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112 constructions of the basic representation of the loop group of E_8.

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In this report, we describe a natural family of vertex constructions of the basic representation of the affine Kac-Moody algebra $\hat{\underline{g}}$ associated to a simple finite-dimensional Lie algebra $\underline{g}$ of type A_n, D_n, E_6, E_7 or E_8. Namely, we show that maximal Heisenberg subalgebras of $\hat{\underline{g}}$ are parametrized, up to conjugacy, by conjugacy classes of elements of the Weyl group W of $\underline{g}$. Given $w \in W$, let $\hat{s}_w$ denote the associated Heisenberg subalgebra of $\hat{\underline{g}}$, and let $\tilde{S}_w$ denote the centralizer of $\hat{s}_w$ in the loop group $\tilde{G}$ of the simply-connected group G whose Lie algebra is $\underline{g}$. We show that the basic representation (V, π_0) of $\hat{\underline{g}}$ remain irreducible under the pair $(\hat{s}_w, \tilde{S}_w)$. This leads to a vertex construction of V, so that for $w = 1$ (resp. w = Coxeter element) we recover the homogeneous [2] (resp. principal [10]) realization; for $w = -1$ we recover the construction of [3]. Thus, to each conjugacy class of W we associate canonically a vertex realization of the basic representation of $\hat{\underline{g}}$. In particular, in the case of $\hat{E}_8$ we obtain 112 such constructions.

The homogeneous realization of $\hat{E}_8$ plays an important role in the construction of the heterotic string [5]. We hope that the large variety of constructions of $\hat{E}_8$ provided by this paper could be useful for the treatment of various symmetry breaking patterns.

1. We feel that it is conceptually appropriate to start with the general framework of Kac-Moody Lie algebras.

Let Γ be a finite graph and let Γ_0 be the set of vertices of Γ. Two vertices can be connected by several lines, but we exclude tadpoles. The <u>Cartan matrix</u> of Γ is the $n \times n$-matrix $A = (a_{ij})_{i,j\in\Gamma_0}$ defined as follows: $a_{ii} = 2$, and $-a_{ij} = -a_{ji}$ is the number of lines connecting vertices i and j.

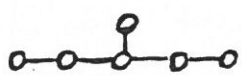
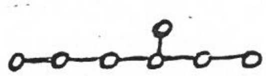
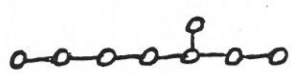
Define the <u>root lattice</u> Q as the lattice on a basis $\{h_i\}_{i\in\Gamma_0}$, with the bilinear form $(h_i|h_j) = a_{ij}$. This is an even integral lattice (i.e. $(h|h)$ is an even integer for $h \in Q$).

Define the <u>Weyl group</u> W as the subgroup of the group of automorphisms of Q generated by the <u>fundamental reflections</u> r_i, $i \in \Gamma_0$, defined by

$$(1.1) \qquad r_i(h_j) = h_j - a_{ji}h_i \ .$$

The following two types of graphs are relevant to our discussion.

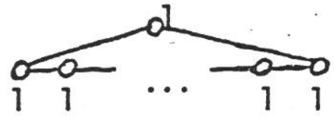
I. <u>Finite-type graphs</u>. These are the Γ with a positive-definite Cartan matrix A , or, equivalently, with a finite Weyl group W . Here is the well-known list of connected finite-type graphs $(|\Gamma_0| = n)$:

Type	Graph Γ	det A
A_n		n + 1
D_n		4
E_6		3
E_7		2
E_8		1

It is well-known that every positive-definite even lattice Λ spanned over $\mathbb{Z}$ by elements α with $(\alpha|\alpha) = 2$ is a direct sum of lattices of type A, D or E. Then the discriminant of Λ (= $(\text{vol }\Lambda)^2$ = square of the volume of a fundamental cell) is the product of the corresponding determinants (given by the table).

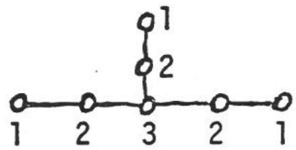
II. <u>Affine graphs</u>. These are the connected graphs with a positive-semidefinite Cartan matrix A such that $\det A = 0$. They are characterized by having a labelling by (relatively prime) positive integers a_i such that each label equals half of the sum of its neighbors. To each connected finite-type graph Γ, one canonically associates an affine graph $\hat{\Gamma}$ by adding a complementary vertex 0 . The list of affine graphs together with the labels a_i is given below $(|\hat{\Gamma}_0| = n+1)$:

$\hat{A}_n$

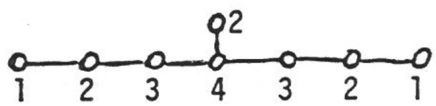


$\hat{D}_n$

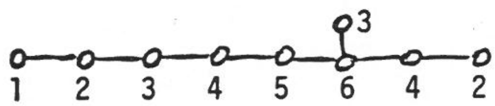
$\hat{E}_6$



$\hat{E}_7$



$\hat{E}_8$



The number $h = \sum_i a_i$ is called the Coxeter number.

2. The Kac-Moody algebra $\underline{g}(\Gamma)$ is the Lie algebra on Chevalley generators e_i, f_i, h_i $(i \in \Gamma_0)$ and the following defining relations:

$$[e_i, f_j] = \delta_{ij} h_i \; ; \; [h_i, h_j] = 0 \; ;$$
$$[h_i, e_j] = a_{ij} e_j \; ; \; [h_i, f_j] = -a_{ij} f_j \; ;$$
$$[\ldots[[e_i, \underbrace{e_j], e_j] \ldots, e_j}_{1-a_{ji} \text{ times}}] = 0 \; ; \; [\ldots[[f_i, \underbrace{f_j], f_j] \ldots, f_j}_{1-a_{ji} \text{ times}}] = 0 \; .$$

An exposition of the theory of Kac-Moody algebras along with some of its many applications to other fields of mathematics may be found in [9]. Here we note only that $\dim \underline{g}(\Gamma) < \infty$ if and only if Γ is of finite type, and $\underline{g}(\Gamma)$ is a simple Lie algebra of type A, D or E if Γ is of that type. (Allowing non-symmetric A, one recovers the algebras of types B, C, G and F as well.)

The commutative subalgebra $\underline{h} = \sum_i \mathbb{C} h_i$ of $\underline{g}(\Gamma)$ is called a Cartan subalgebra; it contains the root lattice Q and the Weyl group W acts on $\underline{h}$ by formula (1.1), preserving Q .

Finally, note that an affine Kac-Moody algebra $\underline{g}(\hat{\Gamma})$ has a 1-dimensional center spanned by the canonical central element $c = \Sigma a_i h_i$.

3. Given a collection of numbers $\bar{\lambda} = \{\lambda_i\}_{i \in \Gamma_0}$, define a highest weight representation $\pi_{\bar{\lambda}}$ of $\underline{g}(\Gamma)$ on a vector space $L(\bar{\lambda})$ as an irreducible represenation which admits a non-zero vector $v_{\bar{\lambda}}$ (heighest weight vector) such that [6]:

$$(3.1) \quad \pi_{\bar{\lambda}}(e_i)v_{\bar{\lambda}} = 0 \ , \ \pi_{\bar{\lambda}}(h_i)v_{\bar{\lambda}} = \lambda_i v_{\bar{\lambda}} \ ; \ i \in \Gamma_0 \ .$$

This representation is unitary (i.e. it carries a positive-definite Hermitian form such that $\pi_{\bar{\lambda}}(e_i)^* = \pi_{\bar{\lambda}}(f_i)$) if and only if all λ_i are non-negative integers [4], [12]. A unitary representation $L(\bar{\lambda})$ is uniquely determined by its highest weight $\bar{\lambda}$ [9, Proposition 10.4].

