fixes $\rho$ mod $\mathbb{C}K$ and normalizes $W$, and that $\varphi_{\lambda,\mu}(\tau)$ depends on $\lambda$ and $\mu$ mod $\mathbb{C}K$. It follows that, defining an action of $\sigma \in$ Aut $\Pi^\vee$ on $P_+$ by $\sigma\sum_i m_i\Lambda_i = \sum_i m_{\sigma(i)}\Lambda_i$, we have:

$$\varphi_{\sigma(\lambda),\sigma(\mu)} = \varphi_{\lambda,\mu}.$$

4.2. Here we establish a connection between the functions $\varphi_{\lambda,\mu}$ and the characters of admissible representations. We keep distinguishing two cases as in §4.1, putting the second case in parenthesis. Let

$$\tilde{G}(\tau) = \begin{cases} G(\tau)^{h^\vee\ (\text{resp.} h)} & \text{if } A \text{ is not of type } A_{2\ell}^{(2)}, \\ (\eta(\tau/2)\eta(2\tau)\eta(\tau)^{\ell-2})^{2\ell} & \text{if} A = A_{2\ell}^{(2)}. \end{cases}$$

By [10, 4.2] we have:

$$\tilde{G}(\tau)G(\tau) = q^{|\rho|^2/2h^\vee} \prod_{\alpha\in\Delta_+\setminus\overline{\Delta}_+} (1-q^{(\Lambda_0|\alpha)})^{\text{mult}\alpha}. \tag{4.2.1}$$

PROPOSITION 4.2. *Let*

$$\Lambda = y.(\Lambda^0-(u-1)(k+h^\vee)\Lambda_0) \in P_{u,y}^k \tag{4.2.2}$$

*be a principal admissible weight and let $z \in \overline{\mathfrak{h}}$ be such that $(\alpha|z) \neq 0$ for all $\alpha \in \overline{\Delta}_+$. Then the limit*

$$\psi_\Lambda(\tau) := \lim_{\varepsilon\to 0}\tilde{G}(\tau)\prod_{\alpha\in\overline{\Delta}_+}(1-e^{-2\pi i(\alpha|\varepsilon z)})\chi_\Lambda(\tau,\varepsilon z,0) \tag{4.2.3}$$

*exists and is equal to $\varphi_{\Lambda^0,\omega_{u,y}}(\tau)$ if*

$$<\Lambda,\alpha>\notin \mathbb{Z} \text{ for all } \alpha\in\overline{\Delta}^\vee, \tag{4.2.4}$$

*and is equal to* 0 *otherwise. (Recall that $\omega_{u,y}$ is defined by (1.5.16).)*

PROOF: If $< \Lambda, \alpha > \in \mathbb{Z}$ for some $\alpha \in \overline{\Delta}^{\vee}$, then $r_\alpha \in W^\Lambda$ and hence $\sum_{w \in W^\Lambda} \varepsilon(w) e^{w(\Lambda+\rho)}(\tau, 0, 0)$ $= 0$ for all Im $\tau > 0$. Since $\prod_{\alpha \in \Delta_+ \setminus \overline{\Delta}_+} (1 - e^{-\alpha})^{\mathrm{mult}\alpha}(\tau, 0, 0) \neq 0$, we see from (1.6.1) that $\varphi_\Lambda(\tau) = 0$. Suppose now that (4.2.4) holds. Then we have by (4.2.1)

$$\lim_{\varepsilon \to 0} \Big( \prod_{\alpha \in \overline{\Delta}_+} (1 - e^{-\alpha})/A_\rho \Big)(\tau, \varepsilon z, 0) = G(\tau)^{-1} \tilde{G}(\tau)^{-1}. \tag{4.2.5}$$

Let $y = t_\beta \overline{y}, \beta \in \tilde{M}, \overline{y} \in \overline{W}$. Then (1.6.1) can be rewritten as follows [13, Theorem 3.5]:

$$\chi_\Lambda(\tau, z, t) = A_{\Lambda^0 + \rho}(u\tau, \tau \overline{y}^{-1} \beta + \overline{y}^{-1} z, u^{-1}(t + (z|\beta) + \tau |\beta|^2/2) \Big/ A_\rho(\tau, z, t).$$

Hence, by (4.2.5),

$$\psi_\Lambda(\tau) = A_{\Lambda^0 + \rho}(u\tau, \tau \overline{y}^{-1} \beta, \tau |\beta|^2 / 2u)/G(\tau) = \varphi_{\Lambda^0, \omega_{u,y}}. \quad \square$$

Let $k \in \mathbb{Q}$ be $A$–admissible with the denominator $u \in \mathbb{N}$, and let

$$p = u(k + h^\vee). \tag{4.2.6}$$

The $A$–admissibility of $k$ is then equivalent to

$$p, u \in \mathbb{N}, p \geq h^\vee, (p, u) = (u, r^\vee) = 1. \tag{4.2.7}$$

Denote by $\tilde{P}^k_{u,y}$ the set of all $\Lambda \in P^k_{u,y}$ satisfying (4.2.4); we shall call these principal admissible weights *nondegenerate*. Given $\overline{y} \in \overline{W}$, let $\tilde{P}^k_{\overline{y}} = \bigcup_{\beta \in \tilde{M}} \tilde{P}^k_{u, t_\beta \overline{y}}$. This set admits the following nice parameterization:

