CONSTRUCTING GROUPS ASSOCIATED TO INFINITE-DIMENSIONAL LIE ALGEBRAS

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In these notes a representation theoretical approach to the construction of groups associated to (possibly infinite-dimensional) "integrable" Lie algebras is discussed. In the first part a general framework is outlined; here most of the discussion consists of definitions, examples and open problems. Deep results are available only in the case of groups associated to Kac-Moody algebras, which are discussed in the second part; it is based on joint work with Dale Peterson [18], [19], [20], [21], [22], [26]. Extension of these results to other classes of groups, like the group of biregular automorphisms of an affine space, would provide a solution to some very difficult open problems of algebraic geometry.

Throughout the paper the base field is the field $\mathbb{C}$ of complex numbers.

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CHAPTER 1. Integrable Lie Algebras and Associated Groups

§1.1 Let V be a (possibly infinite-dimensional) vector space (over $\mathbb{C}$), and let A be an endomorphism of V. A is called locally finite if every $v \in V$ lies in a finite-dimensional A-invariant subspace of V (or, equivalently, $\{A^n(v) \mid n = 0, 1, \ldots\}$ are linearly dependent for every $v \in V$). A is called locally nilpotent if for every $v \in V$ there exists $n > 0$ such that $A^n(v) = 0$. A is called semisimple if V admits a basis of eigenvectors for A. Obviously, locally nilpotent and semisimple elements are locally finite.

A locally finite endomorphism A always admits a Jordan decomposition, i.e. A can be represented in the form $A = A_s + A_n$, where A_s is semisimple and A_n is locally nilpotent and $A_sA_n = A_nA_s$; such a decomposition is unique. This follows from the usual Jordan decomposition in the finite-dimensional case.

If A is a locally finite endomorphism of V, we can form the corresponding 1-parameter group of automorphisms of V:

$$\exp tA = \sum_{n\geqslant 0} \frac{t^n}{n!} A^n, \qquad t \in \mathbb{C}.$$

Let A be a semisimple endomorphism of V so that $A(e_i) = \lambda_i e_i$ for some basis $\{e_i\}$ of V. An endomorphism A' of V, such that $A'(e_i) = \lambda'_i e_i$ for all i, is called a replica of A if a relation $\sum_i n_i\lambda_i = 0$, where $n_i \in \mathbb{Z}$ and all but a finite number of them are 0, implies that $\sum_i n_i\lambda'_i = 0$.

Let A be an arbitrary locally finite endomorphism of V and $A = A_s + A_n$ its Jordan decomposition. All replicas of A_s and the endomorphism A_n are called replicas of A. The linear span of all replicas of A is called the algebraic hull of A.

Lemma. Let A be a locally finite endomorphism of V and let A' be a replica of A. Let $U_1 \subset U_2$ be two subspaces of V such that $A(U_1) \subset U_2$. Then $A'(U_1) \subset U_2$.

Indeed, let $v \in U_1$ and let U' be a finite-dimensional

A-invariant subspace containing v. Put $U_i' = U_i \cap U'$ $(i = 1, 2)$. Then $A(U_1') \subset U_2'$. By [29, p. 6-04], A' restricted to U' is a polynomial with zero constant term in A restricted to U'. Hence $A'(v) \in U_2' \subset U_2$.

<u>**Examples.**</u>

(a) Let R be a commutative associative algebra (over $\mathbb{C}$) with no zero divisors. Let $(r_{ij})_{i,j=1}^n$ be a matrix over R and let $\det(\lambda\delta_{ij} - r_{ij}) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$, $a_i \in R$, be its characteristic polynomial. The matrix (r_{ij}) acts on the free R-module R^n of n-columns over R by left multiplication; regarding R^n as an infinite-dimensional vector space V over $\mathbb{C}$, we get an endomorphism of V, which we denote by r. Then r is locally finite if and only if $a_i \in \mathbb{C}$, $i = 1, \ldots, n$, and is locally nilpotent if and only if $a_i = 0$, $i = 1, \ldots, n$. Indeed, if all the $a_i \in \mathbb{C}$, then, for $v \in V$, we have $r^n(v) \in \sum_{j=0}^{n-1} \mathbb{C}r^j(v)$. Conversely, if r is locally finite, it has at most n eigenvalues $\lambda_1, \ldots, \lambda_s$ (which are the complex roots of $\det(\delta_{ij}\lambda - r_{ij})$), and for any r-invariant subspace U of V, all the eigenvalues of r on V/U are from the set $\{\lambda_1, \ldots, \lambda_s\}$. Taking $U = I^n \subset R^n = V$, where I is a maximal ideal of R (we may assume R to be finitely generated) we deduce that all the $a_i \in \mathbb{C}$.

(b) Let D be a derivation of an algebra R generated by some elements $a_1, a_2, \ldots$. Then D is locally finite (resp. locally nilpotent) if and only if $\dim \sum_i \mathbb{C}D^i(a_j) < \infty$ (resp. $D^{n_j}(a_j) = 0$ for some n_j) for all j. This follows from the Leibnitz rule.

Furthermore, if D is a locally finite derivation of R, then all replicas of D are derivations of R as well. Indeed, let $R = \bigoplus_{\lambda\in\mathbb{C}} R_\lambda$ be the generalized eigenspace decomposition with respect to D. Then $R_\lambda R_\mu \subset R_{\lambda+\mu}$, which implies that D_s, and hence D_n, is a derivation of R. It is also clear that all replicas of D_s are derivations of R as well.

(c) Let $A = \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i}$ be a linear differential operator with polynomial coefficients acting on the vector space $V = \mathbb{C}[x_1, \ldots, x_n]$. In the following two cases, A is evidently locally finite: $\deg P_i \leqslant 1$ for all i (affine differential operator); $\frac{\partial P_i}{\partial x_j} = 0$ for $j \geqslant i$ (triangular differential operator). This follows from (b).

§1.2 Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra (over $\mathbb{C}$) and let V be a $\mathfrak{g}$-module with action π. Let $\mathfrak{r}$ denote the center of $\mathfrak{g}$.

An element $x \in \mathfrak{g}$ is called π-locally finite if $\pi(x)$ is a locally finite endomorphism of the vector space V.

We denote by $F_{\mathfrak{g}}$ the set of all ad-locally finite elements of $\mathfrak{g}$, and by $\mathfrak{g}_{fin}$ the subalgebra of $\mathfrak{g}$ generated by $F_{\mathfrak{g}}$. The Lie algebra $\mathfrak{g}$ is called integrable if $\mathfrak{g} = \mathfrak{g}_{fin}$. Denote by $F_{\mathfrak{g},\pi}$ the set of π-locally finite elements of $F_{\mathfrak{g}}$.

Lemma.

(a) The subalgebra of $\mathfrak{g}$ generated by $F_{\mathfrak{g},\pi}$ is the linear span (over $\mathbb{C}$) of $F_{\mathfrak{g},\pi}$. In particular, $\mathfrak{g}_{fin}$ is spanned by $F_{\mathfrak{g}}$.

(b) Let $\dim \mathfrak{g} < \infty$ and let $\mathfrak{g}$ be generated by $F_{\mathfrak{g},\pi}$. Then V is a locally finite $\mathfrak{g}$-module (i.e. any $v \in V$ is contained in a finite-dimensional $\mathfrak{g}$-submodule).

Proof. Let $\mathfrak{g}_\pi$ denote the $\mathbb{C}$-span of $F_{\mathfrak{g},\pi}$. Let $x \in F_{\mathfrak{g},\pi}$. Using that

(1) $$\pi((\exp \operatorname{ad} x)y) = (\exp \pi(x))\pi(y)(\exp -\pi(x))$$

we deduce that $\mathfrak{g}_\pi$ is invariant with respect to $\exp t(\operatorname{ad} x)$, $t \in \mathbb{C}$. Since

$$\lim_{t\to 0} ((\exp t\,\mathrm{ad}x)(y) - y)/t = [x,y],$$

it follows that $[x,\mathfrak{g}_\pi] \subset \mathfrak{g}_\pi$, proving (a). (b) follows from (a) by the PBW theorem.

<u>Conjecture</u>. Let $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be an integrable $\mathbb{Z}$-graded Lie algebra (we assume that $\dim \mathfrak{g}_j < \infty$), which has no nontrivial graded ideals. Then $\mathfrak{g}$ is isomorphic either to a "centreless" Kac-Moody algebra $\mathfrak{g}'(A)/\mathfrak{c}$ (see §2.2 for the definition) or to a Lie algebra of the Cartan series S_n or H_n (see e.g. [12] for their definition).

<u>Problem</u>. Characterize the <u>"general" integrable Lie algebra</u> $\mathfrak{gl}(V)_{fin}$.

The $\mathfrak{g}$-module (V,π) is called <u>integrable</u> if $F_{\mathfrak{g},\pi} = F_\mathfrak{g}$. Of course the $\mathfrak{g}$-module $(\mathfrak{g},\mathrm{ad})$ is integrable. In general it is difficult to check that a module is integrable since there is not much information about the set $F_\mathfrak{g}$. The matter would simplify considerably if one can prove the following conjecture (which is a strengthening of Lemma 1.2(a)):

<u>Conjecture</u>. If $F_{\mathfrak{g},\pi}$ generates the Lie algebra $\mathfrak{g}_{fin}$, then V is an integrable $\mathfrak{g}$-module.

<u>Problem</u>. Does any Lie algebra admit a faithful integrable module?

Put $V_{fin} = \{v \in V \mid$ for every $x \in F_\mathfrak{g}$ there exists a finite-dimensional $\pi(x)$-invariant subspace of V containing $v\}$. It follows from the Leibnitz rule that V_{fin} is a $\mathfrak{g}$-submodule, which is obviously integrable. The functors $\mathfrak{g} \longmapsto \mathfrak{g}_{fin}$ and $V \longmapsto V_{fin}$ have many nice properties.

§1.3 <u>Examples</u>.

(a) Let R be a complex commutative associative algebra with unity and let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. Then the Lie algebra $\mathfrak{g}_R := R \otimes_\mathbb{C} \mathfrak{g}$ is an integrable Lie algebra (over $\mathbb{C}$). To show this, take a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_\alpha \mathbb{C}e_\alpha$. Then all elements of the form $r \otimes e_\alpha$ are ad-locally finite (even nilpotent), and they generate the Lie algebra $\mathfrak{g}_R$.

Experience shows that the universal central extension $\tilde{\mathfrak{g}}_R \xrightarrow{d\tau} \mathfrak{g}_R$ of $\mathfrak{g}_R$ has a much more interesting representation theory than $\mathfrak{g}_R$ itself. This central extension is constructed as follows [23]:

$$0 \longrightarrow \Omega^1_R/dR \longrightarrow \tilde{\mathfrak{g}}_R: = \mathfrak{g}_R \oplus (\Omega^1_R/dR) \xrightarrow{d\tau} \mathfrak{g}_R \longrightarrow 0,$$

where Ω^1_R is the space of all formal differentials (i.e. expressions of the form fdg, where $f,g \in R$, with relation $d(fg) = fdg + gdf$), and the bracket on $\tilde{\mathfrak{g}}_R$ is defined by

$$[r_1 \otimes g_1, r_2 \otimes g_2] = r_1 r_2 \otimes [g_1, g_2] + (g_1 | g_2) r_2 dr_1 \pmod{dR}$$

where $(\cdot | \cdot)$ is the Killing form on $\mathfrak{g}$. Of course, $\tilde{\mathfrak{g}}_R$ is an integrable Lie algebra as well.

If V is a finite-dimensional $\mathfrak{g}$-module, then one easily shows that $V_R: = R \otimes_{\mathbb{C}} V$ is an integrable $\mathfrak{g}_R$-module. The following special case of the problem stated above is open, however: For which R there exists a faithful integrable $\tilde{\mathfrak{g}}_R$-module?

<u>Remark</u>. Put $\tilde{\mathfrak{g}}_R = (R \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \Omega^1_R \oplus \mathrm{Der}\, R$ and define a bracket by: $[r_1 \otimes g_1, r_2 \otimes g_2] = r_1 r_2 \otimes [g_1, g_2] + (g_1 g_2) r_2 dr_1$; $[D, r \otimes g] = D(r) \otimes g$ for $D \in \mathrm{Der}\, R$; $[\tilde{\mathfrak{g}}_R, \Omega^1_R] = 0$. Then $\tilde{\mathfrak{g}}_R$ is a Lie algebra mod $dR \subset \Omega^1_R$. Define on $\tilde{\mathfrak{g}}_R$ an R-valued symmetric bilinear form $(\cdot | \cdot)_R$ by: $(r_1 \otimes g_1 | r_2 \otimes g_2) = r_1 r_2 (g_1 | g_2)$; $(fdg | D) = fD(g)$ for $D \in \mathrm{Der}\, R$; $(\Omega^1_R + \mathrm{Der}\, R \mid R \otimes_{\mathbb{C}} \mathfrak{g}) = 0$; $(\Omega^1_R | \Omega^1_R) = 0$; $(\mathrm{Der}\, R | \mathrm{Der}\, R) = 0$. It is non-degenerate, and invariant, i.e. $([a,b] | c)_R = (a | [b,c])_R$. Let F be a linear function on R; put $(\mathrm{Der}\, R)_F = \{D \in \mathrm{Der}\, R \mid F(D(\varphi)) = 0 \text{ for all } \varphi \in R\}$. Then $\tilde{\mathfrak{g}}_{R,F}: = (R \otimes_{\mathbb{C}} \mathfrak{g}) \oplus (\Omega^1_R/dR) \oplus (\mathrm{Der}\, R)_F$ is a Lie algebra, and $(a | b): = F((a | b)_R)$ is an invariant bilinear form on it. For example, let M be a compact manifold with a volume form Ω, let R be the algebra of complex valued C^∞-functions on R and let $F(\varphi) = \int_M \varphi \Omega$. Then DerR is the

Lie algebra of vector fields on M, $(\mathrm{Der}\ R)_F$ is the subalgebra of vector fields with zero divergence, and the bilinear form $(\cdot|\cdot)$ on $\tilde{\mathfrak{g}}_{R,F}$ is non-degenerate.

