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Today we start looking at consequences of the vector field problem. First, remember the group $J(X)$. This was the quotient of $KO(X)$ by the equivalence generated on the level of bundles by $V \underset{J}{\simeq} W$ iff $S(V\oplus n\epsilon) \underset{\text{f.h.e.}}{\simeq} S(W\oplus n\epsilon)$.

An immediate consequence of the last few days' work is

Thm.

$$\begin{tikzcd} \widetilde{KO}(\mathbb{R}P^m) \arrow[d, two heads, "\cong"] \arrow[r, phantom, "\cong \mathbb{Z}/2^{\varphi(m)}"] & {} \\ \tilde{J}(\mathbb{R}P^m) & \end{tikzcd}$$

The surjection is in fact an isomorphism.

Proof $\widetilde{KO}(\mathbb{R}P^m)$ is generated by $L-1$. Now $n(L-1) \longmapsto 0$ means nL is stably f.h.t.; earlier we showed that this implies a (stable) splitting

$$\begin{tikzcd} \mathbb{R}P_n^{m+n} \arrow[r] & S^n \\ S^n \arrow[u] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

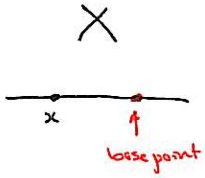
But now we know that this implies that $\nu(n) > \varphi(m)$, so $n \geq 2^{\varphi(m)}$, so $n(L-1)$ was already zero in $\widetilde{KO}(\mathbb{R}P^m)$.

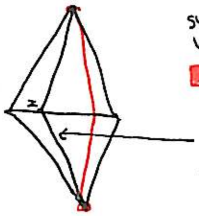
Homework: Compute $\tilde{J}(S^n)$.

The EHP sequence

Relations to problems in unstable homotopy are often mediated by the EHP sequence, so the next topic will be to construct it. Our starting point will be a theorem of Bott and Samelson about $H_*(\Omega\Sigma X)$; for a proof see George Whitehead's book.

There is a map $X \xrightarrow{\alpha} \Omega\Sigma X$ which embeds X as the "straight loops":





Here X is a pointed space, so "ΣX" means pointed suspension, in other words

🟥 crushed to a point, so

this is a loop, the image of x under α.

Well the loop structure makes $H_* \Omega\Sigma X$ into an algebra, so $\bar{H}_*(X) \xrightarrow{\alpha_*} H_*(\Omega\Sigma X)$ yields a unique extension

$$\begin{array}{l} T(\bar{H}_*(X)) \xrightarrow{\bar{\alpha}} H_*(\Omega\Sigma X). \\ \quad \| \\ \bigoplus_{k\geq 0} \bar{H}_*(X)^{\otimes k} \end{array}$$

Then the theorem is:

<u>Theorem</u> If X is connected, R is a p.i.d., and $H_*(X;R)$ is torsion free, then

$$T(\bar{H}^*(X)) \xrightarrow[\cong]{\bar{\alpha}} H_*(\Omega\Sigma X) \quad \text{is an isomorphism,}$$

where all homology is taken with coefficients in R.

Now since $H_*(X;R)$ is torsion free, $\bigoplus_{k\geq 1} \bar{H}_*(X)^{\otimes k} \cong \bigoplus_{k\geq 1} \bar{H}_*(X^{(k)})$ $\cong \bar{H}_*(\bigvee_{k\geq 1} X^{(k)})$, so you might hope idealistically that $\Omega\Sigma X$ splits as a wedge of smash powers of X. Of course that's not true, but amazingly enough

<u>Theorem</u> (James, probably): it does after one suspension; that is, there is a homotopy equivalence, when X is a connected CW complex,

$$\Sigma\Omega\Sigma X \simeq \bigvee_{k\geq 1} \Sigma X^{(k)}$$

and this equivalence reflects the map $\bar{\alpha}$ in the theorem above (or maybe the statement should be the other way around).

So let's prove that:

First, as X is connected, $\Sigma X^{(k)}$ and $\Sigma\Omega\Sigma X$ are simply connected.

If we produce a map giving an isomorphism in homology, then by the Hurewicz theorem we have a weak equivalence. Then if you believe that both spaces are CW complexes and some version of the Whitehead theorem, then we're done.

Now $$\Big[\bigvee_{k\geq 1}\Sigma X^{(k)}, \Sigma\Omega\Sigma X\Big]_* = \prod_{k\geq 1}\big[\Sigma X^{(k)}, \Sigma\Omega\Sigma X\big]_*$$

(which is to say that the wedge is the coproduct in the category of pted. spaces & homotopy classes of maps), so we construct a map on $\Sigma X^{(k)}$ for $k\geq 1$.

$$\Sigma X^{(k)} \xrightarrow[(1)]{} \Sigma X^k \xrightarrow[(2)]{\Sigma\alpha^k} \Sigma(\Omega\Sigma X)^k \xrightarrow[(3)]{\Sigma\mu} \Sigma\Omega\Sigma X.$$

(1) We've seen before in the case $k=2$: we produced a map

$$\begin{tikzcd}
\Sigma(X\times Y) \arrow[r, "\text{pinch}"] & \Sigma(X\times Y)\vee\Sigma(X\times Y) \arrow[r, "\Sigma pr_1\vee\Sigma pr_2"] & \Sigma X\vee\Sigma Y \\
\Sigma(X\vee Y) \arrow[u, hook] \arrow[urr, "\simeq"'] & &
\end{tikzcd}$$

On the other hand the sequence

$$\Sigma(X\vee Y) \longrightarrow \Sigma(X\times Y) \longrightarrow \Sigma(X\wedge Y)$$

combines to provide

$$\begin{tikzcd}
\Sigma(X\vee Y) \arrow[r] \arrow[d, "\simeq"'] & \Sigma(X\times Y) \arrow[r] \arrow[d, shift left] & \Sigma(X\wedge Y) \arrow[d, "\simeq"] \\
\Sigma X\vee\Sigma Y \arrow[r] & \Sigma(X\wedge Y)\vee\Sigma X\vee\Sigma Y \arrow[r] \arrow[u, dotted, shift left] & \Sigma(X\wedge Y)
\end{tikzcd}$$

The two horizontal lines give exact sequences in homotopy, so by the five lemma and assuming X and Y are CW-complexes, we get a map going back. Now the map

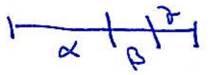
$$\Sigma(X\wedge Y) \longrightarrow \Sigma(X\wedge Y)\vee\Sigma X\vee\Sigma Y \xrightarrow{\simeq} \Sigma(X\times Y) \quad \text{gives}$$

the map (1) $\Sigma X^{(k)} \longrightarrow \Sigma X^k$; from its construction that it splits the homology of ΣX^k as a sum $H_*(\Sigma X^{(k)}) \oplus H_*((\Sigma X)^{\vee k})$.

(2) is the suspension of {the "straight loops" embedding α that started this whole discussion, repeated k times}.

(3) is the suspension of the loop multiplication map. Note that loop multiplication is not associative due to parameterization problems, that is,

$(\alpha\beta)\gamma \rightarrow$ 

$(\alpha)(\beta\gamma) \rightarrow$ 

but it is well-defined as a homotopy class.

Now the Bott-Samelson theorem says that the product of these composites as k ranges over the positive integers is an isomorphism in homology.

Note: the Bott-Samelson theorem required that $H^*(X;R)$ be torsion free and R a p.i.d. However, the claim is that this theorem of James holds for arbitrary coefficients, in particular with $\mathbb{Z}$ coefficients; in fact, we have a general proposition:

Lemma: If $X \longrightarrow Y$ induces $H_*(X;F) \xrightarrow{\cong} H_*(Y;F)$ an isomorphism on homology for $F = \left\{ \begin{matrix} \mathbb{Z}/p\mathbb{Z} & \forall p \text{ prime} \\ \mathbb{Q} & \end{matrix} \right\}$ then

$$H_*(X;\mathbb{Z}) \xrightarrow{\cong} H_*(Y;\mathbb{Z}).$$

Proof:

Use the universal coefficient theorem cleverly many times.

First, for any p, the sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

gives the isomorphism with $\mathbb{Z}/p^2$ coefficients; then

$$0 \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p \to 0$$

in general so for $\mathbb{Z}/p^n$ coefficients, for all n.

Now the limit $\mathbb{Z}_{p^\infty} = \ldots \mathbb{Z}/p^n \hookrightarrow \mathbb{Z}/p^{n+1} \hookrightarrow \cdots = \bigcup \mathbb{Z}/p^n$, the "Prüfer group" also gives an isomorphism because homology commutes with direct limits.

Then in fact $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}$ (this is more or less the method of partial fractions, if you think about it)

so we are done.

