On the v_1-periodicity of the Moore space

by

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B.S., Stanford University (2014)

Submitted to the Department of Mathematics

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2019

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ABSTRACT

We present progress in trying to verify a long-standing conjecture by Mark Mahowald on the v_1-periodic component of the classical Adams spectral sequence for a Moore space M. The approach we follow was proposed by John Palmieri in his work on the stable category of A-comodules. We improve on Palmieri's work by working with the endomorphism ring of M - $End(M)$ thus resolving some of the initial difficulties of his approach and formulating a conjecture of our own that would lead to Mahowald's formulation. We further improve upon a method for calculating differentials via double filtration first used by Miller and apply it to our problem.

Thesis Supervisor: Haynes Miller
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Acknowledgements

1 Introduction

1.1 Motivation and background

Homotopy groups have been one of the cornerstone objects of study in algebraic topology and really something that gave birth to the subject itself. The Freudenthal Suspension theorem gives rise to a stability phenomenon for those groups. More precisely, for an n-connected pointed space X, the suspension map $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k \leq 2n$. This generalizes to an isomorphism $[X, Y] \to [\Sigma X, \Sigma Y]$ given dim $X < 2n - 1$ and Y is $n-1$-connected, and allows us to study homotopy theory in this stable context. We move from working in the category of spaces and homotopy classes of maps to its stable version - the category of spectra.

In this category we have a generalized Adams Spectral Sequence that converges to a certain localiztion of $\pi_*(X)$. This spectral sequence is constructed via a ring spectrum E that needs to satisfy a number of conditions to make sure $E_2 = Ext_{E_*(E)}(E_*, E_*(X))$ and to guarantee convergence. Most common candidates for E are the mod p Eilenberg-MacLane spectrum H or the Brown-Peterson spectrum BP. We get the classical Adams spectral sequence and the Adams Novikov spectral sequence respectively. The later spectral sequence has a striking connection to the theory of formal group laws.

A formal group law over a ring R is a power series of two variables with coefficients in that ring that satisfies certain group-like properties. We can talk about morphisms of group laws in terms of a change-of-base map over R or as arising from a ring map $R \to T$. It's natural to look for universal objects in this setting and (working p-locally) the pair (BP_*BP, BP_*) is one such object. BP_*corepresents p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebra, while BP_*BP correspresents isomorphisms between them. Thus, the pair correpresents objects and morphisms in a groupoid and as such is called a Hopf algebroid. The structure of this Hopf algebroid is present in the world of formal group laws and so we conclude this world "knows" exactly how the E_2 page of the Adams Novikov spectral sequence looks like. One manifestation of this relation is chromatic homotopy theory.

Given $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$, our interpretation of a formal group law over a ring R as a map $f : BP_* \to R$ allows us to define the concept of height associated to the formal group law. The height is the smallest integer n for which $f(v_n) \neq 0$. This "filtration" of formal group laws by height translates to the chromatic filtration in homotopy theory and leads us to talk about v_n-periodicity. Informally speaking, if I_n is the complete information that formal group laws of height n or higher "see" in stable homotopy, then the v_n-periodic phenomena are given by I_n/I_{n+1}. The objects that detect periodicity on the level of spectra are the Morava K-theories $K(n)$. Given a fixed p-local finite spectrum X, let n be the smallest integer such that $K(n)_*(X) \neq 0$. Then we say X is of type n and $\pi_*(X)$ has a non-trivial v_n-periodic part. Furthermore, one can isolate v_n-periodicity by virtue of the Periodicity theorem. The theorem tells us there is an (asymptotically) unique self-map $\beta : \Sigma^{|\beta|}X \to X$ which induces an isomorphism on $K(n)_*$. Hence the fiber of this map has type higher than n and so the v_n-periodic homotopy of X is exactly what (powers of) β detect. The telescope $\beta^{-1}X$ is the geometric manifestation of the v_n-periodic part of X i.e. $\pi_*(\beta^{-1}X) = \beta^{-1}\pi_*(X)$. It's an interesting question how β works on the level of the Adamas Novikov spectral sequence, which is the statement of the telescope conjecture, for instance. More precisely, the telescope conjecture claims that the v_n-localized Adams Novikov spectral sequence of X converges to $\beta^{-1}\pi_*(X)$. Alternatively, it says there is no v_n-periodic element in $\pi_*(X)$ with unbounded Novikov filtration as higher powers of β are applied (there are enough v_n-towers) and there is no v_n-periodic element in the unlocalized spectral sequence that kills off non-periodic elements as higher powers of β are applied (there are not too many v_n-towers). This is a step towards computing $\beta^{-1}\pi_*(X)$.

The connection of BP to formal groups makes it into a computationally effective tool in the study of stable homotopy. However, at least on theory, one can try to play the same game with other spectra and in particular with ordinary mod p homology H. An immediate issue that arises is that homology itself doesn't detect self maps as effectively and we are limited as to what we can construct geometrically. That is to say we don't have an equivalent to the Periodicity theorem or Morava K-theory or at least we don't know what they are supposed to be. For example, the mod 2 Moore space M has a v_1-self map $\alpha : \Sigma^8 M \to M$ and clearly $H(\alpha) = 0$, so ordinary homology doesn't detect α as well as BP. This has to do with the fact that BP (unlike H) detects periodicity at filtration 0 (this is related to the Nilpotence theorem). So what can we do? Can we change our framework so ordinary homology "sees more"?

Before we give an answer we would need to know a bit about the structure theory of A and (co)modules over it. Those were extensively studied by Margolis [4], among others. He introduced elements $P_t^s \in A^*$dual to $\xi_t^{p^s} \in A$. At $p = 2$ we know that $(P_t^s)^2 = 0$ for $s < t$, so one can define

$H(N, P_t^s)$ for a given A-comodule N. The significance of these homology groups becomes apparent by the following results

Theorem 1.1: Let N be a bounded below comodule N such that $H(N, P_t^s) = 0$ for all $s < t$. Then N is cofree.

Theorem 1.2: Given an integer d, if $H(N, P_t^s) = 0$ for all $|P_t^s| < d$ then $Ext_A(\mathbb{F}_2, N)$ has a vanishing line of slope d.

Theorem 1.2 leads us to define the type of a bounded below comodule N to be the smallest $n = |P_t^s|$ such that $H(N, P_t^s) \neq 0$. Naively, following the BP analogy we want to construct a (unique) self map $\beta : N^{|\beta|} \to N$ which induces an isomorphism on $H(-, P_t^s)$. To do that we need to work in the derived category of A-comodules - $Stable(A)$. This is enough to deal with the limitations of H mentioned earlier as compared to BP. To see how, consider again the v_1 self-map $\alpha : \Sigma^8 M \to M$ for the mod 2 Moore space. $H(\alpha) = 0$, but α has to be detected by $Ext_A^{(s,t)}(H_*(M), H_*(M))$ and so it is present in $Stable(A)$ and, in fact, it induces an isomorphism on $H(H_*(M), P_2^0)$. More generally, a type $n = |P_t^s|$ comodule $N = H_*(X)$ has a self-map β with a geometric realization (also called β). If Y is the spectrum that is the fiber of β, we get that $H_*(Y)$ is of type higher than n and so $Ext_A(\mathbb{F}_2, H_*(Y))$ has a vanishing line of slope $m > n$. Hence β induces an isomorphism on $Ext_A(\mathbb{F}_2, H_*(X))$ above a line of slope m. As a result $\beta^{-1}Ext_A(\mathbb{F}_2, H_*(X))$ completely detects $Ext_A(\mathbb{F}_2, H_*(X))$ above a line of slope m. We refer to this as the P_t^s-periodic part of X.

The author is finally in a position to present the problem he will try to tackle. Let N be the stable comodule corresponding to $H_*(M)$. It is a stable comodule of type $|P_2^0|$ and the self-map is induced precisely from the map $\alpha : \Sigma^8 M \to M$. By the above discussion $\alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(M))$ detects completely $E_2^{s,t}(H, M)$ above a line of slope $|P_2^1| - 1 = 5$. This leads to the central problem of this thesis

$$\textit{Problem}: \text{ What is } \alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(M))?$$

An explicit answer was claimed by Mahowald [3], but it was never verified. According to him it is built out of a number of copies of two pieces. Those pieces are $\alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bo \wedge M))$ and $\alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bu \wedge M))$ where bo and bu are connected real and complex K-theory respectively.

It is worth noting that both of these pieces are easily computed by a change of rings isomorphism (occuring due to a colapse of the Cartan-Eilenberg spectral sequence in both cases). To present the answer in an explicit form, we define a polynomial algebra $P = \mathbb{F}_2[x_1, x_2, ...]$ with derivation $d(x_i) = x_1 x_{i-1}^2$. P is bigraded with $|x_i| = (2, 2^{i+2}+1)$. If $H(d)$ and $B(d)$ are the homology and image of d resepectively, then the conjecture takes the following form

$$\begin{aligned} \textit{Conjecture: } \alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(M)) &= \bigoplus_{a \in H(d)} \Sigma^{|a|}\alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bo \wedge M)) \\ &\oplus \bigoplus_{b \in B(d)} \Sigma^{|b|}\alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bu \wedge M)) \end{aligned}$$

We proceed to describe an approach to this conjecture proposed by Palmieri in his book [8]. He first notes the analogy between $Stable(A)$ and the category of spectra allows us to build a generalized Adams Spectral sequence in precisely the same way. Furthermore, there are spectra Q_n (playing the role of Morava K-theories) that detect P_{n+1}^0-periodicity. Recalling N was the stable comodule corresponding to $H_*(M)$ we get a spectral sequence with $E_2 = Ext_{Q_{1*}Q_1}(Q_{1*}, Q_{1*}(N))$ converging to $\alpha^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(M))$. This spectral sequence converges to $v_1^{-1}E_2(M;H)$ and computations seem promising due to the simplicity of $E_2 = \mathbb{F}_2[v_1^{\pm 1}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$ and the fact that $E_3 = E_2$ as for degree reasons nontrivial differentials can only occur at odd pages. It is important to note that since M is not a ring spectrum, E_r is not an algebra and d_r is not a derivation and so what we really mean by the above equality is that E_2 is a $\mathbb{F}_2$-vector space with basis the monomials in $\mathbb{F}_2[v_1^{\pm 1}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$. Palmieri then conjectured what the values of $d_3(h_{n1})$ are and proposed one should be able to extend them in some way to the entire E_3. Moreover he conjectured that the spectral sequence collapses at E_4 and claimed this would imply Mahowald's conjecture. Note it is not immediately obvious how Palmieri's formulation relates to Mahowald's and it is something we address in more detail at a later section of the paper.