(b) Let W_n be the Lie algebra of all linear differential operators with polynomial coefficients in n indeterminates $x_1, \ldots, x_n$. It carries a $\mathbb{Z}$-gradation $W_n = \bigoplus_{j \geq -1} (W_n)_j$, where $(W_n)_j = \{\sum P_i \frac{\partial}{\partial x_i} \in W_n \mid P_i$ are homogeneous of degree $j + 1\}$; then $W_n^0 = \bigoplus_{j \geq 0} (W_n)_j$ is a maximal subalgebra of W_n. Put $CS_n = \{D \in W_n \mid \mathrm{div}\ D \in \mathbb{C}\}$. This is a subalgebra of W_n (recall that $\mathrm{div} \sum P_i \frac{\partial}{\partial x_i} = \sum \frac{\partial P_i}{\partial x_i}$ and that $\mathrm{div}\ [D_1, D_2] = D_1 \cdot \mathrm{div}\ D_2 - D_2 \cdot \mathrm{div}\ D_1$). Put $(CS_n)_j = CS_n \cap W_n$; we have a $\mathbb{Z}$-gradation $CS_n = \bigoplus_{j \geq -1} (CS_n)_j$, where $(CS_n)_{-1} = (W_n)_{-1} = \sum \mathbb{C} \frac{\partial}{\partial x_i}$, $(CS_n)_0 = (W_n)_0 \simeq \mathfrak{gl}_n(\mathbb{C})$. Furthermore, the $(CS_n)_0$-module $(CS_n)_j$ is irreducible if $j \neq 0$ with a highest weight vector $x_1^{j+1} \frac{\partial}{\partial x_n}$, if $n > 1$ (see e.g. [12]). Since the elements of $(CS_n)_0$ and the elements $x_1^j \frac{\partial}{\partial x_n}$ are locally finite if $n > 1$, we obtain that $CS_n \subset (W_n)_{fin}$ and that CS_n is an integrable Lie algebra.

It is clear that $(W_1)_{fin} = CS_1 = \mathbb{C} \frac{\partial}{\partial x} + \mathbb{C}x \frac{\partial}{\partial x}$. Let me show that $(W_n)_{fin} = CS_n$ for any n.

Denote by π the action of W_n on the vector space $\mathbb{C}[x_1, \ldots, x_n]$. Then $D \in W_n$ is ad-locally finite if and only if it is π-locally finite. Indeed, if D is ad-locally finite, then, applying the Leibnitz rule to $D^N(P \frac{\partial}{\partial x_1})$, we see that D is π-locally finite. Conversely, if D is π-locally finite, then $\exp tD$ is an automorphism of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ such that $(\exp tD)x_i = P_i(x_1, \ldots, x_n, t)$, where the degrees of the P_i in the x_j are bounded uniformly for all t. Since the change of indeterminates $\varphi_t: x_i \longmapsto P_i$ is invertible, denoting the inverse by $x_i \longmapsto \bar{P}_i$, we get $(\exp tD)\frac{\partial}{\partial x_i} = \sum_j \frac{\partial \bar{P}_j}{\partial x_i} \frac{\partial}{\partial x_j}$ and the degrees of the $\bar{P}_i$ in the indeterminates x_j are bounded uniformly for all t. It follows that $\sum_j \mathbb{C}(\mathrm{ad}D)^j \frac{\partial}{\partial x_i}$ is finite-dimensional. Hence, D is ad-locally finite. In other words, $\mathbb{C}[x_1, \ldots, x_n]$ is an integrable W_n- (and CS_n-) module.

Furthermore, if D is ad-locally finite, then it is π-locally finite, and we have the change of indeterminates φ_t. But its Jacobian $J(\varphi_t) := \det \left[\frac{\partial P_i}{\partial x_j}\right]$ is an invertible polynomial, hence $J(\varphi_t) \in \mathbb{C}^\times$. Therefore, $\mathrm{div}\, D \in \mathbb{C}$. Thus, $(W_n)_{fin} = CS_n$.

Problem. Is it true that any ad-semisimple element of CS_n is conjugate (by a change of indeterminates) to an element of the form $\sum_i \lambda_i x_i \frac{\partial}{\partial x_i}$, where $\lambda_i \in \mathbb{C}$?

This problem is equivalent to the well-known problem, whether a regular action of $\mathbb{C}^\times$ on $\mathbb{C}^n$ is biregularly equivalent to a linear action.

As we shall see in Chapter 2, the conjugacy problem is intimately related to the problem of existence of non-trivial closed orbits in the projectivized space. Unfortunately, there is no such orbits for the action of Aut $\mathbb{C}^n$ on $\mathbb{C}[x_1, \ldots, x_n]$.

Problem. Compute the closure of the orbit of x_1 in $\mathbb{C}[x_1, \ldots, x_n]$ under the action of Aut $\mathbb{C}^n$ (a set is closed if its intersection with any finite-dimensional subspace U is closed in U).

Finally, let φ: $x_i \longrightarrow P_i$ be a polynomial change of indeterminates with $J(\varphi) \in \mathbb{C}^\times$; we can assume that $P_i(0) = 0$. Then we have the induced (non-zero) homomorphism φ: $W_n \longrightarrow W_n$ which maps W_n^0 into itself. Conversely, any non-zero homomorphism $\varphi: W_n \longrightarrow W_n$ that maps W_n^0 into itself induces an isomorphism $\hat{\varphi}: \hat{W}_n \longrightarrow \hat{W}_n$ of the formal completion and hence is given by a formal change of indeterminates $x_i \longrightarrow P_i$ with $P_i(0) = 0$ [28]. Since $\hat{\varphi}\left(\frac{\partial}{\partial x_i}\right) \in W_n$, we obtain that the inverse change of indeterminates is polynomial. Since $\hat{\varphi}\left(x_i^k \frac{\partial}{\partial x_i}\right) \in W_n$, the P_i are polynomials and $J(\varphi) \in \mathbb{C}^\times$.

Thus, the Jacobian conjecture is equivalent to the question whether a non-zero homomorphism $W_n \longrightarrow W_n$ which maps W_n^0 into itself is an isomorphism (one can replace W_n by CS_n).

§1.4 Let V be a faithful integrable $\mathfrak{g}$-module, so that $\mathfrak{g} \subset \mathfrak{gl}(V)$. If all replicas of any element of $F_{\mathfrak{g}}$ lie in $\mathfrak{g}$, the linear Lie algebra $\mathfrak{g}$ is called <u>algebraic</u>. An integrable $\mathfrak{g}$-module (U,φ) over an algebraic Lie algebra $\mathfrak{g}$ is called <u>rational</u> if for any $x \in F_{\mathfrak{g}}$ one has $\varphi(\bar{x}) = \overline{\varphi(x)}$, where $\bar{x}$ denotes the algebraic hull of x, and $\varphi(x_s) = \varphi(x)_s$, where $x = x_s + x_n$ is the Jordan decomposition of $\pi(x)$. If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is not algebraic, we let $\bar{\mathfrak{g}}$ be the subalgebra of $\mathfrak{gl}(V)$ generated by algebraic hulls of all $x \in F_{\mathfrak{g}}$. Then $\bar{\mathfrak{g}} \subset \mathfrak{gl}(V)$ is an algebraic Lie algebra called the <u>algebraic hull</u> of $\mathfrak{g}$.

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be an algebraic Lie algebra. Then its adjoint representation is rational. Indeed, let $x \in F_{\mathfrak{g}}$ and let $\pi(x) = A_s + A_n$ be the Jordan decomposition. Let $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$ be the eigenspace decomposition for A_s; it is A_n-invariant. Since $\pi(\mathfrak{g}) \subset \operatorname{End} V = \prod_{\lambda,\mu \in \Lambda} \operatorname{Hom}(V_\lambda, V_\mu)$ and $x \in F_{\mathfrak{g}}$, we deduce that $\operatorname{ad} x = \operatorname{ad} A_s + \operatorname{ad} A_n$ is the Jordan decomposition of ad x, and that all the eigenvalues of ad x and ad A_s are $\lambda-\mu$, where $\lambda,\mu \in \Lambda$.

Note that the definition of an algebraic Lie algebra, the Jordan decomposition, etc., are independent of the choice of the rational $\mathfrak{g}$-module. Thus, if the center of $\mathfrak{g}$ is trivial, we can start with its adjoint representation and talk about the Jordan decomposition of $x \in F_{\mathfrak{g}}$.

It follows from Lemma 1.1, that if A is a locally finite endomorphism of V and A' is a replica of A, and if $U_1 \subset U_2$ are two subspaces of $\mathfrak{gl}(V)$, such that $(\text{ad } A)U_1 \subset U_2$, then $(\text{ad } A')U_1 \subset U_2$. As in [29, p. 6-06], one deduces the following easy facts:

(a) Every ideal of $\mathfrak{g}$ remains an ideal in $\bar{\mathfrak{g}}$.

(b) Center of $\mathfrak{g}$ lies in the center of $\bar{\mathfrak{g}}$.

(c) $[\mathfrak{g},\mathfrak{g}] = [\bar{\mathfrak{g}},\bar{\mathfrak{g}}]$, $\mathfrak{g}$ is an ideal in $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}}/\mathfrak{g}$ is abelian.

(d) If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $[\bar{\mathfrak{g}},\bar{\mathfrak{a}}] \subset \mathfrak{g}$.

Problem. Is it true that $[\mathfrak{g},\mathfrak{g}]$ is an algebraic Lie algebra? This is true if $\dim \mathfrak{g} < \infty$. The proof of this and other deeper facts of the theory of finite-dimensional algebraic groups uses the Noetherian property of finite-dimensional algebraic varieties (see e.g. [2]).

The Lie algebra $\mathfrak{g}_R$ acting on V_R (see Example 1.3(a)), where V is a faithful (finite-dimensional) $\mathfrak{g}$-module, is an algebraic Lie algebra, and all $\mathfrak{g}_R$-modules U_R, where U is a finite-dimensional $\mathfrak{g}$-module, are rational. To see this, consider $\mathfrak{g}_{\text{Fr } R}$, where Fr R is the field of fractions of R, and use the uniqueness of the Jordan decomposition.

The Lie algebra Der R of all derivatives of an algebra R is an algebraic linear Lie algebra. This follows from Example 1.1(b). In particular, W_n is an algebraic Lie algebra. Since $(W_n)_{\text{fin}} = CS_n$, it follows that CS_n is an algebraic Lie algebra as well.

§1.5 Let $\mathfrak{g}$ be an integrable Lie algebra. We associate to $\mathfrak{g}$ a group G as follows. Let G^* be a free group on the set $F_{\mathfrak{g}}$. Given an integrable $\mathfrak{g}$-module $(V,d\pi)$, we define a G^*-module $(V,\tilde{\pi})$ by

$$\tilde{\pi}(x) = \exp d\pi(x) := \sum_{n\geqslant 0} (d\pi(x))^n/n!, \qquad x \in F_{\mathfrak{g}}.$$

We put $G = G^*/\bigcap \text{Ker } \tilde{\pi}$, where the intersection is taken over all integrable $\mathfrak{g}$-modules $d\pi$. Thus, the G^*-module $(V,\tilde{\pi})$ is naturally a G-module (V,π), the integrable $\mathfrak{g}$-module $(V,d\pi)$ being its "differential". We call G the group associated to the Lie algebra $\mathfrak{g}$ and (V,π) the G-module associated to the integrable $\mathfrak{g}$-module.

Given an element $x \in F_{\mathfrak{g}}$, we denote its image in G under the canonical homomorphism $G^* \longrightarrow G$ by exp x. Thus, we have by definition:

$$\pi(\exp x) = \exp d\pi(x), \qquad x \in F_{\mathfrak{g}}$$

for an integrable $\mathfrak{g}$-module $(V,d\pi)$. Note also that $\{\exp tx \mid t \in \mathbb{C}\}$ is a 1-parameter subgroup of G.

Put $F_G = \{\exp x \mid x \in F_{\mathfrak{g}}\} \subset G$. A G-module (V,π) is called differentiable if all elements of F_G act locally finitely on V and exp tx restricted to any invariant finite-dimensional subspace is analytic in t ($x \in F_{\mathfrak{g}}$). This definition is justified by the following:

Conjecture. Let (V,π) be a differentiable G-module. Then there exists a unique action $d\pi$ of $\mathfrak{g}$ on V such that $\pi(\exp x) = \exp d\pi(x)$ for all $x \in F_{\mathfrak{g}}$. $(V,d\pi)$ is an integrable $\mathfrak{g}$-module.

Uniqueness follows from Lemma 1.2(a). To show the existence put

$$d\pi(x) := \frac{d}{dt}\, \pi\, (\exp tx)\Big|_{t=0} \text{ for } x \in F_{\mathfrak{g}}.$$

The difficulty is to show that $d\pi$ is linear. This granted, one would have by (1):

$$\pi(\exp tx)\ d\pi(y)\ \pi(\exp -tx) = d\pi(\exp(\operatorname{ad} tx)y), \text{ for } x \in F_{\mathfrak{g}},$$

and therefore,

$$(1 + td\pi(x) + o(t))\ d\pi(y)\ (1-td\pi(x) + o(t)) =$$

$$= d\pi(y) + td\pi([x,y]) + o(t),$$

which would yield $[d\pi(x),d\pi(y)] = d\pi[x,y])$.

Of course, the G-module (V,π) associated to an integrable g-module $(V,d\pi)$ is differentiable. Thus, we would have an invertible functor between the categories of integrable g-modules and differentiable G-modules.

A homomorphism $d\varphi$: $g_1 \longrightarrow g$ of integrable Lie algebras is called <u>integrable</u> if $d\varphi(F_{g_1}) \subset F_g$; then $d\varphi(g_1)$ is called an <u>integrable subalgebra</u> of g. Given an integrable homomorphism $d\varphi$ of Lie algebras, we have a canonically defined homomorphism of the associated groups φ: $G_1 \longrightarrow G$, so that $d(\pi|_{G_1}) = (d\pi)|_{g_1}$. The subgroup $\varphi(G_1)$ of G is called the subgroup corresponding to the integrable subalgebra $\varphi(g_1)$ of g. It is generated by the exp x with x $\in \varphi(g_1) \cap F_g$.

Of course, any isomorphism of integrable Lie algebras is integrable and lifts to an isomorphism of the associated groups.

§1.6 Let $\mathfrak{r}_0$ denote the intersection of kernels of all integrable g-modules; then $\mathfrak{r}_0 \subset \mathfrak{r}$. Replacing g by $g/\mathfrak{r}_0$ we can (and will) assume that $\mathfrak{r}_0 = 0$. Let C denote the center of G. Associated to the integrable g-module (g,ad), we have the <u>adjoint</u> G-module (g,Ad). We denote by Ad G the image of the action of G on g and call it the adjoint group associated to g. Then we have the following exact sequence

(2) $$1 \longrightarrow C \longrightarrow G \longrightarrow \text{Ad } G \longrightarrow 1.$$

This is because, given a faithful integrable g-module $(V,d\pi)$, we can compute the adjoint G-module, thanks to formula (1), by

$$d\pi((\text{Ad } g)x) = \pi(g)d\pi(x)\pi(g)^{-1}, \qquad g \in G.$$

Hence Ad g = 1 iff $\pi(g)$ commutes with $d\pi(g)$, iff $\pi(g)$ commutes with $d\pi(F_g)$, iff $\pi(g)$ commutes with $\pi(F_G)$, iff $\pi(g)$ commutes with $\pi(G)$. Choosing $(V,d\pi)$ such that (V,π) is a faithful G-module, we get that Ker Ad = C. Note that we have shown at the same time that if the g-module $(V,d\pi)$ is faithful, then Ker $\pi \subset C$.