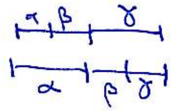
By the way, $\mathbb{Z}/p^\infty$ is the quotient in the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[\tfrac{1}{p}] \longrightarrow \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \to 0.$$

Now the proof of the James' Theorem relied on some heavy stuff: Bott-Samelson, the Hurewicz isomorphism theorem, and the JHC Whitehead theorem. You may think that's too slick, and you would be right, because we missed the

James Construction

The biggest difficulties above were caused by the failure of loop composition to associate:



The idea is to replace ΩX with a homotopy equivalent space, the "Moore loops," where composition does associate.

The "Moore loops" on X is incredibly simple: since scaling caused trouble, don't scale! A Moore loop is a map

$$[0,T] \xrightarrow{\omega} X, \quad T \geq 0, \qquad \omega(0) = \omega(T) = *.$$

And

$$\left.\begin{array}{l} [0,T] \xrightarrow{\omega} X \\ [0,S] \xrightarrow{\tau} X \end{array}\right\} \longrightarrow \quad [0,T+S] \xrightarrow{\tau\cdot\omega} X.$$

ΩX (we'll call this space ΩX too) comes with a basepoint $*$ = the path of length 0, and then ΩX has a strictly associative product with a strict unit $*$. We get a map

$X \xrightarrow{\alpha} \Omega\Sigma X$ in the same way (notice the old ΩX embeds in the Moore loops ΩX).

Now this α factors

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[dr] & & \Omega\Sigma X \\
& J(X) \arrow[ur, "\bar{\alpha}"'] &
\end{tikzcd}$$

through $J(X)$ (J for James), the "free monoid" on X

$$J(X) = \coprod_{k \geq 0} X^k / \sim$$

$\sim$ generated by $x \cdot * = x$,
i.e.
$(x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_k) \sim$
$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$.

And the "real" theorem is

Thm $\bar{\alpha}$ is a homotopy equivalence when X is a connected CW complex.

Notice that this gives a map back, and so a way of constructing maps out of a loop space. In general, this is hard to do; adjointness is no help for maps out of a loop space, for example.

$J(X)$ is filtered by $J_n = \coprod_{k \leq n} X^k/\sim$, and with the obvious maps,

$$\begin{tikzcd}
X^n \arrow[r] \arrow[drrr] & J_n(X) \arrow[r] & J_n(X)/J_{n-1}(X) \arrow[r, phantom, "="] & X^n/F_{n-1}X^n \leftarrow \text{"fat wedge" ...remember?} \arrow[d, equal] \\
& & & X^{(n)}
\end{tikzcd}$$

commutes.

On the other hand, suspending once we get

$$\begin{tikzcd}
\Sigma X^{(n)} \arrow[r] \arrow[dd, "\substack{\text{splitting}\\ \text{above}}"'] & \Sigma\Omega\Sigma X & \\
 & \Sigma J(X) \arrow[u, "\simeq"'] & \\
\Sigma X^{n} \arrow[r] & \Sigma J_n(X) \arrow[u] \arrow[r] & \Sigma X^{(n)}
\end{tikzcd}$$

so after one suspension $J_n(X)$ splits:

$$\Sigma J_n(X) \simeq \bigvee_{k \leq n} \Sigma X^{(k)}$$

Finally, notice that we have a map h_m:

$$\begin{tikzcd}
\bigvee_{k \geq 0} \Sigma X^{(k)} \arrow[r, shift left, "\simeq"] \arrow[d, "\Sigma\left(\substack{\text{smashing}\\ \text{everything}\\ \text{else to a point}}\right)"'] & \Sigma\Omega\Sigma X \arrow[l, shift left, "\exists"] \arrow[dl, "\hat{h}_m"] \\
\Sigma X^{(m)} &
\end{tikzcd}$$

$\uparrow$ epi in homology.

Its adjoint $h_m : \Omega\Sigma X \to \Omega\Sigma X^{(m)}$ is the "mth (James) Hopf invariant."

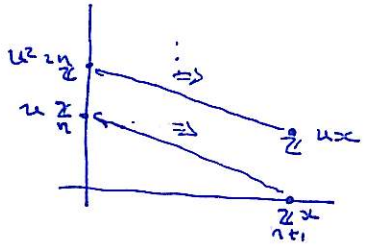
OK, so suppose $X = S^n$ and $m = 2$; then h_2 is a map

$$\Omega S^{n+1} \longrightarrow \Omega\Sigma(S^n)^{(2)} = \Omega S^{2n+1}.$$

This is a piece of the EHP sequence; the rest comes from taking the homotopy fiber and looking at the homotopy long exact sequence.

So now let's try to compute $H_*(\Omega S^{n+1})$. Actually, $S^{n+1} = \Sigma S^n$ so we could use Bott-Samelson; $H_*(\Omega S^{n+1}) = T(\bar{H}_*(S^n)) = T[u]$, $|u| = n$.

Alternatively you could use the path fibration $\Omega S^{n+1} \to \underset{*}{\overset{\simeq}{P S^{n+1}}} \to S^{n+1}$ $(n > 0)$ and the Serre spectral sequence



Note that if you try to remember grading then $u^2 = (-1) u u$ for u odd, so you lose commutativity by trying to remember grading, which is the reverse somehow of the usual situation.

So h_2 induces a map $T[u_n] \longrightarrow T[u_{2n}]$ which preserves degrees, so it couldn't be an algebra map. So we'd better try cohomology.

Theorem With field coefficients:

$$H^*(\Omega S^{2n+1}) \underset{\text{as Hopf Algebras}}{\cong} \Gamma[x_{2n}]$$

"the divided polynomial algebra" on x_{2n}, $|x_{2n}| = 2n$.

$$H^*(\Omega S^{2n}) \underset{\text{as algebras}}{\cong} \underset{\substack{\uparrow \\ \text{exterior algebra}}}{E}[x_{2n-1}] \otimes \Gamma[x_{4n-2}]$$

$$H_*(\Omega S^{2k}) \underset{\substack{\text{as coalgebras} \\ \text{(not as algebras)}}}{\cong} H_*(S^{2k-1}) \otimes H_*(\Omega S^{4k-1}).$$

Digression to explain the statement of the theorem.

We always have a diagonal map $X \xrightarrow{\Delta} X \times X$.

Assume that X is nice enough (or the coefficients are nice enough) so that $H_*(X) \otimes H_*(X) \xrightarrow{\times} H_*(X \times X)$ is an isomorphism, so we get a map going back. The composite of this with Δ_* is another map we also call $\Delta : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$. For obvious reasons this is called a "coproduct." Also, there is the map $H_*(X) \longrightarrow H_*(*) = F$; then these two together give $H_*(X)$ the structure of a coalgebra.

What is a coalgebra? Well, you reverse the diagrams for an algebra:

Maps $C \xrightarrow{\Delta} C \underset{F}{\otimes} C$ and $C \xrightarrow{\varepsilon} F$ satisfying

- coassociativity

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & C \underset{F}{\otimes} C \arrow[d, "\Delta \otimes 1"] \\
C \underset{F}{\otimes} C \arrow[r, "1 \otimes \Delta"'] & C \underset{F}{\otimes} C \underset{F}{\otimes} C
\end{tikzcd}$$

- counit

$$\begin{tikzcd}
c \arrow[d] & & C \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr] & & c \arrow[d] \\
1 \otimes c & F \underset{F}{\otimes} C \arrow[r, phantom, "\circlearrowright"] & C \underset{F}{\otimes} C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] \arrow[r, phantom, "\circlearrowright", shift left=6] & C \underset{F}{\otimes} F & c \otimes 1
\end{tikzcd}$$

- a cocommutative coalgebra also satisfies

$$\begin{tikzcd}
 & C \underset{F}{\otimes} C \arrow[d, "T"] \\
C \arrow[ur, "\Delta"] \arrow[r, "\Delta"'] & C \underset{F}{\otimes} C
\end{tikzcd}$$

← where this involves a sign change for a <u>graded</u> coalgebra.

Suppose X is connected, so $\underset{1 \mapsto 1}{H_0(X) \cong F}$ and there is well-defined class 1 in $H_0(X)$. Then if $x \in H_n(X)$ where $n > 0$,

$$\Delta x = 1 \otimes a + \underbrace{\cdots}_{\substack{\text{tensor terms in}\\ \text{which both have}\\ \text{degree} > 0\\ \text{"middle terms"}}} + b \otimes 1$$

and it is clear that $a = b = x$. If $H_p X$ vanishes for $p < n$, then

$$\Delta x = 1 \otimes x + x \otimes 1$$ and x is called "primitive."

Note that in dimension zero, well, $\Delta(1) = 1 \otimes 1$. This property is called "group-like;" it should be called "set-like" but nobody does that. The reason for this terminology comes from the example $R[G]$; see below.