LEMMA 4.2. *Let $\lambda \in P_+^{p-h^\vee}$, $\mu \in P_+^{u-h^\vee}$ (resp. $\in P_+^{\vee u-h}$).*
*(a) The map associating to the pair $(\lambda, \mu)$ the element*

$$\Lambda_{k,\overline{y}}(\lambda, \mu) = \overline{y}(\lambda + \rho) - \frac{p}{u} \overline{y}(\mu + \rho(resp. + \rho^\vee)) + \frac{p}{u} \Lambda_0 - \rho, \tag{4.2.8}$$

*establishes a bijective correspondence between the set of all such pairs and the set $\tilde{P}^k_{\overline{y}}$. In particular, $\tilde{P}^k_{\overline{y}} \neq \emptyset$ if and only if*

$$k \text{ is principal admissible and } u \geq h^\vee (resp.\ h). \tag{4.2.9}$$

*(b) Let $k$ be principal admissible, let $\overline{y}, \overline{y}' \in \overline{W}$ and let $\overline{\sigma} = \overline{y}^{-1} \overline{y}'$. Then $\Lambda_{k,\overline{y}}(\lambda, \mu) = \Lambda_{k,\overline{y}'}(\lambda', \mu')$ if and only if*

$$p^{-1}(\lambda - \overline{\sigma}.\lambda') = u^{-1}(\mu - \overline{\sigma}.\mu') := \alpha \in \tilde{M} \tag{4.2.10}$$

and

$$t_\alpha \overline{\sigma} \in \tilde{W}_+. \tag{4.2.11}$$

*(c)* $\tilde{P}^k_{\overline{y}}$ *and* $\tilde{P}^k_{\overline{y}'}$ *either are disjoint or coincide, and they coincide if and only if* $\overline{y}^{-1}\overline{y}' \in \overline{W}_+$.

PROOF: Let $\Lambda \in P^k_{u,y}$, $y = t_\beta \overline{y}$, $\beta \in \tilde{M}$, $\overline{y} \in \overline{W}$. Then $\Lambda = y.(\Lambda^0 - (u-1)(k+h^\vee)\Lambda_0)$ (where $\Lambda^0 \in P_+^{p-h^\vee}$) can be rewritten as follows:

$$\Lambda = \Lambda_{k,\overline{y}}(\Lambda^0, \omega_{u,y}). \tag{4.2.12}$$

If $\Lambda \in \tilde{P}^k_{\overline{y}}$, then since (4.2.4) implies (1.5.8), by Lemma 1.5, $\omega_{u,y} \in P_+^{p'-h^\vee}$ (resp. $\in P_+^{\vee p'-h}$). Conversely, this inclusion implies that $\Lambda$ defined by (4.2.12) lies in $\tilde{P}^k_{\overline{y}}$. Finally $\Lambda$ completely determines the pair: $\lambda = \Lambda^0$, and $\mu$ is determined since $\overline{y}$ is given. This proves (a).

Furthermore, if

$$\begin{aligned}\Lambda &= y.(\lambda - (u-1)(k+h^\vee)\Lambda_0) = \Lambda_{k,\overline{y}}(\lambda,\mu) \\ &= y'.(\lambda' - (u-1)(k+h^\vee)\Lambda_0) = \Lambda_{k,\overline{y}'}(\lambda',\mu'),\end{aligned}$$

then, by Proposition 1.5b there exists $\alpha \in \tilde{M}$, such that $y' = yt_{u\alpha}\overline{\sigma}$. Let $\sigma = t_\alpha\overline{\sigma}$. Then we have: $\sigma.\lambda' = \lambda$, hence $\sigma.\mu' = \mu$, and conditions (4.2.10 and 11) hold, and vice versa, (4.2.10 and 11) imply $\Lambda_{k,\overline{y}}(\lambda,\mu) = \Lambda_{k,\overline{y}'}(\lambda',\mu')$, proving (b). (c) follows from (b) and Proposition 1.5b. □

Finally, note

$${}^t\Lambda_{k,\overline{y}}(\lambda,\mu) = \Lambda_{k,{}^t\overline{y}}({}^t\lambda, {}^t\mu). \tag{4.2.13}$$

4.3. We turn now to the discussion of modular invariance. First, by a general result [10, Proposition 4.36] all branching functions $b^\Lambda_\lambda$ and $b^{\Lambda\otimes\mu}_\lambda$ are holomorphic modular functions (of weight 0) in $\tau, \operatorname{Im}\tau > 0$.

In order to describe the explicit transformation formulas for branching functions, we need the transformation formula for characters. Recall the action of $B = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbf{R})$ on $Y$:

$$B\cdot(\tau,z,t) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, (\det B)^{-1}\left(t - \frac{c(z|z)}{2(c\tau+d)}\right)\right),$$

and its right action on functions on $Y$:

$$f(\tau,z,t)|_B = f(B\cdot(\tau,z,t)).$$

Recall the definitions of the congruence subgroups $\Gamma_0(n) = \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv 0 \mod n\}$ and the theta subgroup $\Gamma_\theta = \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{Z}) | ac \text{ and } bd \text{ are even}\}$ of $SL_2(\mathbf{Z})$. Let $\Gamma = \Gamma_0(r)$ if $\mathfrak{g}$ is of type $X_N^{(r)} \neq A_{2\ell}^{(2)}$ and $\Gamma = \Gamma_\theta$ if it is of type $A_{2\ell}^{(2)}$. The following statement is proved in [10] for $r = 1$ and is implicitly contained there in the general case.

