Seminaire de Math. Sup., 102
Systèmes dynamiques non linéaires,
Les Presses de l'Université de Montreal, 1986.

Lectures on the infinite wedge-representation and the MKP hierarchy

V.G. KAC and D.H. PETERSON

These are the notes on a series of lectures given by the first author in August 1985 at the summer school on completely integrable systems in Montreal. The goal is to demonstrate in a simple example, the group GL_∞, the main concepts and constructions of the theory of affine Kac-Moody algebras, and then to link this to the study of the "universal" system of soliton equations, the KP hierarchy. This link was discovered by Sato [12] and developed, making use of the spinor formalism, by Date, Jimbo, Kashiwara and Miwa [1], [2], [4]. In these notes, we use the wedge formalism developed in [7], which we find more elegant and transparent.

In the first part, we discuss two constructions of the fundamental highest weight representations of the group GL_∞, the wedge construction (section 1) and the vertex construction (section 3). The general theory of highest weight representations of GL_∞ (section 2) together with the wedge formalism (section 3) allows one to explicitly give isomorphisms between these constructions (section 4), a kind of boson-fermion correspondence (Theorems 3.1 and 4.1).

In the second part, the results of the first part are applied to give several (six) equivalent definitions of the KP and the modified KP hierarchies (Theorem 5.1). Furthermore, we construct polynomial and soliton solutions of these hierarchies (sections 6 and 7).

In section 8 we recall Sato's parametrization of the solutions of the KP hierarchy by an infinite Grassmann variety and then give a simple geometric interpretation of the MKP-hierarchy (Proposition 8.1) which allows one to parametrize its solutions by an infinite flag variety.

Following the idea of [1], [2], we study in section 9 the reduced MKP hierarchy, using the wedge representation of the "big" group A_∞ discussed in section 8. The main result of section 9 is a very explicit description of all polynomial solutions of the KdV and MKdV hierarchies (Theorem 9.1).

Recall that the key observation of the work of the Kyoto school is that the KP hierarchy describes the GL_∞-orbit of the highest weight vector of a fundamental representation of GL_∞. A new point of this, on the most part expository paper, is the remark that the MKP hierarchy describes the GL_∞-orbit of the sum of highest weight vectors of fundamental representations of GL_∞.

The first author wishes to thank T. Miwa and G. Wilson for valuable discussions. This work was supported in part by the NSF grant DMS-8508953.

1. Representation theoretical interpretation of the Dirac theory of the positron

Let us start with a (somewhat lengthy) quotation from Dirac's book [3]:

... the wave equation for the electron admits of twice as many solutions as it ought to, half of them referring to states with negative values for the kinetic energy ...

... we are led to infer that the negative-energy solutions ... refer to the motion of a new kind of particle having the mass of an electron and the opposite charge. Such particles have been observed experimentally and are called positrons.

... We assume that nearly all the negative-energy states are occupied, with one electron in each state in accordance with the exclusion principle of Pauli. An unoccupied negative-energy state will now appear as something with a positive energy, since to make it disappear, i.e., to fill it up, we should have to add to it an electron with negative energy. We assume that these unoccupied negative-energy states are the positrons.

These assumptions require there to be a distribution of electrons of infinite density everywhere in the world. A perfect vacuum is a region where all the states of positive energy are unoccupied and all those of negative energy are occupied. ... the infinite distribution of negative-energy electrons does not contribute to the electric field.

... there will be a contribution $-e$ for each occupied state of positive energy and a contribution $+e$ for each unoccupied state of negative energy.

The exclusion principle will operate to prevent a positive-energy electron ordinarily from making transitions to states of negative energy. It will still be possible, however, for such an electron to drop into an unoccupied state of negative energy. In this case we should have an electron and positron disappearing simultaneously, their energy being emitted in the form of radiation. The converse process would consist in the creation of an electron and a positron from electromagnetic radiation.

This positron theory of Dirac's may be interpreted as follows.

Let $V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j$ be an (infinite-dimensional) complex vector space with a fixed basis $\{v_j\}_{j\in\mathbb{Z}}$. Each v_j is thought of as a state of an electron of energy $j (\in \mathbb{Z})$. Introduce the *infinite wedge space* $F^{(0)} = \Lambda^{\infty}_{(0)}V$ to be the vector space with the basis consisting of *ensembles* $v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \cdots$ such that

(1.1) $\quad i_0 > i_{-1} > \ldots$ (Pauli exclusion principle), (1.2) $i_k = k$ for $k \ll 0$ (all but a finite number of negative energy states are occupied).

The ensemble $\psi = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \cdots$ is the *perfect vacuum*. We define (following Dirac) the *energy* of an ensemble $v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ to be

(1.3) $$\sum_s (i_s > 0 \text{ which occur}) - \sum_s (i_s \leq 0 \text{ which do not occur}).$$

Denoting by F_k the linear span of all ensembles of energy k, we get the vector space decomposition

(1.4) $$F^{(0)} = \bigoplus_{k\geq 0} F_k .$$

The corresponding q-*dimension* (partition function) $\dim_q F^{(0)} := \sum_{k\geq 0} (\dim F_k) q^k$ is easily seen to be (the j-th summand corresponding to the set of all ensembles with j "holes")

(1.5) $$\dim_q F^{(0)} = 1 + \sum_{j\geq 1} \frac{q^{j^2}}{(1-q)^2 \ldots (1-q^j)^2} .$$

The map $\sum_j c_j v_j \to (c_j)_{j\in\mathbb{Z}}$ identifies V with the space of column vectors whose coordinates are indexed by $\mathbb{Z}$, all but a finite number of them being 0.

Introduce the (infinite complex matrix) group

$GL_\infty = \{A = (a_{ij})_{i,j\in\mathbb{Z}} \mid A$ is invertible and all but a finite number of the $a_{ij} - \delta_{ij}$ are $0\}$.

Its Lie algebra is

$g\ell_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid$ all but a finite number of the a_{ij} are $0\}$,

with the usual bracket.

Both the group GL_∞ and its Lie algebra $g\ell_\infty$ operate on V via the multiplication of a matrix and a column vector. Namely,

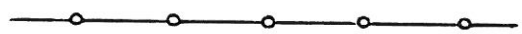
$$E_{ij}(v_j) = v_i ,$$

where E_{ij} denotes the matrix with the (i,j) entry 1 and all the rest 0.

REMARK 1.1. The Lie algebra $g\ell_\infty$ (or rather $s\ell_\infty$) may be thought of as a Kac-Moody algebra (of infinite rank) on Chevalley generators

$$e_i = E_{i,i+1},\quad f_i = E_{i+1,i},\quad h_i = E_{i,i} - E_{i+1,i+1},\quad i \in \mathbb{Z}\ ,$$

its Dynkin diagram being the infinite-in-both-directions chain

...  ...

Define a representation R_0^F of the group GL_∞ and a representation r_0^F of the Lie algebra $g\ell_\infty$ on the space $F^{(0)}$ by:

(1.6) $$R_0^F(A)(v_{i_0} \wedge v_{i_{-1}} \wedge \dots) = Av_{i_0} \wedge Av_{i_{-1}} \wedge \dots\ ,\quad A \in GL_\infty\ ,$$

(1.7) $$r_0^F(A)(v_{i_0} \wedge v_{i_{-1}} \wedge \dots) = Av_{i_0} \wedge v_{i_{-1}} \wedge \dots + v_{i_0} \wedge Av_{i_{-1}} \wedge v_{i_{-2}} \wedge \dots + \\ + \dots\ ,\quad A \in g\ell_\infty\ .$$

In these formulas we assume multilinearity (i.e., $\dots \wedge (\alpha u + \beta v) \wedge \dots = = \alpha(\dots \wedge u \wedge \dots) + \beta(\dots \wedge v \wedge \dots)$) and anticommutativity (i.e., $\dots \wedge u \wedge \dots \wedge v \wedge \dots = - \dots \wedge v \wedge \dots \wedge u \wedge \dots$) to be satisfied. In particular, we have the Pauli exclusion principle: $\dots \wedge u \wedge \dots \wedge u \wedge \dots = 0$. It is easy to see that R_0^F and r_0^F are irreducible representations of GL_∞ and $g\ell_\infty$ on the space $F^{(0)}$, and that they correspond to each other, i.e.

(1.8) $$\exp r_0^F(A) = R_0^F(\exp A),\quad A \in g\ell_\infty\ .$$

Note that the action of $r_0^F(E_{ij})$ on an ensemble exactly corresponds to the effect of the electromagnetic radiation.

Introduce the *principal gradation* $g\ell_\infty = \bigoplus_{j\in\mathbb{Z}} g_j$ by putting

(1.9) $$\deg E_{ij} = j - i\ ,$$

so that $[g_i, g_j] = g_{i+j}$. Then we have

(1.10) $r_0^F(g_i)F_j \subset F_{j-i}$ and $F_0 = \mathbb{C}\psi$ (here we assume that $F_j = 0$ for $j < 0$).

It is easy to see that (1.10) determines uniquely the decomposition (1.4) (since, by the irreducibility of r_0^F we have

$$F_j = \sum_{i_1+\ldots+i_k=-j} r_0^F(g_{i_1})\ldots r_0^F(g_{i_k})\psi) .$$

Thus we obtain a representation theoretical interpretation of Dirac's definition of the energy of a system of electrons and positrons.

In order to perform calculations it is sometimes more convenient to deal with a bigger group:

$\overline{GL}_\infty = \{A = (a_{ij})_{i,j\in\mathbb{Z}}$ | A is invertible and all but a finite number of the $a_{ij} - \delta_{ij}$ with $i \geq j$ are $0\}$.

Its Lie algebra is

$\overline{g\ell}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}}$ | all but a finite number of the a_{ij} with $i \geq j$ are $0\}$.

Both $\overline{GL}_\infty$ and $\overline{g\ell}_\infty$ act on a completion $\overline{V}$ of the space V, where

$$\overline{V} = \{\sum_j c_j v_j \mid c_j = 0 \text{ for } j \gg 0\} .$$

In contrast, both R_0^F and r_0^F extend to representation of $\overline{GL}_\infty$ and $\overline{g\ell}_\infty$ on the same space $F^{(0)} = \Lambda^\infty_{(0)}V = \Lambda^\infty_{(0)}\overline{V}$. Note also that the exponential map is defined on the whole $\overline{g\ell}_\infty$ (with image in $\overline{GL}_\infty$) and formula (1.8) holds for $A \in \overline{g\ell}_\infty$.

In section 3 and further on we shall deal with a yet bigger group and Lie algebra: let

$\overline{a}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}}$ | for each k the number of non-zero a_{ij} with $i \leq k$ and $j \geq k$ is finite$\}$.

It is clear that $\overline{a}_\infty$ still acts on $\overline{V}$ (by matrix multiplication) and is an algebra of endomorphism of the vector space $\overline{V}$. The (associative) algebra $\overline{a}_\infty$ has the following meaning. Let $\overline{V}^{(n)} = \{\sum_j c_j v_j \in \overline{V} | \ c_j = 0$ for $j > n\}$; declaring the $\overline{V}^{(n)}$, $n \in \mathbb{Z}$, a fundamental system of neighborhoods of zero, $\overline{V}$ becomes a topological vector space. Then $\overline{a}_\infty$ is the algebra of all continuous endomorphisms of $\overline{V}$.

We denote by $\overline{A}_\infty$ the group of invertible elements of the (associative) algebra $\overline{a}_\infty$. The "corresponding" Lie algebra is $\overline{a}_\infty$ with the usual bracket. Note that the exponential map $\overline{a}_\infty \to \overline{A}_\infty$ is not everywhere defined. Note also that the representations r_0^F and R_0^F do not extend to representations of $\overline{a}_\infty$ and $\overline{A}_\infty$. We shall tackle this problem in sections 3 and 8, respectively.

The construction of the representation R_0^F of GL_∞ and r_0^F of $g\ell_\infty$ can be generalized as follows. Fix an integer m. Let $F^{(m)} = \Lambda^\infty_{(m)} V = \Lambda^\infty_{(m)} \overline{V}$ be the vector space with the basis consisting of expressions of the form $v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots$ such that $i_m > i_{m-1} > \ldots$ and $i_k = k$ for $k \ll 0$. Let R_m^F (resp. r_m^F) denote the representation of GL_∞ (resp. of $g\ell_\infty$) on $F^{(m)}$ defined by (1.6) (resp. (1.7)). Put

$$\psi_m = v_m \wedge v_{m-1} \wedge \ldots .$$

As before, the space $F^{(m)}$ has a unique vector space decomposition $F^{(m)} = \bigoplus_{j\geq 0} F_j^{(m)}$ satisfying the analogue of (1.10). It is defined by a formula analogous to (1.3) (in which 0 is replaced by m). Note that $\dim_q F^{(m)} = \dim_q F^{(0)}$ and that, moreover, denoting by $\nu_s \in \overline{A}_\infty$ the shift $v_j \mapsto v_{j-s}$, $j \in \mathbb{Z}$, we have:

$$\nu_s r_s^F \nu_s^{-1} = r_0^F, \quad \nu_s R_s^F \nu_s^{-1} = R_0^F . \tag{1.11}$$

Note that we have the following formula for the representation R_m^F of $A \in \overline{GL}_\infty$ on $F^{(m)}$:

$$(1.12)\quad R_m^F(A)(v_{j_m}\wedge v_{j_{m-1}}\wedge\ldots) = \sum_{i_m>i_{m-1}>\ldots}\left(\det A_{i_m,i_{m-1},\ldots}^{j_m,j_{m-1},\ldots}\right) v_{i_m}\wedge v_{i_{m-1}}\wedge\ldots$$

where $A_{i_m,i_{m-1},\ldots}^{j_m,j_{m-1},\ldots}$ denotes the matrix located on the intersection of the rows $i_m,i_{m-1},\ldots$ and columns $j_m,j_{m-1},\ldots$ of the matrix A. Note also that the matrices that occur in (1.12) have only a finite number of non-zero entries under the diagonal and a finite number of entries not equal to 1 on the diagonal; the determinant of such matrices is defined in an obvious way.