If moreover we are looking at ΩX, well, ΩX is an H-space: the map

$\Omega X \times \Omega X \xrightarrow{\mu} \Omega X$ satisfies the requirement that the diagram

$$\begin{tikzcd}
\ast \times Y \arrow[d] \arrow[dr, "="] & \\
Y \times Y \arrow[r, "\mu"] & Y \\
Y \times \ast \arrow[u] \arrow[ur, "="'] &
\end{tikzcd}$$

homotopy commutes, and

$$\begin{tikzcd}
Y \times Y \times Y \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & Y \times Y \arrow[d, "\mu"] \\
Y \times Y \arrow[r, "\mu"] & Y
\end{tikzcd}$$

homotopy commutes

(some people call this an associative H-space). Over field coefficients, this induces

$$\begin{tikzcd}
H_*(Y) \otimes H_*(Y) \arrow[r, "\Psi"] \arrow[d, "\cong"] & H_*(Y) \\
H_*(Y \times Y) \arrow[ur, "\mu_*"'] &
\end{tikzcd}$$

"Pontrjagin Product"

<u>Claim</u>: This is an isomorphism of coalgebras

and so Ψ is a map of coalgebras.

Proof omitted. You have to think about what a tensor product of coalgebras ought to be.

What does all this mean, though? It means that $H_*(Y)$ is a Hopf algebra. So what's a Hopf algebra? Well, its a bunch of structure.

$$\begin{tikzcd} R \arrow[r, "\eta"', "\text{unit}"] & H \\ H\otimes H \arrow[r, "\mu"] & H \end{tikzcd}$$

such that these give H an algebra structure

$$\begin{tikzcd} H \arrow[r, "\epsilon", "\text{counit}"'] & R \\ H \arrow[r, "\Delta"] & H\otimes H \end{tikzcd}$$

so that these give H a coalgebra structure.

Such that η and μ are coalgebra maps; ie they make commutative diagrams

$$\begin{tikzcd} H\otimes H \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] \arrow[dd, bend right=60, "\Delta_{H\otimes H}"'] & H \arrow[dd, "\Delta"] \\ H\otimes H\otimes H\otimes H \arrow[d, "1\otimes T\otimes 1"'] & \\ H\otimes H\otimes H\otimes H \arrow[r, "\mu\otimes\mu"] & H\otimes H \end{tikzcd}$$

$$\begin{tikzcd} H\otimes H \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"] \arrow[dd, bend right=60, "\epsilon_{H\otimes H}"'] & H \arrow[ddl, "\epsilon"] \\ R\otimes R \arrow[d, "\mu"] & \\ R & \end{tikzcd}$$

$$\begin{tikzcd} & R & \\ R \arrow[ur, "id"] \arrow[rr, "\eta"] \arrow[d, "\Delta"'] & & H \arrow[ul, "\epsilon"'] \arrow[d, "\Delta"] \\ R\otimes R \arrow[rr, "\eta\otimes\eta"'] & & H\otimes H \end{tikzcd}$$

Now you can check that these diagrams' commuting also ~~as~~insure that Δ and ϵ are algebra maps, so the symmetric conditions are equivalent.

Before going on, another important example of an Hopf Algebra is, for G a group, the Group algebra $R[G]$: the elements are the free R-module on G; the product generated by $[g][h] = [gh]$. The coalgebra structure comes from $\Delta[g] = [g \otimes g]$; $\epsilon[g] = 1$ for $g \in G$. Notice that this explains the terminology "group-like." In fact, the set of group-like elements is exactly the generators. So the coalgebra structure enables us to recover the generators!

Turning to the Bott-Samelson theorem: if X is connected, R is a p.i.d. and $H_*(X;R)$ is torsion-free, the theorem gave us an algebra isomorphism

$$T\bar{H}_*(X) \xrightarrow{\cong} H_*(\Omega\Sigma X)$$

Now we have established that $H_*(\Omega\Sigma X)$ is a Hopf algebra; the natural question is what the Hopf algebra structure on $T\bar{H}_*(X)$ ought to be in order to make the isomorphism one of Hopf algebras; in particular what we need is a map

$$T \xrightarrow{\Delta} T \otimes T.$$

It has to be an algebra map, so by the universality property of T all we need to do is produce a map $\bar{H}_*(X) \xrightarrow{\bar{\Delta}} T \otimes T$. It works out that $\bar{\Delta}$ is the obvious thing

$$\begin{tikzcd}
T\bar{H}_*(X) \arrow[r, "\Delta"] & T\bar{H}_*(X) \otimes T\bar{H}_*(X) \\
\bar{H}_*(X) \arrow[r, "\bar{\Delta}"] \arrow[u] & \bar{H}_*(X) \otimes \bar{H}_*(X) \arrow[u]
\end{tikzcd}$$

$$x \in \bar{H}_*(X), n > 0$$

$$\Delta x = x \otimes 1 + \overset{\overset{\bar{H}_*(X)}{\cup}}{\bar{\Delta} x} + 1 \otimes x$$

Returning at last to our theorem about $H_*(\Omega S^{n+1})$ and $H^*(\Omega S^{n+1})$, let's compute the coalgebra structure of $H_*(\Omega S^{n+1}) = T[u_n]$.

$$\Delta u_n = u_n \otimes 1 + 1 \otimes u_n \qquad \left\{\begin{array}{l}\text{there is no}\\ \text{room for middle terms}\end{array}\right\}$$

$$\Delta(u_n^k) = (u_n \otimes 1 + 1 \otimes u_n)^k \qquad \{\Delta \text{ is an algebra map}\}$$

n <u>even</u>: then the product of u_n's commutes, and

$$= \sum_{i+j=k} (i,j) \cdot u_n^i \otimes u_n^j$$

$\binom{i+j}{i}$

n odd:

$$= u_n^2 \otimes 1 + u_n \otimes u_n - u_n \otimes u_n + 1 \otimes u^2$$

$= u_n^2 \otimes 1 + 1 \otimes u_n^2$. So u_n^2 is primitive again, and has even degree!

So

$$\Delta(u_n^{2k}) = \sum_{i+j=k} (i,j)\, u_n^{2i} \otimes u_n^{2j}, \text{ and}$$

$$\Delta(u_n^{2k+1}) = [\Delta(u_n^{2k})][\Delta(u_n)]$$

$$= \sum_{i+j=k} (i,j) \left(u_n^{2i+1} \otimes u_n^{2j} + u_n^{2i} \otimes u_n^{2j+1}\right).$$

So $H_*(\Omega S^{\overset{\text{ie } n=2k-1 \text{ above}}{2k}}) \cong H_*(S^{2k-1}) \otimes H_*(\Omega S^{4k-1})$ as <u>coalgebras</u>, though certainly not as algebras.

Now consider $H^*(\Omega S^{n+1})$. With any coefficients, the group structure is

$$H^q(\Omega S^{n+1}; R) = \begin{cases} R\langle x_i \rangle & ni = q \\ 0 & \text{otherwise.} \end{cases}$$

Now with field coefficients, let's compute the ring structure. We can use in this case the pairing of cohomology and homology, so

① n even

$$\langle x_i x_j , u_n^{i+j} \rangle = \langle \Delta^* (x_i \otimes x_j) , u_n^{i+j} \rangle$$

$$= \langle x_i \otimes x_j , \Delta_* u_n^{i+j} \rangle$$

$$= \langle x_i \otimes x_j , \sum_{i+j} (i,j) u_n^i \otimes u_n^j \rangle$$

$$= (i,j) = \langle (i,j) x_{ij} , u_n^{i+j} \rangle$$

So $x_i x_j = (i,j) x_{i+j}$. This algebra is called the "divided polynomial algebra on x_n, $|x_n| = n$," denoted $\Gamma[x_n]$, and we get

$$H^*(\Omega S^{2k+1}) \underset{\substack{\text{Hopf} \\ \text{Algebras}}}{\cong} \Gamma[x_k].$$

Similarly the pairing for n odd gives

$$H^*(\Omega S^{2k}) \cong H^*(S^{2k-1}) \otimes H^*(\Omega S^{4k-1})$$

$$\cong E[x_{2k-1}] \otimes \Gamma[x_{4k-2}],$$

but now since the homology isomorphism was only one of coalgebras, this is only an isomorphism of <u>algebras</u>.