Problem. It is true that for a faithful differentiable G-module (V,π) the corresponding (integrable) $\mathfrak{g}$-module $(V,d\pi)$ is also faithful?

§1.7 Let (V,π) be a $\mathfrak{g}$-module and let σ be an automorphism of the Lie algebra $\mathfrak{g}$. Then we have a new $\mathfrak{g}$-module (V,π_σ) defined by

$$\pi_\sigma(g)v = \pi(\sigma \cdot g)v \qquad \text{for} \qquad g \in \mathfrak{g},\ v \in V.$$

Define the <u>big adjoint group</u> $\widetilde{\mathrm{Ad}}\ G$ by: $\widetilde{\mathrm{Ad}}\ G = \{\sigma \in \mathrm{Aut}\ \mathfrak{g} \mid \pi_\sigma$ is isomorphic to π for any integrable $\mathfrak{g}$-module $\pi\}$.

We have an obvious inclusion $\mathrm{Ad}\ G \subset \widetilde{\mathrm{Ad}}\ G$. It is also clear that $\mathrm{Ad}\ G$ is a normal subgroup of $\widetilde{\mathrm{Ad}}\ G$. We define the group $K_1(\mathfrak{g})$ by the exact sequence

$$\text{(3)} \qquad 1 \longrightarrow \mathrm{Ad}\ G \longrightarrow \widetilde{\mathrm{Ad}}\ G \longrightarrow K_1(\mathfrak{g}) \longrightarrow 1.$$

We put $K_2(\mathfrak{g}) = C$ and define $K_0(\mathfrak{g})$ as the Grothendieck group of the category of all integrable $\mathfrak{g}$-modules. Note that $K_i(\mathfrak{g}_R)$ are closely related to the usual K-functors $K_i(R)$, $i = 1, 2$.

Problem. Compute the groups $K_i(\mathfrak{g}_R)$ and $K_i(\tilde{\mathfrak{g}}_R)$, $i = 0, 1, 2$.

Conjecture. $K_i(CS_n)$ for $i = 1, 2$ are trivial, i.e. the group associated to the Lie algebra CS_n is $\mathrm{Aut}\ \mathbb{C}^n$, the group of biregular automorphisms of $\mathbb{C}^n$.

This conjecture is closely related to the well-known question whether the group $\mathrm{Aut}\ \mathbb{C}^k$ is generated by affine and triangular automorphisms (cf. [30]).

Remarks.

(a) Note that, given an integrable $\mathfrak{g}$-module (V,π), and $\sigma \in \widetilde{\mathrm{Ad}}\ \mathfrak{g}$, we have

$$\pi(\sigma \cdot g) = A_\sigma \pi(g) A_{\sigma^{-1}} \text{ for some } A_\sigma \in \operatorname{Aut} V.$$

Denote by G_V the group generated by all such A_σ. Then we, clearly, have the following exact sequence:

$$1 \longrightarrow \operatorname{Aut}_{\mathfrak{g}} V \longrightarrow G_V \longrightarrow \widetilde{\operatorname{Ad}}\, G \longrightarrow 1.$$

Assuming that (V,π) is a <u>Schur</u> <u>module</u>, i.e. that $\operatorname{Aut}_{\mathfrak{g}} V = \mathbb{C}^\times$, we get a central extension of $\widetilde{\operatorname{Ad}}\, G$.

(b) Given an arbitrary Lie algebra $\mathfrak{g}$ and a category of $\mathfrak{g}$-modules, one can define the associated (adjoint) group as above.

§1.8 If $\mathfrak{g} \subset \mathfrak{gl}(U)$ is a linear algebraic integrable Lie algebra, one can make the definition of the associated group G more algebraic as follows. Let $F^n_{\mathfrak{g}}$ denote the set of all locally nilpotent elements and $F^s_{\mathfrak{g}}$ the set of all semisimple elements with integral eigenvalues; put $F^{alg}_{\mathfrak{g}} = F^n_{\mathfrak{g}} \cup F^s_{\mathfrak{g}}$. Let G^* be the free product of a collection of copies of the additive group $\mathbb{C}$ indexed by $F^n_{\mathfrak{g}}$ and a collection of copies of the multiplicative group $\mathbb{C}^\times$ indexed by $F^s_{\mathfrak{g}}$. Given a rational integrable $\mathfrak{g}$-module $(V,d\pi)$, we define a G^*-module $(V,\tilde{\pi})$ by $\tilde{\pi}(t) = \exp d\pi(tx)$ if $t \in \mathbb{C}_x$, $\tilde{\pi}(t)v_i = t^{k_i}v_i$ if $t \in \mathbb{C}^\times_y$, where v_i is an eigenvector of $d\pi(y)$ with eigenvalue k_i. Put $G = G^*/\cap \operatorname{Ker} \tilde{\pi}$, where intersection is taken over all rational integrable $\mathfrak{g}$-modules $d\pi$. This definition of G coincides with the one in §1.5.

Note that for every $x \in F^n_{\mathfrak{g}}$ (resp. $x \in F^s_{\mathfrak{g}}$) we have a homomorphism ψ_x: $\mathbb{C} \longrightarrow G$ (resp. $\mathbb{C}^\times \longrightarrow G$). Given an ordered finite set $\bar{x} = \{x_1, \ldots, x_n\}$ of $F^{alg}_{\mathfrak{g}}$, we have a map $\psi_{\bar{x}}$ from the product of several copies of $\mathbb{C}$ and $\mathbb{C}^\times$ into G defined by

$$\psi_{\bar{x}}(t_1, \ldots, t_n) = \psi_{x_1}(t_1) \ldots \psi_{x_n}(t_n),$$

where $t_i \in \mathbb{C}$ if $x_i \in F^n_{\mathfrak{g}}$ and $t_i \in \mathbb{C}^\times$ if $x_i \in F^s_{\mathfrak{g}}$. In the general case, given an ordered set $\bar{x} = \{x_1, \ldots, x_n\}$ of elements of $F_{\mathfrak{g}}$, one defines $\psi'_{\bar{x}} : \mathbb{C}^n \longrightarrow G$ by $\psi'_{\bar{x}}(t_1, \ldots, t_n) = (\exp t_1x_1) \ldots$

$(\exp t_n x_n)$.

Now we are in a position to discuss how one introduces various structures on G.

A function $f: G \longrightarrow \mathbb{C}$ is called regular if all the functions $f \circ \psi_{\bar{x}}$ are polynomial. Denote by $\mathbb{C}[G]$ the algebra of all regular functions on G. (Similarly one defines an analytic function on G in the general setup.)

Let $(V, d\pi)$ be an integrable $\mathfrak{g}$-module. Given $v \in V$ and a linear function $v^* \in V^*$, we get a regular function $f_{v^*,v}$ on G, called a matrix coefficient:

$$f_{v^*,v}(g) = \langle \pi(g)v, v^* \rangle.$$

Since the direct sum and tensor product of two integrable $\mathfrak{g}$-modules is again an integrable $\mathfrak{g}$-module, the set of matrix coefficients forms a subalgebra $\mathbb{C}[G]_{m.c.}$ of the algebra $\mathbb{C}[G]$. Note that $\mathbb{C}[G]$ is a $G \times G$-module under the action π_{reg} defined by $(\pi_{reg}(g_1,g_2)f)(g) = f(g_1^{-1} g g_2)$, the subalgebra $\mathbb{C}[G]_{m.c.}$ being a submodule.

Note also that matrix coefficients separate the orbits of G. For if $g \neq 1$, there exists a differentiable G-module (V, π) such that $\pi(g)v \neq v$ for some $v \in V$; choosing $v^* \in V^*$ such that $\langle \pi(g)v, v^* \rangle = 0$ and $\langle v, v^* \rangle = 1$, we get that $f_{v^*,v}(g) = 0$ and $f_{v^*,v}(1) = 1$.

Let $\mathbb{C}[G]_{\bar{x}} = \{f \in \mathbb{C}[G] \mid f \text{ vanishes on the image of } \psi_{\bar{x}}\}$. Taking the $\mathbb{C}[G]_{\bar{x}}$ for a basis of neighborhoods of 0 makes $\mathbb{C}[G]$ into a Hausdorff complete topological ring. We have the canonical inclusion $G \longrightarrow \operatorname{Specm} \mathbb{C}[G]$ (= set of all closed ideals of codimension one).

Problem. Compute $\operatorname{Specm} \mathbb{C}[G]$ and $\operatorname{Specm} \mathbb{C}[G]_{m.c.}$.

Let $\mathcal{M}$ be a subcategory of the category of integrable $\mathfrak{g}$-modules, closed under taking finite direct sums and tensor products. We denote by $\mathbb{C}[G]^{\mathcal{M}}_{s.r.}$ the subalgebra of $\mathbb{C}[G]_{m.c.}$ consisting of functions $f_{v^*,v}$ with $v \in V$, $v^* \in (V^*)_{fin}$, where V is a module from M, and call elements of $\mathbb{C}[G]^{\mathcal{M}}_{s.r.}$ strongly regular functions (with respect to the category $\mathcal{M}$).

Returning to the general setup, we introduce a topology on G as follows. Fix a subset $X \subset F_{\mathfrak{g}}$ such that the set $\{\exp tx \mid x \in X, t \in \mathbb{C}\}$ generates G. We call a subset U of G open if $(\psi'_{\bar{x}})^{-1}(U) \subset \mathbb{C}^n$ is open in the metric topology of $\mathbb{C}^n$ for all $\bar{x}$ such that the x_i of $\bar{x}$ are from X. With this topology G is a Hausdorff topological space (since the matrix coefficients are continuous); it is obviously connected. The inversion map $g \longmapsto g^{-1}$ is obviously continuous. The multiplication map is not continuous in general, however (a counterexample will be given below). One can show (using Milnor's lemma) that if X is countable, then G is a topological group. (It should not be difficult to show that if $\mathfrak{g}$ is countably-dimensional, then G is a topological group for $X = F_{\mathfrak{g}}$.)

§1.9 Let M be a set and let $\mathbb{C}^M$ denote the direct sum of a collection of copies of $\mathbb{C}$ indexed by M. By metric (resp. Zariski) topology on $\mathbb{C}^M$ we mean the finest topology that induces metric (resp. Zariski) topology on finite-dimensional subspaces (i.e. $U \subset \mathbb{C}^M$ is open iff $U \cap V$ is open in V for any finite-dimensional subspace V of $\mathbb{C}^M$).

The additive group of $\mathbb{C}^M$ is the group associated to $\mathbb{C}^M$ viewed as a commutative Lie algebra. If the set M is countable, then the metric topology on $\mathbb{C}^M$ is equivalent to the box topology and hence $\mathbb{C}^M$ is a topological group. If M is uncountable, then $\mathbb{C}^M$ is not a topological group (this has been pointed out to me by D. Wigner).

Let $V = \mathbb{C}^M$ and let x_i, $i \in M$, denote the linear coordinate functions on V. The algebra $\mathbb{C}[V]$ of <u>regular functions</u> on V consists of $\mathbb{C}$-valued functions whose restriction to any finite-dimensional subspace is a polynomial function. The subalgebra $\mathbb{C}[V]_{s.r.}$ of $\mathbb{C}[V]$ of <u>strongly regular functions</u> consists of polynomials in a finite number of the x_i. These definitions agree with the ones in §1.8 for the additive group of V.

The set X of zeros of an ideal of $\mathbb{C}[V]$ in V is called an <u>affine variety</u>; the intersection of X with a finite-dimensional subspace is called a <u>finite</u> subvariety of X. A map $\varphi: X \longrightarrow Y$ of affine varieties is a <u>morphism</u> if for any finite subvariety F of X there exists a finite subvariety F' of Y such that $\varphi(F) \subset F'$ and the map $\varphi: F \longrightarrow F'$ is a morphism of finite-dimensional algebraic varieties. A group in this category is called an affine algebraic group of Shafarevich type [18], [30].

It is easy to see that given an algebra R with a fixed basis $\{v_i\}$, the group Aut R is naturally an affine algebraic group of Shafarevich type. For we have

$$(4) \qquad v_i v_j = \sum_k c^k_{ij} v_k, \qquad c^k_{ij} \in \mathbb{C},$$

and $g \in$ Aut R if and only if $g(v_s) = \sum_t x_{st} v_t$, $g^{-1}(v_s) = \sum_t y_{st} v_t$, with the x_{st}, y_{st} satisfying the following system of equations: g and g^{-1} preserve (4) and $g \cdot g^{-1} = 1$.

Problem. For which integrable Lie algebras the associated group is an affine algebraic group of Shafarevich type? Is it true that the Lie algebra of a group of Shafarevich type (defined in [30]) is an integrable Lie algebra?

Problem. Let R be an arbitrary algebra. Then the Lie algebra Der R contains the following three subalgebras: the Lie algebra of the group Aut R (viewed is an affine algebraic group), the Lie algebra of endomorphism which are locally finite on R and the Lie algebra $(\text{Der } R)_{fin}$. How these subalgebras are related to each other? Interesting examples are: (a) R is a Lie algebra, (b) R is a coordinate ring of a (finite-dimensional) affine algebraic variety, (c) R is the universal enveloping algebra of a finite-dimensional Lie algebra.

§2.1 Let $A = (a_{ij})_{i,j=1}^{n}$ be a generalized Cartan matrix, i.e. $a_{ii} = 2$, a_{ij} are non-positive integers for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. For a pair of indices i,j such that $i \neq j$ put $m_{ij} = 2, 3, 4$ or 6 if $a_{ij}a_{ji} = 0, 1, 2$ or 3 respectively and put $m_{ij} = 0$ otherwise; put $m_{ii} = 1$.

We associate to A a discrete group $\bar{W}(A)$ on n generators $\bar{r}_1, \ldots, \bar{r}_n$ and the following defining relations (r1) and (r2) ($i,j = 1, \ldots, n$):

(r1) $$\bar{r}_j \bar{r}_i^2 \bar{r}_j^{-1} = \bar{r}_i^2 \bar{r}_j^{-2a_{ij}}.$$

(r2) $$\bar{r}_i \bar{r}_j \bar{r}_i \ldots = \bar{r}_j \bar{r}_i \bar{r}_j \ldots \ (m_{ij} \text{ factors on each side}).$$

Conjugating both sides of (r1) by $\bar{r}_j$ we get $\bar{r}_j^2 \bar{r}_i^2 \bar{r}_j^{-2} = \bar{r}_i^2$, i.e. the subgroup $T_{(2)} = \langle \bar{r}_i^2 \mid i = 1, \ldots, n \rangle$ of $\bar{W}(A)$ is a normal commutative subgroup. Also, it follows from (r1) for $i = j$ that $\bar{r}_i^4 = 1$.