THEOREM 4.3. *The* $\mathbb{C}$*–span of the set of normalized characters* $\{\chi_\Lambda\}_{\Lambda\in P_+^k}$ *is* $\Gamma$*–invariant.*

PROOF: In notation of [10], the space $Th_{k+h^\vee}$ is $\Gamma$–invariant by [10, Proposition 4.5a], hence the space $Th^-_{k+h^\vee}$ is $\Gamma$–invariant since the action of $GL_2(\mathbf{R})$ commutes with the action of $\overline{W}$. But $Th^-_{k+h^\vee}$ is the linear span of functions $A_{\lambda+\rho}$, $\lambda\in P_+^k$ and $\dim Th^-_{h^\vee}=1$. Together with (2.1.4) this proves the proposition. □

Let $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}\in SL_2(\mathbf{Z})$. An explicit formula for the action of $S$ on the $\chi_\Lambda$, $\Lambda\in P_+$, was found in [10], and its generalization to the case of the principal admissible weights in [13, Theorem 3.6]:

LEMMA 4.3. *Let* $\mathfrak{g}=\mathfrak{g}(A)$ *be an affine algebra of type* $X_\ell^{(1)}$ *or* $A_{2\ell}^{(2)}$*. Let* $k$ *be a principal admissible rational number with denominator* $u$ *and let* $\lambda\in P^k(A)$ *be a principal admissible weight. Then*

$$\chi_\lambda|_S=\sum_{\lambda'\in P^k(A)}S_{\lambda\lambda'}\chi_{\lambda'}, \tag{4.3.1}$$

*where*

$$\begin{aligned}S_{\lambda\lambda'}&=i^{|\overline{\Delta}_+|}u^{-\ell}(k+h^\vee)^{-\ell/2}|M^*/M|^{-1/2}\varepsilon(\overline{y}\overline{y}')\\ &\times e^{-2\pi i((\lambda^0+\rho|\beta')+(\lambda'^0+\rho|\beta)+(k+h^\vee)(\beta|\beta'))}\sum_{w\in\overline{W}}\varepsilon(w)e^{-\frac{2\pi i}{k+h^\vee}(w(\lambda^0+\rho)|\lambda'^0+\rho)}.\end{aligned} \tag{4.3.2}$$

*(Here* $\lambda\in P^k_{u,y}$*,* $\lambda'\in P^k_{u,y'}$*,* $y=t_\beta\overline{y},y'=t_{\beta'}\overline{y}'$*.)* □

It is easy to check that (see (1.5.12)):

$$S_{\lambda\lambda'}=\varepsilon(\overline{w}^0)\varepsilon(\overline{w}^\lambda)\overline{S}_{\lambda\lambda'}. \tag{4.3.3}$$

Applying (4.3.1) twice and using (1.6.7) and (4.3.3), we deduce

PROPOSITION 4.3. *The matrix* $(S_{\lambda\lambda'})_{\lambda,\lambda'\in P^k(A)}$ *is a unitary (symmetric) matrix.* □

Remark 4.3. (a) Note that $\chi_\lambda|_{T^{a_0}}=e^{2\pi i m_\lambda}\chi_\lambda$, hence, by Lemma 4.3, the $\mathbb{C}$–span of the set $\{\chi_\lambda\}_{\lambda\in P^k(A)}$, which we denote by $CH^k$, is $SL_2(\mathbf{Z})$–invariant in the case $r=1$. Since $S$ and $T^2$ generate $\Gamma_\theta$, we see that $CH^k$ is $\Gamma_\theta$–invariant in the case $A_{2\ell}^{(2)}$. In the remaining cases we have [13, Theorem 3.6]:

$$\chi_\lambda|_S=\sum_{\lambda'\in P'^k(A)}S_{\lambda\lambda'}\chi'_{\lambda'}(\tau/r,z/r,t).$$

Here and further $P'$, $\chi'$ etc. refers to the "adjacent root system" (see [10, §1.5] or [13, §3]. By Proposition 4.3, this can also be written as follows:

$$\chi'_{\lambda'}(-\frac{1}{r\tau},-\frac{z}{r\tau},t-\frac{(z/z)}{2\tau})=\sum_{\lambda\in P^k(A)}\overline{S}_{\lambda\lambda'}\chi_\lambda(\tau,z,t).$$

From these we obtain:

$$\chi_\lambda|_{ST^rS^{-1}} = \sum_{\mu\in P^k(A)}\sum_{\lambda'\in P'^k(A)} e^{2\pi i m_{\lambda'}} S_{\lambda\lambda'}\overline{S}_{\mu\lambda'}\chi_\mu.$$

Thus, $CH^k$ is $ST^rS^{-1}$–invariant. Since $CH^k$ is also $T$– and $S^2$–invariant, and since the elements $ST^rS^{-1}$, $T$ and $S^2$ generate the group $\Gamma_0(r)$ for $r = 2$ or 3, we conclude that $CH^k$ is $\Gamma$–invariant in all cases.