Note that in the finite-type case the unitary $L(\bar{\lambda})$ are precisely all finite-dimensional irreducible representations of $\underline{g}(\Gamma)$.

Now let $\underline{g}(\hat{\Gamma})$ be an affine Kac-Moody algebra, and let $L(\bar{\lambda})$ be a highest weight representation. The number $\sum_i a_i \lambda_i$ is called the level of $L(\bar{\lambda})$. It is clear that the level is the eigenvalue of c and that the level of a unitary $L(\bar{\lambda})$ is a non-negative integer, which is zero if and only if $\bar{\lambda} = \bar{0}$, i.e. $L(\bar{\lambda})$ is a trivial 1-dimensional representation. Note that if λ_i is the i-th fundamental representation, i.e. $\lambda_i = 1$, $\lambda_j = 0$ for $j \neq i$, then the level is a_i . It is thus clear from the table that all level 1 represenations of $\underline{g}(\hat{\Gamma})$ are equivalent to π_0 via an isometry of the graph $\hat{\Gamma}$. Thus, $\underline{g}(\hat{\Gamma})$ has a distinguished highest weight representation π_0 , called the basic representation. We will denote the space of this representation by V for short.

4. Here we will explain how to construct the finite-dimensional Lie algebras $\underline{g}(\Gamma)$, where Γ is A_n, D_n or E_n [2]. First, take the root lattice $Q \subset \mathbb{C}^n = \underline{h}$. An explicit construction of it is well known. For example, in the most non-trivial case, E_8 , which is currently of

primary interest to physicists, Q consists of all 8-vectors such that the sum of coordinates is an even integer and either all the coordinates are integers or all the coordinates are half-integers ($\in \frac{1}{2} + \mathbb{Z}$); the bilinear form is standard: $(\alpha|\beta) = \Sigma\, \alpha_i \beta_i$ (see e.g. [15]).

The set of roots of $\underline{g}(\Gamma)$ is

$$\Delta = \{\alpha \in Q \;(\alpha|\alpha) = 2\} .$$

In the case of E_8, in the standard basis $\{\varepsilon_i\}$ of $\mathbb{C}^8$ we have:

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \text{ with } i \neq j \; ; \; \frac{1}{2}(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8) \text{ with even number of minuses}\}.$$

Then $\underline{g}(\Gamma) = \underline{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{C}\, e_\alpha)$ with the following commutation relations [2]:

$[\alpha,\beta] = 0$ if $\alpha,\beta \in \underline{h}$;
$[\alpha,e_\beta] = (\alpha|\beta)e_\beta$ if $\alpha \in \underline{h}$, $\beta \in \Delta$;
$[e_\beta,e_{-\beta}] = -\beta$ if $\beta \in \Delta$;
$[e_\alpha,e_\beta] = 0$ if $\alpha,\beta \in \Delta$, but $\alpha+\beta \notin \Delta \cup \{0\}$;
$[e_\alpha,e_\beta] = \varepsilon(\alpha,\beta)e_{\alpha+\beta}$ if $\alpha,\beta,\alpha+\beta \in \Delta$.

Here $\varepsilon(\alpha,\beta)$ is a bimultiplicative function on $Q\times Q$ with values in $\{\pm 1\}$ (i.e. $\varepsilon(\alpha+\alpha',\beta) = \varepsilon(\alpha,\beta)\varepsilon(\alpha',\beta)$ and $\varepsilon(\alpha,\beta+\beta') = \varepsilon(\alpha,\beta)\varepsilon(\alpha,\beta')$) satisfying the property:

$$\varepsilon(\alpha,\alpha) = (-1)^{\frac{1}{2}(\alpha|\alpha)}$$

and hence the property $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}$.

5. Now we turn to the construction of affine Lie algebras $\underline{g}(\hat{\Gamma})$, where Γ is a graph of type A, D or E. Normalize the Killing form on the finite-dimensional Lie algebra $\underline{g} = \underline{g}(\Gamma)$ by the condition $(\alpha|\alpha) = 2$ for $\alpha \in \Delta$, as above.

Consider the loop algebra $\tilde{\underline{g}} = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \underline{g}$. This is the Lie algebra of loops on $\underline{g}$, i.e., regular rational maps $\mathbb{C}^\times \to \underline{g}$.

Consider the following central extension of $\tilde{\underline{g}}$ by a 1-dimensional center $\mathbb{C}c$:

(5.1) $\quad \hat{\underline{g}} = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \underline{g} + \mathbb{C}c$,

with the commutation relations:

(5.2) $[t^k \otimes a, t^{k'} \otimes a'] = t^{k+k'} \otimes [a,a'] + k\delta_{k,-k'}(a|a')c$.

We identify $\underline{g}$ with the subalgebra $1 \otimes \underline{g}$ of $\hat{\underline{g}}$.

Choosing an orthonormal basis Q^a of $\underline{g}$ so that $[Q^a,Q^b] = f_{abc}Q^c$, we get a basis $Q^a_k = t^k \otimes Q^a$ of $\mathbb{C}[t,t^{-1}] \otimes \underline{g}$ with the following commutation relations familiar to physicists:

$[Q^a_k,Q^b_{k'}] = f_{abc}\ Q^c_{k+k'} + k\delta_{k,-k'}\ \delta_{a,b}\ c$.

To show that $\hat{\underline{g}}$ is the affine Kac-Moody algebra $\underline{g}(\hat{\Gamma})$, choose a set of simple roots $\{\alpha_i\}_{i \in \Gamma_0}$ of $\underline{g}$ and put $e_i = e_{\alpha_i}$ and $f_i = -e_{-\alpha_i}$ for $i \in \Gamma_0$, $e_0 = t \otimes e_{-\theta}$, $f_0 = -t^{-1} \otimes e_\theta$, $h_0 = c-\theta$, where θ is the highest root of $\underline{g}$. Then the elements e_i,f_i,h_i, $i \in \Gamma_0$, satisfy all the relations of Section 2 with (a_{ij}) being the Cartan matrix of the graph $\hat{\Gamma}$. The fact that there are no further relations is less obvious [9, Theorem 9.11].

6. Recall that the Virasoro algebra is a Lie algebra $\underline{d}$ with a basis $\{d_k\ (k \in \mathbb{Z});\ \tilde{c}\}$ and the following commutation relations:

$[d_k,d_r] = (k-r)d_{k+r} + \frac{1}{12}(k^3-k)\ \delta_{k,-r}\ \tilde{c}$; $[\tilde{c},d_k] = 0$.

This Lie algebra acts by derivations of $\hat{\underline{g}}$ by:

$[d_k,P\otimes a + \lambda c] = -t^{k+1}\frac{dp}{dt} \otimes a$; $[\tilde{c},\hat{\underline{g}}] = 0$.

Thus, we get a semidirect sum $\underline{d} \dotplus \hat{\underline{g}}$ of Lie algebras.

