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Exceptional Hierarchies of Soliton Equations

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Dedicated to Professor M. Sato on his 60th birthday

0. Introduction. The connection between the soliton theory and the classical affine Kac-Moody algebras was developed in the early 1980s by Date, Jimbo, Kashiwara, and Miwa **[3]–[6]**, using the boson-fermion correspondence in the 2-dimensional QFT.

To explain this connection, introduce some representation-theoretical background (see **[3, 15]** for details). Consider the Clifford algebra on generators ψ_j and ψ_j^*, $j \in \mathbb{Z}$, with commutation relations

$$[\psi_i, \psi_j]_+ = [\psi_i^*, \psi_j^*]_+ = 0, \qquad [\psi_i, \psi_j^*]_+ = \delta_{i,j},$$

and consider its spin representation in a vector space F with the vacuum vector $|0\rangle$ such that

$$\psi_j|0\rangle = 0 \quad \text{for } j \leq 0, \qquad \psi_j^*|0\rangle = 0 \quad \text{for } j > 0.$$

The group GL_∞ of automorphisms of the space $\Psi = \sum_{j\in\mathbb{Z}} \mathbb{C}\psi_j$ that leave all but a finite number of the ψ_j fixed acts also on $\Psi^* = \sum_{j\in\mathbb{Z}} \mathbb{C}\psi_j^*$, which is identified with a subspace in the dual of Ψ via $\langle\psi_i, \psi_j^*\rangle = \delta_{ij}$. For $m \in \mathbb{N}$, let

$$|m\rangle = \psi_m \cdots \psi_1|0\rangle \quad \text{and} \quad |-m\rangle = \psi_{-m+1}^* \cdots \psi_0^*|0\rangle.$$

For $g \in \mathrm{GL}_\infty$, let $m \in \mathbb{Z}_+$ be such that $g \cdot \psi_{-j} = \psi_{-j}$ for $j \geq m$; then we can define a representation R of GL_∞ on F by

$$\begin{aligned} &R(g)(\psi_{i_1}\psi_{i_2}\cdots\psi_{j_1}^*\psi_{j_2}^*\cdots|-m\rangle) \\ &\quad = (g\cdot\psi_{i_1})(g\cdot\psi_{i_2})\cdots(g\cdot\psi_{j_1}^*)(g\cdot\psi_{j_2}^*)\cdots|-m\rangle. \end{aligned}$$

This representation preserves the charge decomposition

$$F = \bigoplus_{m\in\mathbb{Z}} F^{(m)}$$

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defined by charge$(\psi_j) = -$ charge$(\psi_j^*) = 1$, charge$(|0\rangle) = 0$.

The main object of interest is the orbit $\mathscr{O} = R(\mathrm{GL}_\infty)|0\rangle$ of the vacuum vector. In order to write down a system of equations defining $\mathscr{O}$, introduce the following operators on the space $F \otimes F$ [15]:

$$S = \sum_{j\in\mathbf{Z}} \psi_j \otimes \psi_j^*, \qquad S^* = \sum_{j\in\mathbf{Z}} \psi_j^* \otimes \psi_j.$$

It is easy to see that S commutes with the diagonal action of GL_∞ on $F \otimes F$ and that $S(|0\rangle \otimes |0\rangle) = 0$ (the same is true for S^*). It follows that any element $\tau \in \mathscr{O}$ satisfies the equation

(0.1) $$S(\tau\otimes\tau) := \sum_{j\in\mathbf{Z}} \psi_j \cdot \tau \otimes \psi_j^* \cdot \tau = 0.$$

(One can show that the converse also holds: if $\tau \neq 0$ satisfies (0.1), then $\tau \in \mathscr{O}$; see [16] for a proof.)

Using the boson-fermion correspondence, one can rewrite (0.1) as a system of partial differential equations as follows. For $n \in \mathbf{Z} \setminus \{0\}$, let

$$\alpha_n = \sum_{j\in\mathbf{Z}} \psi_j \psi_{j+n}^*.$$

One checks that the operators α_n satisfy the canonical commutation relations:

$$[\alpha_m, \alpha_n] = m\delta_{m,-n},$$

and that they act irreducibly on each space $F^{(m)}$. This allows us to establish an isomorphism, the so-called boson-fermion correspondence,

$$\sigma_m : F^{(m)} \xrightarrow{\sim} B^{(m)} = \mathbf{C}[x_1, x_2, \dots]$$

(where $\mathbf{C}[x_1, x_2, \dots]$ is a polynomial algebra on infinitely many indeterminates $x = (x_1, x_2, \dots)$), which is determined by the following conditions:

$$\sigma_m(|m\rangle) = 1; \quad \sigma_m\alpha_n\sigma_m^{-1} = \frac{\partial}{\partial x_n} \quad \text{and} \quad \sigma_m\alpha_{-n}\sigma_m^{-1} = nx_n \quad \text{for } n = 1, 2 \dots.$$

The essential part of the boson-fermion correspondence is the calculation of the operators $\sigma_{m+1}\psi_j\sigma_m^{-1}$: $B^{(m)} \to B^{(m+1)}$. In order to do that, introduce the following generating series

$$\psi(z) = \sum_{j\in\mathbf{Z}} \psi_j z^j, \qquad \psi^*(z) = \sum_{j\in\mathbf{Z}} \psi_j^* z^{-j} \qquad (z \in \mathbf{C}^\times),$$

and the following differential operators of infinite order

$$\Gamma_+(z) = \exp - \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\frac{\partial}{\partial x_n}, \qquad \Gamma_-(z) = \exp \sum_{n=1}^{\infty} z^n x_n.$$

Then a short calculation gives for $\tau(x) \in B^{(m)}$:

(0.2a) $$(\sigma_{m+1}\psi(z)\sigma_m^{-1})\tau(x) = z^{m+1}\Gamma_-(z)\Gamma_+(z)\tau(x) \in B^{(m+1)},$$

(0.2b) $$(\sigma_{m-1}\psi^*(z)\sigma_m^{-1})\tau(x) = z^{-m}\Gamma_-^{-1}(z)\Gamma_+^{-1}(z)\tau(x) \in B^{(m-1)}.$$

The operators of this type are called vertex operators.

Now, a crucial (but very simple) observation is that (0.1) can be rewritten as follows:

(0.3) $$z^0\text{-term of } \psi(z)\tau \otimes \psi^*(z)\tau = 0, \qquad \tau \in F^{(0)}.$$

We think of $B^{(0)} \otimes B^{(0)}$ as of a polynomial algebra $\mathbf{C}[x_1', x_2', \dots; x_1'', x_2'', \dots]$. Then, applying σ_0 to both sides of (0.3), we obtain a system of equations on the orbit $(\sigma_0 R \sigma_0^{-1})(\mathrm{GL}_\infty)\cdot 1 = \sigma_0(\mathscr{O})$ in $B^{(0)}$, which using (0.2) can be written as follows:

(0.4)
$$z^{-1}\text{-term of } \exp\sum_{j\geq 1} z^j(x_j' - x_j'') \exp - \sum_{j\geq 1}\frac{z^{-j}}{j}\left(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\right)\tau(x')\tau(x'') = 0.$$

Making the change of variables

(0.5a) $$x = \tfrac{1}{2}(x' + x''), \qquad y = \tfrac{1}{2}(x' - x''),$$

we have

(0.5b) $$\frac{\partial}{\partial x'} - \frac{\partial}{\partial x''} = \frac{\partial}{\partial y},$$

and (0.4) becomes

$$z^{-1}\text{-term of } \left(\exp\sum_{j\geq 1} 2z^j y_j\right)\left(\exp - \sum_{j\geq 1}\frac{z^{-j}}{j}\frac{\partial}{\partial y_j}\right)\tau(x+y)\tau(x-y) = 0.$$

Introducing the elementary Schur polynomials $p_k(x)$, $k \in \mathbf{Z}$, by

(0.6) $$\sum_{k\in\mathbf{Z}} p_k(x)u^k = \exp\sum_{k=1}^{\infty} u^k x_k,$$

we can rewrite this equation as follows:

(0.7) $$\sum_{j=0}^{\infty} p_j(2y)p_{j+1}\left(-\frac{\tilde\partial}{\partial y}\right)\tau(x+y)\tau(x-y) = 0.$$

Here and further we use the notation

(0.8) $$\tilde x = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots\right), \qquad \frac{\tilde\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \frac{1}{3}\frac{\partial}{\partial x_3}, \dots\right).$$

Using Taylor's formula, we can rewrite (0.7) once more:

(0.9) $$\sum_{j=0}^{\infty} p_j(2y)p_{j+1}\left(-\frac{\tilde\partial}{\partial u}\right) e^{\sum_{r=1}^{\infty} y_r \partial/\partial u_r}\tau(x+u)\tau(x-u)|_{u=0} = 0.$$

(0.9) is a generating series (with $y_1, y_2, \dots$ as free parameters) of a system of Hirota bilinear equations:

(0.10) $$\sum_{j=0}^{\infty} p_j(2y)p_{j+1}(-\tilde D)e^{\sum_{r=1}^{\infty} y_r D_r}\tau\cdot\tau = 0.$$

Recall that for a polynomial P, the corresponding Hirota bilinear equation on functions f and g is defined as follows:

$$P(D)f\cdot g := P(\partial/\partial u)f(x+u)g(x-u)|_{u=0} = 0. \tag{0.11}$$

We can write (0.10) as

$$\sum_{0\le j_1\le\cdots\le j_r} y_1^{j_1}\cdots y_r^{j_r}P_{j_1,\dots,j_r}(D)\tau(x)\cdot\tau(x) = 0,$$

where $P_{j_1,\dots,j_r}(x)$ are certain polynomials. Thus, the orbit $\mathrm{GL}_\infty\cdot 1$ in the space $\mathbb{C}[x_1,x_2,\dots]$ is given by the system of Hirota bilinear equations

$$P_{j_1,\dots,j_r}(D)\tau\cdot\tau = 0.$$

This system is called the KP hierarchy. For example,

$$P_j(x) = 2p_{j+1}(-\tilde{x}) - x_1x_j.$$

In particular, $P_1 = -2x_2$, $3P_2 = x_1^3$, $12P_3 = (x_1^4 - 4x_1x_3 + 3x_2^2) - 6(x_1^2x_2 + x_4)$. Noting that the Hirota bilinear equation $Pf\cdot f = 0$ with $P(-x) = -P(x)$ is trivial, we obtain that the Hirota bilinear equation

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau\cdot\tau = 0 \tag{0.12}$$

is the simplest of the equations of the KP hierarchy.

Taking as a basis of $\mathbb{C}[y_1,y_2,\dots]$ the Schur polynomials $S_\lambda(y)$, one can write down all equations of the KP hierarchy explicitly **[21]**.

Putting

$$x = x_1,\quad y = x_2,\quad t = x_3,\quad u(x,y,t) = (2\log\tau(x,y,t,x_4,x_5,\dots))_{xx},$$

where $x_4, x_5, \dots$ are viewed as free parameters, we see after a calculation that if τ satisfies (0.12), then $u(x,y,t)$ satisfies the classical Kadomtsev-Petviashvili (KP) equation:

$$\tfrac{3}{4}u_{yy} = (u_t - \tfrac{3}{2}uu_x - \tfrac{1}{4}u_{xxx})_x.$$

In order to construct solutions of the KP hierarchy, note that the operator $\psi(z)\psi^*(z')$ lies in the completion of the Lie algebra of GL_∞ acting on the space $F^{(0)}$. Using (0.2), we deduce that the following vertex operator lies in the completion of the Lie algebra of GL_∞ acting on the space $\mathbb{C}[x]$:

$$\Gamma(z,z') = \left(\exp\sum_{j\ge1}(z^j - z'^j)x_j\right)\left(\exp -\sum_{j\ge1}\frac{z^{-j}-z'^{-j}}{j}\frac{\partial}{\partial x_j}\right).$$

Using this, it is not difficult to see that if τ is a solution of the KP hierarchy, then $(1 + a\Gamma(z,z'))\tau$, where $a, z, z' \in \mathbb{C}$, $z, z' \ne 0$, is one as well (see **[16]** for a proof). Since 1 is a solution, we obtain that

$$(1 + a_N\Gamma(z_N,z_N'))\cdots(1 + a_1\Gamma(z_1,z_1'))\cdot 1$$

is a solution as well. This is the so-called N-soliton solution of the KP hierarchy **[3]**.

Using the boson-fermion correspondence, one can find polynomial solutions of the KP hierarchy as well **[10, 15]**. It turns out that all Schur polynomials $S_\lambda(x)$ (attached to linear representations of symmetric groups) are solutions (Sato **[26]**). A similar, but somewhat different, more geometric approach, allows one to obtain "quasiperiodic" solutions **[29]**. It turns out that all theta functions of algebraic curves are solutions, which links the KP hierarchy to the Schottky problems **[27]**.

Using the so-called reduction procedure (see, e.g., **[6, 15]**) one can write down a hierarchy of Hirota bilinear equations for the orbit of the highest weight vector in the basic representation of the loop group of SL_n. We thus obtain the KdV hierarchy ($n = 2$), the Boussinesq hierarchy ($n = 3$), etc., and can in a similar fashion construct their solutions.

A similar approach can be applied to the group O_∞ obtaining the so-called BKP hierarchy **[4]** and its solutions. It turns out that its polynomial solutions are polynomials attached to projective representations of symmetric groups **[32]**. The "quasiperiodic" solutions of BKP turn out to be the Prym theta functions **[28]**. Finally, the reduction procedure applied to BKP produces hierarchies associated to some other classical loop groups **[6]**.

It has remained an open problem, however, how to construct the hierarchies associated to arbitrary loop groups in a unified fashion, including the exceptional ones (an algorithm for constructing low degree equations was given in **[11]** and used in **[25, 30]**,...). In the present paper we are addressing this problem.

The basic observation is very simple. There is no analogue of the operator S in general, but since (0.1) is equivalent to $S^*S(\tau\otimes\tau) = 0$, we may consider the operator S^*S instead, which is the Casimir operator for GL_∞!

Thus we are led to the following general setup. Let $\mathfrak{g}$ be a Lie algebra with an invariant symmetric nondegenerate bilinear form $(\cdot|\cdot)$, and let V be a representation of $\mathfrak{g}$ that "integrates" to a representation of the corresponding group G. Let $\{u_j\}$ and $\{u^j\}$ be dual bases of $\mathfrak{g}$, i.e., $(u_i|u^j) = \delta_{ij}$. We assume that for each $v_1, v_2 \in V$, both $u_j(v_1)$ and $u^j(v_2)$ are nonzero for only a finite number of j. Then we can define the following operator on $V\otimes V$:

$$S = \sum_j u_j\otimes u^j.$$

One easily checks that S commutes with $\mathfrak{g}$ and hence with G. Now if $v^0 \in V$ is such that $v^0\otimes v^0$ is an eigenvector of S with eigenvalue $a \in \mathbb{C}$, then the orbit $G\cdot v^0$ satisfies the equation

$$S(v\otimes v) = a(v\otimes v)\quad \text{for } v \in G\cdot v^0.$$

Applying this observation (and some other simple arguments) to a Kac-Moody algebra $\mathfrak{g}$ with a symmetrizable Cartan matrix, its integrable highest weight representation $L(\Lambda)$, and its highest weight vector $v^0 = v_\Lambda$, one proves the following

THEOREM 0.1 **[24]**. *Let $\mathfrak{g}$ be a Kac-Moody algebra with a symmetrizable Cartan matrix, and let G be the associated group. Let $\{u_j\}$ and $\{u^j\}$ be bases of $\mathfrak{g}$ dual with respect to a nondegenerate invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$, and consistent with the triangular decomposition of $\mathfrak{g}$. Let $L(\Lambda)$ be an integrable representation of $\mathfrak{g}$ with highest weight Λ, and let v_Λ be its highest weight vector. Then*

(a) *A nonzero vector v of $L(\Lambda)$ lies in the orbit $G\cdot v_\Lambda$ if and only if*

$$\sum_j u_j(v)\otimes u^j(v) = (\Lambda|\Lambda)v\otimes v \quad \text{in } L(\Lambda)\otimes L(\Lambda). \tag{0.13}$$

(b) *A vector v of $L(\Lambda)$ satisfies (0.13) if and only if $v\otimes v$ lies in the highest component of $L(\Lambda)\otimes L(\Lambda)$.*

Actually, more is true, equations (0.13) are essentially all equations for $G\cdot v_\Lambda$:

THEOREM 0.2 **[13]**. *Equations (0.13) generate the ideal of all equations of $G\cdot v_\Lambda$ in the symmetric algebra over $L^*(\Lambda)$.*

In this paper we explain how to write down (0.13) in terms of Hirota bilinear equations and its super analogue in the following situation:

DEFINITION. Consider the following data:

(i) an affine Kac-Moody algebra $\mathfrak{g}$,
(ii) an integrable highest weight representation V of $\mathfrak{g}$,
(iii) a vertex operator construction R of V.

Then (0.13) is called the hierarchy of soliton equations associated to the data $(\mathfrak{g}, V, R)$.

Let $\bar{\mathfrak{g}}$ be a simple complex finite-dimensional Lie algebra, and let $\mathfrak{g}$ be the associated affine Kac-Moody algebra. Provided that $\bar{\mathfrak{g}}$ is of A-D-E type, to each conjugacy class w of the Weyl group of $\bar{\mathfrak{g}}$ one can associate a bosionic vertex operator construction R_w of the basic (and, more generally, every level 1) representation V of $\mathfrak{g}$ **[14]**. Then the associated hierarchy of soliton equations can be written as a system of Hirota bilinear equations. In the present paper we write down these systems explicitly in two most interesting cases. First, when $w=\sigma$ is the Coxeter element, the so-called principal picture **[12]** (and, more generally, the case of primitive w **[14]**), and second, when $w=1$, the homogeneous picture **[8]**. Note that the hierarchies thus obtained are very different for different realizations R of the same representation. For example, $\mathfrak{g}=\widehat{sl}_2$ has two realizations, R_σ and R_1; the associated hierarchies are the KdV and the nonlinear Schrödinger hierarchies. Note that even in this case our form of the KdV hierarchy is simpler than that obtained by the reduction procedure in **[5]**, and (as far as we know) only the first few equations of the nonlinear Schrödinger hierarchy have been written down (see **[25, 30]**). We mention that the 2-dimensional Toda lattice hierarchy (see, e.g., **[31]**) is part of the $(\widehat{sl}_3, R_1)$-hierarchy.

If $\bar{\mathfrak{g}}$ is of type B_l, then to each w one still can associate a vertex operator construction of the basic representation, but in this case an additional fermionic field is involved. In the homogeneous picture this has been done by many authors (see **[1, 9, 20]** and references there). In this paper we do this in the principal picture. The case $l=1$ produces a construction of level 2 representations of $\widehat{sl}_2$. The corresponding system of equations turn out to be a hierarchy of super Hirota bilinear equations. Its relation to the known hierarchies of supersoliton equations **[19, 23]** remains unclear.

Of course, the N-soliton solutions of all these hierarchies can be constructed, as above, by iterated application of vertex operators to 1. We use this method, for example, to construct the N soliton, N antisoliton, and N soliton-antisoliton solutions of the nonlinear Schrödinger hierarchy. However, the determination of the polynomial solutions (more precisely, the orbit of the affine Weyl group) remains an open problem. This problem is settled for $(\widehat{sl}_n, R_\sigma)$ by making use of the boson-fermion correspondence **[10, 15]**. Another problem is to determine the theta function solutions. Recent work **[18]** gives a hint that these should be certain Prym-Tjurin theta functions.

Finally, it is well known that the KP and KdV hierarchies are infinite-dimensional Hamiltonian systems: they can be written in a Lax form, as deformation equations of a pseudodifferential operator. We do not know whether this can be done in our general setting. There should be certainly a connection to the work of Drinfeld and Sokolov **[7]**.

Throughout the paper, the base field is the field $\mathbb{C}$ of complex numbers. Symbols $\mathbb{Z}, \mathbb{Z}_{\text{odd}}, \mathbb{N}, \mathbb{N}_{\text{odd}}$, and $\mathbb{Z}_+$ stand for the set of integers, odd integers, positive integers, odd positive integers, and nonnegative integers respectively. Throughout the paper we use notations and basic definition of **[11]** unless otherwise specified.

1. Principal picture in the A-D-E case and exceptional hierarchies of Hirota bilinear equations.

1.1. We start with a (realization-free) description of the principal vertex construction of the basic representation of an affine Kac-Moody algebra $\mathfrak{g}'$ associated to a rank l affine matrix of type $X_N^{(k)}$ (cf. **[11, 12]**). Let $c\in\mathfrak{g}'$ be the canonical central element, and h the Coxeter number.