Thus our problem is three-fold: how does one compute $d_3(h_{n1})$, how does one extend it to the rest of E_3 and why are there no higher degree differentials. We solely address the first two questions, fully answering the second one. We do this by working with the endomorphism ring spectrum of M - $End(M)$. It is the 4 cell complex $M \wedge DM$. The advantage of $End(M)$ is that its spectral sequence is multiplicative and so d_3 is a derivation. At the same time the action $End(M) \wedge M \to M$

makes $E_r(M)$ into a module over $E_r(End(M))$. We will also show Palmieri's originally conjectured values for $d_3(h_{n1})$ can't be true and so we propose a revised conjecture of what those values are. We verify that conjecture modulo knowing that the elements $v_1^m h_{n1}$ don't survive to E_4 for $n \geq 3$, $m \in \mathbb{Z}$.

1.2 The suqare

Computing d_3 on the above elements seems to be significantly harder. An example of a similar computation in the literature can be found in a paper due to Miller [6]. He manages to compute $\alpha^{-1}\pi_*(M)$ in the case of an odd p by analyzing d_2 in the Adams Spectral sequence. This is done by considering the Cartan-Eilenberg spectral sequence arising from the reduced powers in A. This spectral sequence colapses, but its second page coincides with the second page of the Algebraic Novikov spectral sequence which converges to $Ext^{s,t}_{BP_*(BP)}(BP_*, BP_*(M))$. Miller is able to relate d_2 to the differential in the Algebraic Novikov spectral sequence, which is more computationally accessible . This relation determines d_2 modulo higher Cartan-Eilenberg filtration, which is enough to compute $\alpha^{-1}\pi_*(M)$.

We will present an attempt to follow the same strategy refered to as the "square" since one obtains 4 spectral sequnces that form a square diagram. In fact, we will generalize the square construction to any triangulated category (rather than the category of spectra) and obtain information about any d_r (rather than just d_2).

1.3 Organization

This thesis is informally divided into two main parts. In the first part (sections $2-5$) we present the progress regarding the conjecture, while sections $6-7$ are dedicated to the development of the square method as an independant tool and it's use regarding our conjecture.

In section 2 we provide the necessary background about $Stable(A)$ - the stable category of comodules over the Steenrod algebra A, and explicitly write Palmieri's original conjecture and our revised version of it. In section 3 we work out the corresponding spectral sequence for $End(M)$ and its action on the the one for M. Section 4 consists of the meat of the paper as we proceed to show that Mahowald's conjecture would follow as long as a family of elements vanishes at E_4. We conclude the first part with section 5 where we introduce the original conjecture by Mahowlad and show explicitly how it follows from our revised conjecture.

We switch gears in section 6 as we introduce the terminology and basic setting of the square construction. Then in Section 7 we discuss how the square construction fits into the setting of our original problem.

2 The category $Stable(A)$

In this chapter we give a brief description of $Stable(A)$ and any related results of immediate use to us. For more detail the reader is directed to Palmieri's book [8].

Objects in $Stable(A)$ are unbounded cochain complexes of (left) A-comodules. We will identify a comodule L with its injective resolution over A. For two such objects L, N the set of morphisms is $[L,N]_{s,t} = Ext_A^{s,t}(L,N)$. Then $L_{s,t} = \pi_{s,t}(L) = Ext_A^{s,t}(\mathbb{F}_2, L)$. For the sake of clarity we observe L itself is bigraded and one should make a distinction between the elements of degree (s,t) in L and $L_{s,t}$. Note also the sphere spectrum $S \in Stable(A)$ is the injective resolution of $\mathbb{F}_2$, which is in line with our notation of $\pi_{s,t}(L) = [S,L]_{s,t}$ above. $Stable(A)$ is now a triangulated category and for a ring spectrum $X \in Stable(A)$ we can build a generalaized Adams spectral sequence in the usual way. Then assuming certain conditions hold we can identify $E_2(L;X) = Ext_{X_{**}X}(X_{**}, X_{**}L)$ and further conditions would guarantee convergence to $\pi_{**}L$.

We are interested in the case where the spectrum Q_1 plays the role of X. To define Q_1, we first define q_1 to be the injective resolution of $A\Box_{\mathbb{F}_2(\xi_2)/(\xi_2^2)}\mathbb{F}_2$. Q_1 is now obtained from q_1 after working out how to extend the q_1-resolution into the negative dimensions. Then one can check $q_{1**} = \mathbb{F}_2[v_1]$, $Q_{1**} = \mathbb{F}_2[v_1^{\pm 1}]$ [8, p.44] and $Q_{1**}Q_1 = \mathbb{F}_2[v_1^{\pm 1}, \xi_1, \xi_2^2, \cdots \xi_n^2, \cdots]/(\xi_1^4, \xi_2^4, \cdots)$ [8, p.101].

The trigraded spectral sequence of interest is

$$E_2(M;Q_1) = Ext_{Q_{1**}Q_1}(Q_{1**}, Q_{1**}(M)) = \mathbb{F}_2[v_1^{\pm 1}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$$

and it converges to $v_1^{-1}E_2(M;H)$ [8, p.81, 101]. Note the abuse of notation above as what we really mean by $E_2(M;Q_1)$ is $E_2(L;Q_1)$ where L is an injective resolution for $H_*(M)$. Elsewhere M will always refer to the topological Moore spectrum. For degree reasons the only potential non-zero differentials in $E_r(M;Q_1)$ happen at odd pages, so $E_2 = E_3$. Palmieri then conjectured the following differentials:

$$d_3(v_1^2) = h_{11}^3$$
$$d_3(h_{n1}) = v_1^{-2} h_{11} h_{21} h_{n-1,1}^2 \quad \text{for } n \geq 3$$

As we will see later, the conjecture in its current form is incorrect, so we make the following revised conjecture:

$$d_3(v_1^2) = h_{11}^3$$
$$d_3(h_{n1}) = v_1^{-2} h_{11}^3 h_{n1} + v_1^{-2} h_{11} h_{21} h_{n-1,1}^2 \quad \text{for } n \geq 3$$

Though this isn't enough to fully determine d_3, Palmieri goes on to propose that d_3 "looks" as though as $E_2(M;Q_1)$ is an algebra. One reason for this proposal that he notes is we can also compute the E_2 page of the corresponding spectral sequence for the sphere

$$E_2(S;Q_1) = Ext_{Q_{1**}Q_1}(Q_{1**}, Q_{1**}) = \mathbb{F}_2[v_1^{\pm 1}, h_{10}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$$

and use the map $S \to M$ to induce a surjection $E_2(S;Q_1) \to E_2(M;Q_1)$ with $h_{n1} \to h_{n1}$, $h_{10} \to 0$ and $v_1 \to v_1$. Then the identity map $S \wedge M \to M$ turns $E_2(M;Q_1)$ into a cyclic module over $E_2(S;Q_1)$. Now identifying $E_2(M;Q_1)$ with $\mathbb{F}_2[v_1^{\pm 1}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$ becomes justified as both coincide as $E_2(S;Q_1)$-modules:

$$E_2(M;Q_1) \cong E_2(S;Q_1)/(h_{10}) = \mathbb{F}_2[v_1^{\pm 1}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$$

Then information about differentials in $E_r(S;Q_1)$ could directly produce differentials in $E_r(M;Q_1)$ and since S is a ring spectrum, $E_r(S;Q_1)$ is a spectral sequence of algebras, so the differentials in $E_r(S;Q_1)$ are derivations. The problem is differentials in $E_2(S;Q_1)$ are difficult to compute and so we don't know what $E_3(S;Q_1)$ looks like. This is where $End(M)$ enters the picture - it is a ring spectrum that acts on M just as S does, but differentials in $E_2(End(M);Q_1)$ are much more manageable to compute.

3 The Q_1 E_2 term for $End(M)$

We begin by computing $H_*(End(M))$ as a comodule over A. Let x_0 and x_1 denote the two cells of M and y_{-1} and y_0 denote the two cells of $DM = \Sigma^{-1}M$. Then $End(M) = M \wedge DM$ has four cells of the form $x_i y_j$ with $|x_i y_j| = i + j$. As DM is the dual of M we have maps $\eta : S \to M \wedge DM$ and $\epsilon : DM \wedge M \to S$ that specify the ring structure of $End(M)$. More precisely, η is the unit, while multiplication is given by

$$M \wedge DM \wedge M \wedge DM \xrightarrow{1\wedge\epsilon\wedge 1} M \wedge DM$$

and the action of $End(M)$ on M is then given by the map $1\wedge\epsilon : M\wedge DM\wedge M \to M$. If $\iota \in H_*(S)$ is the generator, then $\eta_*(\iota) = x_1 y_{-1} + x_0 y_0$ and $\epsilon_*(y_1 x_{-1}) = \epsilon_*(y_0 x_0) = \iota$. This allows us to compute the multiplicative structure of $H_*(End(M))$

$$(x_i y_j)(x_k y_l) = \begin{cases} x_i y_l & \text{if } j + k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Setting $\alpha = x_0 y_{-1}$ and $\gamma = x_1 y_0$ we get that $H_*(End(M)) = \mathbb{F}_2[\alpha, \gamma]/(\alpha^2, \gamma^2, \alpha\gamma + \gamma\alpha + 1)$. Note this is a 4-dimensional non-commutative $\mathbb{F}_2$-algebra with basis $\langle 1, \alpha, \gamma, \alpha\gamma\rangle$ where $|\alpha| = -1$ and $|\gamma| = 1$. To understand the coaction of A we just need to understand the coaction on α and γ. Since $\psi(x_0) = 1 \otimes x_0$ and $\psi(x_1) = 1 \otimes x_1 + \xi_1 \otimes x_0$ we conclude that

$$\psi(\alpha) = \psi(x_0 y_{-1}) = \psi(x_0)\psi(y_{-1}) = (1 \otimes x_0)(1 \otimes y_{-1}) = 1 \otimes x_0 y_{-1} = 1 \otimes \alpha$$

and

$$\begin{aligned} \psi(\gamma) &= \psi(x_1 y_0) = \psi(x_1)\psi(y_0) = 1 \otimes x_1 y_0 + \xi_1 \otimes (x_1 y_{-1} + x_0 y_0) + \xi_1^2 \otimes x_0 y_{-1} \\ &= 1 \otimes \gamma + \xi_1 \otimes 1 + \xi_1^2 \otimes \alpha \end{aligned}$$