Let $(L(\bar{\lambda}),\pi)$ be a unitary highest weight representation of $\hat{\underline{g}}$ of level m . It is a well-known fact which goes back to Sugawara [16] that the representation $\pi_{\bar{\lambda}}$ of $\hat{\underline{g}}$ on $L(\bar{\lambda})$ can be extended uniquely to the whole semidirect sum $\underline{d} \dotplus \hat{\underline{g}}$. The value of the central charge $\tilde{c}$ is then equal to $(\dim \underline{g})m/(m+h)$ (see e.g. [9, Exercises 12.11 and 12.12]). In particular, the eigenvalue of $\pi_0(\tilde{c})$ is n (the rank of $\underline{g}$).

Furthermore, the energy operator $\pi_{\bar{\lambda}}(d_0)$ is diagonalizable, its eigenvalues have finite multiplicities and are non-negative numbers of the form $k + \frac{b}{2(m+h)}$, where b is the eigenvalue of the Casimir of $\underline{g}$ on $V_{\bar{\lambda}}$ and $k = 0,1,2,\dots$. For the basic representation, $b = 0$ and its multiplicity is 1 .

7. Let (L,π) be the direct sum of all unitary highest weight representations $(L(\bar{\lambda}),\pi_{\bar{\lambda}})$ of $\underline{g}$. When restricted to each 3-dimensional subalgebra $\mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i$, L decomposes into a direct sum of finite-dimensional subrepresentations, and hence can be exponentiated to a represenation $\phi_i : SL_2(\mathbb{C}) \to \mathrm{Aut}\, L$. The group $\hat{G}$ generated by all $\phi_i(SL_2(\mathbb{C}))$, $i \in \hat{\Gamma}_0$, is the Kac-Moody group attached to $\hat{\underline{g}}$. The representation of $\hat{G}$ on $L(\bar{\lambda})$ will be also denoted by $\pi_{\bar{\lambda}}$ (and for the fundamental representations by π_i).

The group $\hat{G}$ can be described as follows [12], [13]. Let G be the simply-connected complex Lie group whose Lie algebra is $\underline{g}$. Let $H = \exp \underline{h} \subset G$. Let $\tilde{G}$ be the loop group, i.e. the group of all rational regular maps $\mathbb{C}^{\times} \to G$ with pointwise multiplication. $\tilde{G}$ may be viewed as the infinite-dimensional group attached to the loop algebra $\tilde{\underline{g}}$. Then $\hat{G}$ is a central extension of $\tilde{G}$ by $\mathbb{C}^{\times}$. More precisely, $\mathbb{C}^{\times}$ imbeds into the center of $\hat{G}$ by $\nu : t \to \prod_i \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{a_i}$, and $1 \to \mathbb{C}^{\times} \xrightarrow{\nu} \hat{G} \xrightarrow{\nu_1} \tilde{G} \to 1$ is an exact sequence.

The subgroup of $\hat{G}$ generated by the $\phi_i(SL_2(\mathbb{C}))$ with $i \in \Gamma_0$ is isomorphic to G, and we will identify G with it. Note that ν_1 identifies the subgroup G of $\hat{G}$ in the obvious way with the subgroup of constant loops of G. Note also that the center C of $\hat{G}$ is $\nu(\mathbb{C}^{\times}) \times (\text{center}\,(G))$ [13], and that C acts on V by:

$$(7.1) \qquad \pi_0\,(\nu(a)) = a \quad \text{for} \quad a \in \mathbb{C}^{\times}\ ; \quad \pi_0\,(\text{center}\,(G)) = 1\ .$$

The operators $\pi_{\bar{\lambda}}(g)$, $g \in \hat{G}$, are not of trace class. The situation is fixed as follows. Let $L(\bar{\lambda}) = \bigoplus_k L(\bar{\lambda})_k$ be the eigenspace decomposition with respect to the energy operator $\pi_{\bar{\lambda}}(d_0)$. It is invariant with respect to G since $[\underline{g},d_0] = 0$. We define the character of the representation $\pi_{\bar{\lambda}}$ as follows:

$$(7.2) \qquad \mathrm{ch}\, L(\bar{\lambda})(g) = \sum_k (\mathrm{tr}_{L(\bar{\lambda})_k} \pi_{\bar{\lambda}}(g))q^k\ , \quad g \in G\ .$$

Thus, to every $g \in G$ we associate a formal power series in $q^{\frac{1}{N}}$, some N (which converges if $|q| < 1$). One knows the Weyl-Kac formula for the

the character $\mathrm{ch}\ L(\bar{\lambda})$ [6] of an arbitrary unitary highest weight representation $\pi_{\bar{\lambda}}$ of $\hat{\underline{g}}$ (and even of any Kac-Moody algebra). For the basic representation (V,π_0), the character formula takes an especially simple form [7]:

$$(7.3)\quad (\mathrm{ch}\ V)(e^h) = \sum_{\gamma\in Q} e^{(\gamma|h)}\, q^{\frac{1}{2}(\gamma|\gamma)}/\phi(q)^n\ ,\ h \in \underline{h} \subset \underline{g}\ .$$

Here $Q \subset \underline{h}$ is the root lattice of $\underline{g}$ and $\phi(q) = \prod_{k=1}^{\infty} (1-q^k)$. Note that the numerator is nothing else but the Riemann theta function, and the denominator is, up to a factor $q^{n/24}$, a power of the Dedekind η-function. In particular, we get the following formula for the partition function $\dim_q V := \sum (\dim V_k)q^k$ $(= (\mathrm{ch}\ V)(1))$:

$$\dim_q V = \sum_{\gamma\in Q} q^{\frac{1}{2}(\gamma|\gamma)}/\phi(q)^n\ ,$$

a special case of which for $\hat{E}_8$ is [8]

$$q^{-\frac{1}{3}} \dim_q V = \sqrt[3]{j(q)}\ ,$$

where j is the celebrated modular invariant [15].

8. Given a complex algebraic variety X, we denote (as above) by $\tilde{X}$ the __loop space__ of X, the space of all regular rational maps of $\mathbb{C}^\times$ into X. Let H $(= G/N_G(H)$, where $N_G(H)$ is the normalizer of the Cartan subgroup H) be the variety of all Cartan subalgebras of $\underline{g}$, and let $\tilde{H}$ be its loop space.

Given a loop $s \in \tilde{H}$ we define the associated __Heisenberg subalgebra__ of $\tilde{\underline{g}}$:

$$\tilde{s} = \{p \in \tilde{\underline{g}} | p(t) \in s(t) \quad \text{for all} \quad t \in \mathbb{C}^\times\}\ .$$

This is a maximal commutative subalgebra of the loop algebra $\tilde{\underline{g}}$. (The name "Heisenberg" will become clear later.) Regarding s as a vector bundle over $\mathbb{C}^\times$ with fiber $s(t)$ over $t \in \mathbb{C}^\times$, we may view $\tilde{s}$ as a space of sections of the bundle s.

__Examples__: Taking s to be a map of $\mathbb{C}^\times$ into one point $\underline{h} \in H$, we get $\tilde{s} = \tilde{\underline{h}} = \mathbb{C}[t,t^{-1}]\otimes\underline{h} \subset \underline{g}$. This is the <u>homogeneous Heisenberg subalgebra</u> of $\tilde{\underline{g}}$. Another important example, the <u>principal Heisenberg subalgebra</u> of $\tilde{\underline{g}}$, corresponds to the loop $s : t \to$ centralizer in $\underline{g}$ of the element $te_{-\theta} + \sum_{i=1}^{n} e_{\alpha_i}$. In the particular case of $\underline{g} = s\ell_n(\mathbb{C})$ putting

$$s(t) = \text{centralizer of } \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix} ; \quad a_{k,m}(t) = t^k \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix}^m ,$$

the principal Heisenberg subalgebra of $\tilde{s\ell}_n(\mathbb{C})$ is the linear span of the elements $a_{k,m}$ for $m,k \in \mathbb{Z}$, $m \not\equiv 0 \bmod n$.