Let $\mathfrak{g}'=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be the principal gradation of $\mathfrak{g}'$. The Lie algebra $\mathfrak{g}'$ can be embedded in a Lie algebra $\hat{\mathfrak{g}}=\mathfrak{g}'\oplus(\bigoplus_{j\in\mathbb{Z}}\mathbb{C}d_j)$ such that

$$[d_n, \mathfrak{g}_0]=0, \tag{1.1a}$$

$$[d_n, \mathfrak{g}_j]\subset \mathfrak{g}_{j+nkh}, \tag{1.1b}$$

$$[d_0, a]=ja \quad \text{for } a\in\mathfrak{g}_j, \tag{1.1c}$$

$$d_n d_m(a)=(j+mkh)d_{m+n}(a) \quad \text{for } a\in\mathfrak{g}_j, \tag{1.1d}$$

$$[d_m, d_n]=kh(n-m)d_{m+n}+(N(kh)^2/12)\delta_{m,-n}(m^3-m)c. \tag{1.1e}$$

We denote by $\mathfrak{g}$ the subalgebra $\mathfrak{g}' + \mathbb{C}d_0$ ($\mathfrak{g}$ is also called an affine Kac-Moody algebra). We let

(1.2a) $$\overline{\mathfrak{g}}_0 = [d_1, \mathfrak{g}_{-kh}] \subset \mathfrak{g}_0 \subset \mathfrak{g}',$$

(1.2b) $$\mathring{\mathfrak{h}} = \sum_{i=1}^{l} \mathbb{C}\alpha_i^\vee .$$

Let $\mathring{\rho}^\vee$ be an element in $\mathring{\mathfrak{h}}$ such that

(1.3) $$\langle \alpha_i, \mathring{\rho}^\vee \rangle = 1 \quad \text{for } i = 1, \dots, l.$$

Here $\{\alpha_0, \dots, \alpha_l\}$ and $\{\alpha_0^\vee, \dots, \alpha_l^\vee\}$ are the sets of simple roots and simple coroots respectively. Then $\mathfrak{g}_0 = \overline{\mathfrak{g}}_0 \oplus \mathbb{C}c = \mathring{\mathfrak{h}} \oplus \mathbb{C}c$. For $a \in \mathfrak{g}_0$ we denote by $\overline{a}$ and $\mathring{a}$ its projections on $\overline{\mathfrak{g}}_0$ and $\mathring{\mathfrak{h}}$ respectively.

The Lie algebra $\mathfrak{g}$ carries a nondegenerate invariant bilinear form $(\cdot|\cdot)$ such that

(1.4a) $$[a, b] = \overline{[a, b]} + h^{-1} j (a|b) c \quad \text{for } a \in \mathfrak{g}_j, b \in \mathfrak{g}_{-j},$$

(1.4b) $$(c|d_0) = h,$$

(1.4c) $$(d_0|d_0) = (\mathring{\rho}^\vee|\mathring{\rho}^\vee).$$

It follows from (1.1c), (1.4a), and (1.4b) that $\overline{\mathfrak{g}}_0$ is orthogonal to c and d_0. Let d be the element of $\mathfrak{g}_0 + \mathbb{C}d_0$ defined by

(1.5a) $$(d|\mathring{\mathfrak{h}}) = 0, \quad (c|d) = a_0, \quad (d|d) = 0,$$

where $a_0 = 2$ for $A_{2l}^{(2)}$ and $= 1$ otherwise. Then we have

(1.6) $$d_0 = a_0^{-1} h d + \mathring{\rho}^\vee .$$

The connection between $\overline{\mathfrak{g}}_0$ and $\mathring{\mathfrak{h}}$ is given by the following formula:

(1.7) $$x = \mathring{x} - h^{-1}(\mathring{\rho}^\vee|\mathring{x})c \quad \text{for } x \in \mathfrak{g}_0 .$$

Indeed, putting $x = \mathring{x} + \xi c$ for some $\xi \in \mathbb{C}$, we obtain, using (1.5) and (1.6):

$$0 = (d_0|x) = (a_0^{-1} h d + \mathring{\rho}^\vee|\mathring{x} + \xi c) = h\xi + (\mathring{\rho}^\vee|\mathring{x}).$$

Let E (resp. E_+) be the set of all (resp. all positive) exponents of $\mathfrak{g}$. For each $j \in E$, one can pick $H_j \in \mathfrak{g}_j$ such that

(1.8a) $$[H_i, H_j] = i\delta_{i,-j} c,$$

(1.8b) $$[d_n, H_j] = j H_{j+nkh}.$$

The subalgebra $\mathfrak{s} = \mathbb{C}c + \sum_{j \in E} \mathbb{C}H_j$ is called the principal Heisenberg subalgebra of $\mathfrak{g}'$. Note that

(1.8c) $$(H_i|H_j) = h\delta_{i,-j}.$$

For each $i \in \mathbb{Z}$ and $r = 1, \dots, l$, there exist elements $X_i^{(r)} \in \mathfrak{g}_i$ such that

(1.9a) $$X_0^{(r)}, r = 1, \dots, l, \text{ form a basis of } \overline{\mathfrak{g}}_0,$$

(1.9b) $$[H_j, X_i^{(r)}] = \beta_{r,j} X_{i+j}^{(r)} \quad \text{for some } \beta_{r,j} \in \mathbb{C},$$

(1.9c) $$[d_n, X_i^{(r)}] = i X_{i+nkh}^{(r)}.$$

Then the elements $\{H_j, X_i^{(r)}, c, d\}$ form a basis of $\mathfrak{g}$. Choose $Y_i^{(r)} \in \mathfrak{g}_i$ ($i \in \mathbb{Z}, r = 1, \dots, l$) such that

(1.10) $$(Y_i^{(r)}|X_j^{(s)}) = \delta_{r,s}\delta_{i,-j}, \quad (Y_i^{(r)}|\mathfrak{s}) = 0, \quad Y_0^{(r)} \in \overline{\mathfrak{g}}_0 .$$

Then it is easy to see that

(1.11a) $$[H_j, Y_i^{(r)}] = -\beta_{r,j} Y_{i+j}^{(r)},$$

(1.11b) $$[d_n, Y_i^{(r)}] = i Y_{i+nkh}^{(r)}.$$

Putting $X^{(r)}(z) = \sum_{i \in \mathbb{Z}} X_i^{(r)} z^{-i}$ and $Y^{(r)}(z) = \sum_{i \in \mathbb{Z}} Y_i^{(r)} z^{-i}$, (1.9b-c) and (1.11a-b) are rewritten as follows:

(1.9b′) $$[H_j, X^{(r)}(z)] = z^j \beta_{r,j} X^{(r)}(z),$$

(1.9c′) $$[d_n, X^{(r)}(z)] = -z^{nkh+1} \frac{\partial}{\partial z} X^{(r)}(z),$$

(1.11a′) $$[H_j, Y^{(r)}(z)] = -z^j \beta_{r,j} Y^{(r)}(z),$$

(1.11b′) $$[d_n, Y^{(r)}(z)] = -z^{nkh+1} \frac{\partial}{\partial z} Y^{(r)}(z).$$

It follows from (1.9b′-c′), (1.11a′-b′) that the basic representation $L(\Lambda_0)$ is constructed on the space $\mathbb{C}[x] = \mathbb{C}[x_j; j \in E_+]$ and that the operators corresponding to each element in $\mathfrak{g}$ are given by vertex operators and Virasoro operators:

(1.12a) $$H_j \mapsto a_j \quad \text{for } j \in E,$$

(1.12b)
$$X^{(r)}(z) \mapsto \widetilde{X}^{(r)}(z) = C_r \left(\exp \sum_{j \in E_+} \frac{\beta_{r,j} a_{-j}}{j} z^j \right) \left(\exp - \sum_{j \in E_+} \frac{\beta_{r,-j} a_j}{j} z^{-j} \right),$$

(1.12c)
$$Y^{(r)}(z) \mapsto \widetilde{Y}^{(r)}(z) = C_r' \left(\exp - \sum_{j \in E_+} \frac{\beta_{r,j} a_{-j}}{j} z^j \right) \left(\exp \sum_{j \in E_+} \frac{\beta_{r,-j} a_j}{j} z^{-j} \right),$$

(1.12d)
$$d_n \mapsto -\frac{1}{2} \sum_{j \in E} a_{nkh-j} a_j \quad \text{for } n \neq 0,$$
$$d_0 \mapsto -\sum_{j \in E_+} a_{-j} a_j,$$

(1.12e) $$c \mapsto 1,$$

where $a_j = \partial/\partial x_j$ and $a_{-j} = jx_j$ for $j \in E_+$, and the coefficients C_r and C'_r are given by the following formulas:

(1.13a) $$C_r = -h^{-1}(\mathring{\rho}^\vee | \mathring{X}_0^{(r)}),$$

(1.13b) $$C'_r = -h^{-1}(\mathring{\rho}^\vee | \mathring{Y}_0^{(r)}).$$

To prove these formulas, we compute the action of the constant term $X_0^{(r)}$ of the vertex operator $X^{(r)}(z)$ on the highest weight vector 1 in $L(\Lambda_0)$, using (1.7):

$$\begin{aligned}\pi_{\Lambda_0}(X_0^{(r)})1 &= \pi_{\Lambda_0}(\mathring{X}_0^{(r)}) \cdot 1 - h^{-1}(\mathring{\rho}^\vee | \mathring{X}_0^{(r)})\pi_{\Lambda_0}(c) \cdot 1 \\ &= \langle \Lambda_0, \mathring{X}_0^{(r)} \rangle \cdot 1 - h^{-1}(\mathring{\rho}^\vee | \mathring{X}_0^{(r)}) \cdot 1 = -h^{-1}(\mathring{\rho}^\vee | \mathring{X}_0^{(r)}) \cdot 1,\end{aligned}$$

since $\langle \Lambda_0, \mathring{\mathfrak{h}} \rangle = 0$, proving (1.13a). The proof of (1.13b) is similar.

Let $\overline{G}$ be the connected simply connected algebraic group over $\mathbb{C}$ with the Lie algebra $\bar{\mathfrak{g}}$, and let $G = \overline{G}(\mathbb{C}[t, t^{-1}])$, so that the Kac-Moody group associated to $\mathfrak{g}$ is a central extension of G by $\mathbb{C}^\times$. The group G acts projectively on each integrable representation $L(\Lambda)$ consistently with the action of $\mathfrak{g}$.

Now we can prove the following theorem:

THEOREM 1.1. *Consider the basic representation of a simply laced or twisted affine Kac-Moody algebra* $\mathfrak{g}$ *of rank* l *and of type* $X_N^{(k)}$ *on the space* $L(\Lambda_0) = \mathbb{C}[x_j; j \in E_+]$. *Then a nonzero element* τ *of* $L(\Lambda_0)$ *lies in the orbit* $G \cdot 1$ *if and only if* τ *satisfies the following hierarchy of Hirota bilinear differential equations*:

(1.14) $$\left\{-2h \sum_{j \in E_+} j y_j D_j + \sum_{r=1}^{l} b_r \sum_{n \geq 1} p_n^{(E)}(2\beta_{r,j} y_j) p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j} D_j\right)\right\} \times e^{\sum_{j \in E_+} y_j D_j} \tau \cdot \tau = 0,$$

where $b_r = (\mathring{\rho}^\vee | \mathring{X}_0^{(r)})(\mathring{\rho}^\vee | \mathring{Y}_0^{(r)})$ *and* $p_n^{(E)}(x)$, $n \in \mathbb{Z}_+$, *are the elementary Schur polynomials of* $\mathfrak{g}$ *defined by*

(1.15) $$\exp \sum_{j \in E_+} x_j z^j = \sum_{n \geq 0} p_n^{(E)}(x) z^n.$$

PROOF. According to Theorem 0.1(a), a nonzero τ lies in $G \cdot 1$ if and only if $S(\tau \otimes \tau) = 0$. One can choose a basis $\{v_i\}$ and its dual basis $\{v^i\}$ of $\mathfrak{g}$ as follows (see (1.8c), (1.10), and (1.5)):

$$\{v_i\}: \quad \frac{1}{\sqrt{h}} H_j \ (j \in E),\ X_n^{(r)} + h^{-1}(\mathring{\rho}^\vee | \mathring{X}_0^{(r)})\delta_{n,0} c \ (1 \leq r \leq l; n \in \mathbb{Z}),\ d,\ c;$$

$$\{v^i\}: \quad \frac{1}{\sqrt{h}} H_{-j} \ (j \in E),\ Y_{-n}^{(r)} + h^{-1}(\mathring{\rho}^\vee | \mathring{Y}_0^{(r)})\delta_{n,0} c \ (1 \leq r \leq l; n \in \mathbb{Z}),\ a_0^{-1} c,\ d.$$

By using these bases, the operator S on $L(\Lambda_0) \otimes L(\Lambda_0) = \mathbb{C}[x'] \otimes \mathbb{C}[x'']$ is computed as follows:[1]

$$\begin{aligned} S &= h^{-1} \sum_{j \in E_+} \{jx'_j \otimes \partial''_j + \partial'_j \otimes jx''_j\} \\ &\quad + \text{constant term of } \sum_{r=1}^{l} \{\widetilde{X}^{(r)\prime}(z) + h^{-1}(\mathring{\rho}^\vee | X_0^{(r)})\} \\ &\qquad \otimes \{\widetilde{Y}^{(r)\prime\prime}(z) + h^{-1}(\mathring{\rho}^\vee | Y_0^{(r)})\} \\ &\quad + h^{-1}(d'_0 - \mathring{\rho}^{\vee\prime}) \otimes 1 + h^{-1} \otimes (d''_0 - \mathring{\rho}^{\vee\prime\prime}) \\ &= S_1 + S_2 + S_3 + S_4, \end{aligned}$$

where

$$\begin{aligned} S_1 &= h^{-1} \sum_{j \in E_+} j(x'_j \partial''_j + x''_j \partial'_j) - h^{-1} \sum_{j \in E_+} j(x'_j \partial'_j + x''_j \partial''_j) \\ &= -h^{-1} \sum_{j \in E_+} j(x'_j - x''_j)(\partial'_j - \partial''_j), \end{aligned}$$

$$S_2 = \text{constant term of } \sum_{r=1}^{l} \widetilde{X}^{(r)\prime}(z) \cdot \widetilde{Y}^{(r)\prime\prime}(z),$$

$$S_3 = h^{-1} \sum_{r=1}^{l} \{(\mathring{\rho}^\vee | Y_0^{(r)}) X_0^{(r)\prime} + (\mathring{\rho}^\vee | \mathring{X}_0^{(r)}) Y_0^{(r)\prime\prime}\},$$

$$S_4 = h^{-2} \sum_{r=1}^{l} b_r - h^{-1}(\mathring{\rho}^{\vee\prime} + \mathring{\rho}^{\vee\prime\prime}) = h^{-2}|\mathring{\rho}^\vee|^2 - h^{-1}(\mathring{\rho}^{\vee\prime} + \mathring{\rho}^{\vee\prime\prime}).$$

From (1.7), one has

$$\begin{aligned} S_3 &= h^{-1} \sum_{r=1}^{l} \{(\mathring{\rho}^\vee | Y_0^{(r)})(\mathring{X}_0^{(r)\prime} - h^{-1}(\mathring{\rho}^\vee | X_0^{(r)})) \\ &\qquad + (\mathring{\rho}^\vee | X_0^{(r)})(\mathring{Y}_0^{(r)\prime\prime} - h^{-1}(\mathring{\rho}^\vee | Y_0^{(r)}))\} \\ &= h^{-1}(\mathring{\rho}^{\vee\prime} + \mathring{\rho}^{\vee\prime\prime} - 2h^{-1}|\mathring{\rho}^\vee|^2), \end{aligned}$$

and so

$$S_3 + S_4 = -h^{-2}|\mathring{\rho}^\vee|^2.$$

Making the change of variables (0.5a), one has

$$S_1 = -2h^{-1} \sum_{j \in E_+} j y_j \frac{\partial}{\partial y_j},$$

[1] Here and further, $'$ and $''$ refer to the operators acting on the first and second factors of the tensor product, respectively.

$$S_2 = \text{constant term of } h^{-2}\sum_{r=1}^{l} b_r \left(\exp 2\sum_{j\in E_+} \beta_{r,j} y_j z^j\right) \times \left(\exp - \sum_{j\in E_+} \beta_{r,-j}\frac{z^{-j}}{j}\frac{\partial}{\partial y_j}\right)$$

$$= h^{-2}\sum_{r=1}^{l} b_r \sum_{n\geq 0} p_n^{(E)}(2b_{r,j}y_j)p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j}\frac{\partial}{\partial y_j}\right).$$

Hence

$$S(f\otimes g) = (S_1+S_2+S_3+S_4)(f\otimes g)$$
$$= \left\{-2h^{-1}\sum_{j\in E_+} jy_j\frac{\partial}{\partial y_j} + h^{-2}\sum_{r=1}^{l} b_r\sum_{n\geq 1} p_n^{(E)}(2\beta_{r,j}y_j)p_n^{(E)}\left(-\frac{\beta_{r,-j}}{j}\frac{\partial}{\partial y_j}\right)\right\} f(x+y)g(x-y).$$

Now, by using Taylor's expansion

$$f(x+y)g(x-y) = \left(\exp\sum_{j\in E_+} y_j\frac{\partial}{\partial \xi_j}\right) f(x+\xi)g(x-\xi)|_{\xi=0}, \tag{1.16}$$

one obtains the desired formula (1.14). □

The hierarchy (1.14) of Hirota bilinear equations is called the *principal hierarchy* of type $X_N^{(k)}$.

REMARK 1.1. According to **[14]**, to every conjugacy class w of Aut $\mathring{Q}$, where $\mathring{Q}$ is the root lattice of A-D-E type, one associates a vertex realization R_w of $L(\Lambda_0)$. One starts with a "good" lifting of w to a finite order automorphism $\tilde{w}$ of $\mathring{\mathfrak{g}}$. This gives a simply laced or twisted affine algebra $\mathfrak{g}'$ with the corresponding $\mathbb{Z}$-gradation. Then one proceeds to construct the corresponding Heisenberg subalgebra $\mathfrak{s}_w$. There are complications for arbitrary w related to the fact that the centralizer S_w of $\mathfrak{s}_w$ in Ad G is nontrivial. If, however, S_w is trivial, which happens iff $\det_{\mathring{\mathfrak{h}}}(1-w) = \det_{\mathring{\mathfrak{h}}}(1-\sigma)$, where σ is the Coxeter element, then the construction of R_w is similar to that of R_σ, the principal realization discussed above. One should replace $\mathring{\rho}^\vee$ defined by (1.3) by an element γ_w defined by $\langle\alpha_i,\gamma_w\rangle = s_i$, $i=1,\dots,l$, where $s=(s_0,\dots,s_l)$ is the type of $\tilde{w}$ (and, of course, the constants $\beta_{r,j}$ will be different). In §3, we will discuss in detail the case $w=1$, the so-called homogeneous picture, for which S_w is as big as possible.

1.2. If P is an odd polynomial, i.e., $P(-x) = -P(x)$, then $P(D)\tau\cdot\tau = 0$ is an identically zero equation. An equation $P(D)\tau\cdot\tau=0$ with even P is called an *even* Hirota bilinear equation (it is always nontrivial). Since the

orthogonal complement of $L(2\Lambda)$ in the symmetric product $S^2L(\Lambda)$ gives rise to all even Hirota bilinear differential equations associated to $L(\Lambda)$, the number N_k of linearly independent such equations of degree k is calculated by the following formula:

$$\sum_{k\geq 0} N_k q^k = \frac{1}{2}[d_\Lambda(q)^2 + d_\Lambda(q^2)] - d_{2\Lambda}(q), \tag{1.17}$$

where $d_\Lambda(q)$ and $d_{2\Lambda}(q)$ are the q-dimensions of $L(\Lambda)$ and $L(2\Lambda)$ respectively. These are given by the following proposition.