Recall we are interested in computing d_3 in $E_2(M; Q_1)$. Since M lacks multiplicative structure, we will work with $End(M)$ and try to understand $E_r(End(M); Q_1)$. We proceed with a direct computation

$$
\begin{aligned}
E_2(End(M);Q_1) &= Ext_{(Q_1)_{**}Q_1}((Q_1)_{**},(Q_1)_{**}(End(M))) \\
&= \mathbb{F}_2[v_1^{\pm 1}]\otimes Ext_{\mathbb{F}_2[\xi_1,\xi_2^2,\cdots]/(\xi_i^4)}(\mathbb{F}_2,\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle) \\
&= \mathbb{F}_2[v_1^{\pm 1}]\otimes \mathbb{F}_2[h_{21},h_{31},...]\otimes Ext_{\mathbb{F}_2[\xi_1]/(\xi_1^4)}(\mathbb{F}_2,\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle)
\end{aligned}
$$

Here we used that the coaction of ξ_i^2 on $\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle$ is trivial for $i\geq 2$. The conormal extension $\mathbb{F}_2(\xi_1^2)/(\xi_1^4)\to\mathbb{F}_2(\xi_1)/(\xi_1^4)\to\mathbb{F}_2(\xi_1)/(\xi_1^2)$ produces a Cartan-Eilenberg spectral sequence that collapses since $H_*(End(M))=\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle$ is cofree over $\mathbb{F}_2(\xi_1)/(\xi_1^2)$. Thus, we get

$$
\begin{aligned}
Ext_{\mathbb{F}_2[\xi_1]/(\xi_1^4)}(\mathbb{F}_2,\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle) &= Ext_{\mathbb{F}_2[\xi_1^2]/(\xi_1^4)}(\mathbb{F}_2,Ext_{\mathbb{F}_2[\xi_1]/(\xi_1^2)}(\mathbb{F}_2,\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle)) \\
&= Ext_{\mathbb{F}_2[\xi_1^2]/(\xi_1^4)}(\mathbb{F}_2,\mathbb{F}_2\langle 1,\alpha\rangle)
\end{aligned}
$$

We conclude that $Ext_{\mathbb{F}_2[\xi_1]/(\xi_1^4)}(\mathbb{F}_2,\mathbb{F}_2\langle 1,\alpha,\gamma,\alpha\gamma\rangle)=\mathbb{F}_2\langle 1,\alpha\rangle\otimes\mathbb{F}_2[h_{11}]$ and so

$$E_2(End(M);Q_1)=\mathbb{F}_2[v_1^{\pm 1},\alpha,h_{11},h_{21},h_{31},...]/(\alpha^2)$$

which (expectedly so) is two copies of $E_2(M;Q_1)$. The degrees of the generators are given by $|v_1|=(0,2,1)$, $|\alpha|=(0,-1,0)$, $|h_{n1}|=(1,2^{n+1}-2,0)$. It is worth noting that even though $H_*(End(M))$ is not commutative, the spectral sequence above ends up with a commutative multiplicative structure.

3.1 $E_2(M;Q_1)$ as a differential module over $E_2(End(M);Q_1)$

The action of $End(M)$ on M extends to an action $E_r(End(M);Q_1)\otimes E_r(M;Q_1)\to E_r(M;Q_1)$ and so $E_r(M;Q_1)$ is a differential module over $E_r(End(M);Q_1)$. The commutative diagram

$$
\begin{tikzcd}
M\wedge DM\wedge M \arrow[r, "1\wedge\epsilon"] & M \\
S\wedge M \arrow[u, "\eta\wedge 1"] \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

implies the action of $E_2(S;Q_1)$ on $E_2(M;Q_1)$ factors through the action of $E_2(End(M);Q_1)$ via the algebra map $\eta_* : E_r(S, Q_1) \to E_r(End(M); Q_1)$, which is just

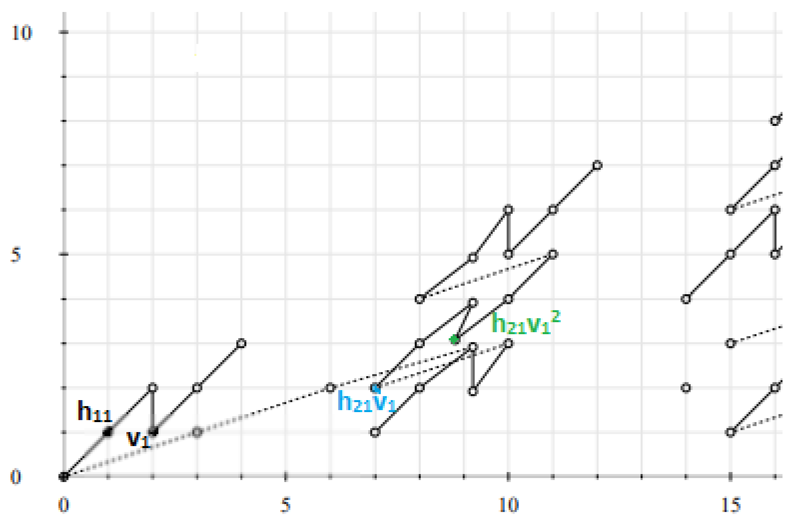
$$\eta_* : \mathbb{F}_2[v_1^{\pm 1}] \otimes \mathbb{F}_2[h_{10}, h_{11}, h_{21}, h_{31}, ...] \to \mathbb{F}_2[v_1^{\pm 1}] \otimes \mathbb{F}_2[h_{11}, h_{21}, h_{31}, ...] \otimes \mathbb{F}_2\langle 1, \alpha \rangle$$

with $\eta_*(v_1) = v_1$ and $\eta_*(h_{n1}) = h_{n1}$. Furthermore we claim $\eta_*(h_{10}) = \alpha h_{11}$. Indeed, since $\psi(\gamma) = 1 \otimes \gamma + \xi_1 \otimes 1 + \xi_1^2 \otimes \alpha$ it follows that $\xi_1 \otimes 1 + \xi_1^2 \otimes \alpha$ vanishes in the homology of the cobar complex of $End(M)$ and so $\alpha h_{11} = \xi_1^2 | \alpha = \xi_1 | 1$, which is the cobar representative of h_{10} in $E_2(S; Q_1)$.

Hence $E_2(M; Q_1)$ is a cyclic module over $E_2(End(M); Q_1)$. Furthermore, we have an isomorphism of $E_2(End(M); Q_1)$-modules:

$$E_2(M; Q_1) \cong E_2(End(M); Q_1)/(\alpha) = \mathbb{F}_2[v_1^{\pm 1}, h_{11}, h_{21}, \cdots, h_{n1}, \cdots]$$

Before we move on to the next section we note that all of the elements $h_{11}, v_1, h_{21}v_1, h_{21}v_1^2$ survive to $E_\infty(M; Q_1)$ as shown by the diagram of $E_2(M; H)$ below. Observe this doesn't guarantee the same is true in $E_r(End(M); Q_1)$, but we will still be able to extract some of the information back to $E_r(End(M); Q_1)$ using the action above.



4 Calculating d_2 and d_3 of $E_2(End(M); Q_1)$

4.1 Low-degree calculations

We begin by calculating d_2 and d_3 on the low-degree elements in $E_r(End(M); Q_1)$ and then proceed to formulating a conjecture for d_2 and d_3 on the remaining elements.

Theorem 4.1.1: The elements $\alpha, h_{11}, v_1\alpha, v_1h_{21}$ survive to $E_4(End(M); Q_1)$. Furthermore,

$d_2(v_1) = \alpha h_{11}^2$

$d_3(v_1^2) = h_{11}^3$

Proof:

Since we will need to distinguish between differentials in $E_r(End(M); Q_1)$ and $E_r(M; Q_1)$, we will denote them by d_r and d_r^M respectively.

In $E_r(M; Q_1)$, h_{11}^3 must be a coboundary at some point and for degree reasons $d_3^M(v_1^2) = h_{11}^3$. Indeed, if $d_r(x) = h_{11}^3$ for some $r \geq 3$ and $x \in E_r(M; Q_1)$ then since $|h_{11}^3| = (3, 6, 0)$ and d_r^M changes degrees by $(r, r-1, 1-r)$ we conclude that $|x| = (3-r, 7-r, r-1)$. Recall $|v_1| = (0, 2, 1)$, $|\alpha| = (0, -1, 0)$, $|h_{n1}| = (1, 2^{n+1}-2, 0)$. Then $3 - r \geq 0$, so $r = 3$ and $|x| = (0, 4, 2)$. The only option now is $x = v_1^2$. Note if v_1 was to survive to $E_3(End(M); Q_1)$ then $d_3(v_1^2) = 0$, which would force $d_3^M(v_1^2) = 0$. Hence $d_2(v_1) \neq 0$ and so for degree reasons $d_2(v_1) = \alpha h_{11}^2$. Given the action of $E_2(End(M); Q_1)$ we must also have $d_2(v_1) = \alpha h_{11}^2$. Either of those differentials could be also seen since $d_2(v_1) = h_{10}h_{11}$ in $E_2(S; Q_1)$ which follows from the same differential in the Cartan-Eilenberg spectral sequence computing $H^*(A(1))$.

Next we claim $d_2(h_{21}) \neq 0$. Indeed, assume that $d_2(h_{21}) = 0$. Then $d_2(v_1^2h_{21}) = 0$ and since $v_1^2h_{21}$ survives in $E_r(M; Q_1)$ it must be that $d_3(v_1^2h_{21}) = 0$ in $E_3(End(M); Q_1)$. By multiplicativity we conclude $d_3(h_{21}) = v_1^{-2}h_{11}^3h_{21}$. But now considering the action $E_3(End(M); Q_1) \otimes E_3(M; Q_1) \to E_3(M; Q_1)$ we have

$$d_3^M(h_{21} \cdot v_1) = d_3(h_{21}) \cdot v_1 + h_{21} \cdot d_3^M(v_1) = v_1^{-1}h_{11}^3h_{21} \neq 0$$

which can't happen since $h_{21}v_1$ survives in $E_r(M; Q_1)$. Note we have to consider the action since $h_{21}v_1$ would not be present in $E_3(End(M); Q_1)$. Hence our assumption was wrong and $d_2(h_{21}) \neq 0$, which by degree reasons means $d_2(h_{21}) = v_1^{-1}\alpha h_{11}^2h_{21}$.