Let W(A) be the corresponding Coxeter group, i.e. the group on generators $r_1, \ldots, r_n$ and the following defining relations ($i,j = 1, \ldots, n$):

$$(r_i r_j)^{m_{ij}} = 1.$$

Then we have a homomorphism $\bar{W}(A) \longrightarrow W(A)$ defined by $\bar{r}_i \longmapsto r_i$ and the exact sequence

$$1 \longrightarrow T_{(2)} \longrightarrow \bar{W}(A) \longrightarrow W(A) \longrightarrow 1.$$

Let $w = r_{i_1} \ldots r_{i_m}$ be a reduced expression of $w \in W$ (i.e. a shortest expression in the r_i); one defines $\ell(w) := m$. Deleting some of the r_i from this expression one gets a new element w' and writes $w' \leqslant w$. The partial ordering $\leqslant$ on W(A) is called the Bruhat order.

One constructs a section of the map $\bar{W}(A) \longrightarrow W(A)$ putting $\bar{w} = \bar{r}_{i_1} \ldots \bar{r}_{i_m}$; one can show that $\bar{w} \in \bar{W}(A)$ is independent of the choice of the reduced expression of w (see e.g. [20]).

We shall construct connected topological groups $G(A) \supset K(A)$ such that they contain $\bar{W}(A)$ as a discrete subgroup and $W(A)$ is their "Weyl group".

§2.2 We first present the necessary material on Kac-Moody algebras and their representations. One may consult the book [14] for details.

Let $(\mathfrak{h},\Pi,\Pi^{\vee})$ be a realization (unique up to isomorphism) of the matrix A, i.e. $\mathfrak{h}$ is a vector space of dimension 2n-rank A, and $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$, $\Pi^{\vee} = \{h_1, \ldots, h_n\} \subset \mathfrak{h}$ are linearly independent sets satisfying $\alpha_j(h_i) = a_{ij}$.

The Kac-Moody algebra $\mathfrak{g}(A)$ associated to the generalized Cartan matrix A is the Lie algebra generated by the vector space $\mathfrak{h}$ and symbols e_i and f_i $(i = 1, \ldots, n)$, with the following defining relations:

(A1) $[\mathfrak{h},\mathfrak{h}] = 0;\ [h,e_i] = \alpha_i(h)e_i,\ [h,f_j] = -\alpha_j(h)f_j\ (h\in\mathfrak{h})$

(A2) $[e_i,f_j] = \delta_{ij}h_i;\ (\operatorname{ad}e_i)^{1-a_{ij}}e_j = 0,\ (\operatorname{ad}f_i)^{1-a_{ij}}f_j = 0\ (i \neq j).$

The derived Lie algebra $\mathfrak{g}'(A)$ is also called the Kac-Moody algebra; it coincides with the subalgebra of $\mathfrak{g}(A)$ generated by e_i, f_i, h_i $(i = 1, \ldots, n)$ and its defining relations are (A2) and

(A'1) $[h_i,h_j] = 0;\ [h_i,e_j] = a_{ij}e_j,\ [h_i,f_j] = -a_{ij}f_j.$

We have the canonical embedding $\mathfrak{h} \subset \mathfrak{g}(A)$ and $\mathfrak{h}' \subset \mathfrak{g}'(A)$, where $\mathfrak{h}' = \sum \mathbb{C}h_i = \mathfrak{h} \cap \mathfrak{g}'(A)$. Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the subalgebra of $\mathfrak{g}(A)$ and $\mathfrak{g}'(A)$ generated by the e_i (resp. f_i), $i = 1, \ldots, n$. Then we have the triangular decompositions $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{g}'(A) = \mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+$.

The center of $\mathfrak{g}(A)$ and $\mathfrak{g}'(A)$ is $\mathfrak{c} = \{h \in \mathfrak{h}' \mid \alpha_i(h) = 0$ for all $i = 1, \ldots, n\}$. (In the non-affine case this follows from the fact that any root $\alpha \in \mathfrak{h}^*$ of $\mathfrak{g}(A)$ restricted to $\mathfrak{h}'$ remains non-zero [14, Chapter 5]; in the affine case this is a consequence of the Gabber-Kac theorem [14, §9.11].) Note that $\mathfrak{c} = 0$ iff $\mathfrak{h} = \mathfrak{h}'$

(which happens iff det $A \neq 0$).

Both $g(A)$ and $g'(A)$ are integrable Lie algebras since the e_i and f_i are ad-locally nilpotent and elements from h are ad-semisimple.

Furthermore, the subalgebras $g_i := \mathbb{C}f_i + \mathbb{C}h_i + \mathbb{C}e_i$ and any subspace of h are, clearly, integrable subalgebras of $g(A)$. This is also true for the subalgebras $h'' + n_+$ and $h'' + n_-$, where h'' is a subspace of h, since such a subalgebra, say p, has the property that for any $x \in p$ and $y \in g$, $(\text{ad } x)^N y \in p$ for sufficiently large N.

Given $\Lambda \in h'^*$, we extend it in some way to a linear function $\tilde{\Lambda} \in h^*$ and define the <u>highest</u> <u>weight</u> <u>module</u> $L(\Lambda)$ over $g(A)$ with action $d\pi_\Lambda$ by the properties

(L1) $L(\Lambda)$ is irreducible;

(L2) there exists a non-zero vector $v_\Lambda \in L(\Lambda)$ such that

$$d\pi_\Lambda(e_i)v_\Lambda = 0,\ i = 1, \ldots, n;\ d\pi_\Lambda(h)v_\Lambda = \tilde{\Lambda}(h)v_\Lambda,\ h \in h.$$

The module $L(\Lambda)$ remains irreducible when restricted to $g'(A)$ and is independent of the extension $\tilde{\Lambda}$ of Λ.

It is easy to see that if $L(\Lambda)$ is an integrable module (in the sense of §1.2), then the $\Lambda(h_i)$ are non-negative integers; we denote the set of such Λ by P_+ ($\subset h'^*$). We put $P_{++} = \{\Lambda \in P_+ \mid \Lambda(h_i) > 0,\ i = 1, \ldots, n\}$. Define fundamental weights $\Lambda_1, \ldots, \Lambda_n \in P_+$ by $\Lambda_i(h_j) = \delta_{ij}$.

A much deeper result is that conversely, if $\Lambda \in P_+$, then $L(\Lambda)$ is an integrable module [26, Corollary 9]. It follows that $r_0 = 0$. This will be discussed in §2.3.

Incidentally, provided that A is a symmetrizable matrix and $\Lambda \in P_+$, the $g'(A)$-module $L(\Lambda)$ is characterized by (L1) and (L2). For the annihilator of $v_\Lambda \in L(\Lambda)$ is a left ideal in the enveloping algebra of $g'(A)$ generated by e_i, $f_i^{\Lambda(h_i)+1}$ and $h_i - \Lambda(h_i)$, $i = 1, \ldots, n$ [14, (10.4.6)]; on the other hand, if (V,π) is a $g'(A)$-module satisfying (L1) and (L2), then, using the gradation of V by eigenspaces of h_i, one checks that $\pi(f_i)^{\Lambda(h_i)+1}v_\Lambda = 0$, $i = 1, \ldots, n$, to get a surjective $\mathfrak{g}'(A)$-module homomorphism $L(\Lambda) \longrightarrow V$.

It is not difficult to show (by making use of the structure of Der $\mathfrak{g}'(A)$) that the linear Lie algebra $\mathfrak{g}'(A)$, acting on $\bigoplus_{\Lambda \in P_+} L(\Lambda)$, is algebraic.

Similarly, one defines the <u>lowest weight module</u> $(L^*(\Lambda), d\pi^*_\Lambda)$ over $\mathfrak{g}(A)$ as the irreducible module for which there exists a non-zero vector v^*_Λ such that

$$d\pi^*_\Lambda(f_i)v^*_\Lambda = 0, \ i = 1, \ldots, n; \ d\pi^*_\Lambda(h)v^*_\Lambda = -\tilde{\Lambda}(h)v^*_\Lambda, \ h \in \mathfrak{h}.$$

This module is integrable if and only if $\Lambda \in P_+$. Actually, one has:

$$L^*(\Lambda) \simeq (L(\Lambda)^*)_{fin}.$$

§2.3 In the remainder of the notes we shall study the group $G(A)$ associated to the (integrable) Lie algebra $\mathfrak{g}'(A)$. (This is a more "canonical" object than the group associated to $\mathfrak{g}(A)$). We have the associated $G(A)$-modules $(L(\Lambda), \pi_\Lambda)$, $\Lambda \in P_+$, and the adjoint $G(A)$-modules $(\mathfrak{g}(A), \mathrm{Ad})$ and $(\mathfrak{g}'(A), \mathrm{Ad})$. The correspondence between the integrable $\mathfrak{g}'(A)$-modules and differentiable $G(A)$-modules (conjectured in §1.5 for an arbitrary integrable Lie algebra) has been established in [18].

Denote by G_i, H_i, H, U_+, U_-, B_+ and B_- the subgroups of $G(A)$ corresponding to the integrable subalgebras $\mathfrak{g}_i$, $\mathbb{C}h_i$, $\mathfrak{h}'$, $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{h}' + \mathfrak{n}_+$ and $\mathfrak{h}' + \mathfrak{n}_-$ respectively of $\mathfrak{g}(A)$. We proceed to give a more explicit description of these groups.

We have an integrable homomorphism $d\varphi_i$: $\mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{g}(A)$ defined by

$$d\varphi_i \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = ah_i + be_i + cf_i.$$

Let φ_i: $SL_2(\mathbb{C}) \longrightarrow G(A)$ be the corresponding homomorphism of groups. Put $H_i(t) = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. The homomorphisms φ_i are injective and one has: $G_i = \varphi_i(SL_2(\mathbb{C}))$; $H_i = \{H_i(t) \mid t \in \mathbb{C}^\times\}$;

$\exp te_i = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $\exp tf_i = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $t \in \mathbb{C}$. Furthermore, H is an abelian group equal to the direct product of the subgroups H_i. We also have $B_\pm = H \ltimes U_\pm$.

The map $\bar{r}_i \longmapsto \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (= $(\exp e_i)(\exp -f_i)(\exp e_i)$) extends to an injective homomorphism ψ: $\bar{W}(A) \longrightarrow G(A)$. We denote by $\bar{W}$ its image and denote the image of $\bar{r}_i$ again by $\bar{r}_i \in G(A)$.

The image of $T_{(2)}$ is a subgroup $\bar{W} \cap H = \{h \in H \mid h^2 = 1\}$. It follows that $T_{(2)} \simeq (\mathbb{Z}/2\mathbb{Z})^n$. The group $\bar{W}$ normalizes H. Denote by N the subgroup of G generated by H and $\bar{W}$. The group N acts on $\mathfrak{h}$ and $\mathfrak{h}'$ via the adjoint action, H acting trivially. The map $r_i \longrightarrow \bar{r}_i H$ extends to an isomorphism $W(A) \xrightarrow{\sim} W: = N/H$; the image of r_i is again denoted by $r_i \in W$. The group W is called the <u>Weyl group</u> of G(A) and the r_i its <u>fundamental</u> <u>reflections</u>. Put $S = \{r_1, \ldots, r_n\}$. The adjoint action of W on $\mathfrak{h}'$ is

$$r_j \cdot h_i = h_i - a_{ij} h_j \qquad (i,j = 1, \ldots, n).$$

All the above facts of this subsection are easily checked by calculating in the adjoint and the integrable highest weight modules. More involved is the proof of the following fundamental result:

<u>Lemma</u> **[26, Corollary 8]**. An element of a Kac-Moody algebra $\mathfrak{g}(A)$ is ad-locally finite (resp. locally nilpotent, resp. semisimple) if and only if it can be conjugated to an (ad-locally finite) subalgebra $\mathfrak{h} + (\mathfrak{n}_+ \cap (\mathrm{Ad}w)\mathfrak{n}_+)$ (resp. $\mathfrak{n}_+ \cap (\mathrm{Ad}w)\mathfrak{n}_+$, resp. $\mathfrak{h}$) for some $w \in \bar{W}$.

The proof of this lemma is based on Borel's fixed point theorem [2] and the Theorem 2.3 stated below.

It follows immediately from the lemma that a $\mathfrak{g}'(A)$-module is integrable if and only if all the e_i and f_i are locally finite (in particular, the $L(\Lambda)$ and $L^*(\Lambda)$ with $\Lambda \in P_+$ are integrable). Therefore, the present definition of G(A) coincides with that of [18]-[21], [26]. Another application of this lemma is the conjugacy of Cartan subalgebras of $\mathfrak{g}'(A)$ and the description of Aut $\mathfrak{g}'(A)$:

Corollary [26].

(a) Every ad-diagonalizable subalgebra of the Kac-Moody algebra $\mathfrak{g}(A)$ (resp. $\mathfrak{g}'(A)$) is Ad $G(A)$-conjugate to a subalgebra of $\mathfrak{h}$ (resp. $\mathfrak{h}'$).

(b) Any automorphism of the Kac-Moody algebra $\mathfrak{g}'(A)$ can be written in the form $\lambda\sigma$ or $\omega\lambda\sigma$ where $\sigma \in$ Ad G; $\lambda(e_i) = \lambda_{i_k} e_{i_k}$, $\lambda(f_i) = \lambda_{i_k}^{-1} f_{i_k}$, $i = 1, \ldots, n$, for some $\lambda_i \in \mathbb{C}^\times$ and a permutation $i \longmapsto i_k$ preserving the matrix A; $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $i = 1, \ldots, n$.

Put $V_\Lambda = \{c\pi_\Lambda(g)v_\Lambda \mid g \in G(A, c \in \mathbb{C}\}$. The following is the key result.

Theorem [26]. V_Λ is a closed affine subvariety of $L(\Lambda)$ (more precisely, V_Λ is the set of zeros of an ideal of $S(L^*(\Lambda))$.

In the case of a symmetrizable generalized Cartan matrix A, one can write down explicit equations for V_Λ. For that choose a non-degenerate invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ ([14, Chapter 2]), choose a basis $\{x_i\}$ of $\mathfrak{g}(A)$ consistent with the triangular decomposition (i.e. a union of bases of $\mathfrak{n}_-$, $\mathfrak{n}_+$ and $\mathfrak{h}$) and let $\{y_i\}$ be the dual basis of $\mathfrak{g}(A)$, i.e. $(x_i|y_j) = \delta_{ij}$. Then $v \in V_\Lambda$ if and only if it satisfies in $L(\Lambda) \otimes L(\Lambda)$ [18]:

$$(1) \qquad (\Lambda|\Lambda)v \otimes v = \sum_i d\pi_\Lambda(x_i)v \otimes d\pi_\Lambda(y_i)v$$

The equations (1) are called generalized Plücker relations (they are identical with the usual Plücker relations in the classical case of the $SL_n(\mathbb{C})$-modules $\Lambda^k\mathbb{C}^n$). One can show that generalized Plücker relations generate the ideal of V_Λ in the symmetric algebra of $L^*(\Lambda)$ [18].