The homogeneous and principal Heisenberg subalgebras are "responsible" for the homogeneous and principal constructions of the basic representation (V,π_0) of $\hat{\underline{g}}$ ([2], [10]). We shall generalize these constructions to the case of an arbitrary Heisenberg subalgebra $\tilde{s}$.

Similarly, given a loop $s \in \tilde{H}$, we define the associated Heisenberg subgroup of $\tilde{G}$:

$$\tilde{S} = \{g \in \tilde{G} | g(t) \in \exp s(t) \quad \text{for all} \quad t \in \mathbb{C}^{\times}\}$$

Note that the subgroup $\tilde{S}$ is the centralizer of $\tilde{s}$ in $\tilde{G}$.

9. Fix $w \in W$ and let m be the order of w . Let $\underline{h} = \bigoplus_{j\in \mathbb{Z}/m\mathbb{Z}} \underline{h}_{j/m}$ be the eigenspace decomposition for w , where $\underline{h}_k$ denotes the eigenspace attached to $e^{-2\pi ik}$, so that $\underline{h}_0$ is the fixed point set of w . Let σ be a lifting of w in G , i.e. σ is such that $\mathrm{Ad}\sigma$ leaves $\underline{h}$ invariant and $\mathrm{Ad}\sigma|_{\underline{h}} = w$. There exists $x \in \underline{g}$ (not unique) such that

(9.1) $\sigma = \exp 2\pi i\, x$ and $[x,\underline{h}_0] = 0$.

Given $a \in \underline{g}$, we write $a = \sum_{\lambda} a_{\lambda}$ as a sum of eigenvectors a_{λ} of σ with distinct eigenvalues λ . For $k \in \mathbb{C}$ such that $e^{-2\pi ik} = \lambda$, we define the loop $a(k)' \in \tilde{\underline{g}}$ to be $e^{i\phi k}\,\mathrm{Ad}(\exp i\phi x)a_\lambda$ at $e^{i\phi}$. It is clear that $a(k)'$ is well-defined. Note also that $\underline{h}(0)' = \underline{h}_0$ and that $\underline{h}(k) \neq 0$ implies that $k \in \frac{1}{m}\mathbb{Z}$.

We put $a(k) = a(k)' + \delta_{k,0}(a|x)c \in \hat{\underline{g}}$. Then we have the following commutation relations:

(9.2) $[a(k),b(\ell)] = [a,b](k\dot{+}\ell)+k\delta_{k,-\ell}(a|b)c$ if $a = a_\lambda$, where $\lambda = \exp(-2\pi ik)$.

We put:

$$\tilde{s}_w = \bigoplus_{k\in\frac{1}{m}\mathbb{Z}} \underline{h}(k)' ;$$

$$\hat{s}_w^{\pm} = \bigoplus_{k>0} \underline{h}(\pm k)\ ,\ \hat{s}_w = \hat{s}_w^- + \mathbb{C}c + \hat{s}_w^+ .$$

Note that $\tilde{s}_w$ is a commutative subalgebra of $\tilde{\underline{g}}$ and $\hat{s}_w$ is a subalgebra of $\hat{\underline{g}}$ isomorphic to an infinite-dimensional Heisenberg subalgebra. The latter fact is clear by (9.2):

(9.3) $[h_1(k),h_2(\ell)] = k\delta_{k,-\ell}(h_1|h_2)c$ if $h_1 \in \underline{h}_k$, $h_2 \in \underline{h}_\ell$.

Here k and ℓ run over $\frac{1}{m}\mathbb{Z}$.

For $\alpha \in \underline{h}_0$ and $\beta \in \underline{h}$ such that $\beta - w(\beta) \in \alpha+Q$, define the loop $h(\alpha,\beta)' \in \tilde{G}$ to be $(\exp i\phi x)(\exp i\phi\alpha+2\pi i\beta)(\exp -i\phi x)$ at $e^{i\phi}$, and let $h(\alpha,\beta)$ be a lift of $h(\alpha,\beta)'$ to $\hat{G}$.

We put $\tilde{S}_w = \{h(\alpha,\beta)'\}$; this is a commutative subgroup of $\tilde{G}$. Let $\hat{S}_w = \nu_1^{-1}(\tilde{S}_w)$ be the preimage of $\tilde{S}_w$ in $\hat{G}$. Then the connected component of unity of $\hat{S}_w$ is $\hat{S}_w^0 = \exp(\underline{h}_0 + \mathbb{C}c)$.

<u>Proposition</u>. (a) $\tilde{s}_w$ (resp. $\tilde{S}_w$) is the Heisenberg subalgebra of $\tilde{\underline{g}}$ (resp. Heisenberg subgroup of $\tilde{G}$), corresponding to the loop $t \to \mathrm{Ad}(\exp(\log t)x)\,\underline{h}$ of $\tilde{H}$.

(b) The Heisenberg subalgebras $\tilde{s}_w$, corresponding to a set of representatives w of conjugacy classes of W, form a complete non-redundant list of Heisenberg subalgebras of $\tilde{\underline{g}}$ up to conjugacy.

<u>Examples</u>. If $w = 1$, then $\tilde{s}_w = \tilde{\underline{h}}$ is the homogeneous Heisenberg

subalgebra of $\tilde{\underline{g}}$ and $\tilde{S}_w = \tilde{H}$. If $w = r_1 \ldots r_n$ is the Coxeter element, then $\tilde{s}_w$ is the principal Heisenberg subalgebra of $\tilde{\underline{g}}$ and $\tilde{S}_w$ is center $(G) \subset \tilde{G}$.

10. Here we introduce the important notion of the <u>defect</u> of $w \in W$. Put

$$M_w = \{\alpha \in \underline{h} \mid \alpha - w(\alpha) \in Q\} ,$$

and define on M_w the following bimultiplicative function:

$$\psi(\alpha,\beta) = \exp 2\pi i(\alpha \mid \beta - w(\beta)) \quad \text{for} \quad \alpha,\beta \in M_w .$$

One easily shows that ψ is an alternating form (i.e. $\psi(\alpha,\alpha) = 1$). Let M_w' be its radical; note that $M_w' = Q^* + \underline{h}_0$, where Q^* is the dual lattice to Q (the so-called weight lattice). Since ψ is alternating, the order of the finite group $\overline{M}_w = M_w/M_w'$ is a square of a positive integer c_w . We call c_w the <u>defect</u> of w . In other words, c_w^2 is the number of connected components of the group Ad H^w (this uses the exponential map $2\pi i\, M_w \to H^w$).

Here are all possible values of the defect:

A_n : $c_w = 1$; D_n : $c_w = 2^k$, where $k \leq [\frac{n-2}{2}]$;

E_6 : $c_w = 1$, 2 or 3 ; E_7 : $c_w = 1$, 2, 3, 4 or 8 ;

E_8 : $c_w = 1$, 2, 3, 4, 5, 6, 8, 9 or 16.