The most convenient space to work with is the *full infinite wedge space*

$$F = \wedge^\infty V = \wedge^\infty \overline{V} = \bigoplus_{m\in\mathbb{Z}} F^{(m)} .$$

Ensembles of the form $v_{i_1}\wedge v_{i_2}\wedge\ldots$, with $i_1 > i_2 > \ldots$ and a finite number of "holes", form a basis of F. We have representations $r^F = \bigoplus_m r_m^F$ of $g\ell_\infty$ and $R^F = \bigoplus_m r_m^F$ of GL_∞ on F.

2. Highest weight representations of GL_∞

Given a collection of numbers $\lambda = \{\lambda_i\}_{i\in\mathbb{Z}}$, called a *highest weight,* we define the *highest weight representation* π_λ of the Lie algebra $g\ell_\infty$ as an irreducible representation on a vector space $L(\lambda)$ which admits a non-zero vector v_λ called a *highest weight* vector, such that

$$(2.1)\qquad \pi_\lambda(E_{ij})v_\lambda = 0 \ \text{ for } i<j;\quad \pi_\lambda(E_{ii})v_\lambda = \lambda_i v_\lambda .$$

It is a simple and standard fact that for every λ, the triple $(L(\lambda),\pi_\lambda,v_\lambda)$ satisfying (2.1) exists and is unique up to a unique isomorphism; also, the properties (2.1) determine v_λ up to a constant factor (see e.g. [5, Chapter 9]).

For $k\in\mathbb{Z}$, let $L(\lambda)_k$ denote the linear span of all elements of $L(\lambda)$ of the form $\pi_\lambda(E_{i_1,j_1})\ldots\pi_\lambda(E_{i_s,j_s})v$ with $(i_1+\ldots+i_s)-(j_1+\ldots+j_s) = k$.

Then we have the decomposition

(2.2) $$L(\lambda) = \bigoplus_{k\geq 0} L(\lambda)_k ,$$

called the *principal gradation* of $L(\lambda)$. Note that in the definition of $L(\lambda)_k$ we could assume that $j_t < i_t$ for all $1 \leq t \leq s$ (by the PBW theorem); it follows that $L(\lambda)_k = 0$ for $k < 0$ and $L(\lambda)_0 = \mathbb{C}v_\lambda$. It is easy to see that $\dim L(\lambda)_k < \infty$ for all k if all but a finite number of the $\lambda_i - \lambda_{i+1}$ are 0. Assuming

(2.3) $$\lambda_i = 0 \text{ for } i \gg 0 \text{ and } \lambda_i = \lambda_{i+1} \text{ for } i \ll 0 ,$$

we may write the formal power series

$$\dim_q L(\lambda) = \sum_{k\geq 0} (\dim L(\lambda)_k) q^k ,$$

called the *q-dimension* (or the partition function) of $L(\lambda)$.

The most interesting of the representations π_λ are those which can be exponentiated to GL_∞. Necessary and sufficient conditions are

(2.4) $$\lambda_i \in \mathbb{Z} \text{ and } \lambda_i \geq \lambda_{i+1} (i \in \mathbb{Z}) .$$

(These are the conditions for $L(\lambda)$ to decompose into a direct sum of finite-dimensional representations with respect to $\mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i \simeq s\ell_2(\mathbb{C})$.)

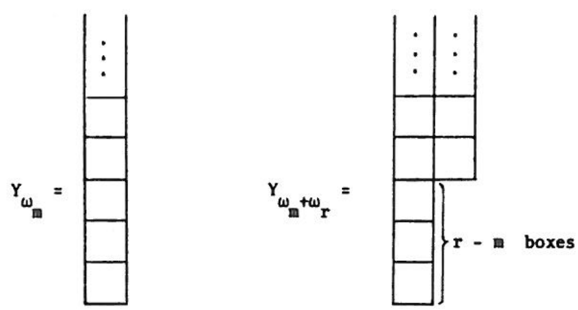
Furthermore, provided that the λ_i are real, $L(\lambda)$ carries a unique Hermitian form $\langle \, , \, \rangle$ satisfying (* denotes the adjoint operator with respect to $\langle \, , \, \rangle$):

(2.5) $$\langle v_\lambda, v_\lambda \rangle = 1 \text{ and } (\pi_\lambda(A))^* = \pi_\lambda({}^t\bar{A}),\ A \in g\ell_\infty ,$$

called the *contravariant Hermitian form* (see e.g. [5, Chapter 11]). Using the fact that $g\ell_\infty$ is an inductive limit of the finite-dimensional $g\ell_N$, one deduces that the contravariant Hermitian form on $L(\lambda)$ is positive definite if and only if (2.4) holds.

We denote by P_+ the set of all λ satisfying (2.3) and (2.4). Given $m \in \mathbb{Z}$, define the fundamental weight $\omega_m = \{\lambda_i = 1 \text{ for } i \leq m, \lambda_i = 0 \text{ for } i > m\}$. Then $P_+ = \{\sum_i k_i\omega_i \mid k_i$ are non-negative integers and all but a finite number of them are $0\}$.

A $\lambda \in P_+$ may be represented by an infinite Young diagram Y_λ; we have $Y_\lambda = Y_\mu$ if μ can be obtained from λ by a shift ν_s of indices, so that $\nu_s\pi_\lambda\nu_{-s} = \pi_\mu$ (cf. (1.11)). For example, we have the following pictures $(m \leq r)$:



By the hook of a box j of Y we mean, as usual, all the boxes to the right of j on the same row and all the boxes downwards from j in the same column (including j itself); the hook length h_j of the box j is the total number of boxes in the hook.

Now we can write the formula for the q-dimension of the representation π_λ with $\lambda \in P_+$ [5]:

(2.6) $$\dim_q L(\lambda) = \prod_{j \in Y_\lambda} (1 - q^{h_j})^{-1}$$ or $\dim_q L(\omega_{s_1}+\dots+\omega_{s_n}) = \frac{\prod_{1\leq i<j\leq n}(1-q^{s_i-s_j+j-i})}{\varphi(q)^n}$

(the product is taken over all boxes of Y_λ).

To prove (2.6) recall that $\dim_q L(\lambda)$ has a product decomposition (see [5, formula (10.10.1)]), which in our case takes the form:

(2.7) $$\dim_q L(\lambda) = \prod_{\substack{i,j \in \mathbb{Z} \\ i<j}} ((1 - q^{\lambda_i-\lambda_j+j-i})/(1 - q^{j-i})) ,$$

Formula (2.6) is a nicer form of (2.7).

The following basic example is a link between section 1 and section 2. It is clear that the representation r_0^F of $g\ell_\infty$ on the space F is the highest weight representation with highest weight ω_0, the highest weight vector being the perfect vacuum ψ. Thus the representation r_0^F is equivalent to the representation π_{ω_0} called the *basic representation* of $g\ell_\infty$. More generally, the representation r_m^F of $g\ell_\infty$ on $F^{(m)}$ is equivalent to π_{ω_m}, called the *fundamental representation* of $g\ell_\infty$, ψ_m being the highest weight vector. Thus, the representation r^F of $g\ell_\infty$ on F is the direct sum of all fundamental representations of $g\ell_\infty$.

Furthermore, it is easy to check that the ensembles form an orthonormal basis of $F^{(m)}$ with respect to the contravariant Hermitian form.

Finally, using (2.6) we get a very simple formula for the q-dimension of the fundamental representations of $g\ell_\infty$:

(2.8) $$\dim_q L(\omega_m) = \dim_q F^{(m)} = \frac{1}{\varphi(q)} ,$$

where $\varphi(q) = \prod_{j\geq 1} (1 - q^j)$.

Comparing (1.5) and (2.8), we get Euler's identity:

$$\frac{1}{\varphi(q)} = 1 + \sum_{k\geq 1} \frac{q^{k^2}}{(1 - q)^2 \ldots (1 - q^k)^2} .$$

Another important example is the following:

(2.9) $$\dim_q L(\omega_m + \omega_n) = \frac{1 - q^{n-m+1}}{\varphi(q)^2} ,$$

where $n,m \in \mathbb{Z}$ and $n \geq m$.

3. Vertex realization of fundamental representations of $g\ell_\infty$

We return to the wedge representation r_0^F of $g\ell_\infty$ on the space $F^{(0)} = \Lambda_{(0)}^\infty V$ constructed in section 1. It was remarked that r_0^F can be extended in the obvious way to a bigger Lie algebra $\overline{g\ell}_\infty$ (which consists of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with a finite number of non-zero entries on and under the diagonal). However, if we try to extend the representation r_0^F to the Lie algebra $\overline{a}_\infty$, we encounter an "anomaly", e.g. in $r_0^F(\mathrm{diag}(\lambda_i)_{i\in\mathbb{Z}})\psi = (\lambda_0 + \lambda_{-1} + \ldots)\psi$, the right-hand side is in general a divergent series. To remove this anomaly, we correct the representation r_0^F as follows. Put:

$$\hat{r}_0^F(E_{ij}) = r_0^F(E_{ij}), \quad \text{if} \quad i \neq j \quad \text{or} \quad i = j > 0 ,$$

$$\hat{r}_0^F(E_{ii}) = r_0^F(E_{ii}) - I, \quad \text{if} \quad i \leq 0 .$$

We correct the representations r_m^F in exactly the same way.

Extending $\hat{r}_0^F$ (resp. $\hat{r}_m^F$) by linearity, we get a projective representation of the Lie algebra $\overline{a}_\infty$. Equivalently, introduce the central extension $a_\infty = \overline{a}_\infty \oplus \mathbb{C}c$ with the center $\mathbb{C}c$ and the bracket

$$[a,b] = ab - ba + \alpha(a,b)c, \quad a,b \in \overline{a}_\infty ,$$

where the 2-cocycle α is defined by:

$$\alpha(E_{ij},E_{ji}) = -\alpha(E_{ji},E_{ij}) = 1, \quad \text{if} \quad i \leq 0, \quad j \geq 1 , \tag{3.1}$$

$$\alpha(E_{ij},E_{mn}) = 0 \quad \text{in all other cases.}$$

Then, extending $\hat{r}_0^F$ to a_∞ by $\hat{r}_0^F(c) = I$, we obtain a linear representation of the Lie algebra a_∞ on the space $F^{(0)}$ (resp. $F^{(m)}$), which we again denote by $\hat{r}_0^F$ (resp. $\hat{r}_m^F$). Of course, we have representations $r^F = \bigoplus_m r_m^F$ and $\hat{r}^F = \bigoplus_m \hat{r}_m^F$ of $\overline{a}_\infty$ and a_∞ on F.

Now, consider the matrix of the shift by k:

$$\Lambda_k = \sum_{i\in\mathbb{Z}} E_{i,i+k} . \tag{3.2}$$

The advantage in introducing a_∞ is that $\Lambda_k \in \bar{a}_\infty \subset a_\infty$. One easily computes in a_∞:

$$[\Lambda_k,\Lambda_n] = k\delta_{k,-n}c . \tag{3.3}$$

Thus, the Lie algebra $s = \sum_{k\neq 0} \mathbb{C}\Lambda_k + \mathbb{C}c \subset a_\infty$ is an infinite-dimensional Heisenberg Lie algebra. It is called the *principal subalgebra* of a_∞.

We restrict the representation $\hat{r}^F_m$ of a_∞ on $F^{(m)}$ to s. We have: $\hat{r}^F_m(\Lambda_k)\psi_m = 0$ for $k > 0$, and all the elements $\hat{r}^F_m(\Lambda_{-k_1})\dots\hat{r}^F_m(\Lambda_{-k_s})\psi_m$ with $k_1 \geq \dots \geq k_s > 0$ are linearly independent (this follows easily from (3.3)). Note also that Λ_k has degree k in the principal gradation. It follows from (2.8) that the elements $\hat{r}^F_m(\Lambda_{-k_1})\dots\hat{r}^F_m(\Lambda_{-k_s})\psi_m$ with $k_1 + \dots + k_s = k$ form a basis of $F^{(m)}_k$.