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4/3

The march to the EHP sequence continues. Recall for a connected CW complex X the James Construction gave a homotopy equivalence

$$\begin{tikzcd}\bigvee_{k\geq 0}\Sigma X^{(k)} \arrow[r, shift left, "\simeq"] & \Sigma\Omega\Sigma X, \arrow[l, dashed, shift left, "\exists"]\end{tikzcd}$$

So in particular we have a map going back. The map

$$\bigvee_{k\geq 0}\Sigma X^{(k)} \xrightarrow{\Sigma\left(\substack{\text{smashing}\\ \text{everything}\\ \text{else to a pt}}\right)} \Sigma X^{(m)}$$

gives a composite

$$\begin{tikzcd}\bigvee_{k\geq 0}\Sigma X^{(k)} \arrow[r, shift left, "\simeq"] \arrow[d, "\Sigma\left(\substack{\text{smash}\\ \text{to a}\\ \text{pt}}\right)"'] & \Sigma\Omega\Sigma X \arrow[l, shift left] \arrow[dl, "\hat{h}_m"] \\ \Sigma X^{(m)} & \end{tikzcd}$$

whose adjoint $\Omega\Sigma X \xrightarrow{h_m} \Omega\Sigma X^{(m)}$ is the "mth James-Hopf invariant." Applying this to the case $X = S^n$ and $m = 2$ gave a map

$$\Omega S^{n+1} \xrightarrow{h_2} \Omega S^{2n+1}.$$

The EHP sequence comes from the long exact sequence of the homotopy we get from taking the homotopy fiber. To start finding out the fiber, we computed the cohomology of ΩS^{n+1}; recall that

$$H^*(\Omega S^{n+1}) = \left\{\begin{array}{ll} E[x_1]\otimes\Gamma[x_2] & \text{if } n \text{ is odd} \\ \Gamma[x_1] & \text{if } n \text{ is even}\end{array}\right\} \quad |x_i| = ni$$

Now the adjoint of the James-Hopf map factors:

$$\begin{tikzcd}
\Sigma\Omega S^{n+1} \arrow[rr, "\hat{h}_2"] \arrow[dr, "\Sigma h_2"'] & & S^{2n+1} \\
& \Sigma\Omega S^{2n+1} \arrow[ur, "\beta = \text{evaluation}"'] &
\end{tikzcd}$$

Which just is the expression of Adjointness between Ω and Σ.

$\hat{h}_2$ came from a splitting, so it splits; i.e. we get

$$S^{2n+1} \hookrightarrow \bigvee_{k\geq 0} \Sigma S^{n(k)} \longrightarrow \Sigma\Omega S^{n+1} \xrightarrow{\hat{h}_2} S^{2n+1}$$

Thus $\hat{h}_2$ is surjective in homology. Looking at the (once-suspended) cohomology of ΩS^{n+1}, it is clear that $H^{2n+1}(\Sigma h_2)$ is an isomorphism, so in dimension $2n$ we get

$$\begin{tikzcd}
H^{2n}(h_2) : H^{2n}(\Omega S^{n+1}) & H^{2n}(\Omega S^{2n+1}) \arrow[l, "\cong"'] \\
u_2 & x_1 \arrow[l, mapsto]
\end{tikzcd}$$

↑ where this is generator of $2n$-dim level of $\Gamma[u_2]$, the divided polynomial algebra.

↑ bottom generator of $H^*(\Omega S^{2n+1})$

To compute the rest of h_2^* in cohomology, there are two cases.

<u>n odd</u>.

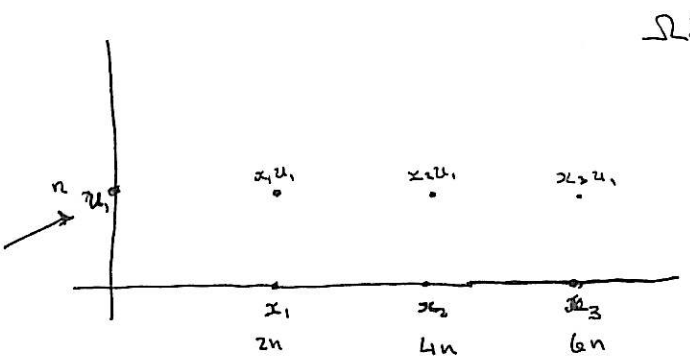
$x_1 \overset{h_2^*}{\longmapsto} u_2$ is an algebra map, so

$k!\,x_k = x_1^k \longmapsto u_2^k = k!\,u_{2k}$, hence

$x_k \overset{h_2^*}{\longmapsto} u_{2k}$

In this case $H^*(\Omega S^{n+1}) = E[u_1] \otimes \Gamma[u_2]$, so we should study the fate of u_1 in the spectral sequence of the (homotopy) fibration

$F \to \Omega S^{n+1} \to \Omega S^{2n+1}$:



need dim. n generator to yield $E[u_1] \otimes \Gamma[u_2]$.

note $u_1^2 = 0$. Moreover there can't be anything else in the fiber, because such a thing couldn't survive to E^∞, but on the other hand isn't allowed to hit anything.

$$\text{So } H^*(F;\mathbb{Z}) = E[u_1], \quad |u_1| = n.$$

On the other hand, we have a map $S^n \xrightarrow{\alpha} \Omega S^{n+1}$ (the map from the Bott-Samelson theorem). In

$$\begin{tikzcd} & u_1 \\ u_1 \arrow[ur, mapsto] & u_1 \arrow[u, mapsto] \arrow[l, mapsto] \end{tikzcd}$$

$$\begin{tikzcd} & S^n \arrow[dl, dashed, "\gamma"'] \arrow[d, "\alpha"] \arrow[dr] & \\ F \arrow[r] & \Omega S^{n+1} \arrow[r, "h_2"] & \Omega S^{2n+1} \end{tikzcd}$$

← this composite is null-homotopic, so the dotted-line map exists.

And γ^* is an isomorphism on cohomology. If $n > 1$, then $\pi_1 F = 0$ and so γ is a homotopy equivalence by the Whitehead theorem. In case $n = 1$, the long exact homotopy sequence shows $\pi_1 F \cong \pi_1 \Omega S^2 = \mathbb{Z}$, so γ is an isomorphism on π_1. Now γ lifts to a map of universal covering spaces

$$\begin{tikzcd} \tilde{F} \arrow[r, "\tilde{\gamma}"] \arrow[d] & \mathbb{R} \arrow[d, "e^{i\theta}"] \\ F \arrow[r, "\gamma"] & S^1 \end{tikzcd}$$

which <u>are</u> homotopic by the JHC Whitehead theorem; since $\tilde{\gamma}$ is equivariant w.r.t. deck transformations, we get $F \xrightarrow[\simeq]{\gamma} S^1$.

So now for n odd we have the (homotopy) fibration

$$S^n \to \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$$

whose long exact sequence is the EHP sequence

← from Bott-Samelson

← from James-Hopf map h_2

$$\begin{tikzcd} {} \arrow[r] & \pi_i S^n \arrow[r, "e"] & \pi_{i+1} S^{n+1} \arrow[r, "h"] & \pi_{i+1} S^{2n+1} \arrow[dll, "P"] \\ & \pi_{i-1} S^n \arrow[r] & \pi_i S^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

← and we will have much occasion to study this map.

The case of n even is even more interesting; then $H^*(\Omega S^{n+1}) = \Gamma[u_1]$ where now $|u_1| = n$. $H^{2n}(h_2)$ is still an isomorphism, so that means

$$\begin{tikzcd} H^{2n}(\Omega S^{2n+1}) \arrow[r, "{H^{2n}(h_2]}", "\cong"'] & H^{2n}(\Omega S^{n+1}) \\ x_1 \arrow[r, mapsto] & u_2 \end{tikzcd}$$

Now however u_2 isn't the bottom class of the divided polynomial algebra, so it is no longer true that $u_2^k = k!\, u_{2k}$; instead

$$2u_2 = u_1^2$$

$$u_1^{2k} = (2k)!\, u_{2k}$$

$$2^k u_2^k = (2k)!\, u_{2k}$$

$$u_2^k = \frac{(2k)!}{2^k}\, u_{2k}.$$

Well, this is a pretty awful number, but if we look at the prime 2 it's not so bad:

$$\frac{(2k)!}{2^k} = 1 \cdot \frac{2}{2} \cdot 3 \cdot \frac{4}{2} \cdot \cdots \frac{(2k)}{2} = k!\,(\text{odd})$$

so over $\mathbb{Z}_{(2)}$ this is $k! \cdot (\text{unit})$. So working over $\mathbb{Z}_{(2)}$,

$$x_k \xmapsto{h_2^*} (\text{unit}) \cdot u_{2k}$$

So the Serre Spectral sequence over $\mathbb{Z}_{(p)}$ looks the same as it did in the odd case, and $H_*(\alpha;\mathbb{Z}_{(2)})$ is an isomorphism. So $S^n \xrightarrow{\gamma} F$ isn't a homotopy equivalence, but it is an isomorphism on π_* localized at 2, by Serre's Mod $\mathcal{C}$ theory:

<u>Theorem</u> $X \xrightarrow{\alpha} Y$ a map of 1-connected spaces. If

$$H_*(X;\mathbb{Z}_{(p)}) \xrightarrow[H_*(\alpha)]{\cong} H_*(X;\mathbb{Z}_{(p)}) \text{ is an isomorphism}$$

then

$$\pi_*(X)\otimes\mathbb{Z}_{(p)} \xrightarrow[\cong]{\alpha_*} \pi_*(Y)\otimes\mathbb{Z}_{(p)} \quad \text{is.}$$

So for n even, the 2-local homotopy groups are the same; only God knows where else but for present purposes we don't care: we get the same EHP sequence for n even, but now localized at 2.