Let $\underline{h}_0^{\perp}$ be the orthocomplement to $\underline{h}_0$ in $\underline{h}$ and let p denote the orthogonal projection of $\underline{h}$ on $\underline{h}_0$. Let w_* denote the restriction of w to $\underline{h}_0^{\perp}$. We have the following alternative descriptions of the defect c_w :

(10.1) $c_w = (\det\,(1-w_*))^{\frac{1}{2}} \operatorname{vol} p(Q)/\operatorname{vol} Q$.

(10.2) $c_w = |\text{torsion}(Q/(1-w)Q^*)|^{\frac{1}{2}}$.

An element w of W is called <u>non-degenerate</u> if $\det(1-w) \neq 0$, and is called <u>primitive</u> if $\det(1-w) = \det A$ (note that $\det(1-w)$ is always non-negative and divisible by $\det A$). Note that w is non-degenerate if and only if the group H^w is finite, and w is primitive if and only if H^w is the center of G . Note that $\det A = |Q^*/Q|$.

Remark. One can see from the classification of the conjugacy classes of W [1], that the number of conjugacy classes of primitive elements for A_n , D_n , E_6 , E_7 , and E_8 is 1 , $[\frac{1}{2}n]$, 3 , 5 and 9 respectively. This equals the number of orbits on $\hat{\Gamma}_0$ of the group of isometries of $\hat{\Gamma}$, as noted by the second author. We have no explanation of this coincidence. Furthermore, the number of conjugacy classes of non-degenerate elements for A_n , E_6 , E_7 and E_8 is 1, 5, 12 and 30 respectively, and the total number of conjugacy classes is $p(n+1)$, 25, 60 and 112 respectively. Note also that the Coxeter element is primitive (and for A_n this is the only primitive conjugacy class).

The discussions of Sections 9 and 10 are linked by the following crucial formula

(10.3) $h(0,\alpha)h(0,\beta) = h(0,\beta)h(0,\alpha)\psi(\alpha,\beta)$,

the proof of which is omitted.

11. Now we can state, and give a sketch of the proof of, the central result of this paper.

Irreducibility Theorem. The basic representation (V,π_0) of an affine Kac-Moody algebra $\hat{\underline{g}}$ of type $\hat{A}_n$, $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$ or $\hat{E}_8$ remains irreducible when restricted to the pair $(\hat{s}_w, S_w)$ for any $w \in W$.

Outline of the proof. First, we pick a "good" lifting $\sigma = \exp 2\pi i x$ of w , as described in [8, Section 4]. Apart from (9.1), it has the properties that $(x|\underline{h}) = 0$ and all eigenvalues of $\operatorname{ad} x$ are in $\frac{1}{2m}\mathbb{Z}$ and are of absolute value ≤ 1 . Put

$$d' = d_0 + x \in \underline{d} \dotplus \hat{\underline{g}} .$$

We have the following useful formulas:

(11.1) $[d',a(k)] = -ka(k)$,

(11.2) $\operatorname{Ad}(h(\beta,\gamma))d' = d' + \beta(0) + \frac{1}{2}|\beta|^2 c$,

(11.3) $\operatorname{Ad}(h(\beta,\gamma))a(k) = e^{2\pi i(\alpha|\gamma)}a(k+(\alpha|\beta)) + \delta_{k,0}(a|\beta)c$,

if $\alpha \in \Delta \cup \{0\}$ and $a \in \underline{g}_\alpha$.

Let V_k denote the eigenspace of $\pi_0(d')$ with the eigenvalue k. Then $V = \bigoplus_{k\geq 0} V_k$ (where k runs over $\frac{1}{2m}\mathbb{Z}_+$). Given a graded subspace U of V, we put $\dim_q U = \sum_{k\geq 0} \dim(U \cap V_k)q^k$.

Let $V^+ = \{v \in V | \pi_0(\hat{s}^+)v = 0\}$. Then an algebraic analogue of the Stone-von Neumann theorem (see e.g. [9, Lemma 14.4]) shows that the theorem holds if V^+ is an irreducible $\hat{S}_w$-module, and that

$$(11.4)\quad \dim_q V = (\dim_q V^+) \prod_{j=1}^{\infty} (1-q^{j/m})^{-\dim \underline{h}_{j/m}} .$$

Let $V^{++} = \{v \in V^+ | \pi_0(\underline{h}_0)v = 0\}$, and let $\tilde{S}_0 = \{h(0,\beta)'\} \subset \tilde{S}_w$, $\hat{S}_0 = \nu_1^{-1}(\tilde{S}_0)$. Then (11.2) and (11.3) show that the theorem holds if V^{++} is an irreducible $\hat{S}_0$-module, and that

$$(11.5)\quad \dim_q V^+ = (\dim_q V^{++}) \sum_{\alpha\in p(Q)} q^{\frac{1}{2}|\alpha|^2} .$$

Finally, the commutation relation (10.3) and a Stone-von Neumann theorem for projective representations of finite abelian groups shows that V^{++} is an irreducible $\hat{S}_0$-module if $\dim V^{++} = c_w$.

We give two methods for showing that $\dim V^{++} = c_w$, both of which depend on the transformation properties of modular forms. Let $q = e^{-2\pi T}$, where $T > 0$.

The first method depends on special character formulas for V. The value of the character ch V at σ may be computed by the following simple formula, entirely in terms of the action of w on $\underline{h}$ [8, eq. (6)]:

$$(11.6)\quad (\mathrm{ch}\ V)(\sigma) = \sum_{\gamma\in Q^w} q^{\frac{1}{2}|\gamma|^2} / \prod_{j=1}^{\infty} \det_{\underline{h}}(1-q^j w) .$$

It is easy to see that, after replacing V by the sum of all level 1 representations of $\hat{\underline{g}}$ and Q by Q^*, formula (11.6) still holds; we denote the resulting formula by (11.6').

Now replace T by $1/T$ in (11.6'). Then, according to [11, Proposition 4.11], (11.6') is transformed into

$$(11.7)\quad \dim_q V = c_w \sum_{\gamma\in p(Q)} q^{\frac{1}{2}|\gamma|^2} / \prod_{j=1}^{\infty} (1-q^{j/m})^{\dim \underline{h}_{j/m}} .$$

Comparing (11.4), (11.5) and (11.7), we get $\dim_q V^{++} = c_W$, proving the theorem.

Following [14], cf. [11, p. 223], the second method depends on the asymptotics of characters and modular forms. We have asymptotically, as $T \downarrow 0$,

$$\dim_q V \sim e^{\pi n/12T}/\mathrm{vol}(Q)$$

$$\prod_{j=1}^{\infty} (1-q^{j/m})^{-\dim \underline{h}_{j/m}} \sim T^{\frac{1}{2}\dim \underline{h}_0}\, e^{\pi n/12T}/\det(1-w_*)^{\frac{1}{2}}$$

$$\sum_{\alpha\in p(Q)} q^{\frac{1}{2}|\alpha|^2} \sim T^{-\frac{1}{2}\dim \underline{h}_0}/\mathrm{vol}(p(Q)) .$$

Comparing these with (10.1), (11.4) and (11.5), we get $\dim_q V^{++} \sim c_W$, so that $\dim V^{++} = c_W$, proving the theorem again.

Since $V_0^{++} \subset V^{++}$ and V^{++} is an irreducible $\hat{S}_0$-module, we get:

(11.8) $V^{++} = V_0$.

(11.9) $\dim V^{++} = c_W$.

<u>Remark</u>. By a remark in Section 3, it is clear that the Irreducibility Theorem holds for all level 1 representations as well.