Thus, by uniqueness, viewed as a representation of the Lie algebra s, the space $F^{(m)}$ may be identified with the space of the canonical commutation relations representation:

$$\sigma_m: F^{(m)} \simeq B^{(m)} = \mathbb{C}[x_1,x_2,\dots] , \tag{3.4}$$

so that the vacuum vectors correspond:

$$\sigma_m(\psi_m) = 1 , \tag{3.5}$$

and the transported representation, which we denote by $\hat{r}^B_m$, of s on the space $B^{(m)}$ is given by the annihilation and creation operators $(k > 0)$:

$$\hat{r}^B_m(\Lambda_k) = \frac{\partial}{\partial x_k} , \quad \hat{r}^B_m(\Lambda_{-k}) = kx_k, \quad \hat{r}^B_m(c) = I . \tag{3.6}$$

The map σ_m which identifies the (irreducible) representations of the Lie algebra s in the spaces $F^{(m)}$ and $B^{(m)}$ may be viewed as a kind of boson-fermion correspondence. This map will be described explicitly in the next section. Here we will show what the principal gradation and the contravariant Hermitian form transported by σ_m look like, and how to extend the representation $\hat{r}_m^B$ of $\mathfrak{s}$ on $B^{(m)}$ to the whole a_∞.

It is clear that the principal gradation $B^{(m)} = \underset{k\geq 0}{\oplus} B_k$ is defined by

(3.7) $$\deg x_j = j$$

(and the degree of a product is the sum of the degrees).

Furthermore, the contravariant Hermitian form $<\ ,\ >$ must satisfy the following conditions (cf. (2.5)):

(3.8) $$<1,1> = 1 \quad \text{and} \quad (\hat{r}_m^B(\Lambda_k))^* = \hat{r}_m^B(\Lambda_{-k}) \ .$$

One checks immediately that the Hermitian form (here $\overline{Q}$ means taking complex conjugates of the coefficients of Q):

(3.9) $$<P,Q> = P(\frac{\partial}{\partial x_1}, \tfrac{1}{2}\frac{\partial}{\partial x_2}, \ldots)\overline{Q}(x)\big|_{x=0}$$

satisfies (3.8). This is therefore (by uniqueness) the Hermitian form transported from $F^{(m)}$. Note that monomials form an orthogonal basis of $B^{(m)}$ and that the square of the length of a monomial is:

(3.10) $$\left\langle x_1^{k_1}\ldots x_n^{k_n}, x_1^{k_1}\ldots x_n^{k_n}\right\rangle = \frac{k_1!\ldots k_n!}{1^{k_1}\ldots n^{k_n}} \ .$$

Put $B = \mathbb{C}[x_1,x_2,\ldots; z,z^{-1}]$. Identifying $B^{(m)}$ with $z^m\mathbb{C}[x_1,x_2,\ldots]$ by $P \leftrightarrow z^mP$, we have:

$$B = \underset{m\in\mathbb{Z}}{\oplus} B^{(m)} \ .$$

Let $r^B = \underset{m\in\mathbb{Z}}{\oplus} r_m^B$ (resp. $\hat{r}^B = \underset{m\in\mathbb{Z}}{\oplus} \hat{r}_m^B$) denote the representation of $g\ell_\infty$

(resp. $\mathfrak{a}_\infty$) on B. The representations r^F and r^B (resp. $\hat{r}^F$ and $\hat{r}^B$) are equivalent via the map $\sigma = \bigoplus_m \sigma_m$. Recall that $\sigma(\psi_m) = 1_m := z^m$.

An element $v \in \overline{V}$ gives rise to a *wedging operator* $\hat{v}$ on F defined by

$$\hat{v}(v_{i_1} \wedge v_{i_2} \wedge \dots) = v \wedge v_{i_1} \wedge v_{i_2} \wedge \dots ,$$

where the right-hand side is interpreted in the obvious way. Let $\overline{V}^*$ denote the space of linear functions f on $\overline{V}$ such that f = 0 on $\overline{V}^{(i)}$ for $i \ll 0$. An element $f \in \overline{V}^*$ gives rise to a *contracting operator* $\check{f}$ defined by:

$$\check{f}(v_{i_1} \wedge v_{i_2} \wedge \dots) = f(v_{i_1})v_{i_2} \wedge v_{i_3} \wedge \dots - f(v_{i_2})v_{i_1} \wedge v_{i_3} \wedge \dots + \dots .$$

We have: $\hat{v}(F^{(m)}) \subset F^{(m+1)}$, $\check{f}(F^{(m)}) \subset F^{(m-1)}$.

Furthermore, we have the following simple formulas:

(3.11) $$\hat{u}\hat{v} + \hat{v}\hat{u} = 0 \text{ for } u,v \in \overline{V}; \quad \check{f}\check{g} + \check{g}\check{f} = 0 \text{ for } f,g \in \overline{V}^* ;$$

(3.12) $$\hat{u}\check{f} + \check{f}\hat{u} = f(u) \text{ for } u \in \overline{V}, \ f \in \overline{V}^* .$$

It follows that all the operators $\hat{u}$ for $u \in \overline{V}$ and $\check{f}$ for $f \in \overline{V}^*$ generate the Clifford algebra, which we denote by $C\ell$, on the space $\overline{V} \oplus \overline{V}^*$ with the symmetric bilinear form $(.|.)$ for which $\overline{V}$ and $\overline{V}^*$ are isotropic and $(u|f) = f(u)$ for $u \in \overline{V}$ and $f \in \overline{V}^*$. The representation of $C\ell$ on F is irreducible.

The operator adjoint to $\hat{v}_i$ is $\check{v}_i^*$, where $v_i^* \in \overline{V}^*$ is defined by $v_i^*(v_j) = \delta_{ij}$, $j \in \mathbb{Z}$. Note that

(3.13) $$r^F(E_{ij}) = \hat{v}_i\check{v}_j^* .$$

Introduce the following two operators on $F \otimes F$ (which are adjoint to each other):

$$S = \sum_{j\in\mathbb{Z}} \hat{v}_j \otimes \check{v}^*_j \quad \text{and} \quad S^* = \sum_{j\in\mathbb{Z}} \check{v}^*_j \otimes \hat{v}_j .$$

The following lemma is immediate by (3.11-13):

LEMMA 3.1. S *and* S^* *commute with* $r^F \otimes r^F(g\ell_\infty)$. □

We turn now to the computation of the "transported" operators $\sigma\hat{v}_j\sigma^{-1}$ and $\sigma\check{v}^*_j\sigma^{-1}$ on B. As we shall see in a moment, each individual $\sigma\hat{v}_j\sigma^{-1}$, etc., is a quite complicated differential operator (of infinite order), whereas their generating series is represented by a very simple differential operator, called a vertex operator.

Let $\hat{F}^{(m)}$ and $\hat{B}^{(m)}$ denote the formal completions of $F^{(m)}$ and $B^{(m)}$ respectively (i.e., arbitrary infinite sums of monomials are allowed), and put $\hat{F} = \bigoplus_{m\in\mathbb{Z}} \hat{F}^{(m)}$, $\hat{B} = \bigoplus_{m\in\mathbb{Z}} \hat{B}^{(m)}$.

Introduce the generating series:

$$X(u) = \sum_{j\in\mathbb{Z}} u^j \hat{v}_j \quad \text{and} \quad X^*(u) = \sum_{j\in\mathbb{Z}} u^{-j} \check{v}^*_j, \quad \text{where} \quad u \in \mathbb{C}^\times .$$

These are operators which map F into $\hat{F}$. The "transported" operators $\sigma X(u)\sigma^{-1}$ and $\sigma X^*(u)\sigma^{-1}$ map B into $\hat{B}$. In order to describe them explicitly, define the linear map $T(u)\colon B \to B$ by:

$$T(u)f(x,z) = uzf(x,uz) .$$

THEOREM 3.2. *One has:*

$$\sigma X(u)\sigma^{-1} = T(u)(\exp \sum_{j\geq 1} u^j x_j)(\exp - \sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j}) ; \tag{3.14}$$

$$\sigma X^*(u)\sigma^{-1} = T(u)^{-1}(\exp - \sum_{j\geq 1} u^j x_j)(\exp \sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j}) . \tag{3.15}$$

PROOF. One checks easily, using (3.11-13), the following relations:

$$[\hat{r}^F(\Lambda_j),X(u)] = u^j X(u) \quad \text{and} \quad [\hat{r}^F(\Lambda_j),X^*(u)] = -u^j X^*(u) \ .$$

Due to (3.6) the first of these two formulas transports into B as follows:

$$(3.16) \qquad [\frac{\partial}{\partial x_k}, \sigma X(u)\sigma^{-1}] = u^*(\sigma X(u)\sigma^{-1}), \quad [x_k, \sigma X(u)\sigma^{-1}] = \frac{u^{-k}}{k}(\sigma X(u)\sigma^{-1}) \ .$$

Note also that $\sigma X(u)\sigma^{-1}$ maps $B^{(m)}$ into $\hat{B}^{(m+1)}$. But up to a constant factor (depending on u) there exists only one operator which maps $B^{(m)} = \mathbb{C}[x_1,x_2,\ldots]$ into $\hat{B}^{(m+1)} = \mathbb{C}[[x_1,x_2,\ldots]]$ and satisfies (3.16), the so called *vertex operator* (see e.g., [5, Lemma 14.6]):

$$\Gamma(u) = (\exp \sum_{j\geq 1} u^j x_j)(\exp - \sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j}) \ .$$

Thus, we obtain:

$$\sigma X(u)\sigma^{-1} = C_m(u) z\Gamma(u) \ .$$

But the coefficient in $X(u)(v_m \wedge v_{m-1} \wedge \ldots)$ of $v_{m+1} \wedge v_m \wedge \ldots$ is u^{m+1}, from which we conclude that $C_m(u) = u^{m+1}$. This completes the proof of (3.14); the proof of (3.15) is similar. □

Further on we will denote, for brevity, the transported operators $\sigma\hat{u}\sigma^{-1}$, $\sigma X(u)\sigma^{-1}$, etc., on B simply by $\hat{u}$, $X(u)$, etc.

It is easy now to compute the representations r_m^B and $\hat{r}_m^B$ of $g\ell_\infty$ and a_∞ on $B^{(m)}$ transported from $F^{(m)}$ via σ_m. Due to (3.11) we have:

$$r^F(\sum_{i,j\in\mathbb{Z}} u^i v^{-j} E_{ij}) = X(u)X^*(v), \quad u,v \in \mathbb{C}^\times \ .$$

Using Theorem 3.1 and the relation

$$(\exp a \frac{\partial}{\partial x})(\exp bx) = (\exp ab)(\exp bx)(\exp a \frac{\partial}{\partial x}) \ ,$$

we deduce (taking $|v| < |u|$):

$$(3.17)\qquad r^B_m\Big(\sum_{i,j} u^i v^{-j} E_{ij}\Big) = \frac{(u/v)^m}{1 - v/u}\,\Gamma(u,v)$$

where $\Gamma(u,v)$ is the following *vertex operator*:

$$(3.18)\qquad \Gamma(u,v) = \Big(\exp \sum_{j\geq 1} (u^j - v^j)x_j\Big)\Big(\exp - \sum_{j\geq 1} \frac{u^{-j} - v^{-j}}{j}\frac{\partial}{\partial x_j}\Big) .$$

Taking into account (3.1), we also have (cf.[1]):

$$(3.19)\qquad \hat{r}^B_m\Big(\sum_{i,j} u^i v^{-j} E_{ij}\Big) = \frac{1}{1 - v/u}\Big(\left(\frac{u}{v}\right)^m \Gamma(u,v) - 1\Big) .$$

Here and further on we do not attempt to justify our manipulations with vertex operators.

In order to calculate $\hat{r}^B_m(E_{ij})$ or $r^B_m(E_{ij})$, we develop $(1 - \frac{v}{u})^{-1} = \sum_{k\geq 0} \left(\frac{v}{u}\right)^k$ and $\Gamma(u,v)$ in formal power series in u and v and then collect the coefficients of $u^i v^{-j}$. As a result we obtain quite complicated differential operators of infinite order in infinitely many variables, which we will not write here as we do not need them.