Well now there's lots to do. Each of these maps has its own personality, so we'll take each in turn. e is most familiar, so we'll start with it.

You could think of the rest of the maps as the obstruction to e's being an isomorphism: since $\pi_{i+1} S^{2n+1} = 0$ for $i \leq 2n-1$, you get

$$\pi_{2n-1} S^n \xrightarrow{e} \pi_{2n} S^{n+1} \longrightarrow 0,$$

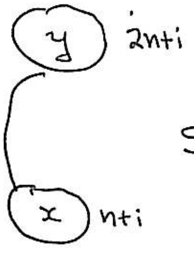
$$\vdots$$

$$0 \longrightarrow \pi_i S^n \xrightarrow{e} \pi_{i+1} S^{n+1} \longrightarrow 0$$

so $\pi_i S^n \xrightarrow{e} \pi_{i+1} S^{n+1}$ is epi if $i = 2n-1$
iso if $i < 2n-1$.

This is precisely the statement of the Freudenthal Suspension Theorem for S^n, so the homotopy groups $\pi_k S^{2n+1}$ and the h and p maps are the obstruction to extending the Freudenthal Suspension theorem to higher dimensions.

By the way, earlier we studied the Hopf invariant 1 problem; there were two main results: using Sq^n we found that if there is an element of Hopf invariant 1 in $\pi_{2n-1} S^n$, then n is a power of 2. Since Sq^n commutes with suspension, this is a stable result:



$Sq^n x = y$ (odd) ⇒ n is a power of 2.

Then using K-theory we showed that n must be 1, 2, 4, or 8. This result is not obviously stable from K-theory because the Adams operations are not stable. But now the EHP sequence gives this to us: $\pi_{2n-1} S^n \xrightarrow{e} \pi_{n-1}^S = \pi_{2n-1+i} S^{n+i}$, i large, is surjective.

So no new elements are born after suspending any number of times. This is an example of the EHP sequence's taking unstable information and giving back stable information.

Now we move up one row and look at h:

$$\begin{tikzcd} \pi_{2n+1} S^{n+1} \arrow[r, "h"] & \pi_{2n+1} S^{2n+1} \arrow[r, "P"] \arrow[d, phantom, "\parallel"] & \pi_{2n-1} S^{n} \arrow[r, "e"] & \pi_{2n} S^{n+1} \\ & \mathbb{Z} & & \end{tikzcd}$$

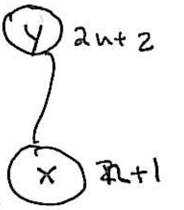
On the other hand we have the Hopf invariant $\pi_{2n+1} S^{n+1} \xrightarrow{H} \mathbb{Z}$ from earlier. To keep them straight, h is the "James-Hopf invariant"; H is the "Hopf Hopf invariant"

<u>Proposition</u>

$H = \pm h$, so the names are well-chosen.

<u>Proof</u>: First take the case n even.

Then



$x^2 = Hy$. But $n+1$ is odd, so $x^2 = 0$ and $H = 0$.

What about the James-Hopf invariant? Remember h comes from $\Omega S^{n+1} \xrightarrow{h_2} \Omega S^{2n+1}$. So a class $f: S^{2n+1} \to S^{n+1}$ composes as

$$S^{2n} \xrightarrow{\hat{f}} \Omega S^{n+1} \xrightarrow{h_2} \Omega S^{2n+1}$$

Now the degree of a map $S^{2n+1} \xrightarrow{h_* f} S^{2n+1}$ is the same as its effect in homology;

we compute the effect in dimension $2n$ of the adjoint of f. Since we know that $H_{2n}(h)$ is an isomorphism, this amounts to computing what happens to a generator $g \in H_{2n}(S^{2n})$ under

$$\begin{tikzcd} H_{2n}(S^{2n}) \arrow[r, "\hat{f}_*"] & H_{2n}(\Omega S^{n+1}) \\ g \arrow[r, mapsto] & h \cdot u_2 \leftarrow \text{generator of } H_{2n}\Omega S^{n+1} \end{tikzcd}$$

$\hat{f}_*$ is a map of coalgebras. $H_{2n}(\Omega S^{n+1})$ is a Hopf algebra whose structure we have computed: $\Delta(u_2) = \Delta(u_1^2) = \Delta(u_1)^2$

$= u_2 \otimes 1 + 2u_1 \otimes u_1 + 1 \otimes u_2$. So

$$\begin{tikzcd} g \arrow[r, mapsto, "\hat{f}_*"] \arrow[d, mapsto] & h \cdot u_2 \arrow[d, mapsto, "\Delta"] \\ g\otimes 1 + 1 \otimes g \arrow[r, mapsto] & h(g\otimes 1 + 1\otimes g) = h \cdot (u_2 \otimes 1 + 2u_1 \otimes u_1 + 1 \otimes u_2) \end{tikzcd}$$

But this equality can hold only if $h=0$, so the James-Hopf invariant is zero as well. At least on an odd sphere S^{n+1}.

In the case that n is odd, there are two steps; first we show $h | H$, then that $H | h$.

To show that $h|H$, study these two Barratt-Puppe sequences and the associated exact sequences in cohomology.

(NB the proof that $h=0$ no longer applies since now $u_2 = u_1^2$ is primitive; remember $H_*(\Omega S^{2k}) \cong H_*(S^{2k-1}) \otimes H_*(\Omega S^{4k-1})$ as coalgebras in this case):

$$\begin{tikzcd}
S^{2n+1} \arrow[r, "f"] \arrow[d, equal] & S^{n+1} \arrow[r] & C(f) \arrow[r] & S^{2n+2} \arrow[d, equal] & \\
S^{2n+1} \arrow[r, "\Sigma\hat{f}"] & \Sigma\Omega S^{n+1} \arrow[r] \arrow[u, "\beta(\text{evolutation})"'] & \Sigma C(\hat{f}) \arrow[r] \arrow[u] & S^{2n+2} \arrow[r, "\Sigma^2\hat{f}"] & \Sigma^2\Omega S^{n+1} \\
 & 0 & \mathbb{Z}/h \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "\cdot h"'] \quad \text{in } H^{2n+2}.
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[dd, mapsto, "\beta^*"'] & x\ {}^{-}y \arrow[l, mapsto] \arrow[d, mapsto] & y \in H^{2n+2} \arrow[l, mapsto] \arrow[d, mapsto] \\
 & y & y \arrow[l, mapsto, "H^{2n+2}"'] \\
x & x \arrow[l, mapsto, "H^{n+1}"'] &
\end{tikzcd}$$

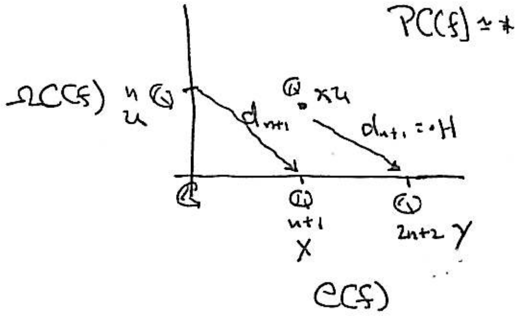
$$x^2 = Hy \text{ in } H^*(C(f)) \Rightarrow x^2 = Hy \text{ in } H^*(\Sigma C(\hat{f})).$$

But recall that $x^2 = 0$ in any cohomology theory on any suspension,

so $Hy = 0$ in $H^*(\Sigma C(\hat{f})) \cong \mathbb{Z}/h$, where $h = \deg \Sigma^2\hat{f}$ in dim $2n+2 = \deg \hat{f}$ in dim $2n$, implies that $h \mid H$.

To show that $H \mid h$, we study $\Omega C(f)$. First we want to show that

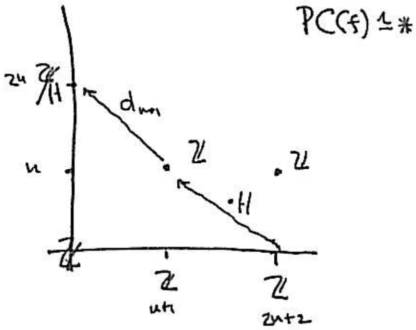
$H_{2n}(\Omega C(f)) \cong \mathbb{Z}/H$. In the rational cohomology spectral sequence for the fibration $\Omega C(f) \longrightarrow PC(f) \longrightarrow C(f)$ we compute a crucial differential:



$$d_{n+1}\, xu = \underset{\overset{\shortparallel}{0}}{dx} \cdot u + x \cdot \underset{\overset{\shortparallel}{x}}{du} = x^2 = Hy$$

Because there is no torsion in these two groups, the integral cohomology SS embeds in the rational cohomology SS, and is dual to the homology SS.

So the same differential is mult. by H in the homology SS.