PROPOSITION 1.1. (a) *Suppose that $\mathfrak{g}$ is an affine Kac-Moody algebra such that its dual ${}^t\mathfrak{g}$ is an affine algebra associated to the simple finite-dimensional Lie algebra $\mathring{\mathfrak{g}}$ of rank l. Let $\mathring{E}$ be the set of exponents of $\mathring{\mathfrak{g}}$, and let $\mathring{h}$ be its Coxeter number. Then for Λ of level* 1 *we have*

$$d_\Lambda(q) = \prod_{j\in\mathring{E}}\prod_{n\in\mathbb{Z}_+}(1-q^{j+n\mathring{h}})^{-1}, \tag{1.18a}$$

$$d_{2\Lambda}(q) = \prod_{j\in\mathring{E}}\prod_{n\in\mathbb{Z}_+}(1-q^{j+n\mathring{h}})^{-1}(1-q^{j+1+n(\mathring{h}+2)})^{-1}. \tag{1.18b}$$

(b) *Formula* (1.18a) *can be rewritten in the following form, which includes the case $\mathfrak{g} = A_{2l}^{(2)}$ as well:*

$$d_\Lambda(q) = \prod_{j\in E_+}(1-q^j)^{-1}. \tag{1.18a$'$}$$

Also we have for $\mathfrak{g} = A_{2l}^{(2)}$

$$d_{2\Lambda_0}(q) = d_{\Lambda_0}(q)\prod_{j=1}^{\infty}\prod_{\substack{n\in\mathbb{N}\\ n\equiv\pm 2j \bmod 4l+6}}(1-q^n)^{-1}. \tag{1.18b$'$}$$

(c) *Denote by n_j the number of positive roots of height j of a simple finite-dimensional Lie algebra $\mathring{\mathfrak{g}}$, and by M_j the multiplicity of j in the set $\mathring{E}$ of its exponents. Then we have*

$$n_j + n_{\mathring{h}-j} = l + M_j, \tag{1.19a}$$

$$n_j + n_{\mathring{h}+1-j} = l, \tag{1.19b}$$

$$n_j + n_{\mathring{h}+2-j} = l - M_{j-1}. \tag{1.19c}$$

PROOF. From **[11**, §10.10**]** it is clear that (a) follows from (c). Formulas (1.19a and b) are well known (see, e.g., **[22]**). Formula (1.19c) seems to be new; the proof given below works for (1.19a and b) as well (cf. **[22]**). The case $\mathfrak{g} = A_{2l}^{(2)}$ is checked by a direct calculation.

In order to prove (1.19c) recall that

$$n_j = n_{j-1} - M_{j-1}, \tag{1.20a}$$

$$M_j = M_{\overset{\circ}{h-j}}. \tag{1.20b}$$

Assume that formula (1.19c) holds for $j-1$: $n_j - 1 + n_{\overset{\circ}{h-j+3}} = l + M_{j-2}$. Subtracting (1.19) from this equation, it suffices to show that $(n_{j-1} - n_j) + (n_{\overset{\circ}{h+3-j}} - n_{\overset{\circ}{h+2-j}}) = M_{j-1} - M_{j-2}$, i.e., using (1.20a), that $M_{j-1} - M_{\overset{\circ}{h+2-j}} = M_{j-1} - M_{j-2}$, which holds due to (1.20b). □

Using (1.17) and (1.18), we deduce the following

PROPOSITION 1.2. *The lowest (principal) degree of an even Hirota bilinear equation of the principal hierarchy of type $X_N^{(k)}$ is given by the following table:*

$$A_l^{(1)}: 4;\quad D_l^{(1)}: 2(l-1);\quad A_{2l-1}^{(2)}, A_{2l}^{(2)}, D_{l+1}^{(2)}: 6;$$
$$E_6^{(1)}, D_4^{(3)}: 8;\quad E_7^{(1)}, E_6^{(2)}: 10;\quad E_8^{(1)}: 14.$$

Such an equation is unique, up to a constant factor, in all cases except for $D_l^{(1)}$, $l \geq 4$. In the latter case there are 2 (resp. 3) linearly independent such equations for $l > 4$ (resp. $l = 4$). □

1.3. The vertex operators for level 1 representations of $A_l^{(1)}$ can be calculated directly as in [**11, 12**] or by using the boson-fermion correspondence discussed in the introduction (cf. [3; **11**, Chapter 14; **15, 16**]). We choose here the second way.

Define a projective representation $\hat{r}$ of gl_∞ on F by (see the introduction)

$$\hat{r}(E_{ij}) = \psi_i\psi_j^* \quad \text{if } i \neq j \text{ or } i = j > 0,$$
$$\hat{r}(E_{ii}) = \psi_i\psi_i^* - 1 \quad \text{if } i \leq 0.$$

This allows one to consider infinite sums. Let $i, j, n \in \mathbb{Z}$ be such that $1 \leq i, j \leq l+1$; and put

$$e_{ij}(n) = \sum_{p \in (l+1)\mathbb{Z}} \hat{r}(E_{i+p,j+(l+1)n+p}).$$

One checks easily that this gives us a representation of level 1 of an affine algebra associated to $\mathrm{gl}_{l+1}(\mathbb{C})$:

$$[e_{ij}(n), e_{pq}(s)] = \delta_{jp}e_{iq}(n+s) - \delta_{iq}e_{pj}(n+s) + \delta_{iq}\delta_{jp}\delta_{n,-s}n.$$

The affine Kac-Moody algebra $\mathfrak{g}'$ of type $A_l^{(1)}$ is the linear span of $e_{ij}(n)$ ($i \neq j, n \in \mathbb{Z}$), $\sum_{i=1}^{l+1} a_i e_{ii}(n)$ ($n \in \mathbb{Z}$, $a_i \in \mathbb{C}$ with $\sum_{i=1}^{l+1} a_i = 0$), and $c = 1$. In this case, the set of exponents is $E = \mathbb{Z} \setminus (l+1)\mathbb{Z}$, and one can choose H_j and $X_j^{(r)}$ as follows. Let k, r, and n be integers such that $0 \leq k \leq l$ and $1 \leq r \leq l$. Then

$$H_{k+(l+1)n} = \sum_{p \in \mathbb{Z}} \hat{r}(E_{p,k+(l+1)n+p}) = \sum_{j-i=k} e_{ij}(n) + \sum_{j-i=k-(l+1)} e_{ij}(n+1), \tag{1.21a}$$

$$\begin{aligned} X_{k+(l+1)n}^{(r)} &= \sum_{p \in \mathbb{Z}} \hat{r}(E_{p,k+(l+1)n+p})\varepsilon^{-r(k+p)} \\ &= \sum_{j-i=k} e_{ij}(n)\varepsilon^{-rj} + \sum_{j-i=k-(l+1)} e_{ij}(n+1)\varepsilon^{-rj}, \end{aligned} \tag{1.21b}$$

where $\varepsilon = \exp(2\pi i/(l+1))$.

Now we fix an integer m, $0 \leq m \leq l$, and consider the representation $L(\Lambda_m)$ of $A_l^{(1)}$. It is known (see [**11, 12**]) that this representation is realized on the subspace $B^{(m;l)} = \mathbb{C}[x_j; j \in E_+]$ of $B^{(m)} = \mathbb{C}[x_j; j \in \mathbb{N}]$ and that the action of $\mathfrak{g}'$ is given as follows:

$$H_j = a_j, \qquad j \in E, \tag{1.22a}$$

$$\sum_{i \in \mathbb{Z}} X_i^{(r)} z^{-i} = \frac{1}{1-\varepsilon^r}[\varepsilon^{-rm}\Gamma_r(z) - 1], \tag{1.22b}$$

where

$$\Gamma_r(z) = \left(\exp \sum_{j \in E_+} (1-\varepsilon^{rj}) z^j x_j\right)\left(\exp - \sum_{j \in E_+} \frac{1-\varepsilon^{-rj}}{j} z^{-j} \frac{\partial}{\partial x_j}\right). \tag{1.23}$$

Let $(\cdot|\cdot)$ be the symmetric invariant bilinear form on $\mathfrak{g}$, so that $(e_{ij}(n)|e_{pq}(s)) = \delta_{n,-s}\delta_{iq}\delta_{jp}$; then one has $(H_j|H_{j'}) = (l+1)\delta_{j,-j'}$ and $(X_i^{(r)}|X_{i'}^{(r')}) = (l+1)\delta_{i,-i'}\delta_{r+r',l+1}$. So one can choose a basis $\{u_i\}$ and its dual basis $\{u^i\}$ of $\mathfrak{g}$ as follows:

$$\{u_i\} = \left\{\frac{1}{\sqrt{l+1}}H_j, \frac{1}{\sqrt{l+1}}X_k^{(r)}, c, d\right\},$$
$$\{u^i\} = \left\{\frac{1}{\sqrt{l+1}}H_{-j}, \frac{1}{\sqrt{l+1}}X_{-k}^{(l+1-r)}, d, c\right\}.$$

Calculating by using these bases, one obtains

$$\begin{aligned} &S - (\Lambda_m|\Lambda_{m'}) \text{ on } L(\Lambda_m) \otimes L(\Lambda_{m'}) \\ &\quad = \left\{-\frac{2}{l+1}\sum_{j \in E_+} jy_jD_j - \frac{1}{l+1}\sum_{r=1}^{l} \frac{\varepsilon^{(m-m'+1)r}}{(1-\varepsilon^r)^2}\right. \\ &\qquad \left. \times \sum_{n \geq 1} p_n^{(l+1)}(2(1-\varepsilon^{rj})y_j)p_n^{(l+1)}\left(-\frac{1-\varepsilon^{rj}}{j}D_j\right)\right\} e^{\sum_{j \in E_+} y_jD_j}, \end{aligned}$$

where $p_n^{(l+1)}(x)$ is defined by

$$\sum_n p_n^{(l+1)}(x)z^n = \exp \sum_{\substack{j \in \mathbb{N} \\ j \not\equiv 0 \bmod l+1}} x_j z^j. \tag{1.24}$$

Now, recall the following extension of Theorem 0.1(a) **[13]**: Let I be a subset of $\{0, \dots, l\}$, and let $V = \bigoplus_{i \in I} L(\Lambda_i)$, $v^0 = \sum_{i \in I} v_{\Lambda_i}$. Then $v = \sum_{i \in I} v_i$ lies in the orbit $G \cdot v^0$ if and only if all $v_i \neq 0$ and

$$\sum_k u_k(v_i) \otimes u^k(v_j) = (\Lambda_i | \Lambda_j) v_i \otimes v_j, \qquad i, j \in I.$$

Applied to $A_l^{(1)}$, this gives the following

THEOREM 1.2. *The element $\bigoplus_{j \in I} \tau_j$ with all $\tau_j \neq 0$, lies in the orbit $\mathrm{GL}_{l+1}(\mathbb{C}[t, t^{-1}]) \cdot (\bigoplus_{j \in I} 1_j)$ if and only if*

$$\text{(1.25)} \qquad \left\{ 2 \sum_{j \in E_+} j y_j D_j + \sum_{r=1}^{l} \frac{\varepsilon^{(m'-m+1)r}}{(1-\varepsilon^r)^2} \times \sum_{n \geq 1} p_n^{(l+1)}(2(1-\varepsilon^{rj}) y_j) p_n^{(l+1)} \left(-\frac{1-\varepsilon^{rj}}{j} D_j \right) \right\} \times e^{\sum_{j \in E_+} y_j D_j} \tau_m \cdot \tau_{m'} = 0 \quad \textit{for all } m, m' \in I.$$

EXAMPLE 1.1. The principal hierarchy (1.14) of type $A_1^{(1)}$ looks as follows:

(1.26)

$$\left\{ -\sum_{j \in \mathbb{N}_{\mathrm{odd}}} j y_j D_j + \frac{1}{8} \sum_{n \in \mathbb{N}} p_n^{\mathrm{odd}}(4 y_j) p_n^{\mathrm{odd}} \left(-\frac{2}{j} D_j \right) \right\} e^{\sum_{j \in \mathbb{N}_{\mathrm{odd}}} y_j D_j} \tau \cdot \tau = 0,$$

where

$$\text{(1.27)} \qquad p_n^{\mathrm{odd}}(x_1, x_3, x_5, \dots) = p_n(x_1, 0, x_3, 0, \dots).$$

The unique nontrivial Hirota bilinear equation of lowest (principal) degree (which is 4) is

$$\text{(1.28)} \qquad (D_1^4 - 4 D_1 D_3) \tau \cdot \tau = 0.$$

Putting $x = x_1$, $t = x_3$, $u(x, t) = 2(\log \tau(x, t, c_5, c_7, \dots))_{xx}$ where the c_i are some constants, after a calculation we see that equation (1.28) implies that the function $u(x, t)$ satisfies the classical KdV equation:

$$u_t = \tfrac{3}{2} u u_x + \tfrac{1}{4} u_{xxx}.$$

Thus, (1.26) is the KdV hierarchy. Using the reduction procedure of the KP hierarchy **[5]**, one obtains a different, less "economical" form of this hierarchy.

The hierarchy (1.25) of type $A_1^{(1)}$ with $I = \{0, 1\}$ is a hierarchy of equations on two functions, τ_0 and τ_1, where both τ_0 and τ_1 satisfy (1.26) and

(1.29)

$$\left\{ \sum_{j \in \mathbb{N}_{\mathrm{odd}}} j y_j D_j + \frac{1}{8} \sum_{n \in \mathbb{N}} p_n^{\mathrm{odd}}(4 y_j) p_n^{\mathrm{odd}} \left(-\frac{2}{j} D_j \right) \right\} e^{\sum_{j \in \mathbb{N}_{\mathrm{odd}}} y_j D_j} \tau_0 \cdot \tau_1 = 0.$$

The simplest equation of the hierarchy (1.29) is

$$\text{(1.30)} \qquad D_1^2 \tau_0 \cdot \tau_1 = 0.$$

Putting $x = x_1$, $t = x_3$, $u(x, t) = 2(\log \tau_0(x, t, c_5, \dots))_{xx}$, and $v(x, t) = (\log(\tau_1 / \tau_0))_x$, we get that (as before) $u(x, t)$ satisfies the classical KdV equations, $v(x, t)$ satisfies the modified KdV equation

$$v_t = -\tfrac{3}{2} v^2 v_x + v_{xxx},$$

and u and v are related by the Miura transformation (cf. **[10]**):

$$u = -v^2 - v_x.$$

1.4. In order to write down explicitly the bilinear differential equations of hierarchies given by Theorem 1.1, one has to calculate the β_{rj} and b_r in (1.14). In the case when $\mathfrak{g}$ is simply laced, the β_{rj} can be calculated by using the Coxeter transformation σ, since the element H_j defined in the subsection 1.1 is an eigenvector of σ with the eigenvalue $\exp(2\pi i j / h)$ (cf. **[2]**). Note that $\beta_{r,j} = \overline{\beta}_{r,-j}$. Once the β_{rj} are known, one can determine the b_r using Proposition 1.2. In this section we shall work out this procedure in the cases of $D_4^{(1)}$ and $E_6^{(1)}$.

PROPOSITION 1.3. (a) *The constants $\beta_{r,j}$ and b_r for the principal hierarchy of type $D_4^{(1)}$ are given by the following table, where $\varepsilon = \exp(i\pi/6)$.*

$\beta_{r,j}$: $j \setminus r$	1	2	3	4
1	$\sqrt{2}$	$\sqrt{6}\varepsilon^{-1}$	$\sqrt{2}$	$\sqrt{2}$
5	$\sqrt{2}$	$\sqrt{6}\varepsilon$	$\sqrt{2}$	$\sqrt{2}$
3	$\sqrt{2}$	0	$\sqrt{2}$	$-2\sqrt{2}$
3′	$\sqrt{6}$	0	$-\sqrt{6}$	0

$$b_1 = b_3 = b_4 = \tfrac{9}{2}, \qquad b_2 = \tfrac{1}{2}.$$

The three linearly independent equations of the lowest degree (= 6) are

$$(D_1^6 + 36 D_1 D_5 - 10 D_3^2 - 10 D_3'^2) \tau \cdot \tau = 0,$$
$$(D_1^3 D_3 + D_3^2 - D_3'^2) \tau \cdot \tau = 0,$$
$$(D_1^3 D_3' - 2 D_3 D_3') \tau \cdot \tau = 0.$$

(b) *The constants $\beta_{r,j}$ and b_r for the principal hierarchy of type $E_6^{(1)}$ are given by the following table, where $\omega = \exp(i\pi/12)$, $\alpha = 3 + \sqrt{3}$, $\beta = 3 - \sqrt{3}$.*

$\beta_{r,j}$: $j \setminus r$	1	2	3	4	5	6
1	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\alpha}\omega$	$\sqrt{\alpha}\omega$	$\sqrt{2\alpha}$	$\sqrt{2\beta}\omega$
11	$\sqrt{\beta}$	$\sqrt{\beta}$	$\sqrt{\alpha}\omega^{-1}$	$\sqrt{\alpha}\omega^{-1}$	$\sqrt{2\alpha}$	$\sqrt{2\beta}\omega^{-1}$
5	$\sqrt{\alpha}$	$\sqrt{\alpha}$	$\sqrt{\beta}\omega^5$	$\sqrt{\beta}\omega^5$	$-\sqrt{2\beta}$	$-\sqrt{2\alpha}\omega^5$
7	$\sqrt{\alpha}$	$\sqrt{\alpha}$	$\sqrt{\beta}\omega^{-5}$	$\sqrt{\beta}\omega^{-5}$	$-\sqrt{2\beta}$	$-\sqrt{2\alpha}\omega^{-5}$
4	$\sqrt{6}$	$-\sqrt{6}$	$\sqrt{6}\omega^4$	$-\sqrt{6}\omega^4$	0	0
8	$\sqrt{6}$	$-\sqrt{6}$	$\sqrt{6}\omega^{-4}$	$-\sqrt{6}\omega^{-4}$	0	0

$$b_1 = b_2 = \tfrac{4}{3}\alpha^2, \quad b_3 = b_4 = \tfrac{4}{3}\beta^2, \quad b_5 = \tfrac{1}{36}\beta^4, \quad b_6 = \tfrac{1}{36}\alpha^4.$$

The equation of the lowest degree (= 8) *is*

$$(D_1^8 - 280\sqrt{6}D_1^3D_5 + 210D_4^2 - 240\sqrt{2}D_1D_7)\tau\cdot\tau = 0.$$

PROOF. The proof is sketched here for $E_6^{(1)}$, the proof for $D_4^{(1)}$ being similar. Let r_i, $i = 1, \dots, 6$, denote reflections with respect to $\alpha_1^\vee, \dots, \alpha_6^\vee$. Then $\sigma = (r_1r_5r_3)(r_2r_4r_6)$ is a Coxeter transformation. The matrix of σ in the basis $\alpha_1^\vee, \dots, \alpha_6^\vee$ is

$$\tilde{\sigma} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

We have the following: (i) ε^m ($m = 1, 5, 7, 11$) is an eigenvalue of σ and the corresponding eigenvector is

(1.32a) $v_m = a(\varepsilon^m)\alpha_1^\vee + b(\varepsilon^m)\alpha_2^\vee + (1+\varepsilon^m)\alpha_3^\vee + b(\varepsilon^m)\alpha_4^\vee + a(\varepsilon^m)\alpha_5^\vee + \alpha_6^\vee,$

where

(1.32b) $$a(x) = \frac{x(1+x)}{1+x+x^2} \quad \text{and} \quad b(x) = \frac{(1+x)^2}{1+x+x^2};$$

(ii) ε^m ($m = 4, 8$) is an eigenvalue of σ and the corresponding eigenvector is

$$v_m = \alpha_1^\vee - \varepsilon^m\alpha_2^\vee + \varepsilon^m\alpha_4^\vee - \alpha_5^\vee.$$

Note that ε^m for $m = 1, 5, 7, 11$ (resp. for $m = 4, 8$) are solutions of equations $x^4 - x^2 + 1 = 0$ (resp. $x^2 + x + 1 = 0$), and that the nonzero inner products of the v_m are given as follows:

(1.33) $$(v_1|v_{11}) = \beta, \quad (v_5|v_7) = \alpha, \quad (v_4|v_8) = 6.$$

Define H_j and β_i ($j = 1, 4, 5, 7, 8, 11$ and $1 \le i \le 6$) as follows:

(1.34) $$\begin{aligned} &H_1 = -i\sqrt{2\alpha}v_1, \quad H_{11} = i\sqrt{2\alpha}v_{11}, \quad H_5 = i\sqrt{2\beta}v_5, \\ &H_7 = -i\sqrt{2\beta}v_7, \quad H_4 = \sqrt{2}\omega^{-2}v_4, \quad H_8 = \sqrt{2}\omega^2 v_8, \end{aligned}$$

(1.35) $\beta_1 = \alpha_1, \quad \beta_2 = \alpha_5, \quad \beta_3 = -\alpha_2, \quad \beta_4 = -\alpha_4, \quad \beta_5 = \alpha_3, \quad \beta_6 = -\alpha_6.$

From (1.33) and (1.34), one has

$$(H_i|H_j) = 12\delta_{i+j,12},$$

which shows that the above choice of H_j's satisfies the conditions of subsection 1.1. The $\beta_{r,j} = \langle\beta_r, H_j\rangle$ are computed immediately from (1.32), (1.34), and (1.35).

Now, the lowest degree of an even equation of the principal $E_6^{(1)}$ hierarchy is 8 (Proposition 1.2); hence all even equations of degree < 8 must vanish. This gives rise to a system of 16 linear equations for 6 unknowns b_r, which determines the b_r completely. □

1.5. The affine algebras $\mathfrak{g}'$ of type $D_4^{(1)}$, $E_6^{(1)}$, and $E_7^{(1)}$ have a diagram automorphism ν of order 4, 3, and 2 respectively, the fixed point set $\mathfrak{g}'^\nu$ being affine algebras of type $A_2^{(2)}$, $D_4^{(3)}$, and $E_6^{(2)}$ respectively; then one can show that the centralizer of $\mathfrak{g}'^\nu$ in $\mathfrak{g}'$ contains the elements H_j of the principal subalgebra $\mathfrak{s}$ with $j \equiv 3 \mod 6$, $\equiv \pm 4 \mod 12$, and $\equiv 9 \mod 18$ respectively. These elements together with c span a Heisenberg subalgebra of $\mathfrak{g}'$, which we denote by $\mathfrak{s}^0$; the span of these elements with $j > 0$ we denote by $\mathfrak{s}_+^0$. The following proposition is now immediate.