Note that any highest weight representation π_λ of $g\ell_\infty$ on $L(\lambda)$ with highest weight $\lambda = \{\lambda_i\}$ satisfying (2.3), extends to a representation $\hat{\pi}_\lambda$ of a_∞ on the same space $L(\lambda)$ as follows. Put $m_\lambda = \max_i(\lambda_i)$; put (cf. formulas (3.1)):

$$\hat{\pi}_\lambda(E_{ij}) = \pi_\lambda(E_{ij}) \quad \text{for} \quad i \neq j \quad \text{or} \quad i = j > 0 ;$$

$$\hat{\pi}_\lambda(E_{ij}) = \pi_\lambda(E_{ii}) - m_\lambda I \quad \text{for} \quad i \leq 0; \quad \pi_\lambda(c) = m_\lambda I .$$

4. Boson-fermion correspondence and Schur polynomials

Introduce the "elementary" Schur polynomials $S_k(x) \in \mathbb{C}[x_1, x_2, \ldots]$ by the following generating series:

$$(4.1) \qquad \sum_{k\in\mathbb{Z}} z^k S_k(x) = \exp \sum_{k=1}^{\infty} z^k x_k .$$

Then we have $S_k(x) = 0$ for $k < 0$, $S_0(x) = 1$, and

$$S_k(x) = \sum_{k_1+2k_2+\ldots=k} \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!} \cdots \quad \text{for } k > 0 .$$

For example, $S_1(x) = x_1$, $S_2(x) = \frac{1}{2}x_1^2 + x_2$, $S_3(x) = \frac{1}{6}x_1^3 + x_1x_2 + x_3$, $S_4(x) = \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 + x_1x_3 + x_4$, etc.

Given a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots\} \in \text{Par}$ (i.e., a finite non-increasing sequence of non-negative integers), define the associated *Schur polynomial* by:

$$(4.2) \qquad S_{\lambda_1,\lambda_2,\ldots}(x) = \det(S_{\lambda_i+j-i}(x))_{i,j} .$$

For example,

$$S_{1,1} = \det\begin{pmatrix} S_1 & S_2 \\ 1 & S_1 \end{pmatrix} = \frac{1}{2}x_1^2 - x_2 ,$$

$$S_{2,1} = \det\begin{pmatrix} S_2 & S_3 \\ 1 & S_1 \end{pmatrix} = \frac{1}{3}x_1^3 - x_3 ,$$

$$S_{2,2} = \det\begin{pmatrix} S_2 & S_3 \\ S_1 & S_2 \end{pmatrix} = \frac{1}{12}x_1^4 - x_1x_3 + x_2^2 ,$$

etc.

Note that $S_{\lambda_1,\lambda_2,\ldots}(x)$ is a homogeneous polynomial with respect to the principal gradation ($\deg x_j = j$) of degree $\lambda_1 + \lambda_2 + \cdots$.

The Schur polynomials have the following simple interpretation in terms of the representations of the group $GL_N(\mathbb{C})$ (big N). One easily shows that $\text{tr}_{S^k\mathbb{C}^N}\pi_k\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_N\end{pmatrix} = S_k(x)$, where $x_j = \dfrac{\varepsilon_1^j+\ldots+\varepsilon_N^j}{j}$. Furthermore, if $(L(\lambda),\pi_\lambda)$

is a representation of $GL_N(\mathbb{C})$ corresponding to a partition λ then

$$\operatorname{tr}_{L(\lambda)} \pi_\lambda \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_N \end{pmatrix} = S_\lambda(x)$$ (see e.g., [10] for a proof of this fact).

Now we are in a position to describe explicitly the isomorphism $\sigma_m : F^{(m)} \simeq B^{(m)} = \mathbb{C}[x_1, x_2, \ldots]$ between the wedge and the polynomial realizations of the fundamental representations $L(\omega_m)$ of $g\ell_\infty$ (and GL_∞) (cf.[5]):

THEOREM 4.1. $\sigma_m(v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots) = S_{i_m - m, i_{m-1}-m+1,\ldots}(x)$.

PROOF. Consider the operator

$$A = \exp \sum_{j \geq 1} y_j \Lambda_j \in \overline{GL_\infty} ,$$

where y_j are some complex numbers. Given an element $v \in L(\omega_m)$, consider the matrix coefficient $F_v(y)$ defined by:

$$\pi_{\omega_m}(A)v = F_v(y) v_{\omega_m} + (\text{terms from } L(\omega_m)_k \text{ with } k > 0) .$$

First we calculate $F_v(y)$ in the polynomial (bosonic) realization for $v = G(x) \in B^{(m)}$:

$$F_v(y) = (\exp \sum_{j \geq 1} y_j \frac{\partial}{\partial x_j}) G(x)|_{x=0} = G(x+y)|_{x=0} = G(y) .$$

Thus, we have

$$(4.3) \qquad F_v(y) = G(y), \quad \text{where } v = G(x) \in B^{(m)} .$$

Second, we calculate $F_v(y)$ in the fermionic picture for $v = v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots$. Note that the matrix of the operator A (acting on $\overline{V}$) is (cf. (4.1)):

$$(S_{j-i}(y))_{i,j \in \mathbb{Z}} .$$

But $R_m^F(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots) = Av_{i_m} \wedge Av_{i_{m-1}} \wedge \ldots$ (cf. (1.8)), and therefore, using (1.12), we get

$$(4.4)\quad R_m^F(A)(v_{i_m}\wedge v_{i_{m-1}}\wedge\dots) = \sum_{j_m>j_{m-1}>\dots} \det(S_{j-i}(y))^{i_m,i_{m-1},\dots}_{j_m,j_{m-1},\dots} v_{j_m}\wedge v_{j_{m-1}}\wedge\dots$$

Thus, $F_v(y) = (\det(S_{j-i}(y)))^{i_m,i_{m-1},\dots}_{m,m-1,\dots}$ and so

$$(4.5)\qquad F_v(y) = S_{i_m-m,\, i_{m-1}-m+1,\dots}(y)\ .$$

Comparing (4.3) and (4.5) completes the proof. □

Theorem 4.1 together with formula (1.12) gives us explicit formulas for the representations R_m^B of GL_∞ (and even $\overline{GL_\infty}$) on $B^{(m)} = \mathbb{C}[x_1,x_2,\dots]$ in the basis of Schur polynomials:

$$(4.6)\qquad R_m^B(A)S_\lambda = \sum_{\mu\in \mathrm{Par}} (\det A^{\lambda_1+m,\lambda_2+m-1,\dots}_{\mu_1+m,\mu_2+m-1,\dots})S_\mu\ .$$

For example, if $A(v_i) = v_{\tau(i)}$, where τ is a permutation of $\mathbb{Z}$ which leaves fixed all but a finite number of elements, then:

$$R_m^B(A)S_\lambda = \varepsilon(\tau,\lambda)S_\mu\ ,$$

where $\varepsilon(\tau,\lambda) = \pm 1$ and $\mu \in$ Par are obtained as follows. Consider the set $M_\lambda = \{\lambda_1 + m, \lambda_2 + m - 1,\dots\}$, and let $a_m > a_{m-1} > \dots$ be a reordering of $\tau(M_\lambda)$. Then $\varepsilon(\tau,\lambda)$ is the sign of this reordering and

$$\mu = \{a_m - m, a_{m-1} - (m - 1),\dots\}\ .$$

Note finally that the following well-known fact is an immediate consequence of Theorem 4.1:

COROLLARY 4.1. *Schur polynomials* S_λ, $\lambda \in$ Par, *form an orthonormal basis of* $\mathbb{C}[x_1,x_2,\dots]$ *with respect to the Hermitian form* (3.9).

5. The KP hierarchy and the modified KP hierarchy

As we shall see, the modified KP hierarchy is the system of equations of the orbit under GL_∞ in B of the sum of highest weight vectors $1_m \in B^{(m)}$, the KP hierarchy being the system of equations of the orbit of 1_m in $B^{(m)}$. However, we find that the most convenient (though the least illuminating) way to introduce these hierarchies is as follows.

Fix $k \in \mathbb{Z}_+$ and consider the following equation:

$$(5.1) \qquad \sum_{i\in\mathbb{Z}} \hat{v}_i f_{m+k} \otimes \check{v}_i^* f_m = 0, \quad \text{where } f_j \in B^{(j)} .$$

The equation (5.1) is called the *k-th modified KP hierarchy* (its form is, obviously, independent of m); the 0-th modified KP hierarchy is called simply the *KP hierarchy*; the union of all k-th modified KP hierarchies is called the *modified KP hierarchy.*

We think of $B^{(m+k)} \otimes B^{(m)}$ as of a polynomial algebra $\mathbb{C}[x_j', x_j'';$ $j = 1,2,\ldots]$. Note that (5.1) can be rewritten as follows:

$$\oint X(u) f_{m+k} \otimes X^*(u) f_m \frac{du}{u} = 0 .$$

Hence, using Theorem 3.1, we can rewrite (5.1) in the form of the bilinear equation of Kashiwara-Miwa [9] (here and further on the integration is taken along a small contour around 0):

$$(5.2) \quad \oint \exp \sum_{j\geq 1} u^j (x_j' - x_j'') \exp - \sum_{j\geq 1} \frac{u^{-j}}{j} \left(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\right) f_{m+k}(x') f_m(x'') u^k du = 0 .$$

This equation can be rewritten in the form of a system (hierarchy) of Hirota bilinear equations as follows [2]. Further on, $\frac{\tilde{\partial}}{\partial u}$ stands for $\left(\frac{\partial}{\partial u_1}, \frac{1}{2}\frac{\partial}{\partial u_2}, \frac{1}{3}\frac{\partial}{\partial u_3}, \ldots\right)$.

Recall that for a polynomial P, the corresponding *Hirota bilinear equation* on functions f and g is

$$Pf \cdot g = P\left(\frac{\tilde{\partial}}{\partial u}\right) f(x - u) g(x + u)\Big|_{u=0} = 0 .$$

Let $x = \frac{1}{2}(x' + x'')$ and $y = \frac{1}{2}(-x' + x'')$; then $\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''} = -\frac{\partial}{\partial y_j}$ and (5.2) becomes:

$$(5.3)\qquad \oint (\exp - \sum_{j\ge 1} 2u^j y_j)(\exp \sum_{j\ge 1} \frac{u^{-j}}{j}\frac{\partial}{\partial y_j}) f_{m+k}(x-y) f_m(x+y) u^k du = 0 .$$

The latter equation is equivalent to (cf. (4.1)):

$$\sum_{j=0}^{\infty} S_j(-2y) S_{j+1+k}\left(\tilde{\frac{\partial}{\partial y}}\right) f_{m+k}(x-y) f_m(x+y) = 0 ,$$

which, by Taylor's formula, is

$$\sum_{j=0}^{\infty} S_j(-2y) S_{j+1+k}\left(\tilde{\frac{\partial}{\partial u}}\right) e^{\sum_{r=1}^{\infty} y_r \frac{\partial}{\partial u_r}} f_{m+k}(x-u) f_m(x+u)\big|_{u=0} = 0 .$$

The last equation can be written as the following generating series of Hirota bi-linear equations

$$(5.4)\qquad \sum_{j=0}^{\infty} S_j(-2y) S_{j+1+k}(x) e^{\sum_{r=1}^{\infty} r y_r x_r} f_{m+k}(x)\cdot f_m(x) = 0 .$$

REMARK 5.1. Taking the coefficient of $S_\lambda(y)$ ($\lambda \in$ Par) in (5.4) we get a Hirota bilinear equation $P_{\lambda;k} f_{m+k}\cdot f_m = 0$, where $P_{\lambda;k}$ is homogeneous (with respect to the principal degree $\deg x_j = j$) of degree $|\lambda| + k + 1$. We will show that the $P_{\lambda;k}$ are linearly independent (for given k).

For example, the coefficient of y_r in (5.4) is

$$P_{r;k} = r x_r S_{k+1}(x) - 2S_{r+k+1}(x) .$$

In particular,

$$-12P_{3;0} = (x_1^4 - 12x_1x_3 + 12x_2^2) + 12(x_1^2x_2 + x_4) .$$

Noting that the Hirota bilinear equation $Pf\cdot f = 0$ with $P(-x) = -P(x)$ is trivial $(0 = 0)$, we obtain that the Hirota bilinear equation

$$(5.5)\qquad (x_1^4 - 12x_1x_3 + 12x_2^2) f\cdot f = 0$$

is one of the equations of the KP hierarchy. Actually, it is easy to see that this is the nontrivial equation of the KP hierarchy of the lowest degree.

Putting

$$x = x_1, \quad y = x_2, \quad t = x_3, \quad u(x,y,t) = 2 \frac{\partial^2}{\partial x^2} \log f , \tag{5.6}$$

we can see that if f satisfies (5.5) then $u(x,y,t)$ satisfies the classical Kadomtsev-Petviashvili (KP) equation:

$$\frac{3}{4} \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2} u \frac{\partial u}{\partial x} - \frac{1}{4} \frac{\partial^3 u}{\partial x^3}\right) .$$

This is a fundamental observation of Hirota.

Furthermore, $-P_{0;1} = x_1^2 + 2x_2$, hence the Hirota bilinear equation

$$(x_1^2 + 2x_2) g \cdot f = 0 \tag{5.7}$$

is one of the equations of the 1'st modified KP hierarchy. Using the change of variables and functions (5.6) together with $v(x,y,t) = \log \frac{g}{f}$, we obtain

$$u = \frac{\partial v}{\partial y} - \left(\frac{\partial v}{\partial x}\right)^2 - \frac{\partial^2 v}{\partial x^2} ,$$

which is the (2-dimensional) Miura transformation (cf.[4]).