Finally both h_{11} and $v_1 h_{21}$ survive d_3^M in $E_3(M;Q_1)$, so they must also survive d_3 in $E_3(End(M);Q_1)$ i.e. $d_3(h_{11}) = d_3(v_1 h_{21}) = 0$. At the same time, for degree reasons $d_r(\alpha) = d_r(\alpha v_1) = 0$ for $r = 2,3$ and neither elements can be a coboundary, which means both α and αv_1 are present in $E_4(End(M);Q_1)$.

□

4.2 Conjectures on $E_r(End(M);Q_1)$

Given the theorem above, in order to compute d_2 completely we just need to know the values on the remaining generators i.e. $d_2(h_{n1})$ for $n \geq 3$. Thus we make the following conjecture:

(Main) Conjecture part 1: $d_2(h_{n1}) = v_1^{-1}\alpha h_{11}^2 h_{n1}$ for $n \geq 3$

Observe then $x_n = v_1 h_{n+1,1}$ is a cycle, and that

$$E_2(End(M);Q_1) = \mathbb{F}_2[x_1, x_2, ...] \otimes \mathbb{F}_2[v_1^{\pm 1}, h_{11}, \alpha]/(\alpha^2)$$

where the first factor has zero differential and the second factor has only $d_2 v_1 = \alpha h_{11}^2$. The homology is thus

$$E_3(End(M);Q_1) = \mathbb{F}_2[x_1, x_2, ...] \otimes \mathbb{F}_2[v_1^{\pm 2}, h_{11}, \alpha, \alpha']/(\alpha^2, \alpha h_{11}^2, \alpha\alpha', \alpha'^2)$$

where α' is the class of $v_1\alpha$. Again Theorem 1 tells us $d_3(x_1) = d_3(\alpha) = d_3(\alpha') = 0$ and $d_3(v_1^2) = h_{11}^3$ and so in order to compute d_3 completely we just need to know the values on the remaining generators i.e. $d_3(x_n)$ for $n \geq 2$. Thus we further conjecture:

(Main) Conjecture part 2: $d_3(x_n) = v_1^{-4} h_{11} x_1 x_{n-1}^2$ for $n \geq 2$

We can prove this conjecture modulo the following assumption

(Smaller) conjecture: $v_1^m x_n$ does not survive to $E_4(End(M);Q_1)$ for $n, m \in \mathbb{Z}$, $n \geq 2$.

Theorem 4.2.1: The smaller conjecture above implies the main one.

Before proving the Theorem observe the converse statement that the main conjecture implies the smaller one also holds. In fact, the main conjecture even specifies what $d_r(v_1^m x_n)$ is, which is what justifies the naming convention of the two conjectures. Thus, the Theorem can be reformulated by saying that the smaller and main conjectures above are equivalent.

Proof:

For $n \geq 3$ $d_2(h_{n1})$ is a linear combination of $v_1^{-1}\alpha h_{11}^2 h_{n1}$ and $v_1^{-1}\alpha h_{21} h_{n-1,1}^2$ for degree reasons, but the later is not in the image of $E_2(S;Q)$. Hence $d_2(h_{n1}) = v_1^{-1}\alpha h_{11}^2 h_{n1}$ or 0. Assume that for some $n \geq 3$ $d_2(h_{n1}) = 0$. For degree reasons, $d_3(h_{n1})$ is a linear combination of $v_1^{-2}h_{11}^3 h_{n1}$ and $v_1^{-2}h_{11}h_{21}h_{n-1,1}^2$, but $v_1^{-2}h_{11}h_{21}h_{n-1,1}^2$ doesn't survive to $E_3(End(M);Q_1)$ since

$$d_2(v_1^{-2}h_{11}h_{21}h_{n-1,1}^2) = d_2(h_{21})v_1^{-2}h_{11}h_{n-1,1}^2 = v_1^{-3}\alpha h_{11}^3 h_{21} h_{n-1,1}^2$$

By our smaller conjecture, $d_3(h_{n1}) \neq 0$ and so $d_3(h_{n1}) = v_1^{-2}h_{11}^3 h_{n1}$. Then

$$d_3(v_1^2 h_{n1}) = d_3(v_1^2)h_{n1} + v_1^2 d_3(h_{n1}) = h_{11}^3 h_{n1} + h_{11}^3 h_{n1} = 0$$

which again contradicts the (smaller) conjecture. We conclude $d_2(h_{n1}) = v_1^{-1}\alpha h_{11}^2 h_{n1}$ for all $n \geq 2$, which is also equivalent to $d_2(v_1 h_{n1}) = 0$ for all $n \geq 2$. Hence the elements $x_n = v_1 h_{n+1,1}$ survive, which justifies their presence in E_3. This completes the d_2 calculation in $E_2(End(M);Q_1)$.

Next for $n \geq 2$ $d_3(x_n)$ is a linear combination of $v_1^{-4}h_{11}x_1x_{n-1}^2$ and $v_1^{-2}h_{11}^3 x_n$, which leaves us with 4 possibilities. $d_3(x_n) = v_1^{-2}h_{11}^3 x_n$ would imply $d_3(v_1^2 x_n) = 0$ and so $d_3(x_n) = 0$ or $v_1^{-2}h_{11}^3 x_n$ are both ruled out as possibilities due to the (smaller) conjecture. Then either $d_3(x_n) = v_1^{-4}h_{11}x_1x_{n-1}^2$ or $d_3(x_n) = v_1^{-4}h_{11}x_1x_{n-1}^2 + v_1^{-2}h_{11}^3 x_n$. However, the latter case would imply

$$d_3(v_1^2 x_n) = d_3(v_1^2)x_n + v_1^2 d_3(x_n) = h_{11}^3 x_n + h_{11}^3 x_n + v_1^{-2}h_{11}x_1x_{n-1}^2 = v_1^{-2}h_{11}x_1x_{n-1}^2$$

and so

$$0 = d_3^2(v_1^2 x_n) = d_3(v_1^{-2}h_{11}x_1x_{n-1}^2) = d_3(v_1^{-2})h_{11}x_1x_{n-1}^2 = v_1^{-4}h_{11}^4 x_1 x_{n-1}^2$$

which is false as $v_1^{-4}h_{11}^4 x_1 x_{n-1}^2$ is present in $E_3(End(M);Q_1)$. We conclude $d_3(x_n) = v_1^{-4}h_{11}x_1x_{n-1}^2$ for $n \geq 2$ as desired.

$\square$

It is worth mentioning that Palmieri's original conjecture would imply that $d_3^M(v_1^m h_{n1}) \neq 0$ for $n \geq 3$, which would guarantee the (smaller) conjecture. However, the smaller conjecture itself is enough to arrive at a different answer than what Palmieri suggested. This proves his original formulation is incorrect, but as we will see in the next section it is close to what we arrive at based on the (smaller) conjecture.

4.3 Completing the calculation of d_3 in $E_3(M; Q_1)$

Now that we have learnt a fair bit about the structure of $E_r(End(M); Q_1)$ we will see how the information about its differentials can translate to information about the differentials in $E_r(M; Q_1)$. Recall for degree reasons $E_2(M; Q_1) = E_3(M; Q_1)$. Observe $E_3(M; Q_1)$ is now generated by $\{1, v_1\}$ as a $E_3(End(M); Q_1)$-module. Since v_1 survives to $E_\infty(M; Q_1)$ we get $d_3^M(v_1) = d_3^M(1) = 0$ and so d_3 now completely determines d_3^M.

For example, to compute $d_3^M(h_{n1})$ for $n \geq 3$ note that $h_{n1} = v_1^{-2} x_{n-1} \cdot v_1$ and so we get

$$d_3^M(h_{n1}) = d_3(v_1^{-2} x_{n-1}) \cdot v_1 = v_1^{-2} h_{11}^3 h_{n1} + v_1^{-2} h_{11} h_{21} h_{n-1,1}^2$$

We conclude that assuming the (smaller) conjecture holds, the differentials in $E_3(M; Q_1)$ are

$d_3^M(v_1^2) = h_{11}^3$

$d_3^M(h_{21}) = v_1^{-2} h_{21} h_{11}^3$

$d_3^M(h_{n1}) = v_1^{-2} h_{11}^3 h_{n1} + v_1^{-2} h_{11} h_{21} h_{n-1,1}^2$ for $n \geq 3$

which is what we conjectured in Section 2.

5 Relation between Palmieri's and Mahowald's notations

In this section we will see how the conjectured differentials for $E_3(M; Q_1)$ imply Mahowald's conjecture assuming there are no higher degree differentials. We begin by stating Mahowald's conjecture explicitly following the original description in [3]. Let $P = \mathbb{F}_2[x_1, x_2, \cdots]$ be a polynomial algebra, which is bigraded with $|x_i| = (2, 2^{i+2} + 1)$. Set a derivation d on P by $d(x_i) = x_1 x_{i-1}^2$ for $i > 1$. Let $H(d)$ be the resulting homology and $B(d)$ the image of d. Then assuming a and b run through an $\mathbb{F}_2$-basis for $H(d)$ and $B(d)$ Mahowald conjectured that

$$\begin{aligned} v_1^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(M)) &= \bigoplus_{a\in H(d)} \Sigma^{|a|} v_1^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bo\wedge M)) \\ &\oplus \bigoplus_{b\in B(d)} \Sigma^{|b|} v_1^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bu\wedge M)) \end{aligned}$$

Here bo and bu are connective real and complex K-theory respectively and we have explicit computations:

$$v_1^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bo\wedge M)) = \mathbb{F}_2[v_1^{\pm 4}]\otimes \mathbb{F}_2(h_{11}, v_1)/(h_{11}^3, v_1^2)$$

$$v_1^{-1}Ext_A^{s,t}(\mathbb{F}_2, H_*(bu\wedge M)) = \mathbb{F}_2[v_1^{\pm 1}]$$

In other words, the conjecture reads that $v_1^{-1}E_2(M;H)$ consists of $|H(d)|$ copies of $\mathbb{F}_2[v_1^{\pm 4}]\otimes \mathbb{F}_2(h_{11}, v_1)/(h_{11}^3, v_1^2)$ and $|B(d)|$ copies of $\mathbb{F}_2[v_1^{\pm 1}]$. To clarify, by $|H(d)|$ we mean the number of basis elements of any given degree in $H(d)$ and even though $H(d)$ is infinite, it is of finite type and so for every basis element $a\in H(d)$ the copy is suspended by the degree of a. The same holds for $B(d)$.