12. In this section, we will establish a commutator formula for the $h(\alpha,\beta)$, generalizing formula (10.3).

Take $\tau \in \mathbb{C}$, write $z = e^{\tau}$. We interpret z^k to mean $e^{k\tau}$ and $\log z$ to mean τ. For $b \in \underline{g}$, define the generating function

$$X_b(z) = \sum_k z^{-k}\, b(k) .$$

Here and further on k runs over the set $\{k \in \mathbb{C} \mid e^{-2\pi ik}$ is an eigenvalue of $\mathrm{Ad}\sigma\}$; if $b \in \underline{h}$, we can assume k to be in $\frac{1}{m}\mathbb{Z}$. In what follows, we are making calculations in the basic representation (V,π_0) of $\hat{\underline{g}}$.

We have the following commutation relations, which follow from (9.2) and (11.3) respectively:

(12.1) $z^{-k}[a(k),X_b(z)] = X_{[a,b]}(z) + k(a|b)c$, if $a = a_\lambda$, where $\lambda = \exp - 2\pi ik$.

(12.2) $\mathrm{Ad}(h(\beta,\gamma))X_a(z) = e^{2\pi i(\alpha|\gamma)}z^{(\alpha|\beta)}X_a(z) + (a|\beta)c$

if $\alpha \in \Delta \cup \{0\}$ and $a \in \underline{g}_\alpha$.

For $\alpha \in \Delta$ and $a \in \underline{g}_\alpha$ put $E_\alpha^{\pm}(z) = \sum_{\pm k>0} -\frac{1}{k} z^{-k}\, \alpha(k)$ (here $k \in \frac{1}{m}\mathbb{Z}$), and put $T_a(z) = (\exp\text{-}E_\alpha^-(z))X_a(z)(\exp\text{-}E_\alpha^+(z))$.
Then we have:

(12.3) $[\hat{s}_w, T_a(z)] = 0$,

(12.4) $h(\beta,\gamma)T_a(z)h(\beta,\gamma)^{-1} = e^{2\pi i(\alpha|\gamma)}z^{(\alpha|\beta)}T_a(z)$.

Furthermore, given $\lambda \in \underline{h}_{0}$, put $V_\lambda^+ = \{v \in V^+ | \pi_0(h(0))v = (\lambda|h)v$ for all $h \in \underline{h}\}$. Then, for $v \in V_\lambda^+$, we have

(12.5) $T_a(z)v = z^{(\lambda|\alpha)+\frac{1}{2}|p(\alpha)|^2}a(-(\lambda|\alpha)-\frac{1}{2}|p(\alpha)|^2)v$.

For $\alpha,\beta \in Q$, define constants $B_{\alpha,\beta}$ and $C_{\alpha,\beta}$ by:

$$B_{\alpha,\beta} = m^{-(\alpha|\beta)} \prod_{r=1}^{m-1} (1-e^{2\pi ir/m})^{(w^r(\alpha)|\beta)} ,$$

$$C_{\alpha,\beta} = (-1)^{(\alpha|\beta)} B_{\beta,\alpha}/B_{\alpha,\beta} = \prod_{r=1}^{m} (-e^{2\pi ir/m})^{(\alpha|w^r(\beta))} .$$

Then we have, for $a \in \underline{g}_\alpha$, $b \in \underline{g}_\beta$:

(12.6) $B_{\alpha,\beta}\, T_a(z)\, T_b(z) = T_{[a,b]}(z)$ if $(\alpha|\beta) = -1$;

(12.7) $B_{\alpha,\beta}\, T_a(z)\, T_b(z) = (a|b)$ if $(\alpha|\beta) = -2$;

(12.8) $T_a(z)T_b(z') = C_{\alpha,\beta}(z/z')^{(p(\alpha)|p(\beta))}T_b(z')T_a(z)$.

Comparing (12.4) with (12.8) allows us to identify $T_a(z)$, up to a multiplicative scalar, with an element of $\pi_0(\hat{S})$:

(12.9) $T_a(z) \in \mathbb{C}^\times \pi_0(h(\beta,\gamma))$,

where $\beta = -p(\alpha)$, and

$$\gamma = (\frac{m}{2} + \frac{1}{2\pi i}\log z)p(\alpha) - \frac{1}{m}\sum_{r=1}^{m-1} rw^r(\alpha)$$

$$= (1-w_*)^{-1}(\alpha-p(\alpha)) + (\frac{1}{2} + \frac{1}{2\pi i}\log z)p(\alpha) .$$

Comparing (12.7) and (12.8), we obtain the following important commutator formula, which is valid in $\hat{G}$:

(12.10) $h(\alpha,\beta)h(\alpha',\beta') = (\exp 2\pi i\, Dc)h(\alpha',\beta')h(\alpha,\beta)$,
where $D = \frac{1}{2}(\alpha|\alpha') + (\alpha|\beta') - (\alpha'|\beta) + (\beta|\beta'-w(\beta'))$.

The following two formulas are useful:

(12.11) $T_{(\mathrm{Ad}\sigma)a}(z) = T_a(e^{2\pi i}z)$.

(Recall that $T_a(z)$ actually depends on $\log z$.)

(12.12) $T_a(z') = T_a(z)\exp((\log(z'/z))(\alpha(0)+\frac{1}{2}|p(\alpha)|^2c))$ if $a \in \underline{g}_\alpha$.

We have obtained the following formula:

(12.13) $$\sum_k z^{-k}\,\pi_0\,(e_\alpha(k)) = \;:\exp \sum_{k\in\frac{1}{m}\mathbb{Z}\setminus\{0\}} -\frac{1}{k}z^{-k}\,\pi_0(\alpha(k)):\; T_{e_\alpha}(z)\;,\quad \alpha\in\Delta\,.$$

Here : : denotes the normal ordering, so that

$$:\exp \sum_{k\neq 0} -\frac{1}{k}z^{-k}\pi_0(\alpha(k)):=(\exp \sum_{k>0} \frac{1}{k}z^{k}\pi_0(\alpha(-k)))(\exp \sum_{k>0} -\frac{1}{k}z^{-k}\pi_0(\alpha(k)))\;,$$

and $T_{e_\alpha}(z)$ is given, up to multiplicative factor, by (12.9). Since $c, e_\alpha(k)$ and $\alpha(k)$ for $\alpha \in \Delta$ and all k , span $\hat{\underline{g}}$, formula (12.13) describes the basic representation (V,π_0) of $\hat{\underline{g}}$ in terms of the operators $\pi_0(\alpha(k))$ and $\pi_0(h(\beta,\gamma))$.

One also easily finds formulas for the representation of the Virasoro algebra in terms of operators $\pi_0(h(k))$, $h \in \underline{h}$. Choose a basis $\{u_i\}$ of $\underline{h}$ and let $\{u^i\}$ be the dual basis, i.e. $(u_i|u^j) = \delta_{ij}$. Define the "twisted" Virasoro operators $D_j = d_j + x(j)'$, $j \in \mathbb{Z}$. Then

(12.14) $$\pi_0(D_j) = \frac{1}{2} \sum_{k\in\frac{1}{m}\mathbb{Z}} \sum_{i=1}^{n} :\pi_0(u_i(-k))\pi_0(u^i(k+j)): .$$

13. Fix $w \in W$. The Irreducibility Theorem and the commutation relations

(9.3), (11.3) and (12.10) allow one to associate to w a vertex construction of the basic representation (V,π_0) described below.