(b) Let $a_0 = 1$ and denote by $CH^k_y$ the $\mathbb{C}$–span of $\{\chi_\lambda\}_{\lambda\in P^k_{u,y}}$. Let $y = t_\beta\overline{y}$, and let $(\tau,z,t)^\wedge = u\overline{y}^{-1}t_{-\beta/u}(\tau, z/u, t/u^2)$. Then

$$STS^{-1}((\tau,z,t)^\wedge) = (ST^uS^{-1}(\tau,z,t))^\wedge.$$

Using this and (1.6.6) one derives the following transformation formula:

$$\chi_\lambda|_{ST^uS^{-1}} = \sum_{\substack{\Lambda\in P^{u(k+h^\vee)-h^\vee}\\ \mu\in P^k_{u,y}}} S_{\lambda^0\Lambda}\overline{S}_{\Lambda\mu^0}e^{2\pi i m_\Lambda}\chi_\mu.$$

Nevertheless, it is not true that $CH^k_y$ is $\Gamma_0(u)$–invariant for each $y$. It is true, however, if $y = 1$. To show this we use formula (1.6.6), which says that the numerator of $\chi_\Lambda$, $\Lambda \in P^k_{u,y}$, is equal to

$$\hat{A}_{\Lambda+\rho}(\tau,z,t) := A_{\Lambda^0+\rho}(u\tau, \tau\overline{y}^{-1}(\beta) + \overline{y}^{-1}(z), u^{-1}(t + (z|\beta) + \tau|\beta|^2/2)).$$

Let $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(u)\cap\Gamma$. Then we have:

$$\hat{A}_{\Lambda+\rho}|_g = A_{\Lambda^0+\rho}(u\tau, (a\tau+b)\overline{y}^{-1}(\beta) + \overline{y}^{-1}(z), u^{-1}(t + a(\beta|z) + \frac{a(a\tau+b)|\beta|^2}{2}))\Big|_{\begin{pmatrix} a & ub\\ c/u & d\end{pmatrix}}.$$

Since $Th^-$ is $\Gamma$–invariant, we see that $\hat{A}_{\Lambda+\rho}|_g$ is a linear combination of functions

$$A_{\mu^0+\rho}(u\tau, a\tau\overline{y}^{-1}(\beta) + \overline{y}^{-1}(z),\ u^{-1}(t + (\beta|z) + \frac{a(a\tau+b)|\beta|^2}{2})),\ \mu^0 \in P^{u(k+h^\vee)-h^\vee}.$$

Assume now that $y = 1$, i.e. $\beta = 0$. Then $\hat{A}_{\Lambda+\rho}|_g$ is a linear combination of functions

$$A_{\mu^0+\rho}(u\tau, \overline{y}^{-1}(z), u^{-1}t) = \hat{A}_{\mu+\rho}(\tau,z,t),\ \mu \in P^k_{u,1}.$$

Note that our argument shows that $CH^k_y$ is invariant with respect to the group $\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma | a \equiv 1 \mod u,\ c \equiv 0 \mod u\}$.

As in [12], Lemma 4.3 together with (3.1.2) give us a transformation law for $b_\lambda^{\Lambda\otimes\mu}$, where $\Lambda \in P^m_+$, $m \in \mathbb{N}$, and $\mu \in P^k(A)$, $\lambda \in P^{k+m}(A)$ are principle admissible weights (and $\mathfrak{g}$ is as in Lemma 4.3):

(4.3.4)
$$b_\lambda^{\Lambda\otimes\mu}(-\frac{1}{\tau}) = \sum_{\substack{\Lambda'\in P^m_+\\ \mu'\in P^k(A)\\ \lambda'\in P^{k+m}(A)}} S_{\Lambda\Lambda'}S_{\mu\mu'}\overline{S}_{\lambda\lambda'}b_{\lambda'}^{\Lambda'\otimes\mu'}(\tau).$$

Since

$$b_\lambda^{\Lambda\otimes\mu}(\tau+1) = e^{2\pi i(m_\Lambda+m_\mu-m_\lambda)}b_\lambda^{\Lambda\otimes\mu}(\tau),$$

we obtain the following corollary of (3.1.2), Remark 4.3a and Proposition 3.2:

COROLLARY 4.3. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine algebra of type $X_N^{(r)}$ and fix $m \in \mathbb{N}$. Then*
*a) Given a principal admissible $k \in \mathbb{Q}$, the $\mathbb{C}$–span of the set*

$$\{b_\lambda^{\Lambda\otimes\mu}(\tau) | \Lambda \in P_+^m,\ \mu \in P^k(A),\ \lambda \in P^{k+m}(A)\}$$