PROPOSITION 1.4. (a) *Let* $V = \mathbb{C}[x_j; j \in E_+]$ *be the space of the basic representation of* $\mathfrak{g}'$ *of type* $D_4^{(1)}$, $E_6^{(1)}$, *and* $E_7^{(1)}$. *Then* $V^{\mathfrak{s}_+^0} = \mathbb{C}[x_j; j \in E_+^\nu]$, *where* E_+^ν *is the set of positive exponents of* $\mathfrak{g}'^\nu$, *is invariant under* $\mathfrak{g}'^\nu$ *and is the principal realization of the basic representation of* $\mathfrak{g}'^\nu$ (*of type* $A_2^{(2)}$, $D_4^{(3)}$, *and* $E_6^{(2)}$ *respectively*). *The corresponding vertex operators are* $\widetilde{X}^{(r)}(z)$ *with* $r = 2$ *for* $A_2^{(2)}$, $r = 5, 6$ *for* $D_4^{(3)}$ *and* $r = 1, 2, 3, 4$ *for* $E_6^{(2)}$ *respectively.*

(b) *The principal hierarchy of type* $A_2^{(2)}$, $D_4^{(3)}$, *and* $E_6^{(2)}$ *is obtained from that of type* $D_4^{(1)}$, $E_6^{(1)}$, *and* $E_7^{(1)}$ *respectively by putting all* D_j *equal to zero for* $j \equiv 3 \mod 6$, $\equiv \pm 4 \mod 12$, *and* $\equiv 9 \mod 18$ *respectively.*

(c) *The* $\beta_{r,j}$ *and* b_r *for* $A_2^{(2)}$ *are as follows*:

$$\beta_{1,1} = \sqrt{6}\varepsilon^{-1}, \quad \beta_{1,5} = \sqrt{6}\varepsilon; \quad b_1 = \tfrac{1}{4}.$$

The $\beta_{r,j}$ *and* b_r *for* $D_4^{(3)}$ *are as follows*:

$$\beta_{1,1} = \beta_{1,11} = \sqrt{2\alpha}, \quad \beta_{1,5} = \overline{\beta}_{1,7} = -\sqrt{2\beta},$$

$$\beta_{2,1} = \overline{\beta}_{2,11} = \sqrt{2\beta}\omega, \quad \beta_{2,5} = \overline{\beta}_{2,7} = -\sqrt{2\alpha}\omega^5; \quad b_1 = \frac{\beta^4}{108}, \quad b_2 = \frac{\alpha^4}{108},$$

where $\varepsilon, \omega, \alpha$, *and* β *are as in Proposition* 1.3.

(d) *The lowest degree even equations for* $A_2^{(2)}$ *and* $D_4^{(3)}$ *are as follows respectively*:

$$(D_1^6 + 36D_1D_5)\tau\cdot\tau = 0;$$
$$(D_1^8 - 280\sqrt{6}D_1^3D_5 - 240\sqrt{2}D_1D_7)\tau\cdot\tau = 0.$$

2. Principal picture for $B_l^{(1)}$ and super bilinear equations.

2.1. Consider the space of polynomials

$$\mathbb{C}[x] = \mathbb{C}[x_j; j \in \mathbb{N}_{\text{odd}}].$$

Let, as before,

$$a_j := \partial/\partial x_j \quad \text{and} \quad a_{-j} := jx_j \quad \text{for } j \in \mathbb{N}_{\text{odd}}.$$

Define the Virasoro operators on $\mathbb{C}[x]$ by

(2.1) $$L_n^{(h)} := \frac{1}{2h}\sum_{j\in\mathbb{Z}_{\text{odd}}} :a_{nh-j}a_j: \qquad (n \in \mathbb{Z}),$$

where h is a (fixed) positive even integer and $:\quad:$ stands for normal ordering. Also introduce a vertex operator

$$V(z;\nu) := \left(\exp \sum_{j\in\mathbf{N}_{\mathrm{odd}}} \frac{\nu_j z^j}{j} a_{-j}\right)\left(\exp - \sum_{j\in\mathbf{N}_{\mathrm{odd}}} \frac{\nu_{-j} z^{-j}}{j} a_j\right), \tag{2.2}$$

where $\nu = (\nu_j)_{j\in\mathbf{Z}_{\mathrm{odd}}}$ is a sequence of complex numbers satisfying $\nu_{j+h} = \nu_j$. Then it is easy to check the following relations:

$$[L_n^{(h)}, a_j] = -(j/h)a_{j+nh}, \tag{2.3}$$

$$[L_n^{(h)}, L_m^{(h)}] = (n-m)L_{n+m}^{(h)} + \left[\frac{h(n^3-n)}{24} + \frac{(h^2+2)n}{24h}\right]\delta_{n,-m}, \tag{2.4}$$

$$[a_j, V(z;\nu)] = \nu_j z^j V(z;\nu), \tag{2.5}$$

$$[L_n^{(h)}, V(z;\nu)] = \frac{z^{nh}}{h}\left\{\frac{1}{2}nR_\nu + z\frac{\partial}{\partial z}\right\} V(z;\nu), \tag{2.6}$$

where $R_\nu = \sum_{1\le j\le h,\, j \text{ odd}} \nu_j\nu_{-j}$.

Next we recall the construction of the irreducible Virasoro module with central charge $\frac{1}{2}$ in terms of the superoscillator algebra. Let $\varepsilon = 0$ or $\frac{1}{2}$, and let $\Lambda^{(\varepsilon)} := \Lambda(\xi_j; j\in \varepsilon+\mathbf{Z}_+)$ be the exterior algebra over $\mathbf{C}$ on generators ξ_j ($j\in\varepsilon+\mathbf{Z}_+$), and ψ_j be the operators on $\Lambda^{(\varepsilon)}$ defined by

$$\psi_j := \partial/\partial\xi_j,\quad \psi_{-j} := \xi_j \quad \text{if } j>0, \tag{2.7a}$$

$$\psi_0 := (1/\sqrt{2})(\xi_0 + \partial/\partial\xi_0). \tag{2.7b}$$

Let

$$\psi_\varepsilon^{(h)}(z) := \sum_{j\in\varepsilon+\mathbf{Z}} \psi_j z^{-hj}, \tag{2.7c}$$

and

$$l_n^{(\varepsilon)} := \frac{1}{2}\sum_{j\in\varepsilon+\mathbf{Z}} j\psi_{-j}\psi_{j+n} \qquad (n\in\mathbf{Z}, n\ne 0), \tag{2.8a}$$

$$l_0^{(\varepsilon)} := \frac{1-2\varepsilon}{16} + \sum_{j\in\varepsilon+\mathbf{Z}_+} j\psi_{-j}\psi_j. \tag{2.8b}$$

Then

$$[l_n^{(\varepsilon)}, l_m^{(\varepsilon)}] = (n-m)l_{n+m}^{(\varepsilon)} + \frac{n^3-n}{24}\delta_{n,-m}, \tag{2.9a}$$

$$[l_n^{(\varepsilon)}, \psi_\varepsilon^{(h)}(z)] = z^{nh}\left\{\frac{n}{2} + \frac{1}{h}z\frac{d}{dz}\right\}\psi_\varepsilon^{(h)}(z). \tag{2.9b}$$

Via the operators $l_n^{(\varepsilon)}$, $\Lambda^{(\varepsilon)}$ turns to be a Virasoro module with central charge $\frac{1}{2}$. We have an obvious decomposition: $\Lambda^{(\varepsilon)} = \Lambda_{\mathrm{even}}^{(\varepsilon)} \oplus \Lambda_{\mathrm{odd}}^{(\varepsilon)}$; it is known that

these Virasoro modules are the following irreducible highest weight modules $V(\frac{1}{2}, h)$, where h is the minimal eigenvalue of $l_0^{(\varepsilon)}$ (see, e.g., **[16]** for a proof):

$$\Lambda_{\mathrm{even}}^{(\frac{1}{2})} \simeq V(\tfrac{1}{2}, 0),\quad \Lambda_{\mathrm{odd}}^{(\frac{1}{2})} \simeq V(\tfrac{1}{2}, \tfrac{1}{2}), \tag{2.10a}$$

$$\Lambda_{\mathrm{even}}^{(0)} \simeq \Lambda_{\mathrm{odd}}^{(0)} \simeq V(\tfrac{1}{2}, \tfrac{1}{16}). \tag{2.10b}$$

2.2. We turn now to the principal vertex construction of level 1 representations of the affine algebra of type $B_l^{(1)}$. Let $\bar{\mathfrak{g}} = \sum_{j\in\mathbf{Z}/2l\mathbf{Z}} \bar{\mathfrak{g}}_j$ be the principal gradation of the finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$ of type B_l (recall that $2l$ is its Coxeter number). Then

$$\mathfrak{g}' = \bigoplus_{j\in\mathbf{Z}} (t^j\otimes\bar{\mathfrak{g}}_{j \bmod 2l}) + \mathbf{C}c$$

is the principal realization of the affine Kac-Moody algebra of type $B_l^{(1)}$, with commutation relations

$$[u(j), v(k)] = [u,v](j+k) + j(u|v)_0\delta_{j,-k}c, \qquad [c, \mathfrak{g}'] = 0,$$

where $u(j) = t^j\otimes u$ and $(\cdot|\cdot)_0$ is the invariant bilinear form on $\bar{\mathfrak{g}}$ normalized by the square-length of long roots being equal to 2.

As in §1, we include $\mathfrak{g}'$ in a Lie algebra $\hat{g} = \mathfrak{g}' \oplus (\sum_{j\in\mathbf{Z}} \mathbf{C}d_j)$ such that

$$[d_n, u(j)] = -ju(j+2nl), \qquad [d_n, c] = 0, \tag{2.11a}$$

$$[d_m, d_n] = 2l(m-n)d_{m+n} + \frac{(2l)^2(l+1/2)}{12}\delta_{m,-n}(m^3-m)c, \tag{2.11b}$$

and denote by $\mathfrak{g}$ its subalgebra $\mathfrak{g}' + \mathbf{C}d_0$.

Let $\bar{\mathfrak{s}}$ be the principal Cartan subalgebra of $\bar{\mathfrak{g}}$ so that $\bar{\mathfrak{s}} = \sum_{j\in\mathbf{Z}/2l\mathbf{Z}} \bar{\mathfrak{s}}_j$, where $\mathfrak{s}_j = \bar{\mathfrak{s}}\cap\bar{\mathfrak{g}}_j$ and $\dim\bar{\mathfrak{s}}_j = 1$ if j is odd and $= 0$ if j is even. Then $\mathfrak{s} = \sum_{j\in\mathbf{Z}_{\mathrm{odd}}} t^j\otimes\bar{\mathfrak{s}}_{j \bmod 2l} + \mathbf{C}c$ is the principal Heisenberg subalgebra of $\mathfrak{g}'$ **[12]**.

Choose an element S_j from each component $\bar{\mathfrak{s}}_j$ (j odd) such that $(S_j|S_k)_0 = \delta_{j+k,0 \bmod 2l}$. Put $H_j = \sqrt{2l}\,t^j\otimes S_{j \bmod 2l}$ for $j\in\mathbf{Z}_{\mathrm{odd}}$; then H_j ($j\in\mathbf{Z}_{\mathrm{odd}}$), c form a basis of $\mathfrak{s}$ and satisfy the commutation relations (1.8a).

Let α be a root of $\bar{\mathfrak{g}}$ with respect to $\bar{\mathfrak{s}}$, and let $X_\alpha = \sum_{j\in\mathbf{Z}/2l\mathbf{Z}} X_{\alpha,j}$ be its (nonzero) root vector. Then, comparing each component of $[S_j, X_\alpha] = \alpha(S_j)X_\alpha$ with this gradation, one has

$$[S_j, X_{\alpha,i}] = \alpha(S_j)X_{\alpha,j+i}. \tag{2.12}$$

Now put $X_j^\alpha := t^j\otimes X_{\alpha, j \bmod 2l}$, and consider the element

$$X^\alpha(z) := \sum_{j\in\mathbf{Z}} X_j^\alpha z^{-j}$$

in $\mathfrak{g}\otimes\mathbf{C}[[z, z^{-1}]]$. From (2.12), one has

$$[H_j, X^\alpha(z)] = \sqrt{2l}\,\alpha(S_j)z^jX^\alpha(z). \tag{2.13}$$

Note also that

$$[d_n, X^\alpha(z)] = z^{nh}[2nl + z\partial/\partial z]X^\alpha(z). \tag{2.14}$$

Now we consider the action of the Heisenberg algebra $\mathfrak{s}$ and the Virasoro operators on a level 1 $\mathfrak{g}$-module $L(\Lambda)$. First take the representation of $\mathfrak{s}$ on $\mathbb{C}[x]$ defined by

(2.15) $$H_j \mapsto a_j, \qquad c \mapsto 1.$$

For each root α of $(\bar{\mathfrak{g}}, \bar{\mathfrak{s}})$, define the vertex operator $V'_\alpha(z)$ on $\mathbb{C}[x]$ as follows:

(2.16)
$$V'_\alpha(z) := \left(\exp\sqrt{2l}\sum_{j\in\mathbb{N}_{\mathrm{odd}}}\alpha(S_j)z^jx_j\right)\left(\exp -\sqrt{2l}\sum_{j\in\mathbb{N}_{\mathrm{odd}}}\frac{\alpha(S_{-j})z^{-j}}{j}\frac{\partial}{\partial x_j}\right).$$

From (2.5), one sees that

(2.17) $$[a_k, V'_\alpha(z)] = \sqrt{2l}\alpha(S_j)z^jV'_\alpha(z)$$

and

(2.18) $$[L_n^{(h)}, V'_\alpha(z)] = \frac{z^{2nl}}{2l}\left\{nl(\alpha|\alpha) + z\frac{d}{dz}\right\}V'_\alpha(z),$$

since $R_{(\alpha(S_j))} = \sum_{l\le j\le 2l-1}\sqrt{2l}\alpha(S_j)\cdot\sqrt{2l}\alpha(S_{-j}) = 2l(\alpha|\alpha)$ in our case. We see also from (2.4) that the Virasoro operators $\dot{L}_n^{(\mathfrak{s})}$ associated to $(\mathfrak{s}, L(\Lambda))$ are

$$\dot{L}_n^{(\mathfrak{s})} := L_n^{(2l)} + \delta_{n,0}\frac{2l^2+1}{48l},$$

and the central charge is l.

On the other hand, the Virasoro operators $L_n^{(\mathfrak{g})}$ of $\mathfrak{g}$ associated to the principal realization of $L(\Lambda)$ can be calculated by using the formulas in [**9**, **17**],.... Among them, the most important are its central charge $z_\mathfrak{g}$ and the energy operator $L_0^{(\mathfrak{g})}$, and they are given as follows:

$$z_\mathfrak{g} = \frac{m}{m+g}\dim\bar{\mathfrak{g}},$$
$$L_0^{(\mathfrak{g})} = \frac{d_0}{2l} + \frac{\Omega}{2(c+g)} + \frac{(l+1)(2l+1)}{48l}m,$$

where m is the level of $L(\Lambda)$, $g = 2l-1$ is the dual Coxeter number, and Ω is the Casimir element of $\mathfrak{g}$, which is the scalar $(\Lambda+2\rho|\Lambda)$.

Now we consider the coset representation of the Virasoro algebra:

$$L_n := L_n^{(\mathfrak{g})} - \dot{L}_n^{(\mathfrak{s})},$$

whose central charge is $z = z_\mathfrak{g} - l$. In our case, the level m of the representation $L(\Lambda)$ is equal to 1; so we have $z = \frac{1}{2}$, and

$$L_0 = \frac{d_0}{2l} + \frac{\Omega}{4l} + \frac{(l+1)(2l+1)}{48l} - \left(L_0^{(2l)} + \frac{2l^2+1}{48l}\right)$$
$$= \frac{d_0}{2l} + \frac{(\Lambda+2\rho|\Lambda)}{4l} + \frac{1}{16} - L_0^{(2l)}.$$

The coset Virasoro algebra acts on the space $L(\Lambda)^{\mathfrak{s}_+}$; since $L_0^{(2l)}$ vanishes on this space, we have

(2.19) $$L_0|_{L(\Lambda)^{\mathfrak{s}_+}} = \frac{d_0}{2l} + \frac{(\Lambda+2\rho|\Lambda)}{4l} + \frac{1}{16}.$$

Recall that $B_l^{(1)}$ has three level 1 integrable highest weight representations: $L(\Lambda_0)$, $L(\Lambda_1)$, and $L(\Lambda_l)$; the fundamental weights Λ_i are chosen such that $\Lambda_i(d_0) = 0$ and $(\Lambda_0|\Lambda_0) = 0$. Now we can calculate the eigenvalue of L_0 on the highest weight vector v_Λ of all $L(\Lambda)$ of level 1:

$$L_0(v_{\Lambda_i}) = \tfrac{1}{16}v_{\Lambda_i} \quad \text{for } i = 0 \text{ or } 1, \qquad L_0(v_{\Lambda_l}) = 0.$$

From this, we have

(2.20a) $$L(\Lambda_i) = \mathbb{C}[x]\otimes[V(\tfrac{1}{2},\tfrac{1}{16}) + \cdots] \quad (i = 0 \text{ or } 1),$$

(2.20b) $$L(\Lambda_l) = \mathbb{C}[x]\otimes[V(\tfrac{1}{2},0) + \cdots].$$

We can determine the structure of $L(\Lambda)^{\mathfrak{s}_+}$ with the help of the q-dimension $d_\Lambda(q) = \dim_q L(\Lambda; B_l^{(1)})$. By a simple calculation using [**11**, §10.10], we have

(2.21a) $$d_{\Lambda_i}(q) = \frac{\phi(q^2)\phi(q^{4l})}{\phi(q)\phi(q^{2l})} \quad (i = 0 \text{ or } 1),$$

(2.21b) $$d_{\Lambda_l}(q) = \frac{\phi(q^2)\phi(q^{2l})^2}{\phi(q)\phi(q^l)\phi(q^{4l})}.$$

It is clear that

(2.22) $$\dim_q\mathbb{C}[x] = \frac{\phi(q^2)}{\phi(q)}$$

and the Virasoro characters are (see subsection 2.1):

(2.23a) $$\mathrm{ch}\,V(\tfrac{1}{2},\tfrac{1}{16}) = x^{1/16}\frac{\phi(x^2)}{\phi(x)}$$

(2.23b) $$\mathrm{ch}[V(\tfrac{1}{2},0)\oplus V(\tfrac{1}{2},\tfrac{1}{2})] = \frac{\phi(x)^2}{\phi(x^{1/2})\phi(x^2)},$$

where $x = q^{2l}$ since the height of the fundamental imaginary root with respect to the principal gradation is equal to $2l$.

From (2.20a), (2.21a), (2.22), and (2.23a), we obtain

(2.24a) $$L(\Lambda_i) = \mathbb{C}[x]\otimes V(\tfrac{1}{2},\tfrac{1}{16}) \quad (i = 0 \text{ or } 1);$$

and moreover, from (2.20b), (2.21b), (2.22), and (2.23b), we obtain

(2.24b) $$L(\Lambda_l) = \mathbb{C}[x]\otimes[V(\tfrac{1}{2},0)\oplus V(\tfrac{1}{2},\tfrac{1}{2})].$$

Thus we have deduced the following identification

(2.25a) $$L(\Lambda_0)\oplus L(\Lambda_1) = \mathbb{C}[x]\otimes\Lambda^{(0)},$$

(2.25b) $$L(\Lambda_l) = \mathbb{C}[x]\otimes\Lambda^{(1/2)}$$

as $\mathfrak{s}\oplus\mathrm{Vir}$-modules.

REMARK 2.1. The principal gradation in (2.25) is given by $\deg x_j = j$, $\deg \xi_j = 2lj$.