Recall $E_3 = E_3(M;Q_1) = \mathbb{F}_2[v_1^{\pm 1}]\otimes \mathbb{F}_2[h_{11}, h_{21}, h_{31}, ...]$ with proposed differentials $d_3(v_1^2) = h_{11}^3$ and $d_3(h_{n1}) = v_1^{-2}h_{11}^3 h_{n1} + v_1^{-2}h_{11}h_{21}h_{n-1,1}^2$ for $n>2$. We will express E_4 in such a way that it takes the form Mahowald suggested. Rewrite $E_3 = \mathbb{F}_2[v_1^{\pm 1}, h_{11}]\otimes \mathbb{F}_2[x_1, x_2...]$ where $x_n = v_1 h_{n+1,1}$ and introduce a grading on E_3 so that $|v_1^i| = \begin{cases} 0 & \text{if } i\equiv 0,1(4) \\ 2 & \text{if } i \equiv 2,3(4) \end{cases}$, $|h_{11}| = 1$ and $|x_n| = 0$. Extend this grading to monomials in the obvious fashion. Then $E_3 = \oplus_{n\geq 0}E_{3,n}$. The reason we are interested in this grading is that now d_3 increases it by 1. But then E_4 is just the homology of the graded chain complex i.e. $E_4 = \oplus_{n\geq 0}\ker(d_3^n)/\mathrm{im}(d_3^{n-1})$.

$$0 \xrightarrow{d_3^{-1}} E_{3,0} \xrightarrow{d_3^0} E_{3,1} \xrightarrow{d_3^1} E_{3,2} \xrightarrow{d_3^2} \cdots$$

We claim that

(1) $\ker(d_3^0)/\mathrm{im}(d_3^{-1}) = \ker(d_3^0) = Z(d)\otimes \mathbb{F}_2[v_1^{\pm 4}]\otimes \mathbb{F}_2[v_1]/(v_1^2)$

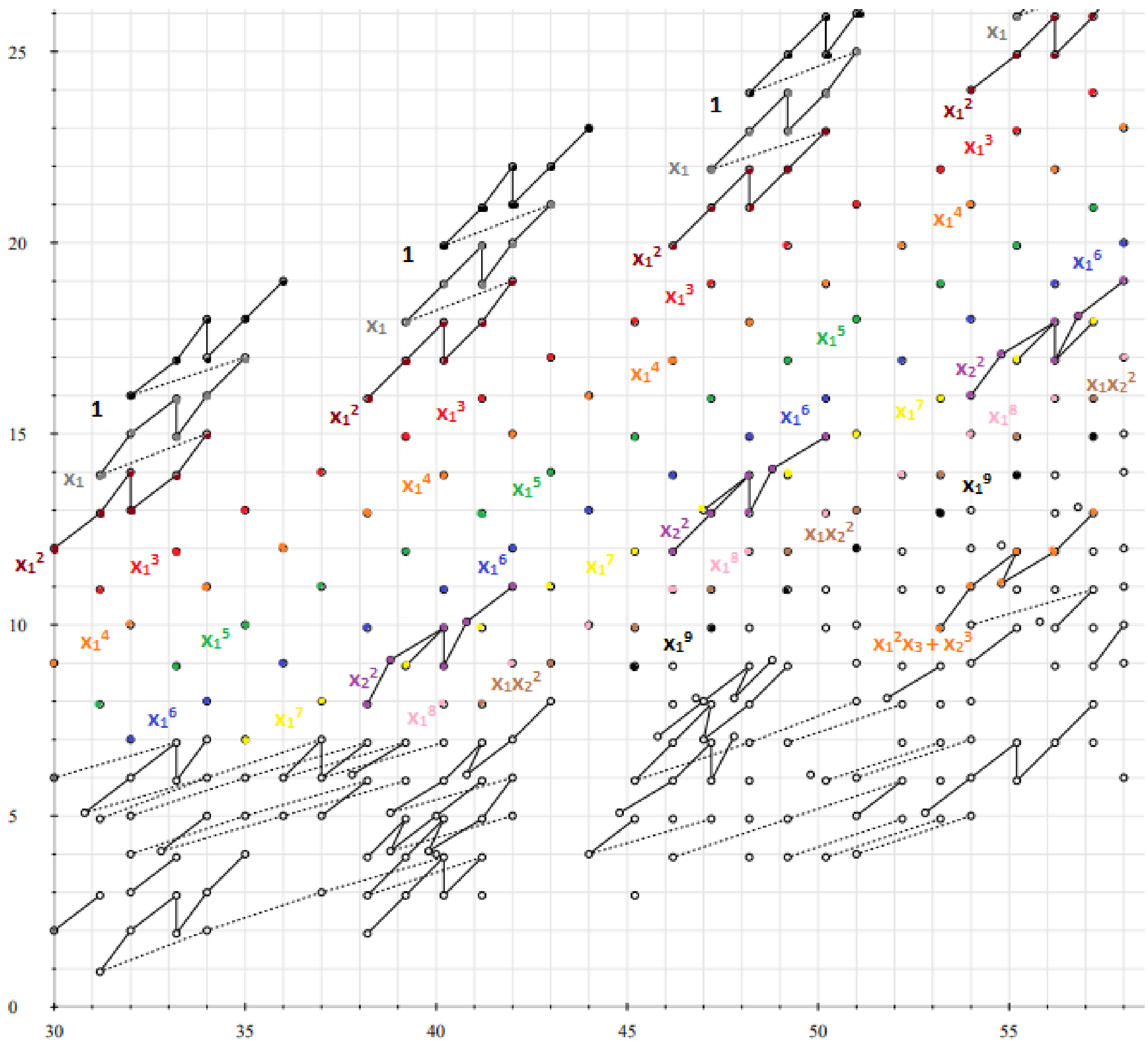
(2) $\ker(d_3^1)/\mathrm{im}(d_3^0) = H(d)\otimes \mathbb{F}_2[v_1^{\pm 4}]\otimes \mathbb{F}_2[v_1]/(v_1^2)\otimes \{h_{11}\}$

$$(3) \quad \begin{aligned} &\Big(\ker(d_3^2)/\mathrm{im}(d_3^1)\Big)/H(d)\otimes\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2[v_1]/(v_1^2)\otimes\{h_{11}^2\}\cong \\ &\qquad\cong B(d)\otimes\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2[v_1]/(v_1^2)\otimes\{v_1^2\} \end{aligned}$$

(4) $\ker(d_3^n)/\mathrm{im}(d_3^{n-1})=0$ for $n\geq 3$

Given the proof of $(1)-(4)$ is not particularly insightful, we leave it for the end of this section. We are left with the task of identifying the expressions above with Mahowald's formulation. The key here is to observe that given (2) and (3) we would need to identify $Z(d)\otimes\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2[v_1]/(v_1^2)$ in (1) with $(H(d)\oplus B(d))\otimes\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2[v_1]/(v_1^2)$. Then from $(1),(2),(3)$ we would get the $|H(d)|$ copies of $\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2(h_{11},v_1)/(h_{11}^3,v_1^2)$. What is left over is $B(d)\otimes\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2[v_1]/(v_1^2)$ from (1) and $B(d)\otimes\mathbb{F}_2[v_1^{\pm 4}]\otimes\mathbb{F}_2[v_1]/(v_1^2)\otimes\{v_1^2\}$ from (3), which combine to produce $|B(d)|$ copies of $\mathbb{F}_2[v_1^{\pm 1}]$. Thus each of $(1),(2)$ and (3) corresponds to a third of the "lightning flash" sequence, while the remainder of (1) and (3) each represent half of the v_1-line.

Below we can see exactly how the elements of $H(d)$ and $B(d)$ correspond to lightning flashes and v_1-lines in $E_2(M;H)$. The first few elements of $H(d)$ appearing are $1,x_1,x_1^2,x_2^2$ and $x_1^2x_3+x_2^3$ and we can see the lightning falshes for each one. Similarly, the first few elements of $B(d)$ appearing are x_1^3 through x_1^9 and $x_1x_2^2$ each corresponding to a copy of $\mathbb{F}_2[v_1^{\pm 1}]$. The colors used have no underlying meaning outside of grouping together the different elements in $E_2(M;H)$ and relating each group to its representing element of $H(d)$ or $B(d)$.



We are left to prove $(1)-(4)$. It is an immediate check to verify they follow from (i) and (ii) below, which is what we set out to show.

$$
(i)\quad
\begin{aligned}
&\ker(d_3^n) = Z(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}^n\} \quad \text{if } n = 0,1 \\
&\ker(d_3^n)/Z(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}^n\} \cong \\
&\cong B(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{v_1^2\} \otimes \{h_{11}^{n-2}\} \quad \text{if } n \geq 2
\end{aligned}
$$

$$
(ii)\ \ \operatorname{im}(d_3^n) = \begin{cases} B(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}^{n+1}\} & \text{if } n = 0,1 \\ \ker(d_3^{n+1}) & \text{if } n \geq 2 \end{cases}
$$

Note that that $E_3^0 = P \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2)$, $E_3^1 = P \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}\}$ and

$d_3^0(y) = d(y)v_1^{-4}h_{11}$ for every $y \in P \subset E_3^0$. Hence $\ker(d_3^0)$ and $\mathrm{im}(d_3^0)$ take the desired form and the same argument holds for $\ker(d_3^1)$ and $\mathrm{im}(d_3^1)$. We proceed to calculate $\ker(d_3^2)$ and the calculation of $\ker(d_3^n)$ for $n > 2$ is analogous. Every element of E_3^2 takes the form $\sum_{i=1}^s v_1^{m_i} y_i + \sum_{j=1}^t v_1^{l_j} z_j h_{11}^2$ where $m_1 < m_2 < \cdots < m_s$, $m_i \equiv 2, 3(4)$, $l_1 < l_2 < \cdots l_t$, $l_j \equiv 0, 1(4)$ and $y_i, z_j \in P$. We also assume $y_i, z_j \neq 0$. Then

$$d_3^2\left(\sum_{i=1}^s v_1^{m_i} y_i + \sum_{j=1}^t v_1^{l_j} z_j h_{11}^2\right) = \sum_{i=1}^s \left(v_1^{m_i-2} y_i h_{11}^3 + v_1^{m_i-4} d(y_i) h_{11}\right) + \sum_{j=1}^t v_1^{l_j-4} d(z_j) h_{11}^3$$

Setting this equal to 0 we observe two cases. First if $s = 0$ then $d(z_j) = 0$ for all j and we get the same component as in $\ker(d_3^0)$, namely $Z(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}^2\} \subset \ker(d_3^2)$. If $s > 0$ then we obtain $d(y_i) = 0$ for all i and we are left with

$$\sum_{i=1}^s v_1^{m_i-2} y_i + \sum_{j=1}^t v_1^{l_j-4} d(z_j) = 0$$

which given the degrees of v_1 can only happen if $s = t$, $m_i - 2 = l_i - 4$ and $y_i = d(z_i)$. Note $y_i = d(z_i)$ already implies $d(y_i) = 0$. Furthermore, for every $y_i \in B(d)$ we have a unique $z_i \in P$ with $y_i = d(z_i)$ modulo $Z(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}^2\} \subset \ker(d_3^2)$. Hence

$$\ker(d_3^2)/Z(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{h_{11}^2\} \cong B(d) \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2) \otimes \{v_1^2\}$$

as desired. In fact, $\ker(d_3^2) \cong P \otimes \mathbb{F}_2[v_1^{\pm 4}] \otimes \mathbb{F}_2[v_1]/(v_1^2)$, but stated this way it does not relate well with Mahowald's conjecture.