The following proposition summarizes the key results on the structure of the group G(A):

<u>Proposition</u> [20], [26].

(a) The group G(A) is generated by the 1-parameter subgroups exp te_i and exp tf_i, i = 1, ..., n.

(b) $(G(A),B_+,N,S)$ is a Tits system (see [4] for the background on Tits systems).

(c) $C = \{H_1(t_1) \ldots H_n(t_n) \mid t_1^{a_{i1}} \ldots t_n^{a_{in}} = 1 \text{ for } i = 1, \ldots, n\}$.

(d) $U_{\pm}$ is generated by the 1-parameter subgroups exp $t(w \cdot e_i)$, where $w \in \bar{W}$ is such that $(\text{Ad}w)e_i \in n_{\pm}$, i = 1, ..., n.

(e) N is the normalizer of H in G(A).

One is referred to [26] for the details of the proof of the Lemma, Theorem and Corollary.

A standard consequence of Proposition 2.3(b) is

(2) $G(A) = \coprod_{w \in W} B_+\bar{w}B_+$ (Bruhat decomposition).

Here and further on $\bar{w}$ denotes a preimage of w in $\bar{W}$ (for example one may take $\bar{w}$ constructed in §2.1). Somewhat less standard is

(3) $G(A) = \coprod_{w \in W} B_-\bar{w}B_+$ (Birkhoff decomposition).

To prove (3) we check that [26]

$$B_-\bar{w}B_+\bar{r}_i \subset B_-\bar{w}B_+ \cup B_-\bar{w}\bar{r}_iB_+.$$

Since also $B_-\bar{w}\ {}_+$ exp te_i = $B_-\bar{w}B_+$, and since the $\bar{r}_i$ and exp te_i generate G(A) (by Proposition 2.3(a)), we get that the right-hand side of (3) is stable under right multiplication by G(A) and hence coincides with G(A). The disjointness in (2) and (3) is easily proved by making use of the G(A)-modules $L(\Lambda)$. For example, if $B_-\bar{w}B_+ = B_-\bar{w}_+B_+$, applying to $v_\Lambda \in L(\Lambda)$, we get $\pi_\Lambda(B_-)\pi_\Lambda(\bar{w})v_\Lambda = \pi_\Lambda(B_-)\pi_\Lambda(\bar{w}_1)v_\Lambda$ and therefore $\mathbb{C}\pi_\Lambda(\bar{w})v_\Lambda = \mathbb{C}\pi_\Lambda(\bar{w}_1)v_\Lambda$. Taking $\Lambda \in P_{++}$, we get $w = w_1$.

We conclude this section by a discussion on presentation problems. It is clear that N is a group on generators $\bar{r}_i$ and $H_i(t)$ where $i = 1, \ldots, n$, $t \in \mathbb{C}^\times$, with the following defining relations:

$$H_i(t)H_j(t') = H_j(t')H_i(t);$$

$$\bar{r}_iH_j(t)\bar{r}_i^{-1} = H_j(t)H_i(t^{-a_{ji}});$$

$$\bar{r}_i^2 = H_i(-1);$$

$$\bar{r}_i\bar{r}_j\bar{r}_i \ldots = \bar{r}_j\bar{r}_i\bar{r}_j \ldots \text{ (m_{ij} factors on each side).}$$

Thus, N, as well as $\tilde{W}$ and W are amalgamated products of subgroups of "rank" 1 and 2. We will see in §2.5 that this is also the case for the "unitary form" K(A). It is also known to be the case for the finite-dimensional G(A) [5] but seems unlikely for general G(A).

Problem. Find a presentation of G(A) and U_+. (These two questions are closely related to each other; a solution is known in the rank 2 case only [20].)

Problem. For which indecomposable A the group G(A)/C is simple. (It is simple if A is of finite type and it is not if A is of affine type. As shown in [25], the formal completion of G(A)/C is always simple.)

§2.4 Recall that a function f: G(A) ⟶ $\mathbb{C}$ is regular (weakly regular in the terminology of [18]) iff the functions $f \circ \psi_{\bar{x}}: \mathbb{C}^n \longrightarrow \mathbb{C}$ are polynomial functions for all $\bar{w} = (x_1, \ldots, x_n)$ with the x_i taken from the set $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ (see §1.8). Very little is known about the structure of the algebra $\mathbb{C}[G(A)]$ of regular functions and the $G(A) \times G(A)$-module $(\mathbb{C}[G(A)], \pi_{reg})$.

Conjecture. The inclusion $G(A) \longrightarrow \operatorname{Specm} \mathbb{C}[G(A)]$ is surjective.

The subalgebra $\mathbb{C}[G(A)]_{s.r.}$ of strongly regular functions is understood better. Recall that a regular function f is called <u>strongly regular</u> [18] if for any $g \in G(A)$ there exists a subgroup U_+' of U_+ (resp. U_-' of U_-) which is an intersection of U_+ (resp. U_-) with a finite number of its conjugates, such that $f(u_- g u_+) = f(g)$ whenever $u_+ \in U_+'$, $u_- \in U_-'$. One has the following analogue of the Peter-Weyl theorem.

Theorem [18]. The linear map $\Phi\colon \bigoplus_{\Lambda \in P_+} L^*(\Lambda) \otimes L(\Lambda) \longrightarrow \mathbb{C}[G(A)]$ defined by $\Phi(v^* \otimes v) = f_{v^*,v}$, is a well-defined injective $G(A) \times G(A)$-module homomorphism onto $\mathbb{C}[G(A)]_{s.r.}$.

The proof of this theorem is fairly simple: we check that the $G(A) \times G(A)$-module $\mathbb{C}[G(A)]_{s.r.}$ is differentiable and hence can be viewed as a $\mathfrak{g}(A) \times \mathfrak{g}(A)$-module, to which we apply a version of the complete reducibility theorem from [17].

Note that $\mathbb{C}[G(A)]_{s.r.} = \mathbb{C}[G(A)]^{\mathcal{M}}_{s.r.}$ (in the terminology of 1.8), where $\mathcal{M}$ is the subcategory of integrable $\mathfrak{g}(A)$-modules from the category $\mathcal{O}$.

For a subgroup P of G(A), let $\mathbb{C}[G(A)]^P$ denote the algebra of all $f \in \mathbb{C}[G(A)]$ such that $f(gp) = f(g)$ for all $p \in P$. G(A) acts on it by $(g \cdot f)(x) = f(g^{-1}x)$. Let θ_Λ be the character of B_+ defined by $\theta_\Lambda((\exp h)u) = e^{\Lambda(h)}$ for $h \in \mathfrak{h}$, $u \in U_+$, and let, for $\Lambda \in P_+$, $S_\Lambda = \{f \in \mathbb{C}[G(A)]_{s.r.} \mid f(gb) = \theta_\Lambda(b)f(g) \text{ for } g \in G,\ b \in B_+\}$. Then we have an immediate corollary of the Theorem 2.

Corollary.

(a) (Borel-Weil-type theorem) The map $L^*(\Lambda) \longrightarrow S_\Lambda$ defined by $v \longmapsto f_{v,v^*}$ is a G-module isomorphism.

(b) $\mathbb{C}[G(A)]^{U_+}_{s.r.} = \bigoplus_{\Lambda\in P_+} S_\Lambda$ and this algebra is isomorphic to $\bigoplus_{\Lambda\in P_+} L^*(\Lambda)$ with the Cartan product μ: $L^*(\Lambda) \otimes L^*(M) \longrightarrow L^*(\Lambda + M)$ characterized by the properties that μ is a $\mathfrak{g}(A)$-module homomorphism and $\mu(v^*_\Lambda \otimes v^*_M) = v^*_{\Lambda+M}$.

The main result of [18] about the algebra structure of $\mathbb{C}[G(A)]_{s.r.}$ is that this algebra is a unique factorization domain. An immediate corollary of this is that the algebra $\mathbb{C}[G(A)]^{U_+}_{s.r.}$ is a unique factorization domain and that the coordinate ring of strongly regular functions on V_Λ is integrally closed.

It is also shown in [18] that G(A) can be given a structure of an affine algebraic group of Shafarevich type by constructing an embedding of G(A) in a vector space as a closed affine subvariety. It has many nice properties, for example, G(A) acts morphically on L(Λ), $\Lambda \in P_+$ and on $\mathfrak{g}(A)$. It is still an open problem, however, whether the coordinate ring for this embedding coincides with $\mathbb{C}[G(A)]$.

Finally, I want to mention one striking difference between the finite- and infinite-dimensional cases. Let A be an indecomposable generalized Cartan matrix. If A is of finite type, then $\mathbb{C}[G(A)] = \mathbb{C}[G(A)]_{s.r.}$ is the coordinate ring of the finite-dimensional affine algebraic variety G(A); in particular, G(A) = Specm $\mathbb{C}[G(A)]_{s.r.}$.

Now let A be of infinite type. Then, of course, $\mathbb{C}[G(A)]$ is much larger than $\mathbb{C}[G(A)]_{s.r.}$. Moreover, the set R: = Specm $\mathbb{C}[G(A)]_{s.r.} \setminus G(A)$ is always non-empty. Namely, the $G(A) \times G(A)$-invariant subspace $\mathfrak{m} = \theta(\bigoplus_{\Lambda\in P_+\setminus\{0\}} L^*(\Lambda) \otimes L(\Lambda))$ is an ideal (in a sharp contrast to the finite-dimensional case), which is an element of R [18]. Recently, D. Peterson has computed the set R. In particular, it turned out that $R = \{\mathfrak{m}\}$ in the affine case. A discussion of the "partial compactification" Specm $\mathbb{C}[G(A)]_{s.r.}$ of G(A) in connection to the theory of singularities of algebraic surfaces may be found in this volume [31].

§2.5 In this section we study the algebraic structure of the <u>unitary form</u> K(A) of the group G(A). If A is a generalized Cartan matrix of finite type, then K(A) is the compact real form of the complex semisimple Lie group G(A). Thus, the groups K(A) are infinite-dimensional analogs of compact Lie groups.

The Kac-Moody algebra g'(A) admits an antilinear involution ω_0 determined by $\omega_0(e_i) = -f_i$, $\omega_0(f_i) = -e_i$, $i = 1, \ldots, n$. Since ω_0 preserves the set of locally finite elements, it can be lifted uniquely to an involution of G(A), which we also denote by ω_0. Let K(A) be the fixed point set of this involution in G(A).

Provided that A is symmetrizable and indecomposable, the Kac-Moody algebra g'(A) carries (a unique up to a constant factor) invariant bilinear form $(\cdot \mid \cdot)$ such that $(e_i \mid f_i) > 0$. Put $(x \mid y)_0 = -(x \mid w_0(y))$. The triangular decomposition is othogonal with respect to the Hermitian form $(\cdot \mid \cdot)_0$ and the main result of [19] is that it is positive definite on n_+ and n_-. Using this, one easily deduces [19] that any G(A)-module $L(\Lambda)$, $\Lambda \in P_+$, carries a unique positive definite Hermitian K(A)-invariant form $H(\cdot \mid \cdot)$ such that $H(v_\Lambda \mid v_\Lambda) = 1$. This is a justification for the term "unitary form".

For an arbitrary generalized Cartan matrix A, it is a simple fact that $L(\Lambda)$, $\Lambda \in P_+$, carries a unique Hermitian form $H(\cdot \mid \cdot)$ such that $H(v_\Lambda \mid v_\Lambda) = 1$. It remains an open problem whether it is positive definite in the non-symmetrizable case.

The involution ω_0 preserves the subgroups G_i, H_i and H; we denote by K_i, T_i and T respectively the corresponding fixed point subgroups. Then $K_i = \varphi_i(SU_2)$, $T_i = \varphi_i(\{\mathrm{diag}(\lambda,\lambda^{-1}) \mid |\lambda| = 1\})$ is a maximal torus of K_i and $T = \prod_i T_i$ is a maximal commutative subgroup of K(A). Put $H_i^+ = \varphi_i(\{\mathrm{diag}(\lambda,\lambda^{-1}) \mid \lambda \in \mathbb{R}, \lambda > 0\})$, $H_+ = \prod_i H_i^+$; then $H = T \times H^+$.

Let D (resp. $\mathring{D}$) $= \{u \in \mathbb{C} \mid |u| \leqslant 1$ (resp. $|u| < 1$)$\}$ be the unit disc (resp. its interior) and let $S^1 = D \setminus \mathring{D}$ be the unit circle. Given $u \in D$, put

$$z(u) = \begin{bmatrix} u & (1-|u|^2)^{1/2} \\ -(1-|u|^2)^{1/2} & \bar{u} \end{bmatrix} \in SU_2,$$

and put $z_i(u) = \varphi_i(z(u))$. We have $\bar{r}_i = z_i(0) \in K_i$, hence $\bar{W} \subset K(A) \subset G(A)$. Put

$$Y_i = \{z_i(u) \mid u \in \overset{\circ}{D}\} \subset K_i.$$

The same argument as in [32, Lemma 43(b)] gives

$$B\bar{r}_i B = Y_i B \text{ (uniquely).} \tag{4}$$

(Here and further on "uniquely" means that any element from the set on the left-hand side is uniquely represented as a product of elements from the factors on the right-hand side.)

Let $w = r_{i_1} \dots r_{i_s}$ be a reduced expression of $w \in W$ and let $\tilde{w}$ be its preimage in $\bar{W}$ defined in §2.1. Using (4), the same argument as in [32, Lemma 15], gives

$$B_+ \tilde{w} B_+ = Y_{i_1} \dots Y_{i_m} B_+ \text{ (uniquely).} \tag{5}$$

Put $K_w = K(A) \cap B_+ \tilde{w} B_+$. Put $Y_w = Y_{i_1} \dots Y_{i_m}$; this is independent of the choice of the reduced expression for w, as follows from the following formula [20]:

$$Y_w = \{k \in K_w \mid H(\pi_{\Lambda_i}(k)v_{\Lambda_i} \mid \pi_{\Lambda_i}(\tilde{w})v_{\Lambda_i}) > 0,\ i = 1, \dots, n\}.$$

We have by (5):

$$K_w = Y_w T \text{ (uniquely).} \tag{6}$$

Put $\bar{K}_w = K_{i_1} \dots K_{i_m} T$; this is also independent of the reduced expression of w, as follows from

$$\bar{K}_w = \coprod_{w' \leqslant w} K_{w'}. \tag{7}$$

Finally, by the Bruhat decomposition, we have

(8) $$K(A) = \coprod_{w \in W} K_w.$$

We obtain, in particular that K(A) is generated by the K_i, $i = 1, \ldots, n$, and the Iwasawa decomposition [26]:

(9) $$G(A) = K(A)H_+U_+ \text{ (uniquely).}$$

We proceed to establish a presentation of the group K(A), which may be viewed as a "real analytic continuation" of the presentation of the group $\bar{W}(A)$.