Now let us consider the action of $X_\alpha(z)$ on the space (2.25). Put

$$V_\alpha^{(\varepsilon)}(z) := \begin{cases} V'_\alpha(z) & \text{if } (\alpha|\alpha) = 2, \\ V'_\alpha(z)\psi_\varepsilon^{(2l)}(z) & \text{if } (\alpha|\alpha) = 1, \end{cases} \tag{2.26}$$

and let

$$l_n := 2l[L_n^{(2l)} + l_n^{(\varepsilon)}]. \tag{2.27}$$

Then from (2.8) and (2.9b), we have

$$[l_n, V_\alpha^{(\varepsilon)}(z)] = z^{2nl}\{2nl + zd/dz\}V_\alpha^{(\varepsilon)}(z). \tag{2.28}$$

In view of (2.10), (2.13), (2.14) and (2.3), (2.17), (2.28), we obtain the following proposition:

PROPOSITION 2.1. *The map*

$$H_j \mapsto a_j; \qquad c \mapsto 1;$$
$$X^\alpha(z) \mapsto c_\alpha V_\alpha^{(\varepsilon)}(z), \qquad c_\alpha \in \mathbb{C};$$
$$d_n \mapsto 2l(L_n^{(2l)} + l_n^{(\varepsilon)}),$$

defines a representation of the extended affine algebra $\hat{\mathfrak{g}}$ *of type* $B_l^{(1)}$ *on the space* $\mathbb{C}[x] \otimes \Lambda^{(\varepsilon)}$, *which as a* $\mathfrak{g}'$*-module is equivalent to* $L(\Lambda_0) \oplus L(\Lambda_1)$ *for* $\varepsilon = 0$ *and to* $L(\Lambda_l)$ *for* $\varepsilon = \frac{1}{2}$. □

2.3. Note that $B_l^{(1)}$ with $l = 1$ becomes $A_1^{(1)}$ and that its level 1 modules $L(\Lambda_0)$, $L(\Lambda_1)$, and $L(\Lambda_l)$ become level 2 modules $L(2\Lambda_0)$, $L(2\Lambda_1)$, and $L(\Lambda_0 + \Lambda_1)$ respectively. In this subsection we use this to construct all level 2 representations of the affine algebra $\mathfrak{g}'$ of type $A_1^{(1)}$.

First recall its principal realization. Take the standard basis of $\mathrm{sl}_2(\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$\mathfrak{g}' = \sum_{j \in \mathbb{Z}_{\text{even}}} \mathbb{C}H(j) + \sum_{j \in \mathbb{Z}_{\text{odd}}} \mathbb{C}X_\pm(j) + \mathbb{C}c,$$
$$[u(j), v(k)] = [u, v](j+k) + \tfrac{1}{2} j \operatorname{tr}(uv)\delta_{j,-k}c, \qquad [c, \mathfrak{g}'] = 0.$$

We embed $\mathfrak{g}'$ into $\hat{\mathfrak{g}} = \mathfrak{g}' \oplus \sum_{j \in \mathbb{Z}} \mathbb{C}d_j$, such that

$$[d_n, u(j)] = -ju(j+2n), \qquad [d_n, c] = 0, \tag{2.29a}$$
$$[d_m, d_n] = 2(m-n)d_{m+n} + \tfrac{1}{2}\delta_{m,-n}(m^3 - m)c. \tag{2.29b}$$

Let $S = \frac{1}{\sqrt{2}}(X_+ + X_-)$. Since $[S, H] = \sqrt{2}(X_- - X_+)$ and $[S, X_- - X_+] = \sqrt{2}H$, the element

$$X(z) := \sum_{j \in \mathbb{Z}_{\text{even}}} H(j)z^{-j} + \sum_{j \in \mathbb{Z}_{\text{odd}}} (X_- - X_+)(j)z^{-j}$$

satisfies the commutation relations

$$[S(j), X(z)] = \sqrt{2}z^j X(z). \tag{2.30}$$

Also it satisfies

$$[d_n, X(z)] = z^{2n}\{2n + zd/dz\}X(z). \tag{2.31}$$

Take the principal Heisenberg algebra $\mathfrak{s} = \sum_{j \in \mathbb{Z}_{\text{odd}}} \mathbb{C}S(j) + \mathbb{C}c$. Its commutation relations are given by

$$[S(j), S(k)] = (j/2)\delta_{j,-k}c.$$

As in subsection 2.2, we consider the action of the coset Virasoro algebra (with respect to the pair $(\mathfrak{g}', \mathfrak{s})$) on a level 2 $\mathfrak{g}'$-module $L(\Lambda)$. Applying the same argument as in the case of $B_l^{(1)}$, we prove the following isomorphism as $\mathfrak{s} \oplus$ Vir-modules:

$$L(2\Lambda_0) \oplus L(2\Lambda_1) = \mathbb{C}[x] \otimes V(\tfrac{1}{2}, \tfrac{1}{16}),$$
$$L(\Lambda_0 + \Lambda_1) = \mathbb{C}[x] \otimes [V(\tfrac{1}{2}, 0) \oplus V(\tfrac{1}{2}, \tfrac{1}{2})].$$

So one can put

$$L(2\Lambda_0) \oplus L(2\Lambda_1) = \mathbb{C}[x] \otimes \Lambda^{(0)}, \tag{2.32a}$$
$$L(\Lambda_0 + \Lambda_1) = \mathbb{C}[x] \otimes \Lambda^{(1/2)}, \tag{2.32b}$$

where the action of $\mathfrak{s}$ and d_n $(n \in \mathbb{Z})$ is as follows:

$$S(j) \mapsto a_j, \qquad c \mapsto 2, \tag{2.33a}$$
$$d_n \mapsto l_n := 2(L_n^{(2)} + l_n^{(\varepsilon)}). \tag{2.33b}$$

Now we calculate the action of $X(z)$ on this space. First, take the vertex operator

$$V'(z) := \left(\exp \sqrt{2} \sum_{j \in \mathbb{N}_{\text{odd}}} x_j z^j\right)\left(\exp -\sqrt{2} \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j}\frac{\partial}{\partial x_j}\right);$$

it satisfies (see notation (2.1)):

$$[a_j, V'(z)] = \sqrt{2}z^j V'(z), \tag{2.34}$$
$$[L_n^{(2)}, V'(z)] = \frac{z^{2n}}{2}\left\{n + z\frac{d}{dz}\right\}V'(z). \tag{2.35}$$

Put $V_\varepsilon(z) := V'(z)\psi_\varepsilon^{(2)}(z)$; then one sees from (2.10) that

$$[l_n^{(\varepsilon)}, \psi_\varepsilon^{(2)}(z)] = \frac{z^{2n}}{2}\left\{n + z\frac{d}{dz}\right\}\psi_\varepsilon^{(2)}(z). \tag{2.36}$$

From (2.35) and (2.36), we obtain

$$[l_n, V_\varepsilon(z)] = z^{2n}\left\{2n + z\frac{d}{dz}\right\}V_\varepsilon(z),$$

which is compatible with the commutation relation (2.31). Thus we have proved (as in subsection 2.2) that the action of $X(z)$ on the space (2.32) is given by

(2.37a) $$X(z) \mapsto a_\varepsilon V_\varepsilon(z)$$

where a_ε is a nonzero constant.

The constant a_ε can be determined by calculating the coefficients of z^j ($j = 0, \pm 1$) in the vertex operator $V_\varepsilon(z)$. Put

(2.37b) $$X(z) = \sum_{n\in\mathbf{Z}} X_n z^n \quad \text{and} \quad a_\varepsilon V_\varepsilon(z) = \sum_{n\in\mathbf{Z}} V_n z^n.$$

First note that one can choose the Chevalley generators of the affine algebra $\mathfrak{g} = A_1^{(1)}$ as follows:

$$\begin{aligned} e_0 &= X_+(1), & e_1 &= X_-(1), \\ f_0 &= X_-(-1), & f_1 &= X_+(-1), \\ \alpha_0^\vee &= H(0) + c/2, & \alpha_1^\vee &= -H(0) + c/2. \end{aligned}$$

Then one has

$$X_0 = \alpha_0^\vee - c/2, \quad X_1 = f_0 - f_1, \quad , X_{-1} = e_1 - e_0.$$

We note that $f_0 + f_1 = \sqrt{2}S(-1)$.

Case (1). $L(2\Lambda_0) + L(2\Lambda_1) = \mathbf{C}[x] \otimes \Lambda^{(0)}$. By a simple calculation, one has

(2.38a) $$V_0(1\otimes 1) = \frac{a_0}{\sqrt{2}}(1\otimes \xi_0),$$

(2.38b) $$V_0(1\otimes \xi_0) = \frac{a_0}{\sqrt{2}}(1\otimes 1).$$

So the eigenvalues of V_0 on the 2-dimensional space spanned by $1\otimes 1$ and $1\otimes\xi_0$ are $\pm a_0/\sqrt{2}$. Now consider the action of X_0 on the highest weight vectors $v_{2\Lambda_i}$ of $L(2\Lambda_i)$ ($i = 0, 1$):

(2.39a) $$X_0 v_{2\Lambda_0} = \langle 2\Lambda_0, \alpha_0^\vee - c/2\rangle v_{2\Lambda_0} = v_{2\Lambda_0},$$

(2.39b) $$X_0 v_{2\Lambda_1} = \langle 2\Lambda_1, \alpha_0^\vee - c/2\rangle v_{2\Lambda_1} = -v_{2\Lambda_1}.$$

By comparing (2.38) and (2.39), one has $a_0 = \pm\sqrt{2}$. We choose $a_0 = \sqrt{2}$; then we have

(2.40) $$v_{2\Lambda_0} = 1\otimes(1+\xi_0), \qquad v_{2\Lambda_1} = 1\otimes(1-\xi_0).$$

Case (2). $L(\Lambda_0 + \Lambda_1) = \mathbf{C}[x]\otimes\Lambda^{(1/2)}$. We compute $(c/2 - \alpha_1^\vee) f_0 v_\Lambda$ in two ways, where we put $\Lambda = \Lambda_0 + \Lambda_1$ and $v_\Lambda = 1\otimes 1$. On the one hand, it is clear that

(2.41) $$(c/2 - \alpha_1^\vee) f_0 v_\Lambda = \langle \Lambda_0 + \Lambda_1 - \alpha_0, c/2 - \alpha_1^\vee\rangle f_0 v_\Lambda = -2 f_0 v_\Lambda.$$

On the other hand, by using the identification (2.37b) we have

$$f_0 = \tfrac{1}{2}(\sqrt{2}S(-1) + X_1) = \tfrac{1}{\sqrt{2}}X_1 + \tfrac{1}{2}V_1, \qquad c/2 - \alpha_1^\vee = H(0) = V_0,$$

so that, using

$$V_1(1\otimes 1) = a_{1/2}(1\otimes\xi_{1/2}), \qquad f_0(1\otimes 1) = \tfrac{1}{\sqrt{2}}x_1\otimes 1 + (a_{1/2}/2)(1\otimes\xi_{1/2}),$$

we obtain

(2.42) $$V_0 f_0(1\otimes 1) = -a_{1/2}(1\otimes\xi_{1/2}) + (a_{1/2}^2/\sqrt{2})(x_1\otimes 1).$$

By comparing (2.41) and (2.42), one sees that $a_{1/2}^2 = -2$. We choose $a_{1/2} = \sqrt{2}i$.

Summing up the above, we obtain the following

Proposition 2.2. *Let X_-, H, X_+ be a standard basis of $\mathrm{sl}_2(\mathbf{C})$, let $S = \frac{1}{\sqrt{2}}(X_+ + X_-)$, and let*

$$X(z) = \sum_{j\in\mathbf{Z}_{\text{even}}} H(j)z^{-j} + \sum_{j\in\mathbf{Z}_{\text{odd}}} (X_- - X_+)(j)z^{-j}.$$

For $\varepsilon = 0$ or $\frac{1}{2}$, let

$$V_\varepsilon(z) = \left(\exp\sqrt{2}\sum_{j\in\mathbf{N}_{\text{odd}}} z^j x_j\right)\left(\exp -\sqrt{2}\sum_{j\in\mathbf{N}_{\text{odd}}} \frac{z^{-j}}{j}\frac{\partial}{\partial x_j}\right)\left(\sum_{j\in\varepsilon+\mathbf{Z}} \psi_j z^{-2j}\right),$$

where ψ_j are defined by (2.7a, b). *Then the map*

$$\begin{aligned} S(j) &\mapsto a_j, \qquad c \mapsto 2, \\ X(z) &\mapsto (-1)^\varepsilon\sqrt{2}V_\varepsilon(z), \\ d_n &\mapsto 2(L_n^{(2)} + l_n^{(\varepsilon)}) \end{aligned}$$

defines a representation of the extended affine Lie algebra $\hat{\mathfrak{g}}$ of type $A_1^{(1)}$ on the space $\mathbf{C}[x]\otimes\Lambda^{(\varepsilon)}$, which as a $\mathfrak{g}'$-module is equivalent to $L(2\Lambda_0) + L(2\Lambda_1)$ for $\varepsilon = 0$ and to $L(\Lambda_0+\Lambda_1)$ for $\varepsilon = \frac{1}{2}$, the highest weight vectors being $1\otimes(1\pm\xi_0)$ and $1\otimes 1$ respectively. □

2.4. Let $\mathfrak{g}$ be the affine Kac-Moody algebra $A_1^{(1)}$ in its principal realization (see subsection 2.3). We choose a basis $\{u_i\}$ and its dual basis $\{u^i\}$ of $\mathfrak{g}$ as follows:

$$\{u_i\} = \left\{\frac{1}{\sqrt{2}}(X_+(k) + X_-(k)), \frac{1}{\sqrt{2}}(X_-(k) - X_+(k)), \frac{1}{\sqrt{2}}\left(H(k) - \frac{c}{2}\delta_{0,k}\right), d, c\right\},$$

$$\{u^i\} = \left\{\frac{1}{\sqrt{2}}(X_+(-k) + X_-(-k)), \frac{-1}{\sqrt{2}}(X_-(-k) - X_+(-k)), \frac{1}{\sqrt{2}}\left(H(-k) - \frac{c}{2}\delta_{0,k}\right), c, d\right\},$$

where $d = -\frac{1}{2}d_0 - \frac{1}{4}\alpha_1^\vee$ is the usual energy operator. Let $\Lambda_{(\varepsilon)} = 2\Lambda_0$ or $\Lambda_0 + \Lambda_1$, and $L_\varepsilon = L(2\Lambda_0) + L(2\Lambda_1)$ or $L(\Lambda_0 + \Lambda_1)$ according as $\varepsilon = 0$ or $\frac{1}{2}$. We have

$$|\Lambda_{(\varepsilon)}|^2 := (\Lambda_{(\varepsilon)}|\Lambda_{(\varepsilon)}) = -\varepsilon.$$

We wish to calculate the operator

$$S - |\Lambda_{(\varepsilon)}|^2 = \left(\sum_i u_i \otimes u^i\right) - |\Lambda_{(\varepsilon)}|^2$$

on the space

$$L_\varepsilon \otimes L_\varepsilon = (\mathbb{C}[x'] \otimes \Lambda^{(\varepsilon)}(\xi')) \otimes (\mathbb{C}[x''] \otimes \Lambda^{(\varepsilon)}(\xi'')).$$

Using Proposition 2.2, S is written as follows:

$$\begin{aligned} S = & \sum_{k \in \mathbf{N}_{\text{odd}}} \partial_k' \otimes k x_k'' + \sum_{k \in \mathbf{N}_{\text{odd}}} k x_k' \otimes \partial_k'' \\ & + \text{coefficient of } z^0 \text{ in } \tfrac{1}{2}(a_\varepsilon V'(z) - 1)(a_\varepsilon V''(-z) - 1) \\ & + 2[-l_0' + \tfrac{1}{4}(X_0' - 1)] + 2[-l_0'' + \tfrac{1}{4}(X_0'' - 1)] \\ = & \sum_{k \in \mathbf{N}_{\text{odd}}} k\{\partial_k' \otimes x_k'' + x_k' \otimes \partial_k''\} \\ & + \text{coefficient of } z^0 \text{ in } \tfrac{1}{2}[a_\varepsilon V'(z) - 1][a_\varepsilon V''(-z) - 1] \\ & + \text{coefficient of } z^0 \text{ in } (a_\varepsilon/2)[V'(z) + V''(-z)] - 2l_0' - 2l_0'' - \tfrac{1}{2} \\ = & - \sum_{k \in \mathbf{N}_{\text{odd}}} k(x_k' - x_k'')(\partial_k' - \partial_k'') - 2 \sum_{j \in \varepsilon + \mathbf{Z}_+} j(\psi_{-j}'\psi_j' + \psi_{-j}''\psi_j'') \\ & + \text{coefficient of } z^0 \text{ in } (a_\varepsilon^2/2)V'(z)V''(-z) - \tfrac{1}{2}, \end{aligned}$$

since

$$l_0' = \frac{1}{2} \sum_{k \in \mathbf{N}_{\text{odd}}} k x_k' \partial_k' + \sum_{j \in \varepsilon + \mathbf{Z}_+} j \psi_{-j}' \psi_j', \qquad l_0'' = \frac{1}{2} \sum_{k \in \mathbf{N}_{\text{odd}}} k x_k'' \partial_k'' + \sum_{j \in \varepsilon + \mathbf{Z}_+} j \psi_{-j}'' \psi_j'',$$

where ∂_k' and ∂_k'' stand for $\partial/\partial x_k'$ and $\partial/\partial x_k''$ respectively. Now making change of variables (0.5), we obtain

$$\begin{aligned} V^{(2)'}(z)V^{(2)''}(-z) = & \left(\exp \sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} (x_j' - x_j'') z^j\right) \\ & \times \left(\exp - \sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} \frac{z^{-j}}{j} (\partial_j' - \partial_j'')\right) \\ = & \left(\exp 2\sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} y_j z^j\right) \left(\exp - \sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial y_j}\right). \end{aligned}$$

So we have

$$\begin{aligned} S = & -2 \sum_{k \in \mathbf{N}_{\text{odd}}} k y_k \frac{\partial}{\partial y_k} - 2 \sum_{j \in \varepsilon + \mathbf{Z}_+} j(\psi_{-j}'\psi_j' + \psi_{-j}''\psi_j'') \\ & + \text{coefficient of } z^0 \text{ in } \frac{a_\varepsilon^2}{2} \left(\exp 2\sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} y_j z^j\right) \\ & \times \left(\exp - \sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial y_j}\right) \psi_\varepsilon'(z)\psi_\varepsilon''(-z) - \tfrac{1}{2}. \end{aligned} \tag{2.43}$$

Note that

$$\begin{aligned} \psi_0''(-z) = \psi_0''(z) \quad & \text{and} \quad a_0^2 = 2 \quad \text{if } \varepsilon = 0, \\ \psi_{1/2}''(-z) = -\psi_{1/2}''(z) \quad & \text{and} \quad a_{1/2}^2 = -2 \quad \text{if } \varepsilon = \tfrac{1}{2}. \end{aligned}$$

So (2.43) can be rewritten as

$$\begin{aligned} S = & -2 \sum_{k \in \mathbf{N}_{\text{odd}}} k y_k \frac{\partial}{\partial y_k} - 2 \sum_{j \in \varepsilon + \mathbf{Z}_+} j(\psi_{-j}'\psi_j' + \psi_{-j}''\psi_j'') \\ & + \text{coefficient of } z^0 \text{ in } \left(\exp 2\sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} y_j z^j\right) \\ & \times \left(\exp - \sqrt{2} \sum_{j \in \mathbf{N}_{\text{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial y_j}\right) \psi_\varepsilon'(z)\psi_\varepsilon''(z) - \frac{1}{2}. \end{aligned} \tag{2.44}$$

We recall the polynomials $p_n^{\text{odd}}(x)$ defined by (0.6) and (1.27). Applying the argument deducing (0.9) from (0.7), to (2.44) we obtain the following

LEMMA 2.1.

$$\begin{aligned} & (S - |\Lambda_{(\varepsilon)}|^2)\tau(x', \xi')\tau(x'', \xi'') \\ & = \Bigg\{-2 \sum_{k \in \mathbf{N}_{\text{odd}}} k y_k D_k + \varepsilon - \frac{1}{2} - 2\widetilde{K}^{(\varepsilon)} \\ & \qquad + \sum_{k \in \mathbf{Z}_+} \sum_{r \in \mathbf{Z}} p_n^{\text{odd}}(2\sqrt{2}y) p_{n-2r}^{\text{odd}}(-\sqrt{2}\widetilde{D}) \widetilde{K}_r^{(\varepsilon)}\Bigg\} e^{\sum_{j \in \mathbf{N}_{\text{odd}}} y_j D_j} \tau \cdot \tau, \end{aligned}$$

where

$$\widetilde{K}_r^{(\varepsilon)} := \sum_{\substack{i, j \in \varepsilon + \mathbf{Z} \\ i + j = r}} \psi_i' \psi_j'', \qquad \widetilde{K}^{(\varepsilon)} := \sum_{j \in \varepsilon + \mathbf{Z}_+} j\{\psi_{-j}'\psi_j' + \psi_{-j}''\psi_j''\}. \quad \square$$

Now let us consider the case of $L(\Lambda_0 + \Lambda_1) = \mathbb{C}[x] \otimes \Lambda^{(1/2)}$. We are going to prove the following

THEOREM 2.1. *For an element* $f = f(x,\xi) = \sum f_{i_1\cdots i_n}(x)\xi_{i_1}\cdots\xi_{i_n}$ *in* $L(\Lambda_0+\Lambda_1) = \mathbb{C}[x]\otimes\Lambda^{(1/2)}$, *put*

$$f^T(x,\xi) := \sum f_{i_1\cdots i_n}(x)\xi_{i_n}\cdots\xi_{i_1}.$$

Then f *is contained in the G-orbit of* $1\otimes 1$ *if and only if* f *satisfies the system of super bilinear differential equations*:

(2.45)

$$\left\{-2\sum_{j\in\mathbb{N}_{\rm odd}} jy_j\frac{\partial}{\partial u_j} - 2K + \sum_{n\in\mathbb{Z}_+}\sum_{r\in\mathbb{Z}} p_n^{\rm odd}(2\sqrt{2}y_j)p_{n-2r}^{\rm odd}\left(-\frac{\sqrt{2}}{j}\frac{\partial}{\partial u_j}\right)K_r\right\}$$
$$\times e^{\sum y_j\partial/\partial u_j}e^{\sum\xi_j\partial/\partial\alpha_j}e^{\sum\eta_j\partial/\partial\beta_j}f^T(x+u,\alpha+\beta)f(x-u,\alpha-\beta)|_{u,\alpha,\beta=0} = 0,$$

where for $r\in\mathbb{Z}_+$:

$$K := \sum_{j\in 1/2+\mathbb{Z}_+} j\left(\xi_j\frac{\partial}{\partial\alpha_j} + \eta_j\frac{\partial}{\partial\beta_j}\right),$$

$$K_r := \frac{1}{2}\sum_{\substack{j,k\in 1/2+\mathbb{Z}_+\\ j+k=r}}\frac{\partial}{\partial\alpha_j}\frac{\partial}{\partial\beta_k} + \sum_{j\in 1/2+\mathbb{Z}_+}\left(\xi_j\frac{\partial}{\partial\alpha_{j+r}} - \eta_j\frac{\partial}{\partial\beta_{j+r}}\right),$$

$$K_{-r} := 2\sum_{\substack{j,k\in 1/2+\mathbb{Z}_+\\ j+k=r}}\eta_j\xi_k + \sum_{j\in 1/2+\mathbb{Z}_+}\left(\xi_{j+r}\frac{\partial}{\partial\alpha_j} - \eta_{j+r}\frac{\partial}{\partial\beta_j}\right).$$

PROOF. By Lemma 2.1, we already have the following formula:

(2.46)
$$(S-|\Lambda|^2)(f\otimes g) = \left[-2\sum_{j\in\mathbb{N}_{\rm odd}} jy_jD_j - 2\widetilde{K} + \sum_{n\in\mathbb{Z}_+}\sum_{r\in\mathbb{Z}} p_n^{\rm odd}(2\sqrt{2}y)p_{n-2r}^{\rm odd}(-\sqrt{2}\widetilde{D})\widetilde{K}_r\right]$$
$$\times e^{\sum y_jD_j}f(x',\xi')\cdot g(x'',\xi''),$$

where $\Lambda = \Lambda_0+\Lambda_1$, $\widetilde{K} = \widetilde{K}^{(1/2)}$, and $\widetilde{K}_r = \widetilde{K}_r^{(1/2)}$.