Next we show $\mathrm{im}(d_3^2) = \ker(d_3^3)$ and the result for $\mathrm{im}(d_3^n)$ follows analogically. As we saw above elements of $\ker(d_3^3)$ are sums of elements of the form $v_1^m y h_{11} + v_1^{m-2} z h_{11}^3$ for $m \equiv 2, 3(4)$ and $y, z \in P$ such that $d(z) = y$. But then $d_3^2(v_1^m z) = v_1^m y h_{11} + v_1^{m-2} z h_{11}^3$ and so $\ker(d_3^3) \subset \mathrm{im}(d_3^2)$ and since the reverse inclusion holds as well the two must coincide. This completes the proof of (i) and (ii) and thus we have successfully identified Mahowald's and Palmieri's formulations of the problem.

6 Introducing the "square" of spectral sequences

In this section we will improve upon a technique originally used by Miller [6] and further refined by Andrews and Miller [1] to obtain information about differentials in a spectral sequence. An informal discussion to the approach below was first presented by Novikov [7]. Most of this section is based on [1] and follows the approach there closely. We will try to set up the machinery of the "square" in a great generality where we are working in any triangulated category, but the reader should keep in mind the goal is to ultimatly use our setup in the category of stable comodules over the dual Steenrod algebra.

Consider resolving a spectrum X by another spectrum B thus obtaining a spectral sequence $E_2(X;B) \Longrightarrow \pi_*(X)$. How can we go about computing the differentials? One approach is to pick a spectrum A and consider the resolutions of X by A and B simultaniously. We can resolve by A first and then by B or vice versa. This would give us 4 different spectral sequences organized as in the figure below - hence a "square" of spectral sequences.

$$\begin{tikzcd} * \arrow[r, Rightarrow, "Mahowald"] \arrow[d, Rightarrow, "May"] & E_2(X;B) \arrow[d, Rightarrow, "B-Adams"] \\ E_2(X;A) \arrow[r, Rightarrow, "A-Adams"] & \pi_*(X) \end{tikzcd}$$

Explaining why would such a diagram make sense and how is it organized is the goal of this section. There are a number of conditions that need to be satisfied by A and B, but perhaps the most vital one - central to the approach - is requiring the existence of a ring map $A \to B$. This guarantees that every element in $\pi_*(X)$ has A-filtration s and B-filtration $s+t$ for some $s, t \geq 0$. Then the diagram gives us two different ways to resolve elements of $\pi_*(X)$ - first by finding $s+t$ and then finding out s or first finding out s and then $s+t$. This condition is at the "heart" of the construction as it will become apparent. The rest of the conditions on A, B are more technical and it is conceivable that one would be able to perform similar (albeit more difficult and less complete) analysis without them.

(C. 1) There exists a ring map $\delta : A \longrightarrow B$

6.1 Setting up the A, B- Adams spectral sequence

We set up the A-Adams spectral sequence for X by considering the canonical A-resolution of $\mathbb{S}$ and smashing it on the left with X. Via the unit map of A we obtain a cofiber sequence $\mathbb{S} \longrightarrow A \longrightarrow \bar{A}$. Smashing it with powers of $\bar{A}$ we obtain an A-resolution for $\mathbb{S}$

$$\begin{tikzcd} \mathbb{S} \arrow[d] & \bar{A} \arrow[l] \arrow[d] & \cdots \arrow[l] & \bar{A}^{\wedge s} \arrow[l] \arrow[d] & \bar{A}^{\wedge(s+1)} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ A^{[0]} & A^{[1]} & & A^{[s]} & A^{[s+1]} & \end{tikzcd}$$

where the top maps are desuspensions and we use the notation $A^{[s]} = \bar{A}^{\wedge s} \wedge A$. Smashing the above diagram with X on the left and taking the LES of homotopy groups for each cofiber sequence results in an exact couple, which is the A- Adams spectral sequence for X.

We perform the exact same construction for B except that $B^{[t]} = B \wedge \bar{B}^{\wedge t}$ and we smash the canonical B-resolution with X on the right instead of on the left. It is crucial to observe that the reason we can simultaniously resolve X by both A and B is precisely because we have a freedom to resolve either on the left or on the right. This will be an important point when we end up performing calculations as the cobar complexes for computing $E_2(X, A)$ and $E_2(X; B)$ would be set up via coaction maps for right and left comodules respectively. An interesting observation is that resolving by more than 2 spectra simultaniously can't be done in that context as we have no more degrees of freedom available (not to mention it is not clear why one would like to deal with such a beast in the first place).

6.2 Setting up the May and Mahowald spectral sequences

We begin by defining the May spectral sequence in our square diagram. Note $E_2(X; A) = H(E_1(X; A), d_1^A) = H(\pi_*(X \wedge A^{[s]}), d_1^A)$, so we consider the B-filtration of $\pi_*(X \wedge A^{[s]})$ we will obtain a spectral sequence converging to $E_2(X; A)$ - the May spectral sequence in our diagram. To be able to perform computations we need the following assumption:

$$\textbf{(C.2)}\ E_r(X \wedge A^{[s]}; B) \Rightarrow \pi_*(X \wedge A^{[s]}) \text{ collapses at } E_2$$

This implies that $E_1^{May} = E_2(X \wedge A^{[s]}; B)$. Another way to express the above condition is by saying that X is a (A, B)-*primary* spectrum.

To define the Mahowald spectral sequence note that **(C.1)** implies that B is A-injective and so A-exact sequences are B-exact. Hence, applying $E_2(-;B)$ to the A-resolution of X would produce a family of LES's that link together to produce an exact couple. The resulting spectral sequence is the Mahowald spectral sequence. It converges to $E_2(X;B)$. Note $E_1^{Mah} = E_2(X \wedge A^{[s]}; B) = E_1^{May}$, which completes our square of spectral sequences.

We will need a final condition stating that the following diagram commutes **(C3)**:

$$\begin{tikzcd} \bar{B} \wedge \bar{A} \arrow[rr, "1\wedge i_A"] \arrow[dr, "1\wedge\delta"'] & & \bar{B} \wedge \bar{B} \arrow[dl, "i_B\wedge 1"] \\ & \Sigma\bar{B} & \end{tikzcd}$$

For simplicity, we introduce the notation $X^{[t][s]} = B^{[t]} \wedge X \wedge A^{[s]}$, $X^{(t)[s]} = \bar{B}^{\wedge t} \wedge X \wedge A^{[s]}$, $X^{[t](s)} = B^{[t]} \wedge X \wedge \bar{A}^{\wedge s}$, $X^{(t)(s)} = \bar{B}^{\wedge t} \wedge X \wedge \bar{A}^{\wedge s}$. We also set i_A, j_A, k_A and i_B, j_B, k_B to be the maps in the exact couple for the A and B Adams Spectral Sequences respectively. For example, $E_r(X \wedge A^{[s]}; B)$ is obtained via the exact couple

$$\begin{tikzcd} \oplus_{t,u}\pi_u(X^{(t)[s]}) \arrow[rr, "i_B"] & & \oplus_{t,u}\pi_u(X^{(t)[s]}) \arrow[dl, "j_B"] \\ & \oplus_{t,u}\pi_u(X^{[t][s]}) \arrow[ul, "k_B"] & \end{tikzcd}$$

with maps

$$i_B : \pi_u(X^{(t+1)[s]}) \to \pi_{u-1}(X^{(t)[s]})$$

$$j_B : \pi_u(X^{(t)[s]}) \to \pi_u(X^{[t][s]})$$

$$k_B : \pi_u(X^{[t][s]}) \to \pi_u(X^{(t+1)[s]})$$

6.3 Main result

Theorem 6.3.1: If an element $x \in E_2^{t+s}(X;B)$ survives to E_4 then its representative $a \in E_1^{May}$ survives to E_3^{May}. More precisely we will see that $d_2^B x = 0$ implies $d_1^{May} a = 0$ and $d_3^B x = 0$ implies

$d_2^{May} a = 0$.

An integral part of the proof is the following lemma due to May [5] following from observations in [2].

Lemma 6.3.2: Let $D \to E \to F$ and $X \to Y \to Z$ be cofiber sequences. Smash them together to get the following commutative diagram of cofiber sequences.

$$\begin{array}{ccccc} D \wedge X & \longrightarrow & D \wedge Y & \longrightarrow & D \wedge Z \\ \downarrow & & \downarrow & & \downarrow \\ E \wedge X & \longrightarrow & E \wedge Y & \longrightarrow & E \wedge Z \\ \downarrow & & \downarrow & & \downarrow \\ F \wedge X & \longrightarrow & F \wedge Y & \longrightarrow & F \wedge Z \end{array}$$

Take $e \in \pi_n(E \wedge Y)$ that maps to 0 in $\pi_n(F \wedge Z)$. Then there are elements $d \in \pi_n(D \wedge Z)$ and $f \in \pi_n(F \wedge X)$ that map to the image of e in $\pi_n(E \wedge Z)$ and $\pi_n(F \wedge Y)$ respectively. Fruthermore, those elements can be chosen so that they have the same image (up to a sign) in $\pi_{n-1}(D \wedge X)$ under the boundary maps associated to the cofiber sequences along the top and left edge of the diagram.