We have the following relations coming from SU_2:

(R1) (i) $z_i(u_1)z_i(u_2) = z_i(u_1u_2)$ if $u_1, u_2 \in S^1$,

(ii) $z_i(u)z_i(-\bar{u}) = z_i(-1)$ if $u \in \overset{\circ}{D}$,

(iii) $z_i(u_1)z_i(u_2) = z_i(u_1')z_i(u_2')$ if $u_1, u_2 \in \overset{\circ}{D}$ and $u_1 \neq -\bar{u}_2$, for some unique $u_1' \in \overset{\circ}{D}$ and $u_2' \in S^1$.

Furthermore, T_i normalizes K_j and the conjugation is given by

(R2) $$z_i(u_1)z_j(u_2)z_i(u_1)^{-1} = z_j(u_1^{a_{ij}}u_2)z_j(u_1^{-a_{ij}}) \text{ if } u_1 \in S^1,\ u_2 \in D.$$

Finally, if $m_{ij} \neq 0$, then $r_ir_jr_i \ldots = r_jr_ir_j \ldots$ (m_{ij} factors on each side). Hence $Y_iY_jY_i \ldots = Y_jY_iY_j \ldots$ (uniquely). In other words, we have

(R3) $z_i(u_1)z_j(u_2)z_i(u_3) \ldots = z_j(u_1')z_i(u_2')z_j(u_3') \ldots$ (m_{ij} factors on each side), if $u_1, u_2, \ldots \in \overset{\circ}{D}$, for some unique $u_1', u_2', \ldots \in \overset{\circ}{D}$.

<u>Theorem</u> [20]. The group K(A) is a group on generators $z_i(u)$ for $i = 1, \ldots, n$; $u \in D$, with defining relations (R1), (R2) and (R3).

Let $\tilde{K}(A)$ be the group on generators $z_i(u)$ ($i = 1, \ldots, n$; $u \in D$) with defining relations (R1), (R2), (R3), let $\alpha: \tilde{K}(A) \longrightarrow K(A)$ be the canonical homomorphism, let $w = r_{i_1} \dots r_{i_m} \in W$ be a reduced expression and let $\tilde{K}_w = \alpha^{-1}(K_w)$. It is not hard to show that any element of $\tilde{K}_w$ can be brought to the form $z_{i_1}(u_1) \dots z_{i_m}(u_m) z_1(v_1) \dots z_n(v_n)$, where $u_i \in \overset{\circ}{D}$, $v_i \in S^1$. Then (6) completes the proof of the Theorem. (The details may be found in [20].)

Note that the groups $\tilde{K}(A)$ have been introduced (in a somewhat different form) in [13] and it was proved there, by a topological argument, that $\mathrm{Ker}\,\alpha$ is a finite central subgroup if A is of finite type.

§2.6 Since G(A) is generated by a finite number of 1-parameter subgroups exp tx, where $x \in X = \{e_i, f_i \mid i = 1, \dots, n\}$, it is a (connected Hausdorff) topological group in the topology defined in §1.8. In this section we discuss some of the results of [21] on the topology of the groups G(A) and K(A) and of the associated flag varieties. The reader is referred to [21] for details.

All the subgroups which have appeared in the discussion are closed. The bijection $K(A) \times H_+ \times U_+ \xrightarrow{\sim} G(A)$ provided by the Iwasawa decomposition is a homeomorphism. Furthermore, H_+ and U_+ are contractible. Thus (as in the finite-dimensional case) G(A) is homotopically equivalent to K(A).

The topology on K(A) can be described explicitly as follows. Given $w \in W$, take its reduced expression $w = r_{i_1} \dots r_{i_m}$ and define a map $(SU_2)^m \times T \longrightarrow K(A)$ by $(k_1, \dots, k_m, t) \longmapsto \varphi_{i_1}(k_1) \dots \varphi_{i_m}(k_m)t$. The image of this map is $\bar{K}_w$, and we take the quotient topology on it. This topology is independent of the choice of the reduced expression and makes $\bar{K}_w$ a connected Hausdorff compact topological space. Then a subset F of K(A) is closed iff $F \cap \bar{K}_w$ is closed in $\bar{K}_w$ for all $w \in W$. It follows that $\bar{K}_w$ is the closure of K_w and that $\bar{K}_{w'} \leqslant \bar{K}_w$ iff $w' \leqslant w$. Thus, as a topological space, K(A) is the inductive limit with respect to the Bruhat order of the compact spaces $\bar{K}_w$.

The most natural way to study the topology of K(A) is to consider the fibration

$$\pi: K(A) \longrightarrow K(A)/T.$$

The topological space $\mathcal{F}(A) := K(A)/T$ is called the flag variety of the group K(A) and of G(A). Put $C_w = \pi(Y_w)$. Then by (6) and (8) we get a cellular decomposition

$$\mathcal{F}(A) = \coprod_{w \in W} C_w$$

To show that this is a CW-complex one has only to construct attaching maps (for some reason this point is routinely omited in the literature on finite-dimensional groups, see e.g. [2]). For that, given $w \in W$, choose a reduced expression $w = r_{i_1} \dots r_{i_s}$ and define a map $\alpha_w: D^s \longrightarrow \mathcal{F}(A)$ by $\alpha_w(u_1, \dots, u_s) = z_{i_1}(u_1) \dots z_{i_s}(u_s) \bmod T$. This gives a homeomorphism of $\overset{\circ}{D}{}^s$ onto Y_w by (5). Since $\bar{K}_w$ is the closure of K_w, by (7) we have:

$$\bar{C}_w = \coprod_{w' \leqslant w} C_{w'}, \tag{10}$$

where $\bar{C}_w$ is the closure of C_w. It is clear that $\alpha_w(D^{k-1} \times S^1 \times D^{s-k}) \subset \bar{C}_{w'}$, where w' is obtained from w by dropping r_{i_k}. Thus, by (10) the image of the boundary under the map α_w lies in the union of cells of lower dimension (this argument is taken from [21]).

Since $\dim C_w = 2\ell(w)$, there are no cells of odd dimension. Thus $H_*(\mathcal{F}(A),\mathbb{Z})$ and $H^*(\mathcal{F}(A),\mathbb{Z})$ are free $\mathbb{Z}$-modules on generators of degree $2\ell(w)$, $w \in W$. Putting $W(q) = \sum_{w \in W} q^{\ell(w)}$, we obtain that the Poincaré series for homology and cohomology of $\mathcal{F}(A)$ over any field is $W(q^2)$. (A simple inductive procedure for computing W(q) may be found in [4].)

Actually, as in the finite-dimensional case, $\mathcal{F}(A)$ can be given a natural structure of a complex projective manifold. For that note that, by the Iwasawa decomposition, we have a homeomorphism G(A)/B

$\xrightarrow{\sim} \mathcal{F}(A)$. But $G(A)/B$ can be identified with the orbit $G \cdot v_\Lambda$ in the projective space $\mathbb{P}L(\Lambda)$ for $\Lambda \in P_{++}$. This is a closed subvariety of $\mathbb{P}L(\Lambda)$ by Theorem 2.3. An equivalent definition, independent of the choice of $\Lambda \in P_{++}$, is $G(A)/B = \mathrm{Proj} \bigoplus_{\Lambda \in P_+} L^*(\Lambda)$ (cf. Corollary 2.4).

As a result, the $\bar{C}_w$ become finite-dimensional projective varieties, called <u>Schubert varieties</u>, and $\mathcal{F}(A)$ is their inductive limit with respect to Bruhat order [18]. The study of singularities of these varieties has many interesting applications. Some of them are discussed in this volume [11].

P. Deligne kindly provided a proof of the following result:

Let X be a projective algebraic variety over $\mathbb{C}$ which is decomposed into a finite disjoint union of subvarieties X^i_j with $\dim_{\mathbb{C}} X^i_j = i$, such that $\bar{X}^i_j \setminus X^i_j \subset \coprod_j \coprod_{s<i} X^s_j$ and there exist morphisms $\coprod^j \mathbb{C}^i \longrightarrow \coprod_j X^i_j$ which are homeomorphisms. Then the topological space X is rationally formal (in the sense of [6]).

Applying this to our situation, we deduce that $\mathcal{F}(A)$ is a rationally formal topological space.

Let $Q^\vee = \sum_i \mathbb{Z}h_i$ and let $P = \{\lambda \in \mathfrak{h}'^* \mid \lambda(h_i) \in \mathbb{Z}, i = 1, \ldots, n\}$ be the dual lattice. Let $S(P) = \bigoplus_{j \geq 0} S^j(P)$ be the symmetric algebra over the lattice P, and $S(P)^+ = \bigoplus_{j>0} S^j(P)$ the augmentation ideal. Given a field $\mathbb{F}$, we denote $S(P)_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{Z}} S(P)$, etc. In order to study the multiplicative structure of $H^*(\mathcal{F}(A),\mathbb{F})$, we define the <u>characteristic</u> <u>homomorphism</u> ψ: $S(P) \longrightarrow H^*(\mathcal{F}(A),\mathbb{Z})$ as follows. Given $\lambda \in P$, we have the corresponding character of T and the associated line bundle $\mathcal{L}_\lambda$ on $\mathcal{F}(A)$. Put $\psi(\lambda) \in H^2(\mathcal{F}(A),\mathbb{Z})$ equal to the Chern class of $\mathcal{L}_\lambda$ and extend by multiplicativity to the whole S(P). Denote by $\psi_{\mathbb{F}}$ the extension of ψ by linearity to $S(P)_{\mathbb{F}}$.

In order to describe the properties of $\psi_{\mathbb{F}}$ define operators Δ_i for $i = 1, \ldots, n$ on S(P) by

$$\Delta_i(f) = (f - r_i(f))/\alpha_i,$$

and extend by linearity to $S(P)_{\mathbb{F}}$. Put $I_{\mathbb{F}} = \{f \in S(P)_{\mathbb{F}}^{+} \mid \Delta_{i_1} \dots \Delta_{i_m}(f) \in S(P)_{\mathbb{F}}^{+}$ for every sequence $i_1, \dots, i_m\}$. This is a graded ideal of $S(P)_{\mathbb{F}}^{+}$.

<u>Proposition</u> [21], [22]. Let $\mathbb{F}$ be a field. Then

(a) Ker $\psi_{\mathbb{F}} = I_{\mathbb{F}}$ (this holds for an arbitrary ring $\mathbb{F}$).

(b) $H^*(\mathcal{F}(A),\mathbb{F})$ is a free module over Im $\psi_{\mathbb{F}}$.

(c) Any minimal system of homogeneous generators of the ideal $I_{\mathbb{F}}$ is a regular sequence.

Let $CH(G(A),\mathbb{F})$ denote the quotient (graded) algebra of $H^*(\mathcal{F}(A),\mathbb{F})$ by the ideal generated by $\psi(P_{\mathbb{F}})$; this is called the <u>Chow algebra</u> of $G(A)$ over $\mathbb{F}$. Notice that, by Theorem 2.6(b) below, $CH(G(A),\mathbb{F}) = \pi^*(H^*(\mathcal{F}(A),\mathbb{F}))$. The terminology is justified by the fact that for A of finite type, the Chow ring of the complex semisimple group $G(A)$ is isomorphic to $CH(G(A),\mathbb{Z})$ (A. Grothendieck).

Denote the degrees of the elements of a minimal system of homogeneous generators of the ideal $I_{\mathbb{F}}$ by $d_1, \dots, d_s (s \leq n)$. These degrees are well-defined; we will call them the <u>degrees of basic generators of</u> $I_{\mathbb{F}}$. Note that $s = n$ if char $\mathbb{F} = p \neq 0$ since W acts on $P \otimes_{\mathbb{Z}} \mathbb{F}$ via a finite group.

Actually, Proposition 2.6 holds in a much more general situation [22]. For example, the part (c) holds for any group generated by reflections over a field $\mathbb{F}$ of arbitrary characteristic. For W finite and $\mathbb{F} = \mathbb{C}$ we recover the classical result of Chevalley-Shepard-Todd.

It is not difficult to deduce from Proposition 2.6 the following results.

<u>Theorem</u> [21]. Let $\mathbb{F}$ be a field. Then:

(a) $CH(G(A),\mathbb{Q})$ is a polynomial algebra on (in general infinite number of) homogeneous generators. The Poincaré series of $CH(G(A),\mathbb{F})$ is equal to $W(q^2)(1-q^2)^n / \prod_{i=1}^{s} (1-q^{2d_i})$. The (graded) algebra $H^*(K/T,\mathbb{Q})$ is (non-canonically) isomorphic to the tensor product of Im $\psi_{\mathbb{Q}}$ and $CH(G(A),\mathbb{Q})$.

(b) The cohomology spectral sequence $E_r(K(A),\mathbb{F})$ of the fibration π: $K(A) \longrightarrow \mathcal{F}(A)$ degenerates at $r = 3$, i.e. $E_3(K(A),\mathbb{F}) = E_\infty(K(A),\mathbb{F})$.

(c) π^* induces an injective homomorphism of $CH(G(A),\mathbb{F})$ into $H^*(K(A),\mathbb{F})$ and into $E_\infty(K(A),\mathbb{F})$, the image being a Hopf subalgebra of $H^*(K(A),\mathbb{F})$.

(d) The algebra $E_\infty(K(A),\mathbb{F})$ is isomorphic to a tensor product of $C(G(A),\mathbb{F})$ and the cohomology algebra of the Koszul complex $(\Lambda(P) \otimes \text{Im}\, \psi_{\mathbb{F}}, d)$, where $d(\lambda \otimes u) = \psi(\lambda) \vee u$. The latter algebra is an exterior algebra on homogeneous generators of degrees $2d_1-1, \ldots, 2d_s-1$. The Poincaré series of $H^*(K(A),\mathbb{F})$ is equal to the product of the Poincaré series of $CH(G(A),\mathbb{F})$ and the polynomial $\prod_{i=1}^{s} (1 + q^{2d_i-1})$.

As an immediate corollary of Theorem 2.6(a) and (d), we deduce the following classical results.

<u>Corollary</u>. Let K be a connected compact Lie group, T its maximal torus, $\mathfrak{h}$ the complexified Lie algebra of T, W the Weyl group, and let $d_1, \ldots, d_n$ be the degrees of the basic homogeneous invariants for the action of W on $S(\mathfrak{h})$. Then:

(a) $W(q) = \prod_{i=1}^{n} ((1-q^{d_i})/(1-q))$.