We have $w^m = 1$ for some positive m. Given $k \in \frac{1}{m}\mathbb{Z}$, let $\hat{k}$ stand for the element $k \bmod \mathbb{Z}$ of $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$. We have the eigenspace decomposition $\underline{h} = \bigoplus_{\hat{k}\in\frac{1}{m}\mathbb{Z}/\mathbb{Z}} \underline{h}_{\hat{k}}$ with respect to w, where $\underline{h}_{\hat{k}}$ corresponds to the eigenvalue $e^{-2\pi i k}$.

For each $k \in \frac{1}{m}\mathbb{Z}$, denote by $\underline{h}(k)$ a copy of $\underline{h}_{\hat{k}}$. Given $\alpha \in \underline{h}$, denote by $\alpha(k)$ the $\underline{h}_{\hat{k}}$-component of α, regarded as an element of the copy $\underline{h}(k)$.

Put $\hat{s}_w^{\pm} = \bigoplus_{\substack{k\in\frac{1}{m}\mathbb{Z}\\ k>0}} \underline{h}(\pm k)$ and define the Heisenberg Lie algebra $\hat{s}_w = \hat{s}_w^{-} \oplus \mathbb{C}c \oplus \hat{s}_w^{+}$ by the commutation relations

$$(13.1)\qquad \begin{aligned} &[h_1(j),h_2(k)] = j\delta_{j,-k}(h_1|h_2)c \quad \text{for } h_1 \in h_{\hat{j}},\ h_2 \in h_{\hat{k}}\ ;\\ &[c,h(j)] = 0\ . \end{aligned}$$

The Lie algebra $\hat{s}_w$ has a unique irreducible representation (F,π_0'), the Fock representation, such that $\pi_0'(c) = 1$ and there exists a non-zero vector, the vacuum vector, which is killed by $\hat{s}_+$. The space F may be identified with the symmetric algebra $S(\hat{s}_w^-)$ over $\hat{s}_w^-$ on which $c = 1$ and, for positive $k \in \frac{1}{m}\mathbb{Z}$, $h(-k)$ acts as a creation operator, i.e. the operator of multiplication by $h(-k)$, and $h(k)$ acts as an annihilation operator, i.e. a derivation of the algebra $S(\hat{s}_w^-)$, which kills 1, subject to relation (13.1), so that 1 is the vacuum vector.

Consider the (additive) group

$$L_w = \{(\alpha,\beta) \in \underline{h}_{\hat{0}} \oplus \underline{h} \,|\, \beta-w(\beta) \in \alpha + Q\}\ ,$$

and define on L_w the following bimultiplicative alternating function:

$$\psi((\alpha,\beta),(\alpha',\beta')) = \exp 2\pi i(\tfrac{1}{2}(\alpha|\alpha')+(\alpha|\beta')-(\alpha'|\beta)+(\beta|\beta'-w(\beta')))\ .$$

The group M_w is embeded in L_w in an obvious way, $\beta \to (0,\beta)$, so that the restriction of Ψ to M_w is the function ψ defined in Section 10. Note that $L_w^0 = \{(0,\beta) | \beta \in \underline{h}_{\hat{0}}\}$ is the identity component of L_w and that Ψ vanishes on $L_w^0 \times L_w^0$.

By a projective representation (U,τ) of a group Γ we mean a vector space U and a map $\tau : \Gamma \to \mathrm{Aut}\, U$ such that $\tau(\gamma_1)\tau(\gamma_2) \in \mathbb{C}^\times \tau(\gamma_1+\gamma_2)$. We say that another projective representation (U',τ') of Γ is equivalent to (U,τ) if there exists an isomorphism $\phi : U \xrightarrow{\sim} U'$ such that $\phi^{-1} \tau'(\gamma)\phi \in \mathbb{C}^\times \tau(\gamma)$ for all $\gamma \in \Gamma$.

The group L_w has a unique, up to equivalence, projective representation (U,π_0'') such that

(i) $\pi_0''(\gamma)\pi_0''(\gamma') = \Psi(\gamma,\gamma')\pi_0''(\gamma')\pi_0''(\gamma)$;

(ii) L_w^0 is diagonalizable and has a fixed vector;

(iii) the only operators commuting with all $\pi_0''(\gamma)$ are scalars.

The group $\tilde{S}_w$ considered earlier is isomorphic to the quotient of L_w by the subgroup $0 \oplus Q$. In the next section, we will construct explicitly a representation of the group $\hat{S}_w$, which induces a projective representation of $\tilde{S}_w$ and hence of L_w . This projective representation, which we denote by $(\overline{V}^+,\pi_0'')$, satisfies the properties (i), (ii) and (iii).

<u>Remark</u>. The radical of Ψ is $0 \oplus Q^*$. If w is non-degenerate, then $L_w = M_w$ and so $\dim \overline{V}^+ = c_w$; otherwise $\dim \overline{V}^+ = \infty$. If w is primitive, then $(\overline{V}^+,\pi_0'')$ is equivalent to the trivial 1-dimensional representation.

Let $\overline{V} = F \otimes_{\mathbb{C}} \overline{V}^+$. Given $\alpha \in \Delta$, introduce the following vertex operator:

$$X(\alpha,z) =: \exp - \sum_{\substack{j \in \frac{1}{m}\mathbb{Z} \\ j \neq 0}} \frac{1}{j} z^{-j} \pi_0'(\alpha(j)): \otimes T_\alpha(z) ,$$

where $T_\alpha(z) = \pi_0''((-p(\alpha),(1-w_*)^{-1}(\alpha-p(\alpha)))\pi_0''((0,(\frac{1}{2} + \frac{1}{2\pi i} \log z)p(\alpha)))$. Here p is the orthogonal projection of $\underline{h}$ on $\underline{h}_{\hat{0}}$ and w_* is the restriction of w to $\underline{h}_{\hat{0}}^{\perp}$; z^k stands for $e^{k\tau}$ and $\log z$ stands for τ , where τ is a complex parameter.

Decomposing by powers of z :

$X(\alpha, z) = \sum_k X_k(\alpha)z^k$, we obtain a collection of operators $X_k(\alpha)$ on $\overline{V}$.

Theorem. The identity operator, the creation and annihilation operators and the components $X_k(\alpha)$ of the vertex operators for $\alpha \in \Delta$, span a Lie algebra of operators on the vector space $\overline{V}$. This Lie algebra is isomorphic to $\hat{\underline{g}}$ and its representation on $\overline{V}$ is equivalent to the basic representation (V,π_0) of $\hat{\underline{g}}$.

14. We proceed to construct the group $\hat{S}_w$ and its representation on the space V^+ explicitly. Recall the bimultiplicative function $\varepsilon(\alpha,\beta)$ and the Chevalley basis elements e_α from Section 4, and the bimultiplicative function $B_{\alpha,\beta}$ from Section 12. Denote by $\mathbb{C}[Q]$ the complex vector space with basis $\{e(\alpha)\}_{\alpha\in Q}$ and by $\mathbb{C}[\exp \underline{h}_0]$ the complex vector space with basis $\{e^h\}_{h\in\underline{h}_0/2\pi i Q^w}$. Introduce the associative (non-commutative) algebra

$$A_w = \mathbb{C}[Q] \otimes_{\mathbb{C}} \mathbb{C}[\exp \underline{h}_0]$$

with the following multiplication:

$$e(\alpha)e(\beta) = \varepsilon(\alpha,\beta)\, B^{-1}_{\alpha,\beta}\, e(\alpha+\beta) ,$$
$$e^h\, e(\alpha) = e^{(\alpha|h)}\, e(\alpha)e^h ,$$
$$e^h\, e^{h'} = e^{h+h'} .$$

Here $e(\alpha)$ stands for $e(\alpha) \otimes 1$ and e^h stands for $1 \otimes e^h$.