We will use a slightly stronger version of this lemma, which states that for every $d' \in \pi_n(D \wedge Z)$ that maps to the image of e in $\pi_n(E \wedge Z)$, there exists a $f' \in \pi_n(F \wedge X)$ that maps to the image of e in $\pi_n(F \wedge Y)$. To see how this stronger version follows from the lemma above, note $d' - d$ maps to 0 in $\pi_n(E \wedge Z)$ and so there is $g \in \pi_{n+1}(F \wedge Z)$ that maps to $d' - d$. But then we can pick $f' \in \pi_n(F \wedge X)$ so that g maps to $f' - f$ in $\pi_n(F \wedge Z)$. Now d' and f' would have the same image (up to a sign) in $\pi_{n-1}(D \wedge X)$. Also note since we are working mod 2, we don't have to worry about signs.

Proof of Theorem 6.3.1: Let x has A-filtration s i.e. it can be lifted to an element $z \in E_2^t(X \wedge \bar{A}^{\wedge s}; B)$. Clearly z survives to E_4 as well. Pick $z' \in E_1^t(X \wedge \bar{A}^{\wedge s}; B) = \pi_*(X^{[t][s]})$ that represents z. Since z' survives to E_4, there must exist $y''' \in \pi_*(X^{(t+4)(s)})$ such that $k_B z' = i_B^3 y'''$ and $j_B y'''$ will be represented by $d_4^B z'$ in $E_4(X \wedge \bar{A}^{\wedge s}; B)$. A central point will be to show we can choose y''' so that it lifts to $E_1(X \wedge \bar{A}^{\wedge s+1}; B)$ via the map δ. With that in mind, note $a' = j_A(z')$ survives to an element $a \in E_2(X \wedge \bar{A}^{\wedge s}; B)$ and so must survive to E_∞ by **(C2)**. Hence there exists $b' \in \pi_*(X^{(t)[s]})$ with $j_B b' = a'$ and so $k_B a' = 0$. Consider $j_A i_B^2 y'''$. We know applying either i_B or j_B to this element produces 0. But note $i_B j_A i_B^2 y''' = 0$ implies we can pull back $j_A i_B^2 y'''$ to an element $w \in \pi_*(X^{[t+1][s]})$ while $j_B j_A i_B^2 y''' = 0$ implies $d_1^B w = 0$ and so w survives to E_∞ and as above $j_A i_B^2 y''' = 0$. By the exact same reason since both i_B and j_B yield 0 on $j_A i_B y'''$ we conclude $j_A i_B y''' = 0$. Hence there exists $y_2 \in \pi_*(X^{(t+3)(s+1)})$ such that $i_A y_2 = i_B y'''$, but $i_A y_2 = i_B \delta y_2$ by **(C3)** and so we can pick $y''' = \delta y_2$. As a side note, observe $j_B y''' = \delta j_B y_2$ and so $d_4^B x$ has A-filtration $s+1$.

Recall we want to show $d_1^{May} a = 0$ and $d_2^{May} a = 0$. $d_1^{May} a$ is obtained by the top of the following diagram.

$$
\begin{tikzcd}
X^{[t][s]} \arrow[rr, "d_1^A"] & & X^{[t][s+1]} & & X^{[t+1][s+1]} \arrow[ll, "i_B"'] \\
X^{(t)[s]} \arrow[u, "j_B"] \arrow[rr, "d_1^A"] \arrow[dr, "k_A"] & & X^{(t)[s+1]} \arrow[u, "j_B"] & & X^{(t+1)[s+1]} \arrow[u, "j_B"] \arrow[ll, "i_B"'] \\
& X^{(t)(s+1)} \arrow[ur, "j_A"] & & X^{(t+1)(s+1)} \arrow[ll, "i_B"] \arrow[ur, "j_A"] &
\end{tikzcd}
$$

Set $y_0 = i_B y_1 = i_B^2 y_2$. Then May's lemma applied to the diagram below guarantees the existence of b' such that $i_B y_0' = k_A b'$. But then $d_1^{May} a$ is represented by $j_B j_A y_0 = 0$ as desired.

$$
\begin{tikzcd}
X^{(t)(s+1)} \arrow[r, "i_A"] \arrow[d, "j_B"] & X^{(t)(s)} \arrow[r, "j_A"] \arrow[d, "j_B"] & X^{(t)[s]} \arrow[d, "j_B"] \arrow[ll, bend right, "k_A"'] \\
X^{[t](s+1)} \arrow[r, "i_A"] \arrow[d, "k_B"] & X^{[t](s)} \arrow[r, "j_A"] \arrow[d, "k_B"] & X^{[t][s]} \arrow[d, "k_B"] \\
X^{(t+1)(s+1)} \arrow[r, "i_A"] \arrow[uu, bend left, "i_B"] & X^{(t+1)(s)} \arrow[r, "j_A"] & X^{(t+1)[s]}
\end{tikzcd}
$$

Similarly to get $d_2^{May} a$ we need to lift $j_A y_0$ via i_B to $\pi_*(X^{(t+2)[s+1]})$ and apply j_B, but y_0 lifts via i_B to y_1 and $j_B y_1 = 0$. Hence $j_B j_A y_1 = 0$ represents $d_2^{May} a$. This concludes the proof.

$$
\begin{tikzcd}
X^{[t][s]} \arrow[rr, "d_1^A"] & & X^{[t][s+1]} & & X^{[t+2][s+1]} \arrow[ll, "i_B^2"'] \\
X^{(t)[s]} \arrow[u, "j_B"] \arrow[rr, "d_1^A"] \arrow[dr, "k_A"] & & X^{(t)[s+1]} \arrow[u, "j_B"] & & X^{(t+2)[s+1]} \arrow[u, "j_B"] \arrow[ll, "i_B^2"'] \\
& X^{(t)(s+1)} \arrow[ur, "j_A"] & & X^{(t+2)(s+1)} \arrow[ll, "i_B^2"] \arrow[ur, "j_A"] &
\end{tikzcd}
$$

$\square$

It is easy to see that the same argument we applied to $d_2^{May} a$ works for higher differentials and we end up with the following generalization:

Theorem 6.3.3 (Generalization): If an element $x \in E_2^{t+s}(X;B)$ survives to E_{n+1} then its representative $a \in E_1^{May}$ survives to E_n^{May}.

It is worth noting the representative a above might not be unique and the result holds for any such choice. Indeed, the element a is obtained uniquely from a representative $z' \in E_1^t(X \wedge \bar{A}^{\wedge s}; B)$ of x and the above proof works for any such z'.

7 Proof of the Smaller Conjecture

7.1 Choice of spectra in the context of the square construction

We will begin this section with an informal discussion that would hopefully shed some light on the reason why the above construction of the square could be useful to our problem as well as problems of that type. Let's recall our goal is to show that a d_3 differential is non-zero on a family of elements of an Adams spectral sequence. We can reformulate this by saying we want to show the family of elements does not survive to E_4. Theorem 2 tells us it is then sufficient to find a spectrum T that together with Q_1 fits into the setting of the square defined above and for which the representatives of the family of elements we are interested in does not survive to E_3^{May}. At first sight this might seem like it introduces an unnecessary level of complexity. It is also not clear how one might go about finding such a T. The advantage we have here is that we know exactly what $d_3^{Q_1}$ should look like. Note all elements h_{n1} have an $(s+t)-$filtration of 1, while d_n^{May} increases $s-$filtration by 1. Hence we want h_{n1} to have s-filtration 1 less than the $s-$filtration of $v_1^{-2}h_{11}h_{21}h_{n-1,1}^2$ for every $n > 2$. For every h_{n1} we have 2 possibilities to the corresponding values of (s,t) as both are non-negative and they sum to 1. Note also the $(s+t)-$filtration of v_1 is 0. Now pick the smallest $n > 2$ (if it exists) such that the $s-$filtration of h_{n1} is 1 (rather than 0). Then the $s-$filtration of $v_1^{-2}h_{11}h_{21}h_{n1}^2$ would be at least 2 and the $s-$filtration of $h_{n+1,1}$ is at most 1, but we want the difference between the two to be exactly 1 and so h_{11} and h_{21} are forced to have an $s-$filtration of 0. However, then the $s-$filtration of $v_1^{-2}h_{11}h_{21}h_{n-1,1}^2$ is 0, which is not 1 more than the $s-$filtration of h_{n1}. Hence we can assume for $n > 2$ h_{n1} has $s-$filtration 0. This forces h_{11} to have $s-$filtration 1 and h_{21} to have $s-$filtration 0. What this means is the elements h_{n1} for $n > 1$ are represented by elements in $T_{**}M$ in the cobar complex that is the E_1 page of the T-Adams spectral sequence for M. At the same time h_{11} should not be present in $T_{**}M$, but rather be represented in $T_{**}M \otimes T_{**}T$.

Recall the May spectral sequence is obtained by applying a Q_1−filtration to the T-cobar complex. Then the calculation of d_n^{May} comes down to calculating the coaction map $T_{**}M \to T_{**}M \otimes T_{**}T$ for the Hopf Algebroid $(T_{**}, T_{**}T)$. For that reason we will choose $T = HC$ for some conormal quotient coalgebra C of the dual Steenrod algebra. Then $(T_{**}, T_{**}T)$ is in fact a split Hopf algebra with $T_{**}T \cong A\Box_C\mathbb{F}_2 \otimes T_{**}$ and $T_{**} = Ext_C(\mathbb{F}_2, \mathbb{F}_2)$ (prop. 1.4.6, p.27, Palmieri) i.e. the map of interest is just the coaction map of $Ext_C(\mathbb{F}_2, \mathbb{F}_2(\xi_1)/(\xi_1^2))$ as a $A\Box_C\mathbb{F}_2$−comodule.

As noted above, for $n > 1$ h_{n1} must be represented in $Ext_C(\mathbb{F}_2, \mathbb{F}_2)$, while h_{11} shouldn't be. This means we can choose any conormal quotient coalgebra$C$ of the dual Steenrod algebra locked between C_0 and C_1 i.e. both $C \to C_0$ and $C_1 \to C$ are quotients, where $C_0 = \mathbb{F}_2(\xi_1, \xi_2, \cdots)/(\xi_1^2, \xi_2^4, \xi_3^4, \cdots)$ and $C_1 = \mathbb{F}_2(\xi_1, \xi_2, \cdots)/(\xi_1^2)$. In other words C_0 and C_1 are the largest and smallest quotients that satisfy the restrictions on h_{n1} listed above.