*is $\Gamma_0(r/a_0)$–invariant.*
*b) If $u \in \mathbb{N}$ is such that $(u, h^\vee) = 1$, then the $\mathbb{C}$–span of the set*

$$\{b_\lambda^\Lambda | \Lambda \in P_+^m,\ \lambda \in P_+^{um}\}$$

*is $\Gamma_0(r/a_0)$–invariant.*
*c) Given $p, p' \in \mathbb{N}$ such that $p \geq h^\vee$, $p' \geq h^\vee$ (resp. $p' \geq h$) and $(p,p') = 1$, the $\mathbb{C}$–span of the set*

$$\{\varphi_{\lambda,\mu}(\tau) | \lambda \in P_+^{p-h^\vee},\ \mu \in P_+^{p'-h^\vee}\ (\text{resp. } \in P_+^{\vee p'-h})\}$$

*is $\Gamma_0(r/a_0)$–invariant.* □

Proposition 2.4 shows that the conclusion of Corollary 4.3b fails if $(u, h^\vee) \neq 1$. In general, however, the subspace considered in Corollary 4.3b is at least $\Gamma_0(u)$–invariant. Indeed, let $\alpha_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$. Then we have:

$$\Gamma_0(u) = SL_2(\mathbb{Z}) \cap \alpha_u^{-1} SL_2(\mathbb{Z}) \alpha_u, \tag{4.3.5}$$

$$\dot{\chi}_\lambda = \chi_\lambda|_{\alpha_u}. \tag{4.3.6}$$

Hence for $\beta = \alpha_u^{-1}\gamma\alpha_u \in \Gamma_0(u)$, where $\gamma \in SL_2(\mathbb{Z})$, we have: $\dot{\chi}_\lambda|_\beta = \chi_\lambda|_{\gamma\alpha_u}$, and by Proposition 4.3, the $\mathbb{C}$–span of the $\dot{\chi}_\lambda$ is $\Gamma_0(u)$–invariant. It follows from (2.1.8) that the $\mathbb{C}$–span of the set $\{b_\lambda^\Lambda | \Lambda \in P_+^m,\ \lambda \in P_+^{um}\}$ is $\Gamma_0(u)$–invariant.

4.4. Formula (4.3.4) and Theorem 4.1b give a transformation formula for the $\varphi_{\lambda,\mu}(\tau)$. However, a much simpler formula may be obtained by using (4.3.1) and Proposition 4.2.

PROPOSITION 4.4. *Under the assumptions of Lemma 4.3 we have:*

$$\psi_\lambda(-\frac{1}{\tau}) = (-i)^{|\overline{\Delta}_+|} \sum_{\lambda' \in P^k(A)} S_{\lambda\lambda'} \psi_{\lambda'}(\tau). \tag{4.4.1}$$

*In particular, in the basis $\{\psi_\lambda\}_{\lambda \in P^k(A)}$ the transformation matrix is unitary.*

PROOF: Let $\tilde{\chi}_\lambda(\tau) = \lim_{\varepsilon\to 0} \prod_{\alpha\in\overline{\Delta}_+}(1 - e^{2\pi i(\alpha|\varepsilon z)})\chi_\lambda(\tau, \varepsilon z, 0)$. Note that (4.3.1) gives:

$$\chi_\lambda(-\frac{1}{\tau}, \frac{\varepsilon z}{\tau}, -\frac{\varepsilon^2 (z|z)}{2\tau}) = \sum_{\lambda' \in P^k(A)} S_{\lambda\lambda'} \chi_{\lambda'}(\tau, \varepsilon z, 0).$$

This can be written as follows:

$$\prod_{\alpha\in\overline{\Delta}_+} \frac{1 - e^{-2\pi i(\alpha|\varepsilon z)}}{1 - e^{-2\pi i(\alpha|\varepsilon z/\tau)}} \prod_{\alpha\in\overline{\Delta}_+} (1 - e^{-2\pi i(\alpha|\varepsilon z/\tau)})\chi_\lambda(-\frac{1}{\tau}, \frac{\varepsilon z}{\tau}, -\frac{\varepsilon^2(z|z)}{2\tau})$$

$$= \sum_{\lambda' \in P^k(A)} S_{\lambda\lambda'} \prod_{\alpha \in \overline{\Delta}_+} (1 - e^{-2\pi i(\alpha|\varepsilon z)}) \chi_{\lambda'}(\tau, \varepsilon z, 0).$$

Taking limit of both sides as $\varepsilon \to 0$ we obtain:

$$\tau^{|\overline{\Delta}_+|} \tilde{\chi}_\lambda(-\frac{1}{\tau}) = \sum_{\lambda' \in P^k(A)} S_{\lambda\lambda'} \tilde{\chi}_{\lambda'}(\tau). \tag{4.4.2}$$

Since, by definition (4.2.3), $\psi_\lambda(\tau) = \tilde{G}(\tau)\tilde{\chi}_\lambda(\tau)$, the proposition follows from (4.4.2) and transformation properties of $\eta(\tau)$. □

Remark 4.4. Proposition 4.4 can be extended to the remaining cases, $a_0 r = 2$ or $3$, using [13, Theorem 3.7]. The result is that on the right–hand side of (4.4.1) one should add the factor $r^{|\overline{\Delta}_{+s}|}$, replace $\tau$ by $\tau/r$, $|M^*/M|$ by $|M^*/\overline{Q}^\vee|$ and $P^k(A)$ by $P'^k(A)$. Here, as before, $P'^k(A)$ refers to the adjacent set of weights associated to the adjacent root system.