Let $L^{(k)} = B^{(m+k)} \otimes B^{(m)}$; because of the unitarity, this space decomposes into an orthogonal direct sum of irreducible highest weight representations of $g\ell_\infty$ (and a_∞). The element $1_{m+k} \otimes 1_m$ is the highest weight vector of one of these representations, which we denote $L^{(k)}_{high}$ and call the highest component of $L^{(k)}$. We, clearly, have with respect to $g\ell_\infty$:

$$L^{(k)}_{high} = L(\omega_{m+k} + \omega_m) . \tag{5.8}$$

We denote the sum of all other irreducible subrepresentation of $L^{(k)}$ by $L^{(k)}_{low}$. Then we have the orthogonal direct sum of representations: $L^{(k)} = L^{(k)}_{high} \oplus L^{(k)}_{low}$.

As before we think of $L^{(k)}$ as of a polynomial algebra:

$L^{(k)} = \mathbb{C}[x'_j, x''_j; j = 1,2,\ldots]$. As before we put

$$x_j = \tfrac{1}{2}(x''_j + x'_j), \quad y_j = \tfrac{1}{2}(x''_j - x'_j)$$

and define the following subspace of $L^{(k)}$:

$$\mathrm{Hir}^{(k)} = \mathbb{C}[y] \cap L^{(k)}_{\mathrm{low}} .$$

It is clear that $\mathrm{Hir}^{(k)}$ is graded with respect to the principal gradation of $L^{(k)}$ ($\deg x'_j = \deg x''_j = j$):

$$\mathrm{Hir}^{(k)} = \bigoplus_{j \geq 0} \mathrm{Hir}^{(k)}_j .$$

Now we are in a position to prove the following proposition.

PROPOSITION 5.1. (a) *The* k*-th modified* KP *hierarchy* (5.1) *is equivalent to the equation*

$$f_{m+k} \otimes f_m \in L^{(k)}_{\mathrm{high}} . \tag{5.9}$$

(b) *The equation* $f \otimes g \in L^{(k)}_{\mathrm{high}}$ *is equivalent to the system of Hirota bilinear equations* $P(\tfrac{1}{2}\tilde{x}) f \cdot g = 0$, *where* $P \in \mathrm{Hir}^{(k)}$.

(c) [2] *The polynomials* $P_{\lambda;k}$ *(see Remark* 5.1*) with* $\lambda \in \mathrm{Par}(s)$ *(the set of partitions of* s*) form a basis of the space* $\mathrm{Hir}^{(k)}_{s+k+1}$. *In particular,*

$$\dim \mathrm{Hir}^{(k)}_j = p(j - k - 1) . \tag{5.10}$$

PROOF. Note that $S(\psi_{m+k} \otimes \psi_m) = 0$ for $k \in \mathbb{Z}_+$ (where S is the operator on $F \otimes F$ defined in section 3). By Lemma 3.1, it follows that $S(GL_\infty(\psi_{m+k} \otimes \psi_m)) = 0$. Since the GL_∞-orbit of $\psi_{m+k} \otimes \psi_m$ spans $L^{(k)}_{\mathrm{high}}$, we conclude that (5.9) implies (5.1). To prove the reverse implication, it suffices to show that if $v = \sum_{\lambda,\mu} a_\lambda \otimes b_\mu \in L^{(k)}_{\mathrm{low}}$ is such that $(r^F \otimes r^F)(E_{ij})v = 0$ for $i < j$, $r^F(E_{ii})a_\lambda = \lambda_i a_\lambda$, $r^F(E_{ii})b_\mu = \mu_i b_\mu$ and $\nu_i := \lambda_i + \mu_i$ is independent of

λ and μ (i.e., v is a highest weight vector in $L^{(k)}_{low}$), then $S^*S(v) \neq 0$ (since S^*S is a diagonalizable operator commuting with $g\ell_\infty$). Note the following:

(5.11) $$\lambda_i \text{ and } \mu_i = 0 \text{ or } 1 ;$$

$$\lambda_i \text{ and } \mu_i \to 0 \text{ (resp. } \to 1) \text{ as } i \to +\infty \text{ (resp. } -\infty) ,$$

(5.12) $$\nu_i \geq \nu_{i+1} .$$

By (3.11) and (3.16) we have:

(5.13) $$S^*S = \sum_{i,j\in\mathbb{Z}}{}^{\cdot} (\delta_{ij} - r^F(E_{ij})) \otimes r^F(E_{ji}) .$$

Hence, if $S^*S(v) = 0$, then $\sum_{\lambda,\mu} \sum_j (1 - \lambda_j)\mu_j = 0$ and we have

(5.14) $$\text{if } \lambda_j = 0, \text{ then } \mu_j = 0 .$$

But (5.11, 12 and 14) imply that $\lambda = \omega_{m+k}$, $\mu = \omega_m$ and $v \in L^{(k)}_{high}$, a contradiction. This proves (a).

To prove (b) note that: $\varphi \in L^{(k)}_{high}$ iff $\langle\varphi, L_{low}\rangle = 0$ and that $L^{(k)}_{low} = \mathbb{C}[x] \otimes \mathrm{Hir}^{(k)}$. It follows that $f \otimes g \in L^{(k)}_{high}$ iff (cf. formula (3.9)):

$$P\left(\tfrac{1}{2}\frac{\tilde\partial}{\partial x}\right) Q\left(\tfrac{1}{2}\frac{\tilde\partial}{\partial y}\right) f(x - y)g(x + y)\Big|_{\substack{x=0\\y=0}} = 0$$

for any $P \in \mathbb{C}[x]$ and any $Q \in \mathrm{Hir}$. Thus, the equation $f \otimes g \in L^{(k)}_{high}$ is equivalent to

$$Q\left(\tfrac{1}{2}\frac{\tilde\partial}{\partial y}\right) f(x - y)g(x + y)\Big|_{y=0} = 0$$

for any $Q \in \mathrm{Hir}^{(k)}$, proving (b).

To prove (c) note that (cf. (5.8)):

(5.15) $$L(\omega_{m+k}) \otimes L(\omega_m) = L(\omega_{m+k} + \omega_m) \oplus (\mathbb{C}[x] \otimes \mathrm{Hir}^{(k)})$$

Taking q-dimensions of both sides, and using (2.8) and (2.9), we get:

$$\varphi(q)^{-2} = \varphi(q)^{-2}(1 - q^{k+1}) + \varphi(q)^{-1} \dim_q \mathrm{Hir}^{(k)} .$$

This proves formula (5.10). Since the polynomials $P_{\lambda;k}$ with $\lambda \in \mathrm{Par}(s)$ span the space $\mathrm{Hir}^{(k)}_{s+k+1}$ and their number is $p(s)$, (a), (b) and formula (5.10) imply (c). □

The following theorem sums up the above discussion.

THEOREM 5.1. *Let* I *be a non-empty finite subset of* $\mathbb{Z}$ *and let* $f = \bigoplus_{m\in I} f_m \in \bigoplus_{m\in I} B^{(m)}$ *be such that all* $f_m \neq 0$. *Then the following are equivalent* $(m,n \in I, m \geq n)$:

(a) $\sum_{i\in\mathbb{Z}} \hat{v}_i f_m \otimes \check{v}_i^* f_n = 0$.

(b) $\oint (\exp \sum_{j\geq 1} u^j (x_j' - x_j''))(\exp - \sum_{j\geq 1} \frac{u^{-j}}{j}\left(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\right) f_m(x') f_n(x'') u^{m-n} du = 0$.

(c) $\sum_{j=0}^{\infty} S_j(-2y) S_{j+1+m-n}(x) e^{\sum_{s=1}^{\infty} s y_s x_s} f_m(x) \cdot f_n(x) = 0$.

(d) $f_m \otimes f_n \in (B^{(m)} \otimes B^{(n)})_{\mathrm{high}}$.

(e) $P(\frac{1}{2}\tilde{y}) f_m \cdot f_n = 0$ *for all* $P \in \mathrm{Hir}^{(m-n)}$.

(f) $\sum_{i,j\in\mathbb{Z}} (\delta_{ij} - r^B(E_{ij})) f_m \otimes r^B(E_{ji}) f_n = 0$.

(g) $f \in R^B(GL_\infty)\cdot(\bigoplus_{j\in I} 1_j)$, *i.e.,* f *lies in the* GL_∞*-orbit of the sum of highest weight vectors.*

PROOF. The equivalence of (a), (b) and (c) was proved at the beginning of this section. The equivalence of (d), (e) and (a) follows from Proposition 5.1 (a). Furthermore, (a) implies (f) as in the proof of Proposition 5.1 (a). (f) implies (a) since $S^*S(f_m \otimes f_n) = 0$ implies $\langle S^*S(f_m \otimes f_n), f_m \otimes f_n\rangle = 0$, which implies

$<S(f_m \otimes f_n), S(f_m \otimes f_n)> = 0$, hence $S(f_m \otimes f_n) = 0$. The proof of the implication (f) $\Rightarrow$ (g) is along the same lines as that of [11, Theorem 1(b)]. (Actually, the proof simplifies if one uses (a) instead of (f).) Finally, the implication (g) $\Rightarrow$ (a) holds, since, by Lemma 3.1, $R^B(GL_\infty)$ commutes with the operator S on $B^{(m)} \otimes B^{(n)}$ and since $f_m = \psi_m$, $f_n = \psi_n$ satisfy (a) (for $m \geq n$). □

REMARK 5.2. The calculation establishing (7.3) shows that on $B^{(m+k)} \otimes B^{(m)}$ the operators S and S^* look as follows:

$$(5.15) \qquad S = \sum_{j \in \mathbb{Z}} S_j(-2y) S_{j+1+k}\left(\tilde{\frac{\partial}{\partial y}}\right) ; \qquad S^* = \sum_{j \in \mathbb{Z}} S_j(2y) S_{j+1-k}\left(-\tilde{\frac{\partial}{\partial y}}\right) .$$

Thus,

$$\mathrm{Hir}^{(k)} = \{\sum_{j \in \mathbb{Z}} S_j(2y) S_{j+1-k}\left(-\tilde{\frac{\partial}{\partial y}}\right)\varphi(y), \text{ where } \varphi(y) \in \mathbb{C}[y_1, y_2, \ldots]\} ,$$

and its orthocomplement in $\mathbb{C}[y]$ is:

$$\{\varphi(y) \in \mathbb{C}[y_1, y_2, \ldots] \mid \sum_{j \in \mathbb{Z}} S_j(-2y) S_{j+1+k}\left(\tilde{\frac{\partial}{\partial y}}\right)\varphi(y) = 0\} .$$

REMARK 5.3. The equations of the modified KP hierarchy (given by Theorem 5.1 in one of the equivalent forms (a) - (f)) generate the ideal of the GL_∞-orbit of the sum of highest weight vectors of B in the symmetric algebra over $B^*_{restr.}$. The proof of this fact is the same as that of [8, Theorem 2], by making use of the following "regularized" Casimir operator:

$$\Omega(v_{\Lambda-\alpha}) = \sum_{i<j} E_{ji} E_{ij}(v_{\Lambda-\alpha}) - 2<\Lambda + \rho - \tfrac{1}{2}\alpha, \nu^{-1}(\alpha)> v_{\Lambda-\alpha} .$$

Here Λ, α and ρ are linear functions on the diagonal subalgebra h of $g\ell_\infty$, all weights are assumed to be of the form $\Lambda - \alpha$, where Λ is fixed and α is a finite sum of positive roots, ρ is defined by $\langle\rho,E_{jj}\rangle = -j$ and $\nu: h \to h^*$ is defined by the trace form. Note that the above fact together with the fact that the projectivized GL_N-orbit of the highest weight vector is closed give a simple proof of the implication (g) $\Rightarrow$ (a - f) of Theorem 5.1.

6. Polynomial solutions of the modified KP hierarchy

Let Σ_∞ denote the group of permutations of $\mathbb{Z}$ leaving all but a finite number of elements fixed. We identify Σ_∞ with a subgroup of GL_∞ via $\sigma(v_i) = v_{\sigma(i)}$. Let M denote the set of all maps $\varphi: \mathbb{Z} \to \mathbb{Z}_+$ such that $\varphi(n) = 0$ for all but a finite number of n. Then we have the following bijection $\Sigma_\infty \to M$: given $\tau \in \Sigma_\infty$, the corresponding map φ_τ is

$$\varphi_\tau(n) = \tau(n) - n + r_n, \quad \text{where} \quad r_n = \#\{s | s<n \text{ and } \tau'(s)>\tau(n)\} .$$

Note that $\sum_n \varphi_\tau(n) = \sum_n r_n$.