The algebra A_w has the following representation τ on $\mathbb{C}[Q]$:

$$\tau(e(\alpha))e(\beta) = e(\alpha)e(\beta) ,$$
$$\tau(e^h)e(\beta) = e^h e(\beta)e^{-h} .$$

Introduce the function $\eta : Q \to \mathbb{C}^\times$ by:

$$e_{w(\alpha)} = \eta(\alpha)(\mathrm{Ad}\sigma)e_\alpha \quad \text{for } \alpha \in \Delta ,$$
$$\eta(\alpha+\beta) = \varepsilon(\alpha,\beta)\varepsilon(w(\alpha),w(\beta))\eta(\alpha)\eta(\beta) .$$

Define an automorphism μ of the algebra A_w by:

$$\mu(e^h) = e^h , \ \mu(e(\alpha)) = \eta(-\alpha)\, e^{-\pi i p(\alpha)} e(w(\alpha)) e^{-\pi i p(\alpha)} .$$

Let A_w be the subgroup of the multiplicative group of the algebra A_w

consisting of the elements of the form $te(\alpha)e^h$, where $t \in \mathbb{C}^\times$, $\alpha \in Q$, $h \in \underline{h}_0$. Define a homomorphism μ_0 of $A_w^\times$ into its center by $\mu_0(a) = a^{-1}\mu(a)$, and let $\overline{S}_w = A_w^\times/\mu_0(A_w^\times)$. We now show that $\overline{S}_w$ is isomorphic to $\hat{S}_w$.

Define homomorphisms $\overline{\nu}_1 : \overline{S}_w \to \tilde{S}_w$ and $\overline{\pi}_0 : \overline{S}_w \to \mathrm{Aut}\, V$ by:

$$\overline{\nu}_1(te(\alpha)e^h) = h(\beta,\gamma)' \exp h ,$$

where $\beta = -p(\alpha)$, $\gamma = (1-w_*)^{-1}(\alpha-p(\alpha))$; $\overline{\pi}_0(te(\alpha)e^h) = tT_\alpha(e^{-\pi i})\exp \pi_0(h)$. Here $T_\alpha(z)$ for $\alpha \in Q$, $z \in \mathbb{C}^\times$ is defined by:

$$T_\alpha(z) = T_{e_\alpha}(z) \quad \text{for} \quad \alpha \in \Delta , \quad T_\alpha(z)T_\beta(z) = \varepsilon(\alpha,\beta)B^{-1}_{\alpha,\beta}T_{\alpha+\beta}(z) .$$

There exists a unique isomorphism $\overline{S}_w \xrightarrow{\sim} \hat{S}_w$ which identifies ν_1 with $\overline{\nu}_1$ and π_0 with $\overline{\pi}_0$; we identify $\overline{S}_w$ with $\hat{S}_w$ using this isomorphism. Note that e^h is identified with $\exp h$.

The space $U = \mathbb{C}[Q]/\tau((1-\mu)A_w)\mathbb{C}[Q]$ is a quotient representation of the representation τ of A_w. This is a representation of $A_w^\times$ on which $\mu_0(A_w^\times)$ acts trivially, thus giving rise to a representation of $\overline{S}_w$. Let U_0 be the fixed point set of center (G) on U.

For $\gamma \in Q$ such that $p(\gamma) = 0$, right multiplication by $e(\gamma)$ on $\mathbb{C}[Q]$ induces an endomorphism of U_0 which we denote by A_γ. The operators A_γ span the commuting algebra of the representation of $\overline{S}_w$ on U_0; this algebra is isomorphic to the algebra of $c_w \times c_w$-matrices over $\mathbb{C}$. Take a rank 1 projector in this algebra, say P, and put $\overline{V^+} = P(U_0)$. The representation of $\overline{S}_w$ on $\overline{V^+}$ is equivalent to that of $\hat{S}_w$ on V^+.

15. Comments and questions.

A. (a) By the proposition in Section 9, the Irreducibility Theorem holds for <u>any</u> pair $(\hat{s},\hat{S})$. This is "philosophically" important.

(b) The Irreducibility Theorem was first proved [10] for the principal Heisenberg, then [2] for the homogeneous Heisenberg, and now for any Heisenberg. These three stages correspond to the appearance first of the Heisenberg Lie algebra $\hat{s}$, then of the "translation group" $\hat{S}/\hat{S}_0$, where $\hat{S}_0$ is the centralizer of the identity component of $\hat{S}$, and finally of the group $\hat{S}_0$, which is, essentially, a finite Heisenberg group.

B. (a) The other fundamental representations of level 1 may be treated in the same way.

(b) One can realize (V,π_0) as a space of functions, and describe the vertex operators, the Virasoro algebra, etc., in this realization (cf. [2]).

(c) Problem: construct intertwining operators $T_{w,w'}$ among the realizations of (V,π_0) .

(d) Taking $w \in \mathrm{Aut}\, Q\backslash W$, one obtains, in a similar way, a construction of all representations of level 1 of all twisted affine Kac-Moody algebras.

C. (a) Formula (12.10) can be generalized to the case where $\beta - w(\beta) \in \alpha + Q^*$, so that $h(\alpha,\beta)' \in \widetilde{Ad(G)}$. This allows one to evaluate the action of the group elements $e(\alpha)$ of Section 14 on all fundamental representations by using the vertex operators.

(b) The embedding of center (G) into the group $\overline{S}_w$ of Section 14 can be given explicitly.

(c) The representation of $\hat{S}_w$ on V^+ can be realized as an induced representation, or as a space of sections of a bundle over H^w .

D. Let K be a simply-connected compact Lie group of type A, D, or E, let $\mathcal{K}$ be the group of C^∞ loops on K , and let $(\mathcal{V},\pi_0)$ be the basic projective unitary representation of $\mathcal{K}$ on a Hilbert space $\mathcal{V}$ [17]. Given a loop S of maximal tori of K , define the associated Heisenberg subgroup of $\mathcal{K}$ to be the subgroup $\tilde{S}$ consisting of all loops $p : S^1 \to K$ such that $p(\theta) \in S(\theta)$. It is reasonable to expect the following version of our irreducibility theorem: $(\mathcal{V},\pi_0)$ remains irreducible when restricted to $\tilde{S}$. (This is known in the case of a 1-point loop [17].)

E. (a) The group $\exp \underline{g}(0)$ is a covering group of G^σ , and V^{++} is a one-dimensional or fundamental representation of it.

(b) The groups $\hat{S}_0$ are of interest in the study of finite abelian and Heisenberg subgroups of semisimple Lie groups. The crucial commutator formula (10.3) was proved in this context.

(c) Problem. Find an integral formula for the commutator in $\hat{G}$ of preimages of two commuting elements of $\tilde{G}$.

F. The second author has given a description of the Heisenberg algebras

$\hat{s}_w$ for primitive w as centralizers of (possibly several) explicitly given loops. This is analogous to the use of Kostant's "cyclic element" in connection with the principal Heisenberg subalgebra (cf. [10]).

G. If $p(\alpha) = 0$, then formula (12.5) says that $T_a(z) = a(0)$ on V^+ . This is a remarkable coincidence: the actions of a group element and a Lie algebra element coincide on V^+ !

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