Now look at the Barratt-Puppe sequence

$$\begin{tikzcd}
\Omega S^{2n+1} \arrow[r, "\Omega f"] & \Omega S^{n+1} \arrow[r] & \Omega C(f). \\
S^{2n} \arrow[u, "\alpha"] \arrow[ur, "{\hat{f},\ \deg h \text{ in dim } 2n.}"'] & &
\end{tikzcd}$$

In dimension $2n$, $H_{2n}(\alpha)$ is an isomorphism, and so

$$\text{gen} \xmapsto{\Omega f} h \cdot \text{gen} \longmapsto 0 \text{ in } H_{2n}(\Omega C(f)) \text{ since}$$

the composite is null-homotopic, so $h = 0$ in $\mathbb{Z}/H$, and $H \mid h$. Notice that in case H or h is zero, the appropriate argument shows the other is as well.

Maybe now we should show that there exists an element of Hopf invariant 1; after all the results we have proven so far have been negative. Let's see that an H-space structure on S^{n-1} yields an element of Hopf invariant one on S^n; then real, complex, quaternionic, and Cayley multiplication will provide elements of Hopf invariant one on S^1, S^2, S^4, and S^8.

We will construct such elements using the "Hopf construction"; to understand this construction, it helps to look at it in extreme generality; remember from the beginning of the term this takes a "multiplication"

$$X\times Y \xrightarrow{\mu} Z \quad \text{and yields a map} \quad X*Y \xrightarrow{H(\mu)} \Sigma Z.$$

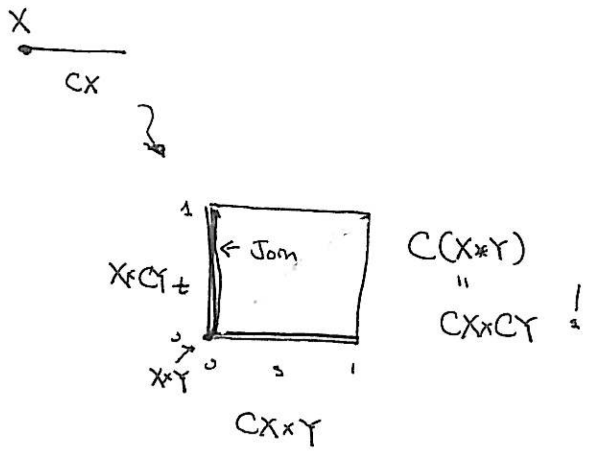
The construction (see first day's notes!) is

$$\begin{tikzcd}
 & X\times Y \arrow[rr, "\mu"] \arrow[dl] \arrow[dd] & & Z \arrow[dl] \arrow[dd] \\
CX\times Y \arrow[rr] \arrow[dd] & & CZ \arrow[dd] & \\
 & X\times CY \arrow[rr] \arrow[dl] & & CZ \arrow[dl] \\
X*Y \arrow[rr, "H(\mu)"'] & & \Sigma Z &
\end{tikzcd}$$

Now call $H(\mu)$ j; since the composite $X*Y \xrightarrow{j} \Sigma Z \to C(j)$ is null-homotopic we get a map k in

$$\begin{tikzcd}
X*Y \arrow[r, "j"] \arrow[d] & \Sigma Z \arrow[d] \\
C(X*Y) \arrow[r, "k"] & C(j)
\end{tikzcd}$$

and it takes some thought to see that k is a relative homeomorphism off of $X*Y$. Finally, note the pictures below illustrate that $C(X*Y) = CX \times CY$:



Now, in our particular case, we have

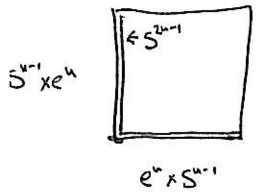
$$\begin{tikzcd}
S^{n-1}\times x_0 \arrow[d] \arrow[dr, "a = \text{degree}"] & & & \\
S^{n-1}\times S^{n-1} \arrow[r, "\mu"] & S^{n-1} \arrow[r, rightsquigarrow] & S^{2n-1} \arrow[r, "j"] & S^n \\
x_0\times S^{n-1} \arrow[u] \arrow[ur, "b = \text{degree}"'] & & &
\end{tikzcd}$$

And the claim is

<u>Lemma</u> Hopf invariant of j is $\pm ab$; in particular if $a=b=1$, then $H(j) = \pm 1$.

Proof

Note $C(S^{n-1}) = e^n$, the closed n-disk. The maps k and j yield:



$$\begin{tikzcd}
(e^n\times e^n, S^{2n-1}) \arrow[r, "k \leftarrow \text{iso. on } H^*"] \arrow[d] & (C(j), S^n) \arrow[d] & C(j) \arrow[l] \arrow[d, "\Delta"] \\
(e^n\times e^n, e^n\times S^{n-1})\times(e^n\times e^n, S^{n-1}\times e^n) \arrow[r] & (C(j), e^n_+)\times(C(j), e^n_-) & C(j)\times C(j) \arrow[l, "\cong"']
\end{tikzcd}$$

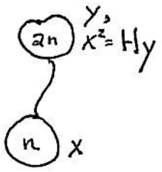
$$\begin{tikzcd}
 & S^n \arrow[d, phantom, "\parallel"] & & S^n \arrow[d, phantom, "\parallel"] \\
S^{n-1} * S^{n-1} \arrow[r, "j"] & \Sigma S^{n-1} \arrow[r] \arrow[d, phantom, "\parallel"] & C(j) = & \Sigma S^{n-1} \cup C(S^{n-1} * S^{n-1}) \\
 & e^n_+ \cup e^n_- & & \cup\ CS^{n-1}\times CS^{n-1} \\
 & S^n \arrow[u, bend left] \arrow[u, bend right] & &
\end{tikzcd}$$

and under this we have

$$\begin{tikzcd}
abx^{2} = H\cdot y & H\cdot y \arrow[l, mapsto, "\text{③}"'] \arrow[r, mapsto, "\text{②}"] & H\cdot y \\
ax\otimes bx \arrow[u, mapsto, "\text{⑤}"] & x\otimes x \arrow[l, mapsto, "\text{④}"'] \arrow[u, mapsto] & x\otimes x \arrow[l, mapsto, "\cong"'] \arrow[u, mapsto, "\text{④}"']
\end{tikzcd}$$

Explanation of this diagram:

① Remember the Hopf construction gives $S^{2n-1} \xrightarrow{j} S^n \longrightarrow C(j) =$



and this diagonal defines the cup-product.

② Inclusion into the pair of $(C(j), S^n)$ affects the cohomology of the bottom cell, but not the $2n$-dim cell. So this map is an isomorphism on H^{2n}.

③ k^* is an isomorphism on H^* because k is a relative homeomorphism.

④ Look at the effect on one of the factors.

$$\begin{tikzcd}
H^n(e^n \times e^n, e^n_+ \times S^{n-1}) \arrow[d, "\cong"', "\cdot"] & H^n(C(j), e^n_+) \arrow[l, "k^*"'] \arrow[d] \\
H^n(x_0 \times e^n_-, x_0 \times S^{n-1}) & H^n(e^n_-, S^{n-1}) \arrow[l] \\
H^{n-1}(S^{n-1}) \arrow[u, "\cong", "\partial"'] & H^{n-1}(S^{n-1}) \arrow[u, "\partial"', "\cong"] \arrow[l, "\mu(x_0,-)^*"', "\cdot b"]
\end{tikzcd}$$

⑤ follows because this is no more than the definition of the smash product map.

4/5,
4/7

OK, so now we'll go back to the EHP sequence, and conclude this look at the map h by seeing what we get from the existence of elements of Hopf invariant 1. Last time we found an element of Hopf invariant one

$$S^7 \xrightarrow{\nu} S^4$$

which came from the Hopf construction on quaternion multiplication

$$S^3 \times S^3 \longrightarrow S^3$$

In the EHP sequence this appears as the π_6 level of

$$S^3 \longrightarrow \Omega S^4 \xrightarrow{h} \Omega S^7$$

We can fit ν into this picture by

$$\begin{tikzcd}
S^3 \arrow[r] & \Omega S^4 \arrow[r, "h"] & \Omega S^7 \\
\Omega S^7 \arrow[ur, "\Omega\nu"] \arrow[urr, bend right=70, "\hbar\ \text{claim} \simeq \text{to} \pm 1."'] \arrow[r, phantom, shift left=4, "\circlearrowleft"] & S^6 \arrow[l, "\alpha"'] \arrow[u, "\hat{\Sigma}"'] \arrow[ur, "\substack{\text{claim.} \\ \pm 1. \\ \iota\ \text{inclusion of} \\ \text{bottom} \\ \text{cell}}"'] &
\end{tikzcd}$$

It suffices to show this in cohomology. But we know, since Hopf-Hopf = James-Hopf, that $\Omega S^7 \xrightarrow{\Omega\nu} \Omega S^4 \xrightarrow{h} \Omega S^7$ is ± 1 in π_6 and so in H^6 by the Hurewicz theorem. And since in ΩS^7 we have generators $u_i \in H^{6i}$, with [illegible] and $u_1 \mapsto u_1$, $u_1^i = i! u_i$

we get $u_i \mapsto u_i$ (there is no torsion around).