We shall write $\Lambda(\xi)$ instead of $\Lambda^{(1/2)}(\xi)$ for short. Note that $L(\Lambda_0+\Lambda_1) = \mathbb{C}[x',x'']\otimes\Lambda(\xi',\xi'')$ is a tensor product of a polynomial algebra in x' and x'' and an exterior algebra in ξ' and ξ''. Define an automorphism T of $\Lambda(\xi',\xi'')$ by

$$T(\xi'_{i_1}\cdots\xi'_{i_n}\xi''_{j_1}\cdots\xi''_{j_m}) = \xi'_{i_n}\cdots\xi'_{i_1}\xi''_{j_1}\cdots\xi''_{j_m}.$$

For operators $X = \psi'_p\psi'_q,\ \psi'_p\psi''_q,\ \psi''_p\psi''_q$, we let $X^\sigma = X$ in all cases except for $X = \psi'_p\psi''_q$ with $p>0$ when we let $X^\sigma = -X$. Then one checks directly that

(2.47)
$$T\circ X = X^\sigma\circ T.$$

Now we make a change of spinor variables:

$$\xi_j = \tfrac{1}{2}(\xi'_j+\xi''_j),\quad \eta_j = \tfrac{1}{2}(\xi'_j-\xi''_j),\quad \psi_j = \tfrac{1}{2}(\psi'_j+\psi''_j),$$
$$\theta_j = \tfrac{1}{2}(\psi'_j-\psi''_j),\qquad j\in\tfrac{1}{2}+\mathbb{Z}_+.$$

The new operators act on $\Lambda(\xi',\xi'') = \Lambda(\xi,\eta)$ as follows:

$$\psi_j = \tfrac{1}{2}\partial/\partial\xi_j,\quad \psi_{-j} = \xi_j,\quad \theta_j = \tfrac{1}{2}\partial/\partial\eta_j,\quad \theta_{-j} = \eta_j,\qquad j\in\tfrac{1}{2}+\mathbb{Z}_+,$$

and satisfy the following commutation relations:

$$[\psi_j,\psi_k]_+ = \tfrac{1}{2}\delta_{j,-k},\quad [\theta_j,\theta_k]_+ = \tfrac{1}{2}\delta_{j,-k},\quad [\psi_j,\theta_k]_+ = 0.$$

Furthermore, we have for $r\in\mathbb{Z}_+$

(2.48a)
$$\widetilde{K}^\sigma = 2\sum_{j\in 1/2+\mathbb{Z}_+} j(\psi_{-j}\psi_j + \theta_{-j}\theta_j),$$

(2.48b)
$$\widetilde{K}_r^\sigma = 2\sum_{\substack{j,k\in 1/2+\mathbb{Z}_+\\ j+k=r}}\psi_j\theta_k + 2\sum_{j\in 1/2+\mathbb{Z}_+}(\psi_{-j}\psi_{j+r} - \theta_{-j}\theta_{j+r}),$$

(2.48c)
$$\widetilde{K}_{-r}^\sigma = 2\sum_{\substack{j,k\in 1/2+\mathbb{Z}_+\\ j+k=r}}\theta_{-j}\psi_{-k} + 2\sum_{j\in 1/2+\mathbb{Z}_+}(\psi_{-j-r}\psi_j - \theta_{-j-r}\theta_j).$$

We also need the following super Taylor formula:

(2.49)
$$e^{\sum\xi_j\partial/\partial\alpha_j}f(\alpha)|_{\alpha=0} = f(\xi).$$

From (2.48) and (2.49) we obtain

(2.50a)
$$\widetilde{K}^\sigma f^T(x',\xi')g(x'',\xi'') = Ke^{\sum\xi_j\partial/\partial\alpha_j}e^{\sum\eta_j\partial/\partial\beta_j}f^T(x',\alpha+\beta)g(x'',\alpha-\beta)|_{\alpha,\beta=0},$$

(2.50b)
$$\widetilde{K}_r^\sigma f^T(x',\xi')g(x'',\xi'') = K_re^{\sum\xi_j\partial/\partial\alpha_j}e^{\sum\eta_j\partial/\partial\beta_j}f^T(x',\alpha+\beta)g(x'',\alpha-\beta)|_{\alpha,\beta=0}.$$

Applying σ to both sides of (2.46) and using (2.50) completes the proof, due to Theorem 0.1(a). □

EXAMPLE 2.1. The coefficients of $y_1^{s_1}y_2^{s_2}\cdots\xi_j^{\varepsilon_j}\eta_k^{\delta_k}$ in (2.45) are as follows:

$$1 \text{ and } \xi_{1/2}:\quad 0;\qquad \eta_{1/2}:\ -2\frac{\partial}{\partial\beta_{1/2}};$$

$$\xi_{1/2}\eta_{1/2} \text{ and } y_1^2:\quad -2\left(D_1^2 - \frac{\partial}{\partial\alpha_{1/2}}\frac{\partial}{\partial\beta_{1/2}}\right);$$

$$\xi_j\eta_k:\quad -2\tilde{p}_{2(j+k)}(D) + 2(j+k)\frac{\partial}{\partial\alpha_j}\frac{\partial}{\partial\beta_k} - \sum_{0<s\le j}\tilde{p}_{2(j-s)}(D)\frac{\partial}{\partial\alpha_s}\frac{\partial}{\partial\beta_k}$$
$$+\sum_{0<s\le k}\tilde{p}_{2(k-s)}(D)\frac{\partial}{\partial\alpha_j}\frac{\partial}{\partial\beta_s};$$

where $\tilde{p}_i(D) := p_i^{\rm odd}(-\sqrt{2}\widetilde{D})$. In particular, we have

$$\xi_{3/2}\eta_{1/2}:\ -\frac{1}{3}\left[D_1^4 + 4D_1D_3 + 3D_1^2\frac{\partial}{\partial\alpha_{1/2}}\frac{\partial}{\partial\beta_{1/2}} - 12\frac{\partial}{\partial\alpha_{3/2}}\frac{\partial}{\partial\beta_{1/2}}\right];$$

$$\xi_{1/2}\eta_{3/2}:\ -\frac{1}{3}\left[D_1^4 + 4D_1D_3 - 3D_1^2\frac{\partial}{\partial\alpha_{1/2}}\frac{\partial}{\partial\beta_{1/2}} - 12\frac{\partial}{\partial\alpha_{1/2}}\frac{\partial}{\partial\beta_{3/2}}\right].$$

Defining a super bilinear equation $P(D_x, D^\alpha, D^\beta)\tau\cdot\tau = 0$ by

$$P\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}\right)\tau^T(x+u, \alpha+\beta)\tau(x-u, \alpha-\beta)|_{u=0,\alpha=\beta=0} = 0,$$

we write the equations of lowest degree of this hierarchy as follows:

$$(D_1^2 - D_{1/2}^\alpha D_{1/2}^\beta)\tau\cdot\tau = 0,$$
$$(D_1^4 + 4D_1D_3 + 3D_1^2D_{1/2}^\alpha D_{1/2}^\beta - 12D_{3/2}^\alpha D_{1/2}^\beta)\tau\cdot\tau = 0,$$
$$(D_1^4 + 4D_1D_3 - 3D_1^2D_{1/2}^\alpha D_{1/2}^\beta - 12D_{1/2}^\alpha D_{3/2}^\beta)\tau\cdot\tau = 0.$$

These may be viewed as super KdV equations. Unfortunately, their relation to the known super generalization of KdV **[19, 23]** is unclear.

3. Homogeneous picture: The nonlinear Schrödinger and the 2-dimensional Toda lattice.

3.1. Let $\mathring{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra of rank l with a symmetric Cartan matrix (type A-D-E). It can be constructed starting from its root lattice $\mathring{Q}$ with the (Weyl group invariant) symmetric bilinear form $(\cdot|\cdot)$, normalized by $(\alpha|\alpha) = 2$ for a shortest nonzero vector α, as follows **[8, 14]**. Choose a (nonsymmetric) bilinear form $R\colon \mathring{Q}\times\mathring{Q}\to\mathbb{Z}$ such that

$$(\alpha|\beta) = R(\alpha,\beta) + R(\beta,\alpha).$$

(R may be constructed, for example, by introducing an orientation on the Dynkin diagram labeled by simple roots α_i, putting $R(\alpha_i,\alpha_i) = 1$, and $R(\alpha_i,\alpha_j) = 0$ for $i\neq j$ in all cases, except for $\alpha_i\to\alpha_j$, when $R(\alpha_i,\alpha_j) = -1$, and then extending R by bilinearity.) Put

$$\varepsilon(\alpha,\beta) = (-1)^{R(\alpha,\beta)}.$$

Let $\mathring{\mathfrak{h}}$ be the complexification of $\mathring{Q}$, and let

$$\mathring{\Delta} = \{\alpha\in\mathring{Q}|(\alpha|\alpha) = 2\}; \qquad \mathring{\mathfrak{g}} = \mathring{\mathfrak{h}}\oplus\sum_{\alpha\in\mathring{\Delta}}\mathbb{C}E_\alpha,$$

with the following commutation relations:

$$[\mathring{\mathfrak{h}},\mathring{\mathfrak{h}}] = 0, \quad [h, E_\alpha] = (\alpha|h)E_\alpha \quad \text{for } h\in\mathring{\mathfrak{h}},$$
$$[E_\alpha, E_\beta] = 0 \quad \text{if } \alpha,\beta\in\mathring{\Delta}, \alpha+\beta\notin\mathring{\Delta}\cup\{0\},$$
$$[E_\alpha, E_{-\alpha}] = -\alpha, \quad [E_\alpha, E_\beta] = \varepsilon(\alpha,\beta)E_{\alpha+\beta} \quad \text{if } \alpha,\beta,\alpha+\beta\in\mathring{\Delta}.$$

The bilinear form $(\cdot|\cdot)$ extends from $\mathring{Q}$ to $\mathring{\mathfrak{h}}$ by bilinearity, and to the whole $\mathring{\mathfrak{g}}$ by

$$\left(\mathring{\mathfrak{h}}|\sum_\alpha\mathbb{C}E_\alpha\right) = 0 \quad\text{and}\quad (E_\alpha|E_\beta) = -\delta_{\alpha,-\beta}.$$

The associated affine Kac-Moody algebra $\mathfrak{g}$ is considered in this section in the following (homogeneous) realization:

$$\mathfrak{g} = \mathbb{C}[t,t^{-1}]\otimes\mathring{\mathfrak{g}}\oplus\mathbb{C}c\oplus\mathbb{C}d,$$

with commutation relations

$$[u(j), v(k)] = [u,v](j+k) + j(u|v)\delta_{j,-k}c,$$
$$[d, u(j)] = ju(j), \qquad [c,\mathfrak{g}] = 0,$$

where as before $u(j)$ stands for $t^j\otimes u$.

Choose a basis $\{u_j\}$ of $\mathring{\mathfrak{h}}$ and let $\{u^j\}$ be the dual basis of $\mathring{\mathfrak{h}}$ with respect to $(\cdot|\cdot)$. The homogeneous realization of the basic representation $L(\Lambda_0)$ of $\mathfrak{g}$ is constructed in the space

$$L(\Lambda_0) = \mathbb{C}[x]\otimes\mathbb{C}[\mathring{Q}] := \mathbb{C}[x_k^{(j)}; 1\le j\le l, k\in\mathbb{N}]\otimes\sum_{\alpha\in\mathring{Q}}\mathbb{C}e^\alpha,$$

where $\mathfrak{g}$ acts as follows:

(3.1a) $\qquad u_j(-k) = kx_k^{(j)}, \quad u^j(k) = \partial/\partial x_k^{(j)}, \qquad 1\le j\le l, k\in\mathbb{N};$

(3.1b) $\qquad H(0)(f\otimes e^\beta) = \beta(H)f\otimes e^\beta \quad \text{for } H\in\mathring{\mathfrak{h}},$

(3.1c) $\qquad d(f\otimes e^\beta) = -\left(\sum_{k\ge1}\sum_{i=1}^{l}kx_k^{(i)}\dfrac{\partial f}{\partial x_k^{(i)}} + \dfrac{1}{2}|\beta|^2 f\right)\otimes e^\beta,$

(3.1d) $\qquad E_\gamma(-k)(f\otimes e^\beta) = \varepsilon(\gamma,\beta)X_k(\gamma)(f\otimes e^\beta) \quad \text{for } \gamma\in\mathring{\Delta},$

where

$$X(\gamma, z) = \sum_{k\in\mathbb{Z}}X_k(\gamma)z^k$$
$$:= z^{|\gamma|^2/2}\left(\exp\sum_{j\ge1}\frac{\gamma(-j)z^j}{j}\right)\left(\exp-\sum_{j\ge1}\frac{\gamma(j)z^{-j}}{j}\right)\otimes e^\gamma z^{\partial_\gamma}$$

is the vertex operator defined generally for any element γ in $\mathring{Q}$. (Recall that $z^{\partial_\gamma}(f\otimes e^\beta) = z^{(\beta|\gamma)}f\otimes e^\beta$.) From (3.1c), one sees that the space $L(\Lambda_0)$ carries a natural $\mathbb{Z}_+$-gradation defined by $\deg e^\beta := \frac{1}{2}(\beta|\beta)$ and $\deg x_k^{(j)} := k$. We let, for brevity, $x = (x_k^{(j)})_{1\le j\le l, k\in\mathbb{N}}$.

For each $\gamma\in\mathring{Q}$ and $n\in\mathbb{Z}_+$, we define polynomial functions P_n^γ and Q_n^γ of degree n in $\mathbb{C}[x]$ by

(3.2) $\qquad \displaystyle\sum_{n\ge0}P_n^\gamma(x)z^n = \exp\sum_{k\ge1}\sum_{j=1}^{l}\langle\gamma, u_j\rangle x_k^{(j)}z^k,$

(3.3) $\qquad \displaystyle\sum_{n\ge0}Q_n^\gamma(x)z^n = \exp\sum_{k\ge1}\sum_{j=1}^{l}\langle\gamma, u^j\rangle x_k^{(j)}z^k.$

3.2. Now we are going to prove the following theorem:

THEOREM 3.1. *An element* $\tau = \sum_{\beta \in \mathring{Q}} \tau_\beta \otimes e^\beta$ *of* $L(\Lambda_0) = \mathbb{C}[x] \otimes \mathbb{C}[\mathring{Q}]$ *is contained in the G-orbit of the vacuum vector* $1 \otimes e^0$ *if and only if it satisfies the following hierarchy of Hirota bilinear differential equations:*

(3.4)
$$-\left[2\sum_{k\geq 1}\sum_{j=1}^{l} k y_k^{(j)} D_k^{(j)} + \frac{1}{2}|\alpha-\beta|^2\right] e^{\Sigma y_k^{(j)} D_k^{(j)}} \tau_\alpha \cdot \tau_\beta + \sum_{\gamma \in \mathring{\Delta}} \varepsilon(\gamma, \alpha-\beta) \sum_{n \geq 0} Q_n^\gamma(2y) P^\gamma_{n-2+(\gamma|\alpha-\beta)}(-\widetilde{D}) e^{\Sigma y_k^{(j)} D_k^{(j)}} \tau_{\alpha-\gamma} \cdot \tau_{\beta+\gamma} = 0$$

for every $\alpha, \beta \in \mathring{Q}$.