(b) $H^*(K/T,\mathbb{C})$ is generated by $H^2(K/T,\mathbb{C})$ and is isomorphic to the quotient of $S(\mathfrak{h})$ by the ideal generated by $(S(\mathfrak{h})^+)^W$.

(c) $H^*(K,\mathbb{C})$ is a Grassmann algebra on homogeneous generators of degrees $2d_1-1, \ldots, 2d_n-1$.

(d) The Chow ring of a complex reductive group is finite.

In fact, using explicit formulas or the cup product [21] (see also the next section), it is easy to show that the third term of the cohomology (resp. homology) spectral sequence over $\mathbb{Z}$ of the fibration π is isomorphic to the homology of the complex (C^*, d^*) (resp. (C_*, d_*)), where $C^* = \mathbb{Z}[W] \otimes_{\mathbb{Z}} \Lambda(P)$, $C_* = \mathbb{Z}[W] \otimes_{\mathbb{Z}} \Lambda(Q^\vee)$, $\deg \delta_w = \deg \delta^w = 2\ell(w)$, $\deg h_i = \deg \Lambda_i = 1$, and

$$d^*(\delta^w \otimes p) = \sum_{w \xrightarrow{\gamma} w'} \delta^{w'} \otimes (\partial_\gamma p),$$

$$d_*(\delta_w \otimes q) = \sum_{w' \xrightarrow{\gamma} w} \delta_{w'} \otimes (\gamma \wedge q).$$

Here $w' \xrightarrow{\gamma} w$ means that $\ell(w') = \ell(w) - 1$ and there exists a positive real coroot $\gamma \in \Sigma\mathbb{Z}h_i$ such that $w = w'r_\gamma$, where r_γ is the reflection with respect to γ; ∂_γ is an antiderivation of $\Lambda(P)$ such that $\partial_\gamma \lambda = \langle \lambda, \gamma \rangle$ for $\lambda \in P$.

__Remark__. If we take a standard cellular decomposition of T, then (8) together with (6) gives us a cellular decomposition of K(A). Unfortunately, it is not a CW-complex; but if it were, then, as one can easily see, the complex (C_*, d_*) would be the corresponding homology complex.

__Conjecture__. $E_\infty(K(A), \mathbb{Z}) = E_3(K(A), \mathbb{Z})$.

Let me state also some corollaries of Theorem 2.6 for arbitrary K(A).

__Corollary__.

(a) K(A) is a connected simply connected topological group;

$H^2(K(A),\mathbb{Z}) = 0.$

(b) Let A be indecomposable and let $\epsilon = 1$ or 0 according as A is symmetrizable or not. Then $H^3(K(A),\mathbb{Q}) = \mathbb{Z}^{\epsilon}$; $\dim_{\mathbb{Q}} H^4(K(A),\mathbb{Q})$ = #(cycles of the Dynkin diagram of A) + 1 − ϵ. $H^*(K(A),\mathbb{Q})$ is completely determined (as a graded vector space) by the Weyl group W regarded as a Coxeter group and by ϵ.

(c) The minimal model (in the sense of [6]) of the topological space $\mathcal{F}(A)$ is a tensor product of an exterior algebra on generators $\xi_1, \ldots, \xi_s$ of degrees $2d_1-1, \ldots, 2d_s-1$, and of a polynomial algebra on n generators $\Lambda_1, \ldots, \Lambda_n$ of degree 2, a_j generators of degree 2j, j = 2, 3, ..., where $d_1, \ldots, d_s$ are the degrees of basic generators of $I_{\mathbb{Q}}$ and the a_j are determined by

$$W(q)(1-q)^n = \prod_{i=1}^{s} (1-q^{d_i}) \prod_{j\geq 2} (1-q^j)^{-a_j}.$$

The differential d of this minimal model is 0 on all even generators and $d\xi_i = P_i(\Lambda_1, \ldots, \Lambda_n)$, where the P_i are basic generators of $I_{\mathbb{Q}} \subset \mathbb{Q}[\Lambda_1, \ldots, \Lambda_n]$.

(d) The minimal model of K(A) is isomorphic to $H^*(K(A),\mathbb{Q})$ with trivial differential, and is a tensor product of an exterior algebra on generators of degrees $2d_1-1, \ldots, 2d_s-1$, and a polynomial algebra on a_j generators of degrees 2j, j = 2, 3, ...

(e) The dimension of the k-th rational homotopy group of $\mathcal{F}(A)$ and K(A) is equal to the number of generators of degree k of their minimal models.

Cohomology and the Chow ring in the finite-dimensional case and arbitrary field $\mathbb{F}$ are discussed in detail (from the presented point of view), in [15]. The affine case will be discussed in the next section. Here I will discuss briefly the case when A is an indecomposable generalized Cartan matrix of non-finite and non-affine type and $\mathbb{F} = \mathbb{Q}$. Put = 1 or 0 according as the matrix A is symmetrizable or not. Then $I_{\mathbb{Q}}$ is generated by ϵ elements of degree 2. Put

$$C(q) = W(q)(1 - q)^n(1 - q^2)^{-\epsilon}.$$

Then we have by Corollary 2.6(c):

(11) $$C(q) = \prod_{j\geq 2} (1 - q^j)^{-a_j}, \text{ where } a_j \geq 0.$$

It would be interesting to find a purely combinatorial proof of this result. By Theorem 2.6(a), the Chow algebra $CH(G(A),\mathbb{Q})$ is a polynomial algebra on a_j generators of degree $2j$, $j = 2, 3, \ldots$. By Theorem 2.6(d), $H^*(K(A),\mathbb{Q})$ is a tensor product of $CH(G(A),\mathbb{Q})$ with the exterior algebra on ϵ generators of degree 3.

A stronger form of (11) is the following:

<u>Conjecture</u>. $C(q) = \frac{1}{1-B(q)}$, where $B(q) = b_2q^2 + b_3q^3 + \ldots$ and $b_i \geq 0$.

For example, if $n = 2$, then $C(q) = 1$. For the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ one has $C(q) = (1 - q^2)(1 - q^3)/(1 - q^2 - q^3)$, and $B(q) = q^5/(1 - q^2)(1 - q^3)$.

If $n = 2$, then $E_3(K(A),\mathbb{Z}) = E_\infty(K(A),\mathbb{Z})$ for trivial reasons, and it is not difficult to compute the homology of the complex (C^*,d^*) explicitly, obtaining the additive structure of $H^*(K(A),\mathbb{Z})$. I state here the result for $A = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$, where $a \geq 2$. Define a sequence of integers c_j for $j \in \mathbb{Z}$ by the following recurrent formula:

$$c_0 = 0,\ c_1 = 1,\ c_{j+2} = ac_{j+1} - c_j.$$

Then $H^{2j}(K(A),\mathbb{Z}) \simeq H^{2j+3}(K(A),\mathbb{Z}) \simeq \mathbb{Z}/c_j\mathbb{Z}$. Notice that $c_j = j$ if $a = 2$, and $c_j = \Phi_{2j}$, the $2j$-th Fibonacci number, if $a = 3$.

§2.7 The basic tool in the study of the cohomology of flag varieties

$\mathcal{F}(A)$ are certain operators introduced in [21] which "extend" the action of the operators Δ_i from the image of ψ to the whole cohomology algebra. (This seems to be a new ingredient even in the finite-dimensional case, cf. [1], as far as "bad primes" are concerned [15].)

The Weyl group W acts by right multiplication on $\mathcal{F}(A) = K(A)/T$, which induces an action of W on homology and cohomology of $\mathcal{F}(A)$. On the other hand, since the odd cohomology of K_i/T_i and $K(A)/K_iT$ is trivial, the spectral sequence of the fibration p_i: $K(A)/T \longrightarrow K(A)/K_iT$ degenerates after the second term. It follows that $H^*(\mathcal{F}(A),\mathbb{Z})$ is generated by Im p_i^*, which is r_i-fixed and the element $\psi(\Lambda_i)$.

We deduce that for each $i = 1, \ldots, n$ there exists a unique $\mathbb{Z}$-linear operator A^i on $H^*(\mathcal{F}(A),\mathbb{Z})$, lowering the degree by 2, such that r_i leaves the image of A^i fixed and

$$u - r_i(u) = A^i(u) \cup \psi(\alpha_i) \text{ for } u \in H^*(\mathcal{F}(a),\mathbb{Z}).$$

Similarly, we introduce operators A_i on $H_*(\mathcal{F}(A),\mathbb{Z})$, raising the degree by 2, such that $r_i(A_i(z)) = -A_i(z)$ and

$$z + r_i(z) = A_i(z) \cap \psi(\alpha_i) \text{ for } z \in H_*(\mathcal{F}(A),\mathbb{Z}).$$

The operators A^i and A_i are dual to each other with respect to the intersection form. One has:

(12) $$A^i(u \cup v) = A^i(u) \cup r_i(v) + u \cup A^i(v);$$

(13) $$A_i(u \cap z) = r_i(u) \cap A_i(z) + A^i(u) \cap z.$$

(14) $$A^i(\psi(\lambda)) = \langle\lambda, h_i\rangle.$$

The operators A_i have the following simple geometric interpretation. Recall the map α_w: $D^{\ell(w)} \longrightarrow \mathcal{F}(A)$ defined for $w \in W$ in §2.6. The relative homology map α_{w*} gives us an element $\delta_w \in H_{2\ell(w)}(\mathcal{F}(A),\mathbb{Z})$. Then $\{\delta_w\}_{w\in W}$ is a $\mathbb{Z}$-basis of

$H_*(\mathcal{F}(A),\mathbb{Z})$; let $\{\delta^w\}_{w\in W}$ be the dual basis of $H^*(\mathcal{F}(A),\mathbb{Z})$. We have the following formulas for the action of the Weyl group in these bases generalizing that from [1] (see [21]):

$$(15)\qquad r_i(\delta^w) = \begin{cases} \delta^w & \text{if } \ell(wr_i) > \ell(w), \\ \delta^w - \sum\limits_{wr_i \xrightarrow{\gamma} w'} \langle\alpha_i,\gamma\rangle\delta^{w'} & \text{otherwise} \end{cases}$$

$$(16)\qquad r_i(\delta_w) = \begin{cases} -\delta_w & \text{if } \ell(w) > \ell(wr_i), \\ -\delta_w + \sum\limits_{w' \xrightarrow{\gamma} wr_i} \langle\alpha_i,\gamma\rangle\delta_{w'} & \text{otherwise} \end{cases}$$

The basic fact that is used to prove these and other formulas is the following lemma which describes the action of the operators A_i and A^i on Schubert cycles δ_w and cocycles δ^w.

<u>Lemma</u> [21].

(a) $A_i(\delta_w) = \delta_{wr_i}$ if $\ell(wr_i) > \ell(w)$ and $= 0$ otherwise.

(b) $A^i(\delta^w) = \delta^{wr_i}$ if $\ell(w) > \ell(wr_i)$ and $= 0$ otherwise.

<u>Corollary</u>.

(a) The subalgebra of W-invariants on $H^*(\mathcal{F}(A),\mathbb{Z})$ coincides with $H^0(\mathcal{F}(A),\mathbb{Z})$.

(b) The operators A^i generate a Hecke algebra, i.e. an associative algebra on the A^i with defining relations: $(A^i)^2 = 0$; $A^iA^jA^i\ldots = A^jA^iA^j\ldots$ (m_{ij} factors on each side).

Note that Corollary 2.7(a) (which means that $A^i(u) = 0$ for all i implies $u \in H^0(\mathcal{F}(A),\mathbb{Z})$) together with (12), (14) and (15) completely determines the multiplicative structure of the algebra $H^*(\mathcal{F}(A),\mathbb{Z})$. Formulas are especially simple when one of the factors is of degree 2; then we get the following formulas, which generalize that in [1] (see

[21]):

$$(17)\qquad \psi(\lambda) \cup \delta^{w} = \sum_{w \xrightarrow{\gamma} w'} \langle\lambda,\gamma\rangle \delta^{w'};$$

$$(18)\qquad \psi(\lambda) \cap \delta_{w} = \sum_{w' \xrightarrow{\gamma} w} \langle\lambda,\gamma\rangle \delta_{w'}.$$

Note that Proposition 2.6(a) follows immediately from the fact that $\psi \circ \Delta_i = A^i \circ \psi$, which is clear from the construction of the A^i.

Furthermore, using the operators A^i, we can compute by induction on the degree of u the action of the total Steenrod power $\mathcal{P}$ on $H^*(\mathcal{F}(A),\mathbb{F}_p)$ by the following formula [21]:

$$(19)\qquad A^i(\mathcal{P}(u)) = \mathcal{P}(A^i(u))(1 + \psi(\alpha_i)^{p-1}).$$

Finally note that the same approach allows us to compute the Lie algebra cohomology $H^*(\mathfrak{g}'(A),\mathbb{C})$ and to show that it is isomorphic to $H^*(K(A),\mathbb{C})$. A differential forms approach to the study of $\mathcal{F}(A)$ is developed by Kumar in [24] and in a paper of this volume.

§2.8 A Kac-Moody algebra $\mathfrak{g}(A)$ is finite-dimensional if and only if A is of finite type (i.e. all principal minors of A are positive). The class of these algebras coincides with the class of finite-dimensional semisimple Lie algebras. The associated group G(A) is the Lie group of $\mathbb{C}$-points of the connected simply connected algebraic group whose Lie algebra is $\mathfrak{g}(A)$. The group K(A) is the compact form of G(A), H is the Cartan subgroup of G(A), B_+ and B_- are "opposite" Borel subgroups, etc. In this case most of the results of Chapter 2, except for some results of §2.6 and 2.7, are well-known.

In this section we discuss in more detail the case when the matrix A is of affine type, i.e. all proper principal minors of A are positive, but det A = 0 (A is then automatically indecomposable and symmetrizable). An example of such a matrix is the extended Cartan matrix of a simple finite-dimensional Lie algebra. This is the "non-twisted" case we will be dealing with. The "twisted" case is then routinely deduced by taking a fixed point set of an automorphism of order 2 or 3 (see [14, Chapter 8] for details).