But this is amazing: it means that ΩS^4 splits; that is,

$$\begin{tikzcd} S^3 \arrow[r] & \Omega S^4 \arrow[r, "h"] & \Omega S^7 \arrow[l, bend right, dotted] \end{tikzcd}, \quad \text{so}$$

$$S^3 \times \Omega S^7 \xrightarrow{e \times \Omega\nu} \Omega S^4 \times \Omega S^4 \xrightarrow{\mu} \Omega S^4$$

is a homotopy equivalence! Similarly $\eta \in \pi_3 S^2$, $\sigma \in \pi_{15} S^8$ with Hopf invariant 1 provide splittings

$$S^1 \times \Omega S^3 \xrightarrow{\simeq} \Omega S^2$$

$$S^7 \times \Omega S^{15} \longrightarrow \Omega S^8$$

and this *never* happens again, as it is equivalent to the existence of elements of Hopf invariant one.

What does this map look like on homotopy groups? Well, when there is an H-map $X \times X \xrightarrow{\mu} X$ and X is connected, you get a homomorphism

$$\begin{array}{ccc} \pi_*(X) \times \pi_*(X) & \longrightarrow & \pi_*(X) \\ (\alpha, \beta) & & \\ \shortparallel & & \\ (\alpha, 0) + (0, \beta) & \longmapsto & \alpha + \beta \end{array}$$

(note that it follows that π_1 is abelian!). So the isomorphism for $S^3 \times \Omega S^7 \simeq \Omega S^4$ is

$$\begin{array}{ccc} \pi_n S^3 \times \pi_{n+1} S^7 & \xrightarrow{\cong} & \pi_{n+1} S^4 \\ (\alpha, \; \beta) & \longmapsto & e\alpha + \nu \circ \beta . \end{array}$$

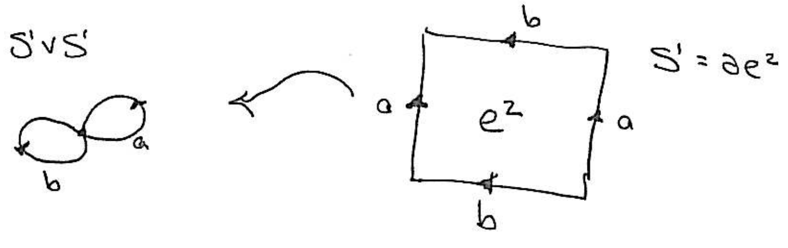
OK, now let's talk about the "P" part of the EHP sequence; "P" stands for product, I guess, so first let's take a step back and talk about Whitehead products. I'm not going to prove everything; for more information refer to George Whitehead's novel.

The Whitehead product is a map $\pi_p X \times \pi_q X \longrightarrow \pi_{p+q-1} X$, and it is easiest to describe it in terms of its universal example, a map $S^{p+q-1} \xrightarrow{W} S^p \vee S^q$; then for $\alpha \in \pi_p X$, $\beta \in \pi_q X$, we define the Whitehead product $[\alpha, \beta] \in \pi_{p+q-1} X$ by

$$\begin{tikzcd}
S^{p+q-1} \arrow[r, "W"] \arrow[rrr, bend right, "{[\alpha,\beta]}"'] & S^p \vee S^q \arrow[r, "\alpha\vee\beta"] & X \vee X \arrow[r, "\Phi \leftarrow \text{fold}"] & X
\end{tikzcd}$$

The map W is constructed in this manner: $S^p \times S^q$ has three non-zero cells; the first two comprise the axes $S^p \vee S^q$, the third is a $p+q$-cell whose attaching map is the map $S^{p+q-1} \xrightarrow{W} S^p \vee S^q$.

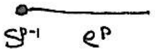
For example, $S^1 \times S^1$:



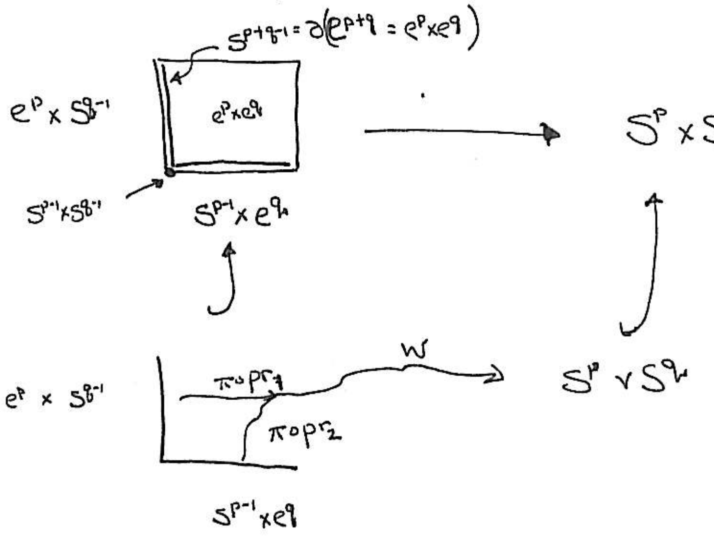
So the top dimensional cell is the mapping cone of W.

Another way to think about is by this schematic picture:

heres a p-cell



Now $S^p = e^p/S^{p-1}$; let the projection $e^p \to S^p$ be denoted π. Then $S^p \times S^q$ is a quotient



$S^p \times S^q$ = pushout of this diagram

$$= \frac{e^p \times e^q \amalg S^p \vee S^q}{\substack{e^p \times S^{q-1} \sim S^p \\ S^{p-1} \times e^q \sim S^q}}$$

Facts about the Whitehead product: once again, for real proofs, see Whitehead's book.

① In the case $p=q=1$ it is a map $\pi_1(X) \wedge \pi_1(X) \to \pi_1(X)$; from the universal case in the picture on the preceding page that

$$[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$$

which justifies the bracket notation.

In case $p=1$ and $q \geq 1$, this product gives the natural action of $\pi_1 X$ on $\pi_q X$.

② $[\beta, \alpha] = (-1)^{|\alpha||\beta|}[\alpha, \beta]$

③ The Jacobi identity : $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $\gamma \in \pi_r(X)$:

$$(-1)^{(r-1)p}[\alpha,[\beta,\gamma]]+(-1)^{(p-1)q}[\gamma,[\alpha,\beta]]+(-1)^{(q-1)(r)}[\beta,[\gamma,\alpha]] = 0$$

④ Interaction with the Hurewicz map.

One way to straighten out the degree shift is to write all the homotopy groups in terms of ΩX; then the whitehead product is a map

$$\pi_p(\Omega X) \times \pi_q(\Omega X) = \pi_{p+1}(X) \times \pi_{q+1}(X) \longrightarrow \pi_{p+q+1}(X) = \pi_{p+q}(\Omega X)$$

On the other hand there is the Hurewicz map

$$\pi_*(\Omega X) \xrightarrow{h} \underset{\substack{\uparrow \\ \text{and this is a non-commutative algebra} \\ \text{with Pontrjagin product,}}}{H_*(\Omega X)},$$

and $h([\alpha, \beta]) = h(\alpha)h(\beta) - h(\beta)h(\alpha)$, i.e. h is a map of Lie algebras.

⑤ The suspension $E[\alpha, \beta] = 0$. We know this fact, that $X \vee Y \longrightarrow X \times X$ splits after one suspension, so $\Sigma W \simeq *$.

⑥ If X is an H-space then $[\alpha, \beta] = 0$.

This follows from the commutative diagram

$$\begin{tikzcd}
S^{p+q-1} \arrow[r, "W"] \arrow[dr] & S^p \vee S^q \arrow[r, "\alpha \vee \beta"] \arrow[d] & X \vee X \arrow[r, "\Phi"] \arrow[d] & X \\
 & S^p \times S^q \arrow[r, "\alpha \times \beta"] & X \times X \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Now consider the Whitehead product in the case $p=q=n$; here the most interesting case is the "Whitehead square" $[\alpha,\alpha]$. It can be computed in terms of its universal example $[\imath_n,\imath_n]=W_n$, where $\imath_n$ is the fundamental class in $\pi_n S^n$.

$$(x,y) \longmapsto (\pi x, \pi y)$$

pushout =

$$\begin{tikzcd}
e_n\times e_n \arrow[r] & S^n\times S^n \arrow[r] & \frac{S^n\times S^n}{(*,x)\sim(x,*)} = J_2 S^n \\
\partial(e_n\times e_n) = S^{2n-1} = e^n\times S^{n-1} \cup S^{n-1}\times e^n \arrow[u, hook] \arrow[r] \arrow[rr, bend right, "W_n"'] & S^n\vee S^n \arrow[u, hook] \arrow[r, "\Phi"] & S^n \arrow[u]
\end{tikzcd}$$

$$(x,y) \longmapsto \left\{ \begin{matrix} \pi x, & y\in S^{n-1} \\ \pi y, & x\in S^{n-1} \end{matrix} \right\} \longmapsto \pi x$$

one of these

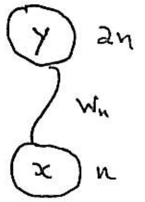
meaning choose whichever applies from here

But this all means that the mapping cone $C(W_n)=J_2S^n$, the second filtration of the free monoid on S^n, JS^n, from the James construction.