Given $\varphi \in M$, construct a sequence of partitions $\{\Phi_k^\varphi\}_{k\in\mathbb{Z}}$ as follows. Put $\Phi_n^\varphi = \emptyset$ if $\varphi(j) = 0$ for all $j \le n$. If $\Phi_k^\varphi = \{a_0 \ge a_1 \ge \dots \ge a_r\} \in \text{Par}$, let

$$\Phi_{k+1}^\varphi = \text{reordering of } \{\varphi(k+1), b_0, \dots, b_r\} \in \text{Par} ,$$

where $b_i = a_i - 1$ if $a_i > \varphi(k+1)$ and $b_i = a_i$ otherwise. Define a sequence $\{\varepsilon_k^\varphi\}_{k\in\mathbb{Z}}$ of ± 1 as follows. Put $\varepsilon_n^\varphi = 1$ if $\varphi(j) = 0$ for all $j \le n$, and $\varepsilon_{k+1}^\varphi = (-1)^s \varepsilon_k^\varphi$, where $s = \#\{i | a_i > \varphi(k+1)\}$. It is easy to see inductively that if $k \in \mathbb{Z}$ and $\Phi_k^{\varphi_\tau} = \{a_o \ge a_1 \ge \dots \ge a_r\}$, then

$$R^F(\tau)\psi_k = \varepsilon_k^{\varphi_\tau} v_{k+a_0} \wedge v_{k-1+a_1} \wedge \dots \tag{6.1}$$

It is a famous result of Sato [12] that all Schur functions S_λ are solutions of the KP hierarchy. This, of course, follows immediately from

Theorems 4.1 and 5.1: the orbit of $1 \in B^{(0)}$ under Σ_∞ consists of all Schur functions (with suitable signs, which may be changed by sign changes of the v_i).

The following proposition describes the orbit of a sum of highest weight vectors under Σ_∞. According to Theorem 5.1, this provides solutions of the modified KP hierarchy. The proof of the proposition follows from (6.1) and the boson-fermion correspondence.

PROPOSITION 6.1. *Let* $\tau \in \Sigma_\infty$ *be such that* $\tau(n) = n$ *for* $|n| > N$. *Put* $\lambda^j = \Phi_j^{\varphi_\tau} \in \mathrm{Par}$ $(j \in \mathbb{Z})$. *Then*

$$R^B(\tau) \bigoplus_{j=-N}^{N} 1_j = \bigoplus_{j=-N}^{N} \varepsilon_j^{\varphi_\tau} S_{\lambda^j}(x) . \qquad \square$$

REMARK. We have for every $j \in \mathbb{Z}$ (in notation of Proposition 6.1):

$$\sum_{i=-\infty}^{j} (\tau(i) - i) = |\lambda^j| .$$

7. Soliton solutions of the KP and the MKP hierarchies

Recall that by the vertex construction of the representation r^B of $g\ell_\infty$ on B (given by (3.17)), the operator $M(u,v)$ defined by

$$(7.1) \qquad M(u,v)(\sum_m z^m f_m) = \sum_m (zu/v)^m \Gamma(u,v) f_m$$

is contained in the completion of $r^B(g\ell_\infty)$. Here $\Gamma(u,v)$ is the vertex operator defined by (3.18).

Note that by the Taylor formula we have:

$$\Gamma(u,v)f(x) = (\exp \sum_{j\geq 1} (u^j - v^j)x_j)f(\ldots, x_j - \frac{u^{-j} - v^{-j}}{j}, \ldots) .$$

Using $\log(1 - c) = - \sum_{j\geq 1} \frac{c^j}{j}$ for $|c| < 1$, we deduce

$$(7.2)\quad \Gamma(u',v')\Gamma(u,v)f(x) = \frac{(u'-u)(v'-v)}{(v'-u)(u'-v)} \exp \sum_{j\geq 1} (u^j - v^j + u'^j - v'^j)x_j$$

$$\cdot\ f(\ldots, x_j - \frac{u^{-j} - v^{-j} + u'^{-j} - v'^{-j}}{j}, \ldots)$$

provided that $|u|,|v| < \min|u'|,|v'|$. By induction on N we deduce from (7.2):

$$(7.3)\quad \Gamma(u_N,v_N) \ldots \Gamma(u_1,v_1)\cdot 1 =$$

$$= \prod_{1\leq i<j\leq N} \frac{(u_j - u_i)(v_j - v_i)}{(u_j - v_i)(v_j - u_i)} \exp \sum_{k\geq 1} \sum_{j=1}^{N} (u_j^k - v_j^k)x_k ,$$

provided that $|u_i|,|v_i| < \min|u_j|,|v_j|$ for $i < j$.

We can prove now the following lemma.

LEMMA 7.1. *If* $f = \bigoplus_m f_m$ *is a solution of the* MKP *hierarchy then so is the function*

$$\bigoplus_m \left(1 + a\left(\frac{u}{v}\right)^m \Gamma(u,v)\right)f_m \quad (\textit{here} \quad a \in C, \quad u,v \in \mathbb{C}^\times) .$$

PROOF. We use the definition of the MKP hierarchy given by Theorem 5.1 (d). Let f be a solution of the MKP. We have to show that the function

$$\left(1 + a\left(\frac{u}{v}\right)^m \Gamma(u,v)\right)f_m \otimes \left(1 + a\left(\frac{u}{v}\right)^n \Gamma(u,v)\right)f_n$$

is in the completion of $L_{high}^{(m-n)}$. Since $M(u,v)$ is contained in the completion of $r^B(g\ell_\infty)$, we have

$$(7.4)\quad \left(\left(\frac{u}{v}\right)^m \Gamma(u,v)f_m\right) \otimes f_n + f_m \otimes \left(\left(\frac{u}{v}\right)^n \Gamma(u,v)f_n\right) \in L_{high}^{(m-n)} .$$

It remains to show that

$$(7.5)\quad \Gamma(u,v)f_m \otimes \Gamma(u,v)f_n \in L_{high}^{(m-n)} .$$

For that take $|u| = |v|$ and choose u',v' such that $|u| = |v| < |u'| = |v'|$, apply $M(u',v')$ to both sides of (7.4) and take a limit as $u' \to u$, $v' \to v$.

Formula (7.2) implies then (7.5). □

Let now $a_1,\dots,a_N,u_1,\dots,u_N,v_1,\dots,v_N$ be some complex numbers such that $\max\{|u_i|,|v_i|\} < \min\{|u_j|,|v_j|\}$ for $i < j$ and $u_i \neq 0$, $v_i \neq 0$. We let

$$(7.6) \qquad f_{m;a;u,v}(x) = \left(1 + a_N\left(\frac{u_N}{v_N}\right)^m \Gamma(u_N,v_n)\right) \dots \left(1 + a_1\left(\frac{u_1}{v_1}\right)^m \Gamma(u_1,v_1)\right)\cdot 1 .$$

Then, by Lemma 7.1, $\underset{m}{\oplus} f_{m;a;u,v}(x)$ is a solution of the MKP hierarchy (here $I \subset \mathbf{Z}$ is a finite set). This function is called an *N-soliton solution* of the MKP hierarchy. Using (7.3), we obtain an explicit formula:

$$(7.7) \quad f_{m;a;u,v}(x) = \sum_{\substack{0\le r\le N \\ 1=j_1<\dots<j_r\le N}} \prod_{\nu=1}^{r} \left(\frac{u_{j_\nu}}{v_{j_\nu}}\right)^m a_{j_\nu} \cdot$$

$$\cdot \prod_{1\le\nu<\mu\le r} \frac{(u_{j_\nu} - u_{j_\mu})(v_{j_\nu} - v_{j_\mu})}{(u_{j_\nu} - v_{j_\mu})(v_{j_\nu} - u_{j_\mu})} \exp \sum_{k\ge 1} \sum_{\nu=1}^{r} (u_{j_\nu}^k - v_{j_\nu}^k) x_k .$$

The N-soliton solutions of the KP hierarchy were constructed along these lines by Date-Jimbo-Kashiwara-Miwa [1].

For example, the 1-soliton solution of the classical KP equation is

$$u(x,y,t) = 2(\log(1 + a\exp((u - v)x + (u^2 - v^2)y + (u^3 - v^3)t + \text{const})))_{xx}$$

$$= \tfrac{1}{2}(u - v)^2(\cosh \tfrac{1}{2}((u - v)x + (u^2 - v^2)y + (u^3 - v^3)t + \text{const}))^{-2} ,$$

which describes a solitary wave (soliton) during time t in coordinates x,y. The N-soliton solution of the classical KP equation describes the interaction of N solitons.

8. More on the infinite wedge representation

We return to the full infinite wedge space $F = \Lambda^{\infty}\overline{V}$ over the topological space $\overline{V}$. Recall that with respect to the representation R of GL_∞ on F

the space F decomposes into a direct sum of fundamental representations of GL_∞: $F = \bigoplus_{m\in\mathbb{Z}} F^{(m)}$, the element $\psi_m = v_m \wedge v_{m-1} \wedge \dots$ being the highest weight vector of $F^{(m)}$ (see section 1).

Recall subspaces $\overline{V}^{(n)}$ of $\overline{V}$ and denote by Grass $\overline{V}$ the set of all subspaces U of $\overline{V}$ such that $U \supset \overline{V}^{(n)}$ for some n and $\dim U/\overline{V}^{(n)} < \infty$, called the Grassmannian of $\overline{V}$. Put $\mathrm{Grass}_m\overline{V} = \{U \in \mathrm{Grass}\,\overline{V} | \dim U/\overline{V}^{(n)} = m - n$ for $n \ll 0\}$. The group GL_∞ acts on Grass $\overline{V}$, the decomposition $\mathrm{Grass}\,\overline{V} = \coprod_{m\in\mathbb{Z}} \mathrm{Grass}_m\overline{V}$ being the orbit decomposition. The bigger group $\overline{A}_\infty$ acts on Grass $\overline{V}$ as well, and this action is transitive since the shifts $\nu_s \in \overline{A}_\infty$ permute the components $\mathrm{Grass}_{(m)}\overline{V}$.

Let $U \in \mathrm{Grass}_m\overline{V}$, so that $U \supset \overline{V}^{(n)}$ for some n; let $u_1,\dots,u_s$ be a basis of $U \bmod \overline{V}^{(n)}$ and put $\alpha_U = u_1 \wedge \dots \wedge u_s \wedge \psi_n$. It is clear that $\alpha_U \in F^{(m)}$ and up to a constant factor is independent of the choice of basis of $U \bmod \overline{V}_n$ and n. Thus, we obtain a map μ_m: $\mathrm{Grass}_{(m)}F \to \mathbb{P}F^{(m)}$ defined by $\mu_m(U) = \mathbb{C}\alpha_U$. We have:

$$\mu_m(\mathrm{Grass}_m F) = \mathbb{P}(GL_\infty \cdot \psi_m) .$$

According to Theorem 5.1, the set of non-zero solutions of the KP hierarchy on $F^{(m)}$, viewed up to a constant factor, coincides with $\mathbb{P}(GL_\infty\cdot\psi_m)$ and hence is parametrized by $\mathrm{Grass}_{(m)}F$. This is the basic observation of the Sato philosophy.

We claim that the k-th modified KP hierarchy has a very simple geometrical interpretation as well. Given $\alpha = u_m \wedge u_{m-1} \wedge \dots \in F^{(m)}$, put $U_\alpha = \{\sum_{i=1}^{\infty} \lambda_i u_i, \lambda_i \in \mathbb{C}\} \in \mathrm{Grass}_m\overline{V}$.

PROPOSITION 8.1. *The non-zero elements* $\alpha \in F^{(m)}$ *and* $\beta \in F^{(n)}$ *satisfy the* (m-n)*-th modified* KP *hierarchy*

$$\sum_{i\in\mathbb{Z}} \hat{v}_i\alpha \otimes \check{v}_i^*\beta = 0$$

if and only if $U_\alpha \supset U_\beta$.