Let $\mathring{\mathfrak{g}}$ be a complex simple finite-dimensional Lie algebra with Chevalley generators $\mathring{e}_i$, $\mathring{f}_i$, $\mathring{h}_i$, $i = 1, \ldots, \ell$, and let $M = \sum \mathbb{Z}\mathring{h}_i$, $\mathring{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} M$. Let $\mathring{A} = (a_{ij})_{i,j=1}^{\ell}$ be the Cartan matrix and $A = (a_{ij})_{i,j=0}^{\ell}$ the extended Cartan matrix of $\mathring{\mathfrak{g}}$. We may identify the affine Lie algebra $\mathfrak{g}'(A)$ with the Lie algebra $\tilde{\mathring{\mathfrak{g}}}_{\mathbb{C}[z,z^{-1}]}$ (see §1.3 for its definition) via the isomorphism determined by:

$$e_i \longmapsto 1 \otimes \mathring{e}_i,\ f_i \longmapsto 1 \otimes \mathring{f}_i,\ i = 1, \ldots, \ell;$$

$$e_0 \longmapsto z \otimes e_{-\theta},\ f_0 \longmapsto z^{-1} \otimes e_{\theta},$$

where θ is the highest root of $\mathring{\mathfrak{g}}$, and $e_{-\theta}$ and e_{θ} are root vectors normalized such that for $\mathring{h}_0 := [e_{\theta}, e_{-\theta}]$ one has: $\theta(\mathring{h}_0) = 2$. Since $\Omega^1_{\mathbb{C}[z,z^{-1}]} = \mathbb{C}\frac{dz}{z} + d\mathbb{C}[z,z^{-1}]$, this construction coincides with the customary one (see e.g. [14, Chapter 7]). In particular dim $\mathfrak{c} = 1$ and $\mathfrak{c} = \mathbb{C}c$, where $c = \sum_{i=0}^{\ell} a_i^{\vee} h_i$, $a_i^{\vee}$ are positive relatively prime integers. Thus, we have an exact sequence:

$$0 \longrightarrow \mathbb{C}c \longrightarrow \mathfrak{g}'(A) \xrightarrow{d\tau} \mathring{\mathfrak{g}}_{\mathbb{C}[z,z^{-1}]} \longrightarrow 0. \tag{20}$$

Taking $F(P(z))$ = constant term of $P(z) \in \mathbb{C}[z,z^{-1}]$, one easily sees that $\mathfrak{g}(A) = \hat{\mathring{\mathfrak{g}}}_{\mathbb{C}[z,z^{-1}],F}$ (see §1.3 for the definition).

As in the case of the affine Lie algebra theory, our objective is to describe the structure of the affine group $G(A)$ in terms of the "underlying" finite-dimensional group $G(\mathring{A})$.

Let G be connected simply connected algebraic group over $\mathbb{C}$ whose Lie algebra is $\mathring{\mathfrak{g}}$. We will denote by G_R the group of points of G over a commutative algebra R in a fixed finite-dimensional faithful G-module V.

First of all, we identify the group $G(\mathring{A})$ with the group $\mathring{G} :=$

$G_{\mathbb{C}}$. Using the notation of §2.3, we have injective homomorphisms $\mathring{\varphi}_i$: $SL_2(\mathbb{C}) \longrightarrow \mathring{G}$, the subgroups $\mathring{G}_i$, $\mathring{H}_i$, exp $t\mathring{e}_i$, exp $t\mathring{f}_i$ and elements $\mathring{\bar{r}}_i$ for $i = 1, \ldots, \ell$. Then $\mathring{H} := \prod_{i=1}^{\ell} \mathring{H}_i$ is the Cartan subgroup of $\mathring{G}$, the subgroup $\mathring{U}_+$ (resp. $\mathring{U}_-$), generated by the exp $t\mathring{e}_i$ (resp. exp $t\mathring{f}_i$), $t \in \mathbb{C}$, $i = 1, \ldots, n$, are maximal unipotent subgroups of $\mathring{G}$.

Let $\mathring{\bar{W}}$ (resp. $\mathring{N}$) be the subgroup of $\mathring{G}$ generated by the $\mathring{\bar{r}}_i$, $i = 1, \ldots, n$ (resp. by $\mathring{\bar{W}}$ and $\mathring{H}$). Then $\mathring{N}$ is the normalizer of $\mathring{H}$ in $\mathring{G}$ and $\mathring{N}/\mathring{T} = \mathring{W}$, the Weyl group of $\mathring{G}$. Let $\mathring{C}$ denote the center of $\mathring{G}$ (it is finite).

It is not difficult to see that the group associated to the integrable Lie algebra $\mathring{\mathfrak{g}}_{\mathbb{C}[z,z^{-1}]}$ is $\tilde{G} := G_{\mathbb{C}[z,z^{-1}]}$, and that associated to the exact sequence (20), we have an exact sequence of groups:

$$1 \longrightarrow \mathbb{C}^{\times} \xrightarrow{\ \mu\ } G(A) \xrightarrow{\ \tau\ } \tilde{G} \longrightarrow 1. \tag{21}$$

We have a canonical embedding $\mathring{G} \longrightarrow \tilde{G}$; the exact sequence (21) splits uniquely over $\mathring{G}$, hence we have a canonical embedding $\mathring{G} \longrightarrow G(A)$, so that $\varphi_i = \mathring{\varphi}_i$, $G_i = \mathring{G}_i$, $H_i = \mathring{H}_i$ and $\bar{r}_i = \mathring{\bar{r}}_i$ for $i = 1, \ldots, \ell$. Furthermore, associated to the integrable homomorphism $s\ell_2(\mathbb{C}) \longrightarrow \mathring{\mathfrak{g}}_{\mathbb{C}[z,z^{-1}]}$ defined by $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longmapsto -a\mathring{h}_0 + bz^{-1}e_{\theta} + cze_{-\theta}$, we have an injective homomorphism $SL_2(\mathbb{C}) \longrightarrow \tilde{G}$, which lifts uniquely to φ_0: $SL_2(\mathbb{C}) \longrightarrow G(A)$. The homomorphism μ is defined by $\mu(t) = \prod_{i=0}^{\ell} H_i(t)$, $t \in \mathbb{C}^{\times}$, and we have $C = \mu(\mathbb{C}^{\times}) \times \mathring{C}$.

Define an embedding $M \longrightarrow \tilde{G}$ by $\mathring{h}_i \longmapsto \mathring{H}_i(z)$, $i = 1, \ldots, \ell$. Then we get the subgroup $\mathring{\bar{W}} \ltimes M$ of $\tilde{G}$. Restricting τ to the subgroup $\bar{W}$ of G(A), we get from (21) the following exact sequence:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \bar{W} \longrightarrow \bar{\mathring{W}} \ltimes M \longrightarrow 1.$$

This sequence of course splits over $\bar{\mathring{W}}$, but over M it gives a non-split exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow L \longrightarrow M \longrightarrow 1.$$

It is not hard to show using the results of [7], that this central extension is determined by the property that for any preimages $\tilde{\alpha}$ and $\tilde{\beta}$ of $\alpha, \beta \in M$, one has

$$\tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1}\tilde{\beta}^{-1} = (-1)^{(\alpha \mid \beta)},$$

when the bilinear form $(\cdot \mid \cdot)$ is the W-invariant form on $\mathring{\mathfrak{h}}^*$ normalized by the condition $(\theta \mid \theta) = 2$. Of course, $W(A) \simeq W(\mathring{A}) \ltimes M$.

The invariant bilinear form on $\mathfrak{g}(A)$ (defined in §1.3) is non-degenerate and invariant under Ad G(A) and the adjoint action via $\tilde{G}$ is (see e.g. [19]):

$$(\mathrm{Ad}\ a(z))x(z) = a(z)x(z)a(z)^{-1} + \mathrm{Res}\ \mathrm{tr}\ \frac{da(z)}{dz}\ x(z)a(z)^{-1}.$$

Put $\tilde{U}_+ = \{a(z) \in G_{\mathbb{C}[z]} \mid a(0) \in \mathring{U}_+\}$, $\tilde{U}_- = \{a(z^{-1}) \in G_{\mathbb{C}[z^{-1}]} \mid a(\infty) \in \mathring{U}_-\} \subset \tilde{G}$. The exact sequence (21) splits over $\tilde{U}_+$ and $\tilde{U}_-$, but not uniquely. The subgroups U_+ and U_- of G(A) are the (unique) sections which fix $v_\Lambda \in L(\Lambda)$ for all $\Lambda \in P_+$. Put $\tilde{U}^k = \{a(z^{\pm 1}) \in G_{\mathbb{C}[z^{\pm 1}]} \mid a(z^{\pm 1})-I_V \in z^{\pm k}G_{\mathbb{C}[z^{\pm 1}]}\}$, and let U^k be the preimage of $\tilde{U}^k$ in $U_{\pm}$.

The Bruhat and Birkhoff decompositions (2) and (3) give the following decompositions:

$$G_{\mathbb{C}[z,z^{-1}]} = G_{\mathbb{C}[z^{\pm 1}]}\ M\ G_{\mathbb{C}[z]},$$

various versions of which play an important role in geometry and analysis (see e.g. [9], [10]).

Among the integrable highest weight modules the basic module $L(\Lambda_0)$ is especially important. It is realized in [16] in the space of polynomials in infinitely many indeterminates. The main idea behind the work of the Kyoto school on the KdV-type hierarchies is that the generalized Plücker relations can be written in this realization in terms of Hirota bilinear equations, which are PDE of certain special form which include many important PDE of mathematical physics; the variety V_{Λ_0} thus becomes the totality of polynomial solutions of these PDE (see [14] for a discussion of these results). A somewhat different approach is discussed in this volume by A. Pressley [27].

Of course, the matrix coefficients of the G(A)-module $V_{\mathbb{C}[z,z^{-1}]}$ are regular functions. None of them, except constants, are strongly regular functions, however, since by Theorem 2.4, a strongly regular function f, such that $f(cg) = f(g)$ for all $c \in C$ and $g \in G(A)$, is constant. Notice that f is a strongly regular function iff for every $g \in G(A)$ there exists $k > 0$ such that $f(u_-gu_+) = f(g)$ for any $u_\pm \in U^k$.

The topology on G(A) is the unique topology such that (20) is an exact sequence of topological groups, $\mathbb{C}^\times$ carries the metric topology and $\tilde{G}$ the topology induced by the box topology on $\mathbb{C}[z,z^{-1}]$.

Now we turn to the discussion of the unitary form K(A) of G(A). Let $\mathring{\omega}_0$ be the involution of the group $\mathring{G}$ which leaves the $\mathring{G}_i$ invariant and induces on it the standard involution of $SL_2(\mathbb{C})$: $a \longmapsto {}^t\bar{a}^{-1}$. The fixed point set of $\mathring{\omega}_0$ is a compact form of $\mathring{G}$ denoted by $\mathring{K}$. The involution $\mathring{\omega}_0$ lifts to an involution $\tilde{\omega}_0$ of $\tilde{G}$ via the antilinear involution of the algebra $\mathbb{C}[z,z^{-1}]$ which maps z to z^{-1}. In turn, $\tilde{\omega}_0$ lifts (uniquely) to the involution ω_0 of G(A) by requiring $\omega_0(\mu(t)) = \mu(\bar{t}^{-1})$, $t \in \mathbb{C}^\times$.

Note that $\tilde{G}$ may be viewed as the group of polynomial maps $\mathbb{C}^\times \longrightarrow \mathring{G}$. The fixed point set of $\tilde{\omega}_0$ on $\tilde{G}$ are those maps for which the image of the unit circle is contained in $\mathring{K}$; these are called polynomial loops on $\overset{\circ}{K}$. We denote the group of polynomial loops $S^1 \longrightarrow \overset{\circ}{K}$ by $\tilde{K}$. Exact sequence (20) gives, by restriction, the following exact sequence:

$$1 \longrightarrow S^1 \xrightarrow{\mu} K(A) \xrightarrow{\tau} \tilde{K} \longrightarrow 1.$$

Identifying $\overset{\circ}{K}$ with the subgroup of constant loops of $\tilde{K}$ and denoting by $\Omega(\overset{\circ}{K})$ the subgroup of based loops (i.e. 1 goes to 1), we have $\tilde{K} = \overset{\circ}{K} \ltimes \Omega(\overset{\circ}{K})$.

Consider the map $K(A) \longrightarrow \mathbb{P}V_{\Lambda_0}$ defined by $k \longmapsto \pi_{\Lambda_0}(k)v_{\Lambda_0}$. It is not difficult to see that $\overset{\circ}{K}$ is the stabilizer of v_{Λ_0} and hence the above map induces a homeomorphism $\Omega(\overset{\circ}{K}) \xrightarrow{\sim} \mathbb{P}V_{\Lambda_0}$. It is a well-known fact (see [8]) that the space of all continuous based loops on a compact Lie group $\overset{\circ}{K}$ is homotopically equivalent to the space of poynomial loops $\Omega(\overset{\circ}{K})$. Thus, classical results on loop space cohomology [3] fall into the general framework of §2.6. Moreover using that $\pi_i(\Omega(X)) \simeq \pi_{i+1}(X)$, we deduce from Corollary 2.6(d) and (e) that for the affine Weyl group W one has [3]:

$$W(q) = \overset{\circ}{W}(q) \prod_{i=1}^{\ell} (1-q^{2m_i})^{-1},$$

where $m_1 + 1 < m_2 + 1 \leqslant \dots < m_\ell + 1$ are the degrees of the basic W-invariants, and $H^*(\Omega\overset{\circ}{K},\mathbb{Q})$ is a polynomial algebra on generators of degrees $2m_1, \dots, 2m_\ell$.

Put $\Omega(\overset{\circ}{K})\langle 2\rangle := \tau^{-1}(\Omega(\overset{\circ}{K}))$. This is a standard notation of the 2-connected cover of $\Omega(\overset{\circ}{K})$. This means that the map $\tau\colon \Omega(\overset{\circ}{K})\langle 2\rangle \longrightarrow \Omega(\overset{\circ}{K})$ kills the second homotopy group (which is $\mathbb{Z}$) and induces isomorphism of higher homotopy groups (this property of τ can be easily checked). Thus, we have

$$K(A) = \overset{\circ}{K} \ltimes \Omega(\overset{\circ}{K})\langle 2\rangle.$$

Since the cohomology of $\overset{\circ}{K}$ is by now well understood [15], it remains (and is of independent interest) to compute the cohomology of $\Omega(\overset{\circ}{K})\langle 2\rangle$. Theorem 2.6 leads to the following result.

<u>Theorem</u> [21]. Let $\overset{\circ}{K}$ be a connected simply connected simple compact Lie group, and let $m_1 + 1, \dots, m_\ell + 1$ be the degrees of the basic invariants of its Weyl group. Then

(a) $H^*(\Omega(\overset{\circ}{K})\langle 2\rangle, \mathbb{Q})$ is a polynomial algebra on generators of degrees $2m_2, \dots, 2m_\ell$.

(b) The Poincaré polynomial of $H^*(\Omega(\overset{\circ}{K})\langle 2\rangle, \mathbb{F})$, where $\mathbb{F}$ is a field of characteristic $p > 0$, is

$$(1 + q^{2p^a-1})(1 - q^{2p^a})^{-1} \prod_{i=2}^{\ell} (1 - q^{2m_i})^{-1}.$$

Here a is the minimal positive integer such that $\Lambda_0^{p^a} \in I_{\mathbb{F}}$. One has: $a = 1$ if $p > m_\ell$. The number a for $p \leqslant m_\ell$ has been computed recently (at my request) by A. Kono using topological arguments.

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