PROOF. We write simply $\tau_\beta e^\beta$ for $\tau_\beta \otimes e^\beta$. We shall calculate here the operator S on the space
$$L(\Lambda_0) \otimes L(\Lambda_0) = (\mathbb{C}[x'] \otimes \mathbb{C}[\mathring{Q}']) \otimes (\mathbb{C}[x''] \otimes \mathbb{C}[\mathring{Q}'']).$$
We have the following dual bases of $\mathfrak{g}$:
$$\{v_i\}: u^j(k), k>0, 1 \leq j \leq l; \quad u_j(-k), k \geq 0, 1 \leq j \leq l;$$
$$E_\gamma(k), \gamma \in \mathring{\Delta}, k \in \mathbb{Z}; \quad c; \quad d;$$
$$\{v^i\}: u_j(-k), k>0, 1 \leq j \leq l; \quad u^j(k), k \geq 0, 1 \leq j \leq l;$$
$$-E_{-\gamma}(-k), \gamma \in \mathring{\Delta}, k \in \mathbb{Z}; \quad d; \quad c.$$
Take $f = \sum f_{\beta'} e^{\beta'}$ and $g = \sum g_{\beta''} e^{\beta''}$. Let $S = \sum_i v_i \otimes v^i$; then
$$\begin{aligned} S(f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''}) &= \sum_{\substack{k\geq 1\\ 1\leq j\leq l}} (u^j(k) f_{\beta'} \otimes u_j(-k) g_{\beta''}) e^{\beta'} \otimes e^{\beta''} \\ &+ \sum_{\substack{k\geq 1\\ 1\leq j\leq l}} (u_j(-k) f_{\beta'} \otimes u^j(k) g_{\beta''}) e^{\beta'} \otimes e^{\beta''} \\ &+ \sum_{1\leq j\leq l} (u_j(0) \otimes u^j(0))(f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''}) \\ &- \sum_{\gamma\in\mathring{\Delta}}\sum_{k\in\mathbb{Z}} (E_\gamma(k) \otimes E_{-\gamma}(-k))(f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''}) \\ &+ f_{\beta'}e^{\beta'} \otimes d(g_{\beta''}e^{\beta''}) + d(f_{\beta'}e^{\beta'}) \otimes g_{\beta''}e^{\beta''} \\ &= \left\{ \sum_{\substack{k\geq 1\\ 1\leq j\leq l}} \frac{\partial}{\partial x_k^{(j)'}} f_{\beta'} \otimes k x_k^{(j)''} g_{\beta''} + k x_k^{(j)'} f_{\beta'} \otimes \frac{\partial}{\partial x_k^{(j)''}} g_{\beta''} \right\} e^{\beta'} \otimes e^{\beta''} \\ &+ \sum_{1\leq j\leq l} \beta'(u_j)\beta''(u^j) f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''} \end{aligned}$$

$$\begin{aligned} &- \text{coefficient of } z^0 \text{ in } \sum_{\gamma\in\mathring{\Delta}} \varepsilon(\gamma,\beta')\varepsilon(-\gamma,\beta'') \\ &\qquad \times (X(\gamma,z) \otimes X(-\gamma,z))(f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''}) \\ &- \left\{ \sum_{\substack{k\geq 1\\ 1\leq j\leq l}} k\left(x_k^{(j)'} \frac{\partial}{\partial x_k^{(j)'}} + x_k^{(j)''} \frac{\partial}{\partial x_k^{(j)''}} \right) + \frac{1}{2}(|\beta'|^2 + |\beta''|^2) \right\} (f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''}) \\ &= (\mathrm{I}) + \sum_{\gamma\in\mathring{\Delta}} \text{coefficient of } z^0 \text{ in } (\mathrm{II})_\gamma, \end{aligned}$$

where we put
$$(\mathrm{I}) = -\left\{ \sum_{\substack{k\geq 1\\ 1\leq j\leq l}} k(x_k^{(j)'} - x_k^{(j)''}) \left(\frac{\partial}{\partial x_k^{(j)'}} - \frac{\partial}{\partial x_k^{(j)''}} \right) + \frac{1}{2}|\beta' - \beta''|^2 \right\} (f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''})$$

and
$$(\mathrm{II})_\gamma = -\varepsilon(\gamma, \beta'-\beta'')(X(\gamma,z) \otimes X(-\gamma,z))(f_{\beta'}e^{\beta'} \otimes g_{\beta''}e^{\beta''}).$$

Changing the variables

(3.5)
$$x_k^{(j)} := \tfrac{1}{2}(x_k^{(j)'} + x_k^{(j)''}), \qquad y_k^{(j)} := \tfrac{1}{2}(x_k^{(j)'} - x_k^{(j)''}),$$

we have

(3.6)
$$\frac{\partial}{\partial x_k^{(j)}} = \frac{\partial}{\partial x_k^{(j)'}} + \frac{\partial}{\partial x_k^{(j)''}}, \qquad \frac{\partial}{\partial y_k^{(j)}} = \frac{\partial}{\partial x_k^{(j)'}} - \frac{\partial}{\partial x_k^{(j)''}},$$

and so

(3.7)
$$(\mathrm{I}) = -\left\{ 2\sum_{\substack{k\geq 1\\ 1\leq j\leq l}} k y_k^{(j)} \frac{\partial}{\partial y_k^{(j)}} + \frac{1}{2}|\beta' - \beta''|^2 \right\} (f_{\beta'}(x+y) g_{\beta''}(x-y)) e^{\beta'} \otimes e^{\beta''}.$$

By using (3.2), (3.5), and (3.5), $(\mathrm{II})_\gamma$ can be rewritten as

$$
\begin{aligned}
(\mathrm{II})_\gamma = &-\varepsilon(\gamma, \beta'-\beta'')\left(\exp 2\sum_{k\geq 1}\frac{\tilde{\gamma}(-k)z^k}{k}\right)\left(\exp -\sum_{k\geq 1}\frac{\tilde{\gamma}(k)z^{-k}}{k}\right)(f_{\beta'}\otimes g_{\beta''})\\
&\times (z\cdot z^{(\gamma|\beta')}e^{\gamma}\otimes z\cdot z^{-(\gamma|\beta'')}e^{-\gamma})(e^{\beta'}\otimes e^{\beta''})\\
=&-\varepsilon(\gamma,\beta'-\beta'')z^{2+(\gamma|\beta'-\beta'')}\left(\exp 2\sum_{k\geq 1}\frac{\tilde{\gamma}(-k)z^k}{k}\right)\\
&\times\left(\exp -\sum_{k\geq 1}\frac{\tilde{\gamma}(k)z^{-k}}{k}\right)(f_{\beta'}\otimes g_{\beta''})(e^{\beta'+\gamma}\otimes e^{\beta''-\gamma}),
\end{aligned}
$$

where

$$
\tilde{\gamma}(-k) := k\sum_{j=1}^{l}\langle\gamma, u^j\rangle y_k^{(j)},\quad \tilde{\gamma}(k) := \sum_{j=1}^{l}\langle\gamma, u_j\rangle\frac{\partial}{\partial y_k^{(j)}}\quad \text{for } k\in\mathbb{N}.
$$

By using (3.3), this can be rewritten as

$$
\begin{aligned}
(\mathrm{II})_\gamma = &-\varepsilon(\gamma,\beta'-\beta'')z^{2+(\gamma|\beta'-\beta'')}\left(\sum_{n\geq 0}Q_n^\gamma(2y)z^n\sum_{m\geq 0}P_m^\gamma(-\tilde{\partial}_y)z^{-m}\right)\\
&\times(f_{\beta'}\otimes g_{\beta''})(e^{\beta'+\gamma}\otimes e^{\beta''-\gamma})\\
=&-\varepsilon(\gamma,\beta'-\beta'')z^{2+(\gamma|\beta'-\beta'')}\left(\sum_{n,m\geq 0}Q_n^\gamma(2y)P_m^\gamma(-\tilde{\partial}_y)z^{n-m}\right)\\
&\times(f_{\beta'}(x+y)g_{\beta''}(x-y))(e^{\beta'+\gamma}\otimes e^{\beta''-\gamma}),
\end{aligned}
$$

where

$$
\tilde{\partial}_y := \left(\frac{1}{k}\frac{\partial}{\partial y_k^{(j)}}; 1\leq j\leq l, k\geq 1\right).
$$

From this, one sees that

$$
\begin{aligned}
&\text{the coefficient of } z^0 \text{ in } (\mathrm{II})_\gamma\\
(3.8)\qquad &= -\varepsilon(\gamma,\beta'-\beta'')\sum_{n\geq 0}Q_n^\gamma(2y)P^\gamma_{n+2+(\gamma|\beta'-\beta'')}(-\tilde{\partial}_y)\\
&\quad\times(f_{\beta'}(x+y)g_{\beta''}(x-y))(e^{\beta'+\gamma}\otimes e^{\beta''-\gamma}).
\end{aligned}
$$

Thus, from (3.7) and (3.8), one obtains

(3.9)

$$
\begin{aligned}
&S(f\otimes g)\\
&=\sum_{\beta',\beta''\in\mathring{Q}}\Bigg\{\left(-2\sum_{\substack{k\geq 1\\1\leq j\leq l}}ky_k^{(j)}\frac{\partial}{\partial y_k^{(j)}}+\frac{1}{2}|\beta'-\beta''|^2\right)f_{\beta'}(x+y)g_{\beta''}(x-y)\\
&\qquad-\sum_{\gamma\in\mathring{\Delta}}\varepsilon(\gamma,\beta'-\beta''-2\gamma)\sum_{n\geq 0}Q_n^\gamma(2y)P^\gamma_{n+2+(\gamma|\beta'-\beta''-2\gamma)}(-\tilde{\partial}_y)\\
&\qquad\times f_{\beta'-\gamma}(x+y)g_{\beta''+\gamma}(x-y)\Bigg\}e^{\beta'}\otimes e^{\beta''}\\
&=\sum_{\beta',\beta''\in\mathring{Q}}\Bigg\{-\left(2\sum_{\substack{1\leq j\leq l\\k\geq 1}}ky_k^{(j)}\frac{\partial}{\partial y_k^{(j)}}+\frac{1}{2}|\beta'-\beta''|^2\right)f_{\beta'}(x+y)g_{\beta''}(x-y)\\
&\qquad-\sum_{\gamma\in\mathring{\Delta}}\varepsilon(\gamma,\beta'-\beta'')\sum_{n\geq 0}Q_n^\gamma(2y)P^\gamma_{n-2+(\gamma|\beta'-\beta'')}(-\tilde{\partial}_y)\\
&\qquad\times f_{\beta'-\gamma}(x+y)g_{\beta''+\gamma}(x-y)\Bigg\}e^{\beta'}\otimes e^{\beta''}.
\end{aligned}
$$

By Theorem 0.1, we know that $\tau=\sum\tau_\beta e^\beta$ belongs to the G-orbit of the vacuum $1\otimes e^0$ if and only if $S(\tau\otimes\tau)=0$ (recall that $(\Lambda_0|\Lambda_0)=0$), which, due to (3.9), is equivalent to

$$
\begin{aligned}
(3.10)\qquad&\left(2\sum_{\substack{1\leq j\leq l\\k\geq 1}}ky_k^{(j)}\frac{\partial}{\partial y_k^{(j)}}+\frac{1}{2}|\beta'-\beta''|^2\right)\tau_{\beta'}(x+y)\tau_{\beta''}(x-y)\\
&+\sum_{\gamma\in\mathring{\Delta}}\varepsilon(\gamma,\beta'-\beta'')\sum_{n\geq 0}Q_n^\gamma(2y)P^\gamma_{n-2+(\gamma|\beta'-\beta'')}(-\tilde{\partial}_y)\\
&\times\tau_{\beta'-\gamma}(x+y)\tau_{\beta''+\gamma}(x-y)=0\quad\text{for every }\beta',\beta''\in\mathring{Q}.
\end{aligned}
$$

Using (1.16), (3.10) gives us the desired formula (3.4). □

The hierarchy (3.4) is called the *homogeneous hierarchy* of type $X_l^{(1)}$.

REMARK 3.1. Given $\lambda\in\mathring{Q}$, the transformation $\alpha\to\alpha+\lambda$, $\beta\to\beta+\lambda$ leaves the hierarchy (3.4) unchanged. Such type of transformations are called in the soliton theory the Schlesinger transformations.

Lower degree equations of hierarchy (3.4) are given below. The constant term:

$$\mathrm{I}_{\alpha,\beta} \qquad \frac{1}{2}|\alpha-\beta|^2\tau_\alpha\tau_\beta + \sum_{\gamma\in\mathring{\Delta}} \varepsilon(\gamma,\alpha-\beta)P^{\gamma}_{(\gamma|\alpha-\beta)-2}(-\widetilde{D})\tau_{\alpha-\gamma}\cdot\tau_{\beta+\gamma} = 0.$$

The coefficient of $y_k^{(j)}$:

$$\mathrm{II}^{(j)}_{k;\alpha,\beta} \qquad \begin{aligned} &\left(2k+\frac{1}{2}|\alpha-\beta|^2\right)D^{(j)}_{\tau}\tau_\alpha\cdot\tau_\beta \\ &+\sum_{\gamma\in\mathring{\Delta}}\varepsilon(\gamma,\alpha-\beta)\{2\langle\gamma,u^j\rangle P^{\gamma}_{(\gamma|\alpha-\beta)+k-2}(-\widetilde{D}) \\ &\qquad + P^{\gamma}_{(\gamma|\alpha-\beta)-2}(-\widetilde{D})D^{(j)}_k\}\tau_{\alpha-\gamma}\cdot\tau_{\beta+\gamma} = 0. \end{aligned}$$

The coefficient of $y_k^{(j)2}$:

$$\mathrm{III}^{(j)}_{k;\alpha,\beta} \quad \begin{aligned} &(2k+\tfrac{1}{4}|\alpha-\beta|^2)D^{(j)2}_k\tau_\alpha\cdot\tau_\beta \\ &+2\sum_{\gamma\in\mathring{\Delta}}\varepsilon(\gamma,\alpha-\beta)\Big\{\langle\gamma,u^j\rangle P^{\gamma}_{(\gamma|\alpha-\beta)+k-2}(-\widetilde{D})D^{(j)}_k \\ &\qquad + \langle\gamma,u^j\rangle^2 P^{\gamma}_{(\gamma|\alpha-\beta)+2k-2}(-\widetilde{D}) \\ &\qquad + \tfrac{1}{4}P^{\gamma}_{(\gamma|\alpha-\beta)-2}(-\widetilde{D})D^{(j)2}_k\Big\}\tau_{\alpha-\gamma}\cdot\tau_{\beta+\gamma} = 0. \end{aligned}$$

The coefficient of $y_k^{(j)}y_{k'}^{(j')}$, $(j,k)\neq(j',k')$:

$\mathrm{IV}^{(j,j')}_{k,k';\alpha,\beta}$

$$\begin{aligned} &(2(k+k')+\tfrac{1}{2}|\alpha-\beta|^2)D^{(j)}_kD^{(j')}_{k'}\tau_\alpha\cdot\tau_\beta \\ &+\sum_{\gamma\in\mathring{\Delta}}\varepsilon(\gamma,\alpha-\beta)\{2\langle\gamma,u^j\rangle P^{\gamma}_{(\gamma|\alpha-\beta)+k-2}(-\widetilde{D})D^{(j')}_{k'} \\ &\qquad + 2\langle\gamma,u^{j'}\rangle P^{\gamma}_{(\gamma|\alpha-\beta)+k'-2}(-\widetilde{D})D^{(j)}_k \\ &\qquad + 4\langle\gamma,u^j\rangle\langle\gamma,u^{j'}\rangle P^{\gamma}_{(\gamma|\alpha-\beta)+k+k'-2}(-\widetilde{D}) \\ &\qquad + P^{\gamma}_{(\gamma|\alpha-\beta)-2}(-\widetilde{D})D^{(j)}_kD^{(j')}_{k'}\}\tau_{\alpha-\gamma}\cdot\tau_{\beta+\gamma} = 0. \end{aligned}$$

We use these equations to derive a system of partial differential equations for the functions

$$u := \log\tau_0 \quad\text{and}\quad q_\alpha := \tau_\alpha/\tau_0, \quad \alpha\in\mathring{\Delta}.$$

For simplicity of notation we let $(k=1,2,\dots)$:

$$x_k = \sum_{j=1}^{l} x^{(j)}_k u^j \in \mathring{\mathfrak{h}}, \qquad D^{[\alpha]}_k = \sum_{j=1}^{l}\langle\alpha,u_j\rangle\frac{\partial}{\partial^{(j)}_k}.$$

Below we write explicitly some special cases of equations I-IV (noting that $P^{\alpha}_2(-\tilde{D}) = \frac{1}{2}(D^{[\alpha]^2}_1 - D^{[\alpha]}_2)$):

$$\mathrm{I}_{\alpha,-\alpha}: \quad \frac{1}{2}(D^{[\alpha]^2}_1 - D^{[\alpha]}_2)\tau_0\cdot\tau_0 + 4\tau_\alpha\tau_{-\alpha} + \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha+\gamma\in\mathring{\Delta}}}\tau_\gamma\tau_{-\gamma} = 0.$$

$$\mathrm{I}_{\alpha,\beta}(\alpha+\beta\in\mathring{\Delta}): \quad \begin{aligned} &(D^{[\alpha]}_1 - D^{[\beta]}_1)\tau_0\cdot\tau_{\alpha+\beta} + 3\varepsilon(\alpha,\beta)\tau_\alpha\tau_\beta \\ &+\sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha-\gamma\in\mathring{\Delta}\\ \beta+\gamma\in\mathring{\Delta}}}\varepsilon(\alpha-\gamma,\beta+\gamma)\tau_{\alpha-\gamma}\tau_{\beta+\gamma} = 0. \end{aligned}$$

$$\mathrm{II}_{1;\alpha,0} := \sum_{j=1}^{l}\langle\alpha,u_j\rangle\mathrm{II}^{(j)}_{1;\alpha,0}: \qquad \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha+\gamma\in\mathring{\Delta}}}\varepsilon(\gamma,\alpha)\tau_{\alpha+\gamma}\tau_{-\gamma} \equiv 0.$$

$$\mathrm{II}_{2;\alpha,0} := \sum_{j=1}^{l}\langle\alpha,u_j\rangle\mathrm{II}^{(j)}_{2;\alpha,0}:$$

$$(2D^{[\alpha]}_2 + D^{[\alpha]^2}_1)\tau_0\cdot\tau_\alpha + \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha+\gamma\in\mathring{\Delta}}}\varepsilon(\gamma,\alpha)D^{[\alpha+\gamma]}_1\tau_{\alpha+\gamma}\cdot\tau_{-\gamma} = 0.$$

$$\mathrm{III}^{(j)}_{1;0,0}: \quad D^{(j)^2}_1\tau_0\cdot\tau_0 + \sum_{\gamma\in\mathring{\Delta}}\langle\gamma,u^j\rangle^2\tau_\gamma\tau_{-\gamma} = 0.$$

$$\mathrm{III}^{(j)}_{2;0,0}: \quad 2D^{(j)^2}_2\tau_0\cdot\tau_0 + \sum_{\gamma\in\mathring{\Delta}}\left[\langle\gamma,u^j\rangle D^{(j)}_2 + \langle\gamma,u^j\rangle^2\frac{1}{2}(D^{[\gamma]^2}_1 - D^{[\gamma]}_2)\right]\tau_{-\gamma}\cdot\tau_\gamma = 0.$$

$$\mathrm{IV}^{(j,j')}_{1,1;0,0}(j\neq j'): \quad D^{(j)}_1D^{(j')}_1\tau_0\cdot\tau_0 + \sum_{\gamma\in\mathring{\Delta}}\langle\gamma,u^j\rangle\langle\gamma,u^{j'}\rangle\tau_{-\gamma}\tau_\gamma = 0.$$

$$\mathrm{IV}^{(j,j')}_{2,2;0,0}(j\neq j): \quad \begin{aligned} &4D^{(j)}_2D^{(j')}_2\tau_0\cdot\tau_0 \\ &+\sum_{\gamma\in\mathring{\Delta}}[\langle\gamma,u^j\rangle D^{(j')}_2 + \langle\gamma,u^{j'}\rangle D^{(j)}_2 + 2\langle\gamma,u^j\rangle\langle\gamma,u^{j'}\rangle P^{\gamma}_2(-\tilde{D})]\tau_{-\gamma}\cdot\tau_\gamma = 0. \end{aligned}$$

$$\mathrm{IV}^{(j,j')}_{2,1;0,0}: \quad \begin{aligned} &3D^{(j)}_2D^{(j')}_1\tau_0\cdot\tau_0 \\ &+\sum_{\gamma\in\mathring{\Delta}}[\langle\gamma,u^j\rangle D^{(j')}_1 + 2\langle\gamma,u^j\rangle\langle\gamma,u^{j'}\rangle P^{\gamma}_1(-\tilde{D})]\tau_{-\gamma}\cdot\tau_\gamma = 0. \end{aligned}$$

Letting

$$(\mathrm{III}+\mathrm{IV})_{k,k;0,0} := \sum_{\substack{1\le j,j'\le l\\ j\ne j'}} \langle\alpha,u_j\rangle\langle\beta,u_{j'}\rangle \mathrm{IV}^{(j,j')}_{k,k;0,0} + \sum_{1\le j\le l}\langle\alpha,u_j\rangle\langle\beta,u_j\rangle \mathrm{III}^{(j)}_{k;0,0}$$

and $\mathrm{IV}_{2,1;0,0} := \sum_{j=1}^{l}\sum_{j'=1}^{l}\langle\alpha,u_j\rangle\langle\beta,u_{j'}\rangle\mathrm{IV}^{(j,j')}_{2,1;0,0}$, we have:

$$(\mathrm{III}+\mathrm{IV})_{1,1;0,0}:\quad D_1^{[\alpha]}D_1^{[\beta]}\tau_0\cdot\tau_0 + \sum_{\gamma\in\mathring{\Delta}}(\alpha|\gamma)(\beta|\gamma)\tau_\gamma\tau_{-\gamma} = 0,$$

$(\mathrm{III}+\mathrm{IV})_{2,2;0,0}$:

$$4D_2^{[\alpha]}D_2^{[\beta]}\tau_0\cdot\tau_0 + \sum_{\gamma\in\mathring{\Delta}}\left[(\alpha|\gamma)D_2^{[\beta]} + (\beta|\gamma)D_2^{[2]} + 2(\alpha|\gamma)(\beta|\gamma)\frac{1}{2}(D_1^{[\gamma]^2} - D_2^{[\gamma]})\right]\tau_{-\gamma}\cdot\tau_\gamma = 0,$$

$$\mathrm{IV}_{2,1;0,0}:\quad 3D_2^{[\alpha]}D_1^{[\beta]}\tau_0\cdot\tau_0 + \sum_{\gamma\in\mathring{\Delta}}(\alpha|\gamma)(D_1^{[\beta]} - 2(\beta|\gamma)D_1^{[\gamma]})\tau_{-\gamma}\cdot\tau_\gamma = 0.$$

Thus we have deduced a system of bilinear differential equations for τ_α, $\alpha\in\mathring{\Delta}\cup\{0\}$. (Note that $\mathrm{I}_{\alpha,-\alpha}$ is a special case of $(\mathrm{III}+\mathrm{IV})_{1,1;0,0}$, since $D_2^{[\alpha]}\tau_0\cdot\tau_0 = 0$ is trivial.)