PROOF. Using Lemma 3.1, we can assume that $\alpha = \psi_m$; then we have for $\beta = u_n \wedge u_{n-1} \wedge \dots$:

$$\sum_{i>m} (v_i \wedge v_m \wedge v_{m-1} \wedge \dots)\otimes v_i^*(u_n \wedge u_{n-1} \wedge \dots) = 0 .$$

It follows that $v_i^*(u_j) = 0$ for all $i > m$ and $j \le n$, hence all u_j lie in $V^{(m)}$ and $U_\beta \subset \overline{V}^{(m)} = U_{\psi_m}$. □

COROLLARY 8.1. (a) *The* MKP *hierarchy is equivalent to the union of the* KP *hierarchy and the* 1'*st modified* KP *hierarchy.*

(b) *Let* Flag $\overline{V}$ *denote the set of all flags of the form* $\dots \supset U_m \supset U_{m-1} \supset \dots$, *where* $U_m \in \mathrm{Grass}_m\overline{V}$. *Then the map* $\mathrm{Flag}\,\overline{V} \to \prod_{m\in\mathbb{Z}} \mathbb{P}F^{(m)}$ *defined by*

$$\{\dots \supset U_m \supset U_{m-1} \supset \dots\} \mapsto (\dots,\alpha_{U_m},\alpha_{U_{m-1}},\dots)$$

is a bijection between the set Flag $\overline{V}$ *and the set of solutions of the* MKP *hierarchy whose components are non-zero and are viewed up to constant factors.* □

As was pointed out in section 1, the representation R^F of GL_∞ on F extends to the bigger group $\overline{GL_\infty}$. However, as in the Lie algebra case (see section 3) it extends only to a projective (non-linearizable) representation of the group $\overline{A}_\infty$, which we denote again by R^F. The representation $R^F(g)$, $g \in \overline{A}_\infty$, is uniquely determined by the following three properties:

$$R^F(g)\psi_0 = \alpha_{g\cdot\overline{V}^{(0)}} , \tag{8.1}$$

$$R^F(g)vR^F(g)^{-1} = (g\cdot v)\hat{} \quad \text{for} \quad v \in \overline{V} , \tag{8.2}$$

$$R^F(g)\check{f}R^F(g)^{-1} = (g\cdot f)\check{} \quad \text{for} \quad f \in \overline{V}^* . \tag{8.3}$$

The uniqueness (as long as the choice of a constant in (8.1) is made) is clear by irreducibility of $C\ell$ on F. Formulas (8.1-3) allow us to construct $R^F(g)$

effectively by induction, proving the existence. It is easy to see that the representations $\hat{r}^F$ of a_∞ and R^F of $\bar{A}_\infty$ correspond to each other in the following sense:

$$R^F(g)\hat{r}^F(a)R^F(g)^{-1} = \hat{r}^F(gag^{-1}),\quad g \in \bar{A}_\infty,\quad a \in a_\infty . \tag{8.4}$$

Note that (in contrast with GL_∞ and $\overline{GL}_\infty$) the subspaces $F^{(m)}$ are not invariant under $\bar{A}_\infty$. In fact, we have:

$$R^F(\nu_s)\psi_m = \psi_{m+s} , \tag{8.5}$$

from which one easily concludes that F is irreducible under $R^F(\bar{A}_\infty)$.

Let $\bar{A}^o_\infty = \{g \in \bar{A}_\infty \mid \dim g\bar{V}_0/\bar{V}_n = -n \text{ for } n \ll 0\}$. Then $\bar{A}_\infty = \bar{A}^o_\infty \rtimes \langle \nu_s \mid s \in \mathbb{Z} \rangle$ and the subspaces $F^{(m)}$ are invariant and irreducible under $\bar{A}^o_\infty$.

Further on, instead of talking about the projective representation R^F of the groups $\bar{A}^o_\infty \subset \bar{A}_\infty$, we shall often talk about the linear representation $\hat{R}^F$ of the corresponding central extentions $A^o_\infty \subset A_\infty$ by $\mathbb{C}^\times$.

We can lift elements $\nu_s \in \bar{A}_\infty$ to $\nu_s \in A_\infty$ such that $\hat{R}^F(\nu_s)\psi_m = \psi_{m+s}$. Then we have: $A_\infty = A^o_\infty \rtimes \langle \nu_s \mid s \in \mathbb{Z} \rangle$.

9. Reduced KP and MKP hierarchies

Let $L = \mathbb{C}[t,t^{-1}]$ be the algebra of Laurent polynomials in the indeterminate t. Fix a positive integer n. Denoting by $\{u_i\}$ the standard basis of $\mathbb{C}^n$, we identify the vector space L^n over $\mathbb{C}$ with the space V by

$$v_{nk+j} = t^{-k}u_j . \tag{9.1}$$

The matrix algebra $\mathrm{Mat}_n(L)$ acts in a usual way on L^n. The identification (9.1) gives us an embedding of associative algebras $\varphi: \mathrm{Mat}_n(L) \to \bar{a}_\infty$. Explicitly:

$\varphi(\sum_j A_j t^j) = (A_{j-i})_{i,j\in\mathbb{Z}}$. In particular, we have:

(9.2) $$\varphi(t^k E_{ij}) = \sum_{s\in\mathbb{Z}} E_{n(s-k)+i,ns+j} ;$$

(9.3) $$\varphi\left(\begin{pmatrix} 0 & 1 & \dots & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ t & & & & 0 \end{pmatrix}^j\right) = \Lambda_j, \quad j \in \mathbb{Z} .$$

The embedding φ of associative algebras gives rise to the embedding of the corresponding Lie algebras $\varphi: g\ell_n(L) \to \overline{a}_\infty$ and the groups of invertible elements $\varphi: GL_n(L) \to \overline{A}_\infty$.

An easy calculation using (9.2) shows that the restriction of the cocycle α on $\overline{a}_\infty$ (defined by (3.1)) to $\varphi(g\ell_n(L))$ induces the following cocycle on $g\ell_n(L)$:

$$\alpha(A(t),B(t)) = \text{Res}_0 \text{ tr } \frac{dA(t)}{dt} B(t) ,$$

where $\text{Res}_0 \sum a_j t^j = a_{-1}$. This gives us a central extension $\hat{g\ell}_n = g\ell_n(L) \oplus \mathbb{C}c$, where the bracket is defined by $(A,B \in g\ell_n(\mathbb{C}))$:

$$[t^k A, t^m B] = t^{k+m}(AB - BA) + k\delta_{k,-m}(\text{tr } AB)c .$$

The Lie algebra $\hat{g\ell}_n$ is called the affine (Kac-Moody) algebra associated to the Lie algebra $g\ell_n(\mathbb{C})$. Thus, we have an embedding $\varphi: \hat{g\ell}_n \to a_\infty$. Of course, $\hat{g\ell}_n$ contains $\hat{s\ell}_n = s\ell_n(L) \oplus \mathbb{C}c$, the affine algebra associated to $s\ell_n(\mathbb{C})$.

Recall the subalgebra s of a_∞ defined by (3.3). Let $s_{(n)} = \sum_{k\neq 0} \mathbb{C}\Lambda_{nk} + \mathbb{C}c \subset s$. The following simple lemma is the key to the reduction procedure developed in [2], [4].

LEMMA 9.1. *The centralizer of* $s_{(n)}$ *in* a_∞ *is* $\varphi(\hat{s\ell}_n) + \mathbb{C}I$, *where* I *is the identity matrix in* $\overline{a}_\infty$.

PROOF. The centralizer of Λ_n in $\overline{a}_\infty$ consists of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ satisfying condition $a_{ij} = a_{i+n,j+n}$. It follows that the centralizer of Λ_n is in $\varphi(\hat{g\ell}_n)$. But $\varphi(X(t))$ is in the centralizer of $s_{(n)}$ iff $[\varphi(X(t)),\Lambda_{nj}] = 0$ for all $j \in \mathbb{Z}$, which is equivalent to $\mathrm{Res}_0 n \ \mathrm{tr}\ t^{n-1}X(t) = 0$, which means that $\mathrm{tr}\ X(t) \in \mathbb{C}$. □

Now we turn to the representation theory of the affine algebra $\hat{s\ell}_n$ (see [5] for details). Let h be the diagonal subalgebra of $s\ell_n(\mathbb{C})$; the (commutative) subalgebra $\hat{h} = h + \mathbb{C}c$ of $\hat{s\ell}_n$ is called a *Cartan subalgebra*. We choose the following basis of $\hat{h}$:

$$h_0 = c - E_{11} + E_{nn},\ h_1 = E_{11} - E_{22},\dots,h_{n-1} = E_{n-1,n-1} - E_{n,n}\ .$$

The linear functions $\hat{\omega}_j$ on $\hat{h}$ defined by $\hat{\omega}_j(h_i) = \delta_{ij}(i,j = 0,\dots,n-1)$ are called fundamental weights; put $\hat{P}_+ = \{\sum_i k_i\hat{\omega}_i | k_i \in \mathbb{Z}_+\}$.

Putting $\hat{n} = n + t\ s\ell_n(\mathbb{C}[t])$, where n is the subalgebra of strictly upper triangular matrices of $s\ell_n(\mathbb{C})$, we have

(9.4) $$\varphi(\hat{n}) \subset \text{strictly upper triangular subalgebra of } \overline{a}_\infty\ .$$

Given $\lambda \in \hat{P}_+$, we define the *highest weight representation* τ_λ of $\hat{s\ell}_n$ as an irreducible representation admitting a non-zero vector v_λ (highest weight vector) such that

(9.5) $$\tau_\lambda(\hat{n})v_\lambda = 0;\ \tau_\lambda(h)v_\lambda = \lambda(h)v_\lambda \quad \text{for } h \in \hat{h}\ .$$

(Due to (9.4), the definition (9.5) is consistent with the definition of the highest weight representation of $g\ell_\infty$). One can show that (9.5) determines an irreducible representation uniquely [5, 2nd edition, Proposition 10.4]. We proceed to give a vertex construction of the fundamental representations τ_{ω_j}, $j = 0,\dots,n-1$, of $\hat{s\ell}_n$ (cf.[6]).

Note that the subspace $B_{(n)}^{(m)} = \mathbb{C}[x_j | j \not\equiv 0 \bmod n]$ of $B^{(m)}$ is the intersection of kernels of all operators $\hat{r}^B_m(\Lambda_{jn})$ with $j > 0$. Hence, thanks to Lemma 9.1, the subspace $B^{(m)}_{(n)}$ is invariant under $\hat{r}^B_m(\hat{s\ell}_n)$.

PROPOSITION 9.1. (a) *The representation* $\hat{r}^B_m$ *of* $\hat{s\ell}_n$ *on* $B^{(m)}_{(n)}$ *is equivalent to the fundamental representation* $\tau_{m'}$, *where* m' *is an element of* $\{0,\dots,n-1\}$ *congruent to* $m \bmod n$.

(b) *Let* $\varepsilon = \exp 2\pi i/n$, *and for* $k \in \mathbb{Z}$, $s \in \{0,\dots,n-1\}$ *such that* $k \not\equiv 0 \bmod n$ *when* $s = 0$, *put*

$$A_{k,s} = \sum_{\substack{i,j=1 \\ i-j=k+rn}}^{n} t^r E_{ij} \varepsilon^{-sj} \ ;$$

these elements form a basis of $s\ell_n(\mathbb{C}[t,t^{-1}])$. *Then the representation of* $\hat{s\ell}_n$ *on* $B^{(m)}_{(n)}$ *is defined by the following formulas:*

$$\sum_{k\in\mathbb{Z}} A_{k,s}u^k \mapsto (1-\varepsilon^s)^{-1}(\varepsilon^{-ms}(\exp \sum_{j>0}(1-\varepsilon^{sj})u^j x_j) \cdot$$

$$\cdot (\exp - \sum_{j>0} \frac{1-\varepsilon^{-sj}}{j} u^{-j} \frac{\partial}{\partial x_j}) - 1) \qquad (s \neq 0)$$

$$A_{k,0} \mapsto \frac{\partial}{\partial x_n}, \quad A_{-k,0} \mapsto kx_k \quad \textit{for} \quad k > 0, \ k \not\equiv 0 \bmod n; \quad c \mapsto 1 \ .$$

(c) *All the representations* τ_λ *of* $\hat{s\ell}_n$ *are unitary in the sense that there exists a positive definite Hermitian form on the representation space such that*

$$(\tau_\lambda(A(t)))^* = \tau_\lambda({}^t\overline{A}(t^{-1})) \ .$$

PROOF. Thanks to (9.3), $\hat{s\ell}_n$ contains all Λ_j with $j \not\equiv 0 \bmod n$, and therefore $B^{(m)}_{(n)}$ is irreducible under $\hat{s\ell}_n$ (see (3.6)). Using (9.4), we see that $\hat{r}^B_m(\hat{h})\cdot 1 = 0$. Using (9.2) one easily sees in the fermionic picture that $\hat{r}^B_m(h)\cdot 1 = \hat{\omega}_{m'}(h)\cdot 1$. This proves (a). Furthermore, (b) follows immediately from (3.19) by putting in there $v = \varepsilon^s u$ and using (9.2), (9.3). Finally, the unitarity of the fundamental representations of $\hat{s\ell}_n$ follows from the unitarity of the fundamental representations of a_∞, using

(9.6) $$^t\overline{\varphi(A(t))} = \varphi(^t\overline{A}(t^{-1})) ,$$

where $\overline{\sum a_j t^j} = \sum \overline{a}_j t^j$. The unitarity of all representations τ_λ now follows by taking highest components of tensor products of fundamental representations. □

Recall the group $\overline{A}_\infty$, its subgroup $\overline{A}^0_\infty$ and the corresponding central extensions A_∞ and A^0_∞ by $\mathbb{C}^\times$ (see section 8). Note that

(9.7) $$\varphi(SL_n(L)) \subset \overline{A}^0_\infty .$$

To show this, it suffices to check that $\varphi(SL_n(\mathbb{C}[t])) \subset \overline{A}^0_\infty$ (which is obvious) and that $\varphi(\mathrm{diag}(t,t^{-1},1,\dots,1)) \in \overline{A}^0_\infty$, which is straightforward. The inclusion φ gives us a central extension of $SL_n(L)$ by $\mathbb{C}^\times$ which we denote by $\hat{SL}_n$.