As we proceed with the formal application of the square construction in our setup, observe there is a bit of care we need to exercise when translating the statements. Specifically, maps in $Stable(A)$ are bigraded and our construction will essentially ignore the second grading. Also as a matter of convention, cofiber sequences in $Stable(A)$ have the form $E \to R \to F \to \Sigma^{-1,0}E$ and so while the general arguments remain unchanged, **C.3** takes the following slightly different form:

$$\begin{tikzcd}
\bar{Q_1} \wedge \bar{HC} \arrow[rr, "1\wedge i_{HC}"] \arrow[dr, "1\wedge\delta"'] & & \bar{Q_1} \wedge \bar{Q_1} \arrow[dl, "i_{Q_1}\wedge 1"] \\
& \Sigma^{-1,0}\bar{Q_1} &
\end{tikzcd}$$

7.2 Condition $C.1$

In the next sections we will address what choice of C would fit in the setup of the square so that the pair (HC, Q_1) would satisfy conditions **C.1** − **C.3**. Condition **C.1** is in fact trivial as $Q_1 = H\mathbb{F}_2(\xi_2)/(\xi_2^2) = A\Box_{\mathbb{F}_2(\xi_2)/(\xi_2^2)}\mathbb{F}_2$ and so the quotient map $C \to \mathbb{F}_2[\xi_2]/(\xi_2^2)$ produces a ring map $HC \to Q_1$. So far this imposes no further restrictions on our choice of C.

7.3 Condition $C.2$

Condition **C.2** is essential for the construction of the May spectral sequence. More precisely we have that $E_2(M; HC) = H(\pi_{**}(M \wedge HC^{[s]}), d_1^{HC})$ and we would like to filter this complex via Q_1. This would produce a filtration spectral sequence which is the May spectral sequence. Condition **C.2** now allows us to identify $E_1^{May} = E_2(M \wedge HC^{[s]}; Q_1)$. An important point is that $E_2(M \wedge HC^{[s]}; Q_1)$ converges to $v_1^{-1}\pi_{**}(M \wedge HC^{[s]}) = \pi_{**}(v_1^{-1}M \wedge HC^{[s]})$, where the equality is just the Telescope conjecture in the setting of $Stable(A)$, which is known to hold [8, Prop.3.1.10]. Hence in order to construct the May spectral sequence we should be working with $v_1^{-1}M$ instead of M.

Proposition 7.3.1: The Q_1-Adams spectral sequences converging to $v_1^{-1}\pi_{**}(M \wedge HC_0)$ and $v_1^{-1}\pi_{**}(M \wedge HC_1)$ collapse.

Proof: This holds for degree reasons (p102-103, Palmieri).

Proposition 7.3.2: The Q_1-Adams spectral sequences converging to $v_1^{-1}\pi_{**}(M \wedge HC_0^{[s]})$ and $v_1^{-1}\pi_{**}(M \wedge HC_1^{[s]})$ collapse.

Proof: In fact this proposition holds for any conormal C as long as *Prop.*7.3.1 holds. Indeed, since C is conormal

$$v_1^{-1}\pi_{**}(M \wedge HC^{[s]}) = v_1^{-1}\pi_{**}(M \wedge HC) \otimes A\square_{\bar{C}}\mathbb{F}_2{}^{\otimes s}$$

Furthermore

$$E_2(M \wedge HC^{[s]}; Q_1) = E_2(M \wedge HC; Q_1) \otimes A\square_{\bar{C}}\mathbb{F}_2{}^{\otimes s}$$

and so the result follows from the previous proposition.

7.4 Condition $C.3$

Recall condition **C.3** states that the following diagram commutes:

$$\begin{tikzcd}
\bar{Q}_1 \wedge \bar{H}C \arrow[rr, "1\wedge i_{HC}"] \arrow[dr, "1\wedge\delta"'] & & \bar{Q}_1 \wedge \bar{Q}_1 \arrow[dl, "i_{Q_1}\wedge 1"] \\
& \Sigma^{-1,0}\bar{Q}_1 &
\end{tikzcd}$$

This would follow from the stronger statement that $Q_1^{-1,0}(\bar{Q}_1 \wedge \bar{H}C) = 0$ as observed by Andrews and Miller in [1]. More precisely, if we compose either of the two maps $\bar{Q}_1 \wedge \bar{H}C \to \Sigma^{-1,0}\bar{Q}_1$ in the diagram with $i_{Q_1} : \Sigma^{-1,0}\bar{Q}_1 \to \Sigma^{-2,0}\mathbb{S}$ we will obtain $i_{Q_1} \wedge i_{HC} : \bar{Q}_1 \wedge \bar{H}C \to \Sigma^{-2,0}\mathbb{S}$. Hence the difference between the two maps lifts to the fiber of i_{Q_1}, which is just $\Sigma^{-1,0}Q_1$, so to prove condition

C.3 it suffices to show $[\bar{Q_1} \wedge \bar{HC}, \Sigma^{-1,0}Q_1] = Q_1^{-1,0}(\bar{Q_1} \wedge \bar{HC})$ vanishes.

$$\begin{tikzcd}
\bar{Q_1} \wedge \bar{HC} \arrow[rr, "1\wedge i_{HC}"] \arrow[ddr, "1\wedge\delta"'] \arrow[dr, dotted] & & \bar{Q_1} \wedge \bar{Q_1} \arrow[ddl, "i_{Q_1}\wedge 1"] \\
& \Sigma^{-1,0}Q_1 \arrow[d, "k_{Q_1}"] & \\
& \Sigma^{-1,0}\bar{Q_1} \arrow[d, "i_{Q_1}"] & \\
& \Sigma^{-2,0}\mathbb{S} &
\end{tikzcd}$$

To prove $Q_1^{-1,0}(\bar{Q_1} \wedge \bar{HC}) = 0$ first note we have a duality statement relating the Q_1-homology and cohomology. This follows since $Q_{1**} = \mathbb{F}_2[v_1^{\pm 1}]$ is a field and so $Q_1^{a,b} \cong \mathrm{Hom}^{a,b}{}_{Q_{1**}}(Q_{1**}, Q_{1**}) \cong (Q_1)_{a,b}$. Furthermore $\mathrm{Hom}_{Q_{1**}}(-, Q_{1**})$ is exact and so inductively we get that for every finite type stable comodule N over the dual Steenrod algebra it holds that $Q_1^{a,b}(N) \cong (Q_1)_{a,b}(N)$. Thus it suffices to show $(Q_1)_{-1,0}(\bar{Q_1} \wedge \bar{HC}) = 0$. Indeed, we claim that for a suitable choice of C that $(Q_1)_{-1,0}(Q_1 \wedge HC)$ is an $\mathbb{F}_2$-vector space of dimension 2 with elements coming from $(Q_1)_{-1,0}(Q_1)$ and $(Q_1)_{-1,0}(HC)$ each of dimension 1. In other words smashing the two spectra produces no further homology and so $(Q_1)_{-1,0}(\bar{Q_1} \wedge \bar{HC})$ is trivial. Note this is exactly the same reasoning one uses in the ordinary category of stable cell complexes.

We directly compute $Q_{1**}(HC) = H(A\Box_C\mathbb{F}_2, Q_1) \otimes Q_{1**}$. Note $H(A\Box_C\mathbb{F}_2, Q_1)$ has bidegree $(0,*)$. Hence as long as $\xi_1^2 \in A\Box_C\mathbb{F}_2$ we have that

$$(Q_1)_{-1,0}(HC) = H_2(A\Box_C\mathbb{F}_2, Q_1) \otimes \{v_1^{-1}\} = \mathbb{F}_2\langle \xi_1^2 \otimes v_1^{-1}\rangle$$

Similarly

$$(Q_1)_{-1,0}(Q_1) = H_2(A\Box_{\mathbb{F}_2(\xi_2)/(\xi_2^2)}\mathbb{F}_2, Q_1) \otimes \{v_1^{-1}\} = \mathbb{F}_2\langle \xi_1^2 \otimes v_1^{-1}\rangle$$

and

$$(Q_1)_{-1,0}(Q_1 \wedge HC) = H_2(A\Box_{\mathbb{F}_2(\xi_2)/(\xi_2^2)}\mathbb{F}_2 \otimes A\Box_C\mathbb{F}_2, Q_1) \otimes \{v_1^{-1}\} = \mathbb{F}_2\langle (1\otimes\xi_1^2)\otimes v_1^{-1}, (\xi_1^2\otimes 1)\otimes v_1^{-1}\rangle$$

as desired.

7.5 Calculating d_2^{May}

Recall we start with $E_2(M;HC) = H(\pi_{**}(HC \wedge \bar{HC}^s \wedge M), d_1^{HC})$. Note following our construction we are smashing on the left and not on the right. That's because the Q_1-spectral sequence we already have is obtained by smashing on the right. Furthermore, the part of d_1^{HC} we are interested in is exactly the coaction map for $\pi_{**}(HC \wedge M)$ as a $A\Box_C\mathbb{F}_2$-comodule. However, this coaction map is one for a **right** comodule i.e. we are interested in $\pi_{**}(HC \wedge M) \to \pi_{**}(HC \wedge M) \otimes A\Box_C\mathbb{F}_2$. Recall

$$\pi_{**}(HC \wedge M) = Ext_{\mathbb{F}_2[\xi_2,\cdots]/(\xi_i^4)}(\mathbb{F}_2, \mathbb{F}_2) = \otimes_{n\geq 2}\mathbb{F}_2[h_{n0}, h_{n1}]$$

So what is $d_1^{HC}(h_{n1})$? Well, the representative in the cobar complex for $\mathbb{F}_2[\xi_2,\cdots]/(\xi_i^4)$ is just $\xi_n^2|1$. We have that $\triangle\xi_n^2 = \sum_{i=0}^{n} \xi_{n-i}^{2^{i+1}} \otimes \xi_i^2$ and so we are interested in those indices $0 < i \leq n$ for which $\xi_{n-i}^{2^{i+1}} \in \mathbb{F}_2[\xi_2,\cdots]/(\xi_i^4)$ and $\xi_i^2 \in A\Box_C\mathbb{F}_2$, but this can't happen and so $d_1^{HC}(h_{n1}) = 0$ and h_{n1} is primitive. But then all May differentials for h_{n1} vanish, which while not being what we want, at least doesn't contradict the square construction.

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