LEMMA 4.4.. *Let $p$ and $p'$ be relatively prime positive integers and let $\lambda \in P_+^p, \lambda' \in P_+^{p'}$ be regular weights. Then $\mu := p'\lambda - p\lambda' + p\Lambda_0$ is a regular weight.*

PROOF: We have to show that $(\mu|\alpha) \neq 0$ for any $\alpha \in \Delta_+$. Let $\alpha = \gamma + nK$, where $\gamma \in \overline{\Delta}_+$, $n \in \mathbb{Z}_+$. Then:

$$(\mu|\alpha) = p'(\lambda|\gamma) - p(\lambda'|\gamma) + pn,$$

hence $p$ divides $(\lambda|\gamma)$ if $(\mu|\alpha) \neq 0$, which is impossible since $0 < (\lambda|\gamma) < p$ for any $\gamma \in \overline{\Delta}_+$. □

Let $p, p'$ be relatively prime integers $\geq h^\vee$. We define a map of $P_+^{p-h^\vee} \times P_+^{p'-h^\vee}$ into itself, denoted by $(\mu, \mu') \longmapsto (\tilde{\mu}, \tilde{\mu}')$, and $\varepsilon_{\mu,\mu'}$, $\varepsilon'_{\mu,\mu'} = \pm 1$ as follows. Due to Lemma 4.4, there exist a unique $\tilde{\mu} \in P_+^{p-h^\vee}$, $\tilde{\mu}' \in P_+^{p'-h^\vee}$ and unique $w, w' \in \overline{W}$ such that

$$p'(\mu + \rho) - p(\mu' + \rho) - w(\overline{\tilde{\mu}} + \overline{\rho}) \in pM \text{ and}$$
$$p(\mu' + \rho) - p'(\mu + \rho) - w'(\overline{\tilde{\mu}}' + \overline{\rho}) \in p'M,$$

and we let

$$\varepsilon_{\mu,\mu'} := \varepsilon(w), \ \varepsilon'_{\mu\mu'} := \varepsilon(w').$$

In particular, for the pair $(\mu, p'\Lambda_0)$, we denote $\mu_{p'} = \tilde{\lambda}, \varepsilon_{p'}(\mu) = \varepsilon_{\mu,p'\Lambda_0}$.

Note also that for an integer $p \geq h^\vee$ and $\lambda, \lambda' \in P_+^{p-h^\vee}$ formula (4.3.2) turns into

$$S_{\lambda\lambda'} = i^{|\overline{\Delta}_+|} p^{-\ell/2} |M^*/M|^{-1/2} \sum_{w \in \overline{W}} \varepsilon(w) e^{-\frac{2\pi i}{p}(w(\overline{\lambda}+\overline{\rho})|\overline{\lambda}'+\overline{\rho})}. \tag{4.4.3}$$

We can rewrite now Proposition 4.4 in a form more suitable for applications.

THEOREM 4.4. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine algebra associated either to a symmetric matrix $A$ or to $A = A_{2\ell}^{(2)}$. Let $p$ and $p'$ be relatively prime integers such that $p, p' \geq h$ and let $\lambda \in P_+^{p-h^\vee}$, $\lambda' \in P_+^{p'-h^\vee}$. Then*

$$\varphi_{\lambda,\lambda'}(-\frac{1}{\tau}) = \sum_{(\mu,\mu') \in P_+^{p-h^\vee} \times P_+^{p'-h^\vee} \mod \tilde{W}_+} S_{(\lambda\lambda')(\mu,\mu')} \varphi_{\mu\mu'}, \tag{4.4.4}$$

*where*

$$\begin{aligned} S_{(\lambda\lambda')(\mu\mu')} &= (-1)^{|\overline{\Delta}_+|} \varepsilon_{p'}(\lambda) \varepsilon_p(\mu') |J|^{1/2} e^{2\pi i((\lambda+\rho|\mu'+\rho)+(\mu+\rho|\lambda'+\rho))} S_{\lambda_{p'}\mu} S_{\lambda'\mu'_p} \\ &= (-1)^{|\overline{\Delta}_+|} |J|^{1/2} \varepsilon_{\mu\mu'} \varepsilon'_{\mu\mu'} S_{\lambda\tilde{\mu}} S_{\lambda'\tilde{\mu}'}. \end{aligned} \tag{4.4.5}$$

PROOF: Using that $\varphi_{\lambda,\lambda'} = \varphi_{\lambda',\lambda}$ we may assume that $p'$ is odd (we shall need this in the proof for $A_{2\ell}^{(2)}$). Note that (4.2.3) can be written as follows, using (4.2.8):

$$\varphi_{\lambda,\lambda'} = \psi_{\Lambda_{k,\overline{y}}(\lambda,\lambda')}(\tau) \text{ for any } \overline{y} \in \overline{W}. \tag{4.4.6}$$