Thanks to (9.7), the subspace $F^{(m)}$ is invariant under the representation R^F of $\hat{SL}_n$. Via the boson-fermion correspondence σ, we get a representation of $\hat{SL}_n$ on $B^{(m)} = \mathbb{C}[x_1,x_2,\dots]$.

Since the subspace $B^{(m)}_{(n)}$ of $B^{(m)}$ is invariant and irreducible under the Lie algebra $\hat{s\ell}_n$, one deduces without difficulty that the same holds for the representation of $\hat{SL}_n$ on $B^{(m)}$. This irreducible representation of $\hat{SL}_n$ on $B^{(m)}_{(n)}$ is denoted by $\hat{R}^B_m$.

Since ν_n commutes with $\hat{SL}_n$, it follows that $\hat{R}^B_m(1_m)$ is the same polynomial as $\hat{R}^B_{m\pm n}(1_{m\pm n})$. This justifies the following definition.

We let $B_{(n)} = \bigoplus_{m=0}^{n-1} B^{(m)}_{(n)}$; this is a subspace of the space $B = \bigoplus_m B^{(m)}$. The MKP (resp. KP) hierarchy of equations on B (resp. $B^{(m)}$) restricted to $B_{(n)}$ (resp. $B^{(m)}_{(n)}$) is called the n-*th reduced* MKP (resp. KP) *hierarchy* [4]. One can show, along the lines of [11], that the $\hat{SL}_n$-orbit of a sum of highest weight vectors $1_I = \bigoplus_{j\in I} 1_j$, where $I \subset \{0,\dots,n-1\}$, coincides with the set of solutions of the n-th reduced MKP hierarchy of the form $\underset{j\in I}{\oplus} f_j$, where all f_j are non-zero. Thus, the $\hat{SL}_n$-orbit of 1_I coincides with the intersection of the GL_∞-orbit of 1_I with $B_{(n)}$.

In the reminder of the paper we shall discuss in more detail the 2-nd reduced KP and MKP hierarchies. They are called the KdV and the *modified* KdV *hierarchies* for the following reason. The simplest equation of the 2-nd reduced KP is (cf. (5.5)):

$$(x_1^4 - 12x_1x_3)f\cdot f = 0 . \tag{9.8}$$

Furthermore the simplest equation of the 2-nd reduced 1-st MKP is (cf. (5.7)):

$$x_1^2 g\cdot f = 0 . \tag{9.9}$$

Putting $x_1 = x$, $x_3 = t$, $u(x,t) = 2\frac{\partial^2}{\partial x^2}(\log f)$, $v(x,t) = \frac{\partial}{\partial x}(\log \frac{g}{f})$, we obtain from (9.8) and (9.9):

$$\frac{\partial u}{\partial t} = \frac{3}{2} u \frac{\partial u}{\partial x} + \frac{1}{4}\frac{\partial^3 u}{\partial x^3} , \tag{9.10}$$

$$u = - v^2 - \frac{\partial v}{\partial x} , \tag{9.11}$$

$$\frac{\partial v}{\partial t} = - \frac{3}{2} v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} . \tag{9.12}$$

The equation (9.10) is the classical KdV equation, (9.11) is the Miura transformation and (9.12) is the classical modified KdV equation (see e.g., [4]).

As has been pointed out in the proof of Proposition 9.1, the vertex operator for the fundamental representations of $\hat{s\ell}_2$ is obtained from $\Gamma(u,v)$ by putting $v = -u$. Hence the N-soliton solution for the KdV (resp. MKdV) hierarchy will be $f_{0;a;u,-u}(x)$ (resp. $f_{0;a;u,-u}(x) \oplus f_{1;a;u,-u}(x)$), where $f_{m;a;u,v}(x)$ is defined by (7.6) and given explicitly by (7.7). Note that

$f_{m;a;n,-n}$ is a function of $x_1, x_3, \ldots$ only.

In order to study polynomial solutions of the KdV and MKdV hierarchies, recall the Bruhat decomposition (see e.g., [8]):

$$SL_2(L) = \bigcup_{k\in\mathbb{Z}} (Mt_kM \cup Mt_krM) , \tag{9.13}$$

where

$$M = \{A(t) \in SL_2(C[t]) \,|\, A(0) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} ,$$

$$t_k = \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

and the union is disjoint. Put $s_k(x) = S_{k,k-1,\ldots,1}(x)$ for short.

We want to describe the $SL_2(L)$-orbit of the highest weight vector 1 (resp. of the sum of highest weight vectors $\mathbf{1} = 1_o \oplus 1_1$) in the projectivized fundamental representation $\mathbb{P}B^{(0)}_{(2)}$ (resp. in the sum $\mathbb{P}B^{(0)}_{(2)} \oplus \mathbb{P}B^{(1)}_{(2)}$). These will be (up to arbitrary constant factors) the polynomial solutions of the KdV (resp. MKdV) hierarchies. In view of (9.13), we shall need for that

LEMMA 9.2.

$t_k\cdot\mathbf{1} = s_{2k}(x) \oplus s_{2k-1}(x)$ *if* $k \geq 1$;

$t_{-k}\cdot\mathbf{1} = s_{2k-1}(x) \oplus s_{2k}(x)$ *if* $k \geq 1$;

$t_kr\cdot\mathbf{1} = s_{2k}(x) \oplus s_{2k+1}(x)$ *if* $k \geq 0$;

$t_{-k}r\cdot\mathbf{1} = s_{2k-1}(x) \oplus s_{2k-2}(x)$ *if* $k \geq 1$.

PROOF. Using (8.2) we obtain the following formula for the action of t_k on $\mathbb{C}^\infty$:

$$t_k\cdot v_{2j} = v_{2(k+j)}; \quad t_k\cdot v_{2j+1} = v_{2(-k+j)+1} .$$

The corresponding projective action of t_k on $\Lambda^\infty\mathbb{C}^\infty$ is:

$$t_k(v_{i_1} \wedge v_{i_2} \wedge \ldots) = \pm \text{ reordering of } ((t_k\cdot v_{i_1}) \wedge (t_k\cdot v_{i_2}) \wedge \ldots) .$$

For example,

$$t_k(v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots) = v_{2k} \wedge v_{2k-2} \wedge \dots \wedge v_{-2k} \wedge v_{-2k-1} \wedge \dots .$$

The boson-fermion correspondence (Theorem 4.1) completes the proof. □

We can prove now the main result of this section.

THEOREM 9.1. (a) *The orbit* $SL_2(L)\cdot 1$ *in* $\mathbb{P}\mathbb{C}[x_1,x_3,\dots]$ *is a disjoint union of cells* $\{S_{k,k-1,\dots,1}(x_1+c_1,x_3+c_3,\dots)\,|\,c_i\in\mathbb{C}\}$ *of dimension* $k\in\mathbb{Z}_+$. (b) *The orbit* $SL_2(L)\cdot(1\oplus 1)$ *in* $\mathbb{P}\mathbb{C}[x_1,x_3,\dots]\oplus\mathbb{P}\mathbb{C}[x_1,x_3,\dots]$ *is a disjoint union of cells* $\{S_{k,k-1,\dots,1}(x_1+c_1,x_3+c_3,\dots)\oplus S_{r,r-1,\dots,1}(x_1+c_1,x_3+c_3,\dots)\,|\,c_i\in\mathbb{C}\}$ *of dimension* $\max\{k,r\}$, *where* $k,r\in\mathbb{Z}_+$ *and* $|k-r|=1$ *or* $k=r=0$.

PROOF. We sketch here a proof of (a); the proof of (b) is exactly the same. Since $M\cdot 1 = 1$, and $r\cdot 1 = 1$, we have, using the Bruhat decomposition, the following disjoint union:

$$SL_2(L)\cdot 1 = \bigcup_{k\in\mathbb{Z}} Mt_k\cdot 1 . \tag{9.14}$$

Recall that $\hat{s\ell}_2$ acting on $\mathbb{C}[x_1,x_3,\dots]$ contains all $\frac{\partial}{\partial x_{2j-1}}$, $j=1,2,\dots$. Hence the transformation $\exp\lambda\frac{\partial}{\partial x_{2j-1}}$ can be approximated by elements from M. Since $(\exp\lambda\frac{\partial}{\partial x})f(x) = f(x+\lambda)$, we deduce from Lemma 8.2 and (9.14):

$$\left.\begin{aligned} Mt_k\cdot 1 &\supset \{S_{2k,2k-1,\dots,1}(x_1+c_1,x_3+c_3,\dots)\}, \\ Mt_{-k}\cdot 1 &\supset \{S_{2k-1,2k-2,\dots,1}(x_1+c_1,x_3+c_3,\dots)\}, \end{aligned}\right\} k>0 . \tag{9.15}$$

Note that the inclusion in (9.15) is a rational regular map of affine spaces. An easy algebro-geometric fact states that if these spaces have the same dimensions, then the inclusion is a bijection. Thus, to complete the proof, we need to show that the algebraic varieties on both sides of (9.15) have the same dimension. But one knows that [8]:

(9.16) $$\dim Mt_k \cdot 1 = 2k, \quad \dim Mt_{-k} \cdot 1 = 2k - 1 (k > 0) .$$

On the other hand, putting $\mathscr{s}_+ = \sum_{j>0} \mathbb{C}\Lambda_{2j-1}$, we have (see Proposition 9.1 (c)):

$$\mathscr{s}_+(t_k \cdot 1) = \sum_{j>0} \mathbb{C}t^j \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} t_k \cdot 1 = t_k \cdot \sum_{j>0} \begin{pmatrix} 0 & t^{j-2k} \\ t^{j+1+2k} & 0 \end{pmatrix} \cdot 1 .$$

Hence

(9.17) $$\dim \mathscr{s}_+(t_k \cdot 1) = 2k, \quad \dim \mathscr{s}_+(t_{-k} \cdot 1) = 2k - 1 (k > 0) .$$

Comparing (9.16) with (9.17) completes the proof. □

COROLLARY 9.1. (a) *The rational functions*

$$u(x,t) = 2 \frac{\partial^2}{\partial x^2} \log S_{k,k-1,\ldots,1}(x + c_1, t + c_3, c_5, \ldots)$$

are solutions of the classical KdV *equation.*

(b) *The rational functions*

$$v(x,t) = \pm \frac{\partial}{\partial x} \log(S_{k,k-1,\ldots,1}(x + c_1, t + c_3, c_5, \ldots)/S_{k+1,k,\ldots,1}(x+c_1, t+c_3, c_5, \ldots))$$

are solutions of the classical MKdV *equation.*

REMARK. Whereas Theorem 9.1 (a) was pointed out by several authors [4], [5], [13], Theorem 9.1 (b) is probably new.

REFERENCES

[1] E. DATE, M. JIMBO, M. KASHIWARA, T. MIWA, Operator approach to the Kadomtsev-Petviashuili equation. Transformation groups for soliton equations III, *J. Phys. Soc. Japan* 50(1981), 3806-3812.

[2] E. DATE, M. JIMBO, M. KASHIWARA, T. MIWA, Transformation groups for soliton equations, in: *Proceedings of RIMS Symposium*, M. Jimbo and T. Miwa, eds., World Scientific, 1983, 34-120.

[3] P.A.M. DIRAC, *The Principles of Quantum Mechanics*, 4th edition, Oxford University Press, 1958.

[4] M. JIMBO, T. MIWA, Solitons and infinite dimensional Lie algebras, *Publ. RIMS* 19(1983), 943-1001.

[5] V.G. KAC, *Infinite Dimensional Lie Algebras*, Progress in Mathematics 44, Birkhäuser, Boston, 1983. Second edition, Cambridge University Press, 1985.

[6] V.G. KAC, D.A. KAZHDAN, J. LEPOWSKY, R.L. WILSON, Realization of the basic representation of the Euclidean Lie algebras, *Advances in Math.* 42(1981), 83-112.

[7] V.G. KAC, D.H. PETERSON, Spin and wedge representations of infinite dimensionae Lie algebras and groups, *Proc. National Acad. Sci. USA* 78(1981), 3308-3312.

[8] V.G. KAC, D.H. PETERSON, Regular functions on certain infinite dimensional groups, *Progress in Math.* 36, Birkhäuser, Boston, 1983.

[9] M. KASHIWARA, T. MIWA, The τ-function of the Kadomtsev-Petviashuili equation, *Proc. Japan Acad.* 57A(1981), 342-347.

[10] I.G. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1979.

[11] D.H. PETERSON, V.G. KAC, Infinite flag varieties and conjugacy theorems, *Proc. National Acad. Sci. USA* 80(1983), 1778-1782.

[12] M. SATO, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, *RIMS Kokyuroku* 439(1981), 30-40.

[13] G. SEGAL, G. WILSON, Loop groups and equations of KdV type, *Publ. Math. IHES* 63(1985), 1-64.