Now we rewrite these bilinear equations as partial differential equations by putting

$$u(x,t) := \log\tau_0(x,t,c_3,c_4,\dots),\qquad q^\alpha(x,t) := \frac{\tau_\alpha}{\tau_0}(x,t,c_3,c_4,\dots),$$

where $x,t\in\mathring{\mathfrak{h}}$ and $c_3,c_4,\dots$ are any (constant) elements in $\mathring{\mathfrak{h}}$.

We use the following notation for a function $f(x,t)$, $(x,t\in\mathring{\mathfrak{h}})$ and $\mu\in\mathring{\mathfrak{h}}$:

$$f_\mu(x,t) := \frac{d}{d\xi}f(x+\xi\mu,t)\Big|_{\xi=0}\quad\text{and}\quad f_{\dot\mu}(x,t) := \frac{d}{d\xi}f(x,t+\xi\mu)\Big|_{\xi=0},$$

and a lemma from [25, Appendix D]:

Lemma. *Let $q(x,t) := \frac{G}{F}$ and $u(x,t) := \log F$; then*

(1) $$\frac{F'}{F} = u_x,\quad \frac{G'}{F} = q_x + qu_x,$$

(2) $$\frac{F''}{F} = u_{xx} + u_x^2,\quad \frac{G''}{F} = q_{xx} + 2q_xu_x + q(u_{xx}+u_x^2),$$

(3) $$\frac{\dot G'}{F} = q_{xt} + q_tu_x + qu_{xt} + q_xu_t + qu_xu_t,$$

(4) $$\frac{\dot F'}{F} = u_{xt} + u_xu_t,$$

where $' := \frac{\partial}{\partial x}$ *and* $\cdot := \frac{\partial}{\partial t}$.

Using these formulas, we obtain the following

Theorem 3.2. *Functions u and $q^\alpha(\alpha\in\mathring{\Delta})$ satisfy the following system of partial differential equations (in these equations $\alpha,\beta\in\mathring{\Delta}$):*

$$\mathrm{I}_{\alpha,\beta}(\alpha+\beta\in\mathring{\Delta}):\quad q^{\alpha+\beta}_{\alpha-\beta} = 3\varepsilon(\alpha,\beta)q^\alpha q^\beta + \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha-\gamma\in\mathring{\Delta}\\ \beta+\gamma\in\mathring{\Delta}}}\varepsilon(\alpha-\gamma,\beta+\gamma)q^{\alpha-\gamma}q^{\beta+\gamma},$$

$$\mathrm{II}_{2;\alpha,0}:\quad 2q^\alpha_{\dot\alpha} - [q^\alpha_{\alpha\alpha} + 2q^\alpha u_{\alpha\alpha}] + \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha+\gamma\in\mathring{\Delta}}}\varepsilon(\alpha,\gamma)q^{\alpha+\gamma}_\alpha q^{-\gamma} = 0,$$

$$(\mathrm{III}+\mathrm{IV})_{1,1;0,0}:\quad 2u_{\alpha\beta} + \sum_{\gamma\in\mathring{\Delta}}(\alpha|\gamma)(\beta|\gamma)q^\gamma q^{-\gamma} = 0,$$

$$(\mathrm{III}+\mathrm{IV})_{2,2;0,0}:\quad 4u_{\dot\alpha\dot\beta} - \sum_{\gamma\in\mathring{\Delta}}[(\alpha|\gamma)q^\gamma_{\dot\beta}q^{-\gamma} + (\beta|\gamma)q^\gamma_{\dot\alpha}q^{-\gamma}] + \sum_{\gamma\in\mathring{\Delta}}(\alpha|\gamma)(\beta|\gamma)[q^\gamma_{\gamma\gamma}q^{-\gamma} + q^\gamma_{\dot\gamma}q^{-\gamma} - q^\gamma_\gamma q^{-\gamma}_\gamma + q^\gamma q^{-\gamma}u_{\gamma\gamma}] = 0,$$

$$\mathrm{IV}_{2,1;0,0}:\quad 3u_{\dot\alpha\beta} - \sum_{\gamma\in\mathring{\Delta}}(\alpha|\gamma)q^\gamma_\beta q^{-\gamma} + 2\sum_{\gamma\in\mathring{\Delta}}(\alpha|\gamma)(\beta|\gamma)q^\gamma_\gamma q^{-\gamma} = 0.\quad\square$$

Plugging $u_{\alpha\alpha}$ given by $(\mathrm{III}+\mathrm{IV})_{1,1;0,0}$ for $\alpha=\beta$ (in the form given by $\mathrm{I}_{\alpha,-\alpha}$) into $\mathrm{II}_{2;\alpha,0}$ we deduce

Theorem 3.3. *Functions $q^\alpha(\alpha\in\mathring{\Delta})$ satisfy the following system of partial differential equations ($\mathrm{I}_{\alpha,\beta}$ with $\alpha+\beta\in\mathring{\Delta}$ and $(\mathrm{I}+\mathrm{II})_{2;\alpha,0}$):*

$$q^{\alpha+\beta}_{\alpha-\beta} = 3\varepsilon(\alpha,\beta)q^\alpha q^\beta + \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha-\gamma,\beta+\gamma\in\mathring{\Delta}}}\varepsilon(\alpha-\gamma,\beta+\gamma)q^{\alpha-\gamma}q^{\beta+\gamma},$$

$$2q^\alpha_{\dot\alpha} - q^\alpha_{\alpha\alpha} + 8(q^\alpha)^2q^{-\alpha} + \sum_{\substack{\gamma\in\mathring{\Delta}\\ \alpha+\gamma\in\mathring{\Delta}}}q^{-\gamma}(2q^\alpha q^\gamma + \varepsilon(\alpha,\gamma)q^{\alpha+\gamma}_\alpha) = 0.\quad\square$$

Example 3.1. $\mathfrak{g} = A_1^{(1)}$. Then $\mathring{Q} = \mathbb{Z}\alpha_1$, and $\varepsilon(m\alpha_1,n\alpha_1) = (-1)^{mn}$. We take $u_1 = \alpha_1$, $u^1 = \frac{1}{2}\alpha_1$, and put $\tau_n(x) = \tau_{n\alpha_1}(x)$, $(x) = (x_1,x_2,\dots)$. We have

$$P_n^{\pm\alpha_1}(x) = p_n(\pm 2x),\qquad Q_n^{\pm\alpha_1}(x) = p_n(\pm x),$$

where $p_n(x)$ are given by (0.6). Then the hierarchy (3.4) looks as follows $(m, n \in \mathbb{Z})$:

(3.11)

$$\left(2\sum_{k\in\mathbb{N}} ky_kD_k + (m-n)^2\right) e^{\sum_{k\in\mathbb{N}} y_kD_k}\tau_n\cdot\tau_m$$
$$+(-1)^{m-n}\sum_{k\in\mathbb{Z}_+} p_k(2y)p_{k-2(m-n+1)}(-2\tilde{D})e^{\sum_{k\in\mathbb{N}} y_kD_k}\tau_{n-1}\cdot\tau_{m+1}$$
$$+(-1)^{m-n}\sum_{k\in\mathbb{Z}_+} p_k(-2y)p_{k+2(m-n-1)}(2\tilde{D})e^{\sum_{k\in\mathbb{N}} y_kD_k}\tau_{n+1}\cdot\tau_{m-1} = 0.$$

The simplest equations of this hierarchy are as follows. The constant term:

$$(\mathrm{I})_{n,m}:\quad (n-m)^2\tau_n\cdot\tau_m + (-1)^{m-n}p_{2(n-m-1)}(-2\tilde{D})\tau_{n-1}\cdot\tau_{m+1}$$
$$+(-1)^{m-n}p_{2(m-n-1)}(2\tilde{D})\tau_{n+1}\cdot\tau_{m-1} = 0.$$

The coefficient of y_k:

$$(\mathrm{II})_{k;n,m}:\quad (2k+(n-m)^2)D_k\tau_n\cdot\tau_m$$
$$+(-1)^{m-n}\{2p_{2(n-m)+k-2}(-2\tilde{D}) + p_{2(m-n)-2}(-2\tilde{D})D_k\}\tau_{n-1}\cdot\tau_{m+1}$$
$$+(-1)^{m-n}\{-2p_{2(m-n)+k-2}(2\tilde{D}) + p_{2(m-n)-2}(2\tilde{D})D_k\}\tau_{n+1}\cdot\tau_{m-1} = 0.$$

The coefficient of y_k^2:

$$(\mathrm{III})_{k;n,m}:\quad (2k+\tfrac{1}{2}(n-m)^2)D_k^2\tau_n\cdot\tau_m$$
$$+(-1)^{m-n}\{2p_{2(n-m)+k-2}(-2\tilde{D})D_k + 2p_{2(n-m)+2k-2}(-2\tilde{D})$$
$$+\tfrac{1}{2}p_{2(n-m)-2}(-2\tilde{D})D_k^2\}\tau_{n-1}\cdot\tau_{m+1}$$
$$+(-1)^{m-n}\{-2p_{2(m-n)+k-2}(2\tilde{D})D_k + 2p_{2(m-n)+2k-2}(2\tilde{D})$$
$$+\tfrac{1}{2}p_{2(m-n)-2}(2\tilde{D})D_k^2\}\tau_{n+1}\cdot\tau_{m-1} = 0.$$

The coefficient of y_jy_k, $j\neq k$:

$$(\mathrm{IV})_{j,k;n,m}:\quad (2(j+k)+(n-m)^2)D_jD_k\tau_n\cdot\tau_m$$
$$+(-1)^{m-n}\{4p_{2(n-m)+j+k-2}(-2\tilde{D}) + 2p_{2(n-m)+j-2}(-2\tilde{D})D_k$$
$$+2p_{2(n-m)+k-2}(-2\tilde{D})D_j + p_{2(n-m)-2}(-2\tilde{D})D_jD_k\}\tau_{n-1}\cdot\tau_{m+1}$$
$$+(-1)^{m-n}\{4p_{2(m-n)+j+k-2}(2\tilde{D}) - 2p_{2(m-n)+j-2}(2\tilde{D})D_k$$
$$-2p_{2(m-n)+k-2}(2\tilde{D})D_j$$
$$+p_{2(m-n)-2}(2\tilde{D})D_jD_k\}\tau_{n+1}\cdot\tau_{m-1} = 0.$$

Note that (I) and (II) give trivial equations for $k=1$ and $m=n$ or $n+1$. Taking $k=1$, $j=2$ and $m=n$ or $n+1$ in (III) and (IV), we get the following bilinear differential equations:

$$(\mathrm{III})_{1;n,n}:\quad D_1^2\tau_n\cdot\tau_n + 2\tau_{n-1}\tau_{n+1} = 0,$$
$$(\mathrm{III})_{1;n,n+1}:\quad (D_1^2+D_2)\tau_n\cdot\tau_{n+1} = 0,$$
$$(\mathrm{IV})_{2,1;n,n}:\quad D_1D_2\tau_n\cdot\tau_n - 2D_1\tau_{n-1}\cdot\tau_{n+1} = 0.$$

Note that in some cases, the constant term $(\mathrm{I})_{n,m}$ gives a nontrivial equation; for example, by putting $m=n+2$ in (I), one has

$$(\mathrm{I})_{n,n+2}:\quad (2D_1^2+D_2)\tau_{n+1}\cdot\tau_{n+1} + 4\tau_n\cdot\tau_{n+2} = 0,$$

which is just the same as $(\mathrm{III})_{1;n,n}$ since $D_2\tau_{n+1}\cdot\tau_{n+1}=0$ trivially.

Now we fix an integer n and, following **[25]**, put

$$q(x,t) = \frac{\tau_{n+1}}{\tau_n}(x,t,c_3,c_4,\dots),$$
$$q^*(x,t) = \frac{\tau_{n-1}}{\tau_n}(x,t,c_3,c_4,\dots),$$
$$u(x,t) = \log\tau_n(x,t,c_3,c_4,\dots).$$

Then just by the same calculation as in **[25]**, the above bilinear differential equations take the following form:

$$(\mathrm{III})'_{1;n,n}:\quad u_{xx} = -qq^*,$$
$$(\mathrm{III})'_{1;n,n-1}:\quad -q_t + q_{xx} + 2qu_{xx} = 0,$$
$$(\mathrm{III})'_{1;n-1,n}:\quad q_t^* + q_{xx}^* + 2q^*u_{xx} = 0,$$
$$(\mathrm{IV})'_{2,1;n,n}:\quad u_{xt} - qq_x^* + q_xq^* = 0.$$

From the first three equations, one obtains

$$-q_t = -q_{xx} + 2q^2q^*, \tag{3.12a}$$
$$q_t^* = -q_{xx}^* + 2qq^{*2}. \tag{3.12b}$$

It is easy to see that the last equation is compatible with (3.12a), (3.12b), and $(\mathrm{III})'_{1;n,n}$.

Imposing an additional constraint

$$q^*(x,it) = \overline{q(x,it)}\quad (\text{resp. } = \overline{-q(x,it)}),$$

and letting $g(x,t) = q(x,it)$, we see that both equations (3.12a) and (3.12b) turn into the classical nonlinear Schrödinger equation:

$$ig_t = -g_{xx} + 2\kappa|g|^2g, \tag{3.13}$$

where $\kappa = 1$ (resp. $= -1$). For this reason, (3.11) is called the NLS hierarchy.

We note here that $(\mathrm{III})_{1;n,n}$ gives also the Hirota bilinear differential equations of the 1-dimensional Toda lattice; actually by putting

$$u_n(x) = \log\frac{\tau_{n+1}}{\tau_n}(x,c_2,c_3,\dots),$$

one obtains (cf. **[30]**):

$$(u_n)_{xx} = e^{u_n-u_{n-1}} - e^{u_{n+1}-u_n}. \tag{3.14}$$

We shall explain now how to construct soliton type solutions of the NLS hierarchy. First, one checks that if $\tau(x)$ is a solution, then $(1+aX(\pm\alpha,z))\tau(x)$

is one as well ($a \in \mathbb{C}, z \in \mathbb{C}^\times$). The proof is the same as, for example, in [**16**, p. 78]. Thus the functions

$$\tau^{\pm}(x) := (1 + a_N X(\pm\alpha, z_N)) \cdots (1 + a_1 X(\pm\alpha, z_1)) \cdot 1 \otimes 1,$$

$$\tau^{+-}(x) := (1 + b_N X(-\alpha, w_N))(1 + a_N X(\alpha, z_N)) \cdots (1 + b_1 X(-\alpha, w_1))(1 + a_1 X(\alpha, z_1)) \cdot 1 \otimes 1,$$

are solutions of the NLS hierarchy, where $a_i, b_i \in \mathbb{C}$, $z_i, w_i \in \mathbb{C}^\times$ are complex parameters such that $|z_1| < |w_1| < |z_2| < |w_2| < \cdots$. The function τ^+ (resp. τ^- or τ^{+-}) is called the N soliton (resp. N antisoliton, or N soliton-antisoliton) solution. Here are explicit expressions for these solutions (parameters z_i and w_i can be arbitrary such that $z_i \neq w_j$, using analytic continuation):

$$\tau^{\pm} = \sum_{n \in \mathbb{Z}} e^{n\alpha} \otimes \tau_n^{\pm}(x),$$

etc., where

$$\tau_{\pm n}^{\pm} = \begin{cases} \sum_{1 \le i_1 < \cdots < i_n \le N} f^{\pm}_{i_1, \ldots, i_n} & \text{if } 0 < n \le N, \quad = 1 \quad \text{if } n = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_n^{+-} = \begin{cases} \sum_s \sum_{\substack{1 \le i_1 < \cdots < i_{s+n} \le N \\ 1 \le j_1 < \cdots < j_s \le N}} f^{+-}_{i_1, \ldots, i_{s+n}; j_1, \ldots, j_s} & \text{if } |n| \le N, \\ 0 & \text{otherwise}, \end{cases}$$

and the functions $f^{\pm}$, f^{+-} are defined as follows:

$$f_i^+ = a_i z_i \exp\left(\sum_{k \ge 1} x_k z_i^k\right), \qquad f_i^- = b_i w_i \exp\left(-\sum_{k \ge 1} x_k w_i^k\right),$$

$$f^+_{i_1, \ldots, i_n} = \begin{cases} \left(\prod_{1 \le p < q \le n} (z_{i_p} - z_{i_q})^2\right) f^+_{i_1} \cdots f^+_{i_n} & \text{if } n > 0, \\ 1 & \text{if } n = 0; \end{cases}$$

$$f^-_{j_1, \ldots, j_n} = \begin{cases} \prod_{1 \le p < q \le n} (w_{j_p} - w_{j_q})^2 f^-_{j_1} \cdots f^-_{j_n} & \text{if } n > 0, \\ 1 & \text{if } n = 0; \end{cases}$$

$$f^{+-}_{i_1, \ldots, i_r; j_1, \ldots, j_s} = \left(\prod_{q=1}^{r} \prod_{q=1}^{s} (z_{i_p} - w_{j_q})^{-2}\right) f^+_{i_1, \ldots, i_r} f^-_{j_1, \ldots, j_s}.$$

In particular, putting

$$\hat{f}_j(x,t) = a_j \exp(z_j x + i z_j^2 t), \hat{f}_{j_1, \ldots, j_n}(x,t) = \left(\prod_{1 \le p < q \le n} (z_{j_p} - z_{j_q})^2\right) \hat{f}_{j_1} \cdots \hat{f}_{j_n},$$

and

$$\hat{f}_{i_1, \ldots, i_r; j_1, \ldots, j_s}(x,t) = \left(\prod_{p=1}^{r} \prod_{q=1}^{s} (z_{i_p} + \bar{z}_{j_q})^{-2}\right) \hat{f}_{i_1, \ldots, i_r} \overline{\hat{f}}_{j_1, \ldots, j_s},$$

we obtain the following solutions

$$g_{\pm}^{[N]}(x,t) = \frac{\sum_s \sum_{\substack{1 \le i_1 < \cdots < i_{s+1} \le N \\ 1 \le j_1 < \cdots < j_s \le N}} (\pm 1)^s \hat{f}_{i_1, \ldots, i_{s+1}; j_1, \ldots, j_s}}{\sum_s \sum_{\substack{1 \le i_1 < \cdots < i_s \le N \\ 1 \le j_1 < \cdots < j_s \le N}} (\pm 1)^s \hat{f}_{i_1, \ldots, i_s, j_1, \ldots, j_s}} \tag{3.15}$$

of the NLS equation (3.13) with the coupling constant $\kappa = \pm 1$.

EXAMPLE 3.2. $A_l^{(1)}$ ($l \ge 1$). Choose a basis and its dual basis of $\mathring{\mathfrak{h}}$ as follows:

$$u_i = \alpha_i^\vee \quad \text{and} \quad u^i = \overline{\Lambda}_i \qquad (1 \le i \le l);$$

then, as the coefficients of the (principal) degree 2 terms in $y_k^{(j)}$, we get the following bilinear differential equations:

(3.16) The coefficient of $y_2^{(i)}$: $\displaystyle\sum_{\substack{\gamma \in \mathring{\Delta}_+ \\ \operatorname{supp} \gamma \ni i}} \tau_{\alpha+\gamma} \cdot \tau_{\alpha-\gamma} = 0,$

The coefficient of $y_1^{(i)2}$: $D_1^{(i)2} \tau_\alpha \cdot \tau_\alpha = 0,$

The coefficient of $y_1^{(i)} y_1^{(j)}, i \neq j$:

$$D_1^{(i)} D_1^{(j)} \tau_\alpha \cdot \tau_\alpha + 2 \sum_{\substack{\gamma \in \mathring{\Delta}_+ \\ \operatorname{supp} \gamma \ni i,j}} \tau_{\alpha+\gamma} \cdot \tau_{\alpha-\gamma} = 0.$$

We consider the special case $\alpha = n\mu$ and $(i,j) = (1,l)$, where $\mu = \alpha_1 + \cdots + \alpha_l$ is the highest root in $\mathring{\Delta}$. Then (3.16) turns into

$$D_1^{(1)} D_1^{(l)} \tau_{n\mu} \cdot \tau_{n\mu} + 2\tau_{(n+1)\mu} \tau_{(n-1)\mu} = 0. \tag{3.17}$$

Put $\tau_n(x,y) = \tau_{n\mu}$, $x_1^{(1)} = x$, $x_1^{(l)} = y$, and other $x_k^{(j)}$ to be constant; then the τ_n satisfy the equation

$$D_x D_y \tau_n \cdot \tau_n + 2\tau_{n+1} \cdot \tau_{n-1} = 0, \tag{3.18}$$

which is known to be the Hirota equation of the 2-dimensional Toda lattice; actually by putting [**31**]

$$u_n(x,y) = \log \frac{\tau_{n+1}(x,y)}{\tau_n(x,y)},$$

one sees easily that (3.18) is equivalent to the classical 2-dimensional Toda lattice equation:

$$(u_n)_{xy} = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}.$$

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