Hence we have by Proposition 4.4:

$$\varphi_{\lambda,\lambda'}(-1/\tau) = (-i)^{|\overline{\Delta}_+|} \sum_{(\mu,\mu') \in P^{p-h^\vee} \times P_+^{p'-h^\vee} \mod \tilde{W}_+} \sum_{\overline{y} \in \overline{W}} S_{\Lambda_{k,1}(\lambda,\lambda'),\Lambda_{k,\overline{y}}(\mu,\mu')} \varphi_{\mu,\mu'}(\tau).$$

Hence we need to prove

$$S_{(\lambda\lambda')(\mu\mu')} = (-i)^{|\overline{\Delta}_+|} S_{\Lambda_{k,1}(\lambda,\lambda'),\Lambda_{k,\overline{y}}(\mu,\mu')}. \tag{4.4.7}$$

By (4.3.2) the right-hand side of (4.4.7) is equal to

$$\begin{aligned} &\sum_{y \in \overline{W}} \varepsilon(y)(pp')^{-\ell/2} |M^*/M|^{-1/2} e^{-2\pi i((\lambda+\rho|\beta')+(\mu+\rho|\beta)+(p/p')(\beta|\beta'))} \\ &\times \sum_{w \in \overline{W}} \varepsilon(w) e^{-\frac{2\pi i p'}{p}(w(\lambda+\rho)|\mu+\rho)}, \end{aligned} \tag{4.4.8}$$

where $\beta = p'\Lambda_0 - \lambda' - \rho$, $\beta' = p'\Lambda_0 - y(\mu' + \rho)$. Substituting $\beta$ and $\beta'$ in (4.4.8), we obtain:

$$\begin{aligned} &(pp')^{-\ell/2} |M^*/M|^{-1/2} e^{2\pi i((\lambda+\rho|\mu'+\rho)+(\lambda'+\rho|\mu+\rho))} \\ &\times \sum_{y \in \overline{W}} \varepsilon(y) e^{-2\pi i(p/p')(y(\lambda'+\rho)|\mu'+\rho)} \sum_{w \in \overline{W}} \varepsilon(w) e^{-2\pi i(p'/p)(w(\lambda+\rho)|\mu+\rho)}. \end{aligned}$$

This gives us (4.4.4) with the first expression of $S_{(\lambda\lambda')(\mu\mu')}$ in (4.4.5). To get the second expression, we rewrite $e^{2\pi i(\lambda+\rho|\mu'+\rho)} = e^{2\pi i(w(\lambda+\rho)|\mu'+\rho)} = e^{\frac{2\pi i}{p}(pw(\lambda+\rho)|\mu'+\rho)}$, and similarly for $e^{2\pi i(\lambda'+\rho|\mu+\rho)}$. By (4.4.3), (1.3.5) and the definition of the map $(\mu, \mu') \longmapsto (\tilde{\mu}, \tilde{\mu}')$ and the $\varepsilon_{\mu\mu'}, \varepsilon'_{\mu\mu'}$, this is equal to $(-1)^{|\overline{\Delta}_+|} |J|^{1/2} \varepsilon_{\mu\mu'} \varepsilon'_{\mu\mu'} S_{\lambda\tilde{\mu}} S_{\lambda'\tilde{\mu}'}$. □

Using Theorem 4.1a and the fact that $S_{00} = 1$, we obtain the following corollary of Theorem 4.4:

COROLLARY 4.4. *Let* $\mathfrak{g}$ *be as in Theorem 4.4 and let* $u \in \mathbb{N}$ *be relatively prime to* $h$. *For each* $\lambda \in P_+^u$ *choose the unique* $\Lambda \in P_+^1$ *such that* $\lambda - \Lambda \in (u-1)\Lambda_0 + Q$, *and let* $b_\lambda(\tau) = b_\lambda^\Lambda(\tau)$. *Then*

$$b_\lambda(-\frac{1}{\tau}) = \varepsilon_u(0)\varepsilon_h(\lambda)|J|^{1/2} \sum_{\mu \in P_+^u} e^{2\pi i(\lambda+\mu|\rho)} S_{\lambda_h,\mu} b_\mu(\tau). \qquad \square$$

Note that $\varepsilon_u(0)$ is the generalized Legendre symbol [7]:

$$\varepsilon_u(0) = \prod_{\alpha \in \overline{\Delta}_+} \sin \frac{\pi u(\overline{\rho}|\alpha)}{h^\vee} \Big/ \sin \frac{\pi(\overline{\rho}|\alpha)}{h^\vee}.$$

In a similar way, using Proposition 4.4 and Remark 4.4 one can derive the following transformation formula valid in all cases:

$$\varphi_{\lambda,\lambda'}(-\frac{1}{\tau}) = \sum_{\substack{\mu \in P_+^{\prime p - h^\vee} \\ \mu' \in P^{\prime p' - h^\vee} (\text{resp} \in P^{\vee p' - h})}} S_{(\lambda\lambda')(\mu\mu')} \varphi'_{\mu,\mu'}(\frac{a_0 \tau}{r}), \tag{4.4.9}$$

where

$$\begin{aligned} S_{(\lambda\lambda')(\mu\mu')} &= (-1)^{|\overline{\Delta}_+|}(\frac{r}{a_0})^{|\overline{\Delta}_{+,s}|} \varepsilon_{p'}(\lambda) S_{\lambda_{p'},\lambda'} \\ &\times |M^*/\overline{Q}^\vee|^{1/2} \varepsilon_p(\mu') e^{2\pi i((\lambda+\rho|\mu'+\rho')+(\mu+\rho'|\lambda'+\rho)} S_{\mu\mu'_p} \\ &(\text{resp. } \times |M^*/M|^{1/2} \varepsilon_p^\vee(\mu') e^{2\pi i((\lambda+\rho|\mu'+\rho^\vee)+(\mu+\rho|\lambda'+\rho^\vee)} S^\vee_{\mu\mu_p'^\vee}). \end{aligned}$$