Now $w_n \in \pi_{2n-1}(S^n)$ so we should compute its Hopf invariant, and we already have a good start since we know that $C(w_n) = J_2 S^n$. Now from the James construction we know that

$$J_2(S^n) \xhookrightarrow{2n\text{-skeleton}} \Omega S^{n+1} \quad \text{moreover } H^*(\Omega S^{n+1}) \cong \begin{cases} \Gamma[x_1] & n \text{ even} \\ E[x_1] \otimes \Gamma[x_2] & n \text{ odd.} \end{cases}$$

So in



we have $H(w_n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

Well, this is pretty nice, in fact, it's pretty amazing: what we've done is look at

$$\pi_{2n-1}(S^n) \xrightarrow{h} \pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$$

and show that the image contains $2\mathbb{Z}$ if n is even. So for n even we get to split off $\pi_{2n-1}(S^n) = \mathbb{Z} \oplus \boxed{?}$.

This and $\pi_n S^n = \mathbb{Z}$ are in fact the only free abelian summands in higher homotopy groups!

Now if you're away from 2, 2 is as good as one; in other words a corollary of the above and our calculation of $H^*(\Omega S^{n+1})$ is

<u>Corollary</u> For n even, away from 2 $\Omega S^n \simeq S^{n-1} \times \Omega S^{2n-1}$, i.e.

$$\pi_{*+1}(S^n) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \left(\pi_*(S^{n-1}) \oplus \pi_{*+1}(S^{2n-1}) \right) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Now remember the h map appeared in the long exact sequence

$$\begin{tikzcd} {} \arrow[r] & \pi_{k+n} S^n \arrow[r, "e"] & \pi_{k+n+1} S^{n+1} \arrow[d, "h"] \\ & & \pi_{k+n+1} S^{2n+1} \end{tikzcd}$$

as the obstruction to desuspending a class in $\pi_{k+n+1} S^{n+1}$, so

we get

<u>Corollary</u> For n even, $w_n \in \pi_{2n-1}(S^n)$ doesn't desuspend, and for n odd it desuspends at least once.

It might desuspend more times: you might ask where it was "born" in the sequence

$$\xrightarrow{e} \pi_{k+n} S^n \xrightarrow{e} \pi_{k+n+1} S^{n+1} \xrightarrow{e} \pi_{k+n+2} S^{n+2} \longrightarrow \dots$$

And here we see the (as yet) mysterious rebirth of the vector field problem:

Theorem $W_{n-1} \in \pi_{2n-3}(S^{n-1})$ desuspends to an element in $\pi_{2n-p(n)-2}\left(S^{n-p(n)}\right)$ and no further.

Recall $p(n)$ was the number from the vector field problem

n	1	2	4	8	16	32	64	...
$p(n)$	1	2	4	8	9	10	12	...

Now in order to prove this theorem, we have to relate W_n to the EHP sequence, here's one way:

Theorem

$$\begin{tikzcd} S^{2n-1} \arrow[r, "e^2"] \arrow[dr, "\pm W_n"'] & \Omega^2 S^{2n+1} \arrow[d, "P"] \\ & S^n \end{tikzcd}$$

P ← from EHP sequence for S^n, i.e.

$$\begin{tikzcd} S^n \arrow[r] & \Omega S^{n+1} \arrow[r] & \Omega S^{2n+1} \arrow[r, "f"] & {} \\ \Omega^2 S^{2n+1} \arrow[u, "P"] \end{tikzcd}$$

in particular on $\pi_k S^{2n+1} \longrightarrow \pi_{k-2} S^n$ $\quad P(e^2 \alpha) = \pm W_n \circ \alpha$

This is why the P map is often called the "Whitehead product", although its behavior on a class which is not a double suspension is more erratic.

<u>Proof</u> $e \circ w_n$ is null-homotopic so we get

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "w_n"] & S^n \arrow[r] \arrow[d, equal] & C(w_n) \arrow[d] \\
 & S^n \arrow[r, "e"] & \Omega S^{n+1}
\end{tikzcd}$$

representing the null-homotopy. On the other hand the exactness of the EHP sequence gives a map for the dotted arrow

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "w_n"] \arrow[d, dashed] & S^n \arrow[r] \arrow[d, equal] & C(w_n) \arrow[d] \\
\Omega^2 S^{2n+1} \arrow[r, "p"] & S^n \arrow[r, "e"] & \Omega S^{n+1}
\end{tikzcd}$$

and we need to show this map is $\pm e^2$. This follows from the diagram:

$$\begin{tikzcd}
\bar{H}_{2n}(C(w_n)) \arrow[r, "\cong", "①"'] & \bar{H}_{2n}(\Omega S^{n+1}) \\
H_{2n}(S^n, w_n(S^{2n-1})) \arrow[u, "\cong"] \arrow[r] \arrow[d, "\partial", "\cong"'] & H_{2n}(S^n, \Omega^2 S^{2n+1}) \arrow[u, "\cong ②"'] \arrow[d, "\partial", "\cong"'] \\
H_{2n-1}(S^{2n-1}) \arrow[r, "?"] & H_{2n-1}(\Omega^2 S^{2n+1})
\end{tikzcd}$$

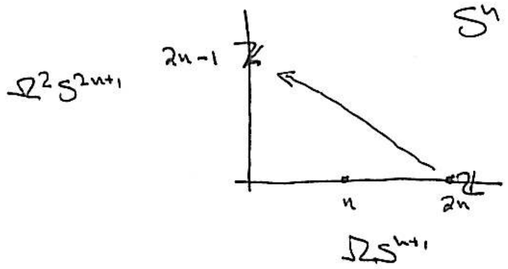
if we verify the isomorphisms ① and ②.

!!

① is the fact we already showed, that $C(w_n) \longrightarrow \Omega S^{n+1}$

$J_2 S^n \hookrightarrow \Omega S^{n+1}$

is the $2n$-skeleton. ② follows from the spectral sequence of the fibration $\Omega^2 S^{2n+1} \longrightarrow S^n \longrightarrow \Omega S^{n+1}$



One consequence of this is

Corollary $P(\iota_{2n+1}) = \pm w_n$

i.e.

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "id."] & S^{2n-1} \arrow[r, "e^2"] \arrow[dr, "\pm w_n"'] & \Omega^2 S^{2n+1} \arrow[d, "P"] \\
& & S^n
\end{tikzcd}$$

and [illegible] this corresponds to $\iota_{2n+1} \in [S^{2n-1}, \Omega^2 S^{2n+1}] = [S^{2n+1}, S^{2n+1}]$.

and so we get

Theorem (G. Whitehead)

$$\ker\left(\pi_{2n-1}(S^n) \xrightarrow{e} \pi_{2n}(S^{2n+1})\right) = \left\{ \begin{array}{ll} 0 & n = 1,3,7 \\ \mathbb{Z}/2\langle w_n\rangle & n \text{ odd otherwise} \\ \mathbb{Z}\langle w_n\rangle & n \text{ even} \end{array} \right\}$$

OK, now in order to address the desuspension problem, and in order to do that we'll link up all the EHP sequences, I mean, here they are:

$$S^n \xrightarrow{e} \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1};$$

apply Ω^n to get $\Omega^n S^n \xrightarrow{e} \Omega^{n+1} S^{n+1} \xrightarrow{h} \Omega^{n+1} S^{2n+1}$.

Now these link together:

$$\begin{tikzcd}
* \arrow[r, "e"] \arrow[dr, bend left, "\text{fiber}"'] & \Omega S^1 \arrow[r, "e"] \arrow[d, "h"] \arrow[dr, bend left, "\text{fiber}"'] & \Omega^2 S^2 \arrow[r, "e"] \arrow[d, "h"] \arrow[dr, bend left, "\text{fiber}"'] & \Omega^3 S^3 \arrow[r] \arrow[d, "h"] & \cdots \\
& \Omega S^1 & \Omega^2 S^3 & \Omega^3 S^5 &
\end{tikzcd}$$

$\bigcup \Omega^n S^n = QS^0$, weak topology so, $\pi_k QS^0 =$ $\pi_k \Omega^n S^n$ $= \pi_k^S$ for n large.

So we've filtered stable homotopy with unstable homotopy. But each leg $\urcorner$ is a fibration and the [illegible] corners match, so if you apply homotopy something wonderful happens: you get a sequence of exact triangles whose ends match up:

This is an exact couple, so you get a spectral sequence.