Here for $\lambda \in P_+^{\vee p' - h}$ we define $\lambda_p^\vee$ and $\varepsilon_p^\vee(\lambda)$ as follows: there exists a unique $w \in \overline{W}$ and a unique $\lambda_p^\vee \in P_+^{\vee p - h}$ such that $p'(\overline{\lambda} + \overline{\rho}^\vee) - w(\overline{\lambda}_p^\vee + \overline{\rho}) \in pM$, and we let $\varepsilon_p^\vee(\lambda) = \varepsilon(w)$; for $\lambda, \lambda' \in P_+^{\vee p' - h}$ we let (cf. (4.4.3)):

$$S^\vee_{\lambda\lambda'} = i^{|\overline{\Delta}_+|} p'^{-\ell/2} |M^*/M|^{1/2} \sum_{w \in \overline{W}} \varepsilon(w) e^{-\frac{2\pi i}{p'}(w(\overline{\lambda}+\overline{\rho})|\overline{\mu}+\overline{\rho}^\vee)}.$$

In conclusion, note the following useful property of the map $(\lambda, \lambda') \longmapsto (\tilde{\lambda}, \tilde{\lambda}')$:

LEMMA 4.4. *Let* $p, p' \in \mathbb{N}$ *be such that* $p, p' \geq h^\vee$ *and* $(p,p') = (p, r^\vee) = (p', r^\vee) = 1$. *Let* $\lambda, \mu \in P_+^{p-h^\vee}$ *and* $\lambda', \mu' \in P_+^{p'-h^\vee}$. *Then* $(\tilde{\lambda}, \tilde{\lambda}') = (\tilde{\mu}, \tilde{\mu}')$ *if and only if there exists* $\sigma \in \tilde{W}_+$ *such that*

$$\lambda + \rho = \sigma(\mu + \rho) \quad \text{and} \quad \lambda' + \rho = \sigma(\mu' + \rho).$$

PROOF: The "if" part is clear by the definition. Suppose now that $\tilde{\lambda} = \tilde{\mu}$. Consider the principal admissible weights

$$\Lambda = t_\beta.(\lambda - (p'-1)(k+h^\vee)\Lambda_0), \; \Lambda' = t_{\beta'}.(\mu - (p'-1)(k+h^\vee)\Lambda_0), \tag{4.4.10}$$

where $\beta = -(\overline{\lambda}' + \overline{\rho})$, $\beta' = -(\overline{\mu}' + \overline{\rho})$. Then by the definition of $\tilde{\lambda}$ and $\tilde{\mu}$, $p'(\Lambda + \rho) \in W(\tilde{\lambda} + \rho)$ and $p'(\Lambda' + \rho) \in W(\tilde{\mu} + \tilde{\rho})$, hence

$$\Lambda' + \rho = w(\Lambda + \rho) \text{ for some } w \in W. \tag{4.4.11}$$

But the corresponding simple coroot bases are $t_{\beta'}(\Pi_{[u]})$ and $wt_{\beta}(\Pi_{[u]})$. Hence, by Proposition 1.5b, there exists $\sigma = t_\alpha \overline{\sigma} \in \tilde{W}_+$ such that

$$t_{-\beta} w^{-1} t_{\beta'} = t_{p'\alpha} \overline{\sigma}. \tag{4.4.12}$$

From (4.4.10–12) we obtain:

$$t_{p'\alpha}\overline{\sigma}.(\mu - (p'-1)(k+h^\vee)\Lambda_0) = \lambda - (p'-1)(k+h^\vee)\Lambda_0.$$

But the left–hand side of the last equality is equal to $\sigma.\mu - (p'-1)(k+h^\vee)\Lambda_0$, hence $\lambda = \sigma.\mu$. Similarly, $\lambda' = \sigma'.\mu'$, and it remains to show that $\sigma = \sigma'$.

Letting $w = t_\gamma \overline{w}$, we obtain from (4.4.12):

$$t_{-\beta + \overline{w}^{-1}(\beta' - \gamma)} \overline{w}^{-1} = t_{p'\alpha}\overline{\sigma}.$$

Since $\gamma \in M$, it follows that $p'\alpha + \beta - \beta' \in M$. Since also $\beta' - \beta = \sigma'(\mu' + \rho) - (\mu' + \rho) = \overline{\sigma}'(\mu' + \rho) + p'\alpha'$, we derive that $p'(\alpha - \alpha') \in M$. Similarly, $p(\alpha - \alpha') \in M$, and since $(p, p') = 1$, we deduce that $\alpha - \alpha' \in M$. It follows that $\sigma = \sigma'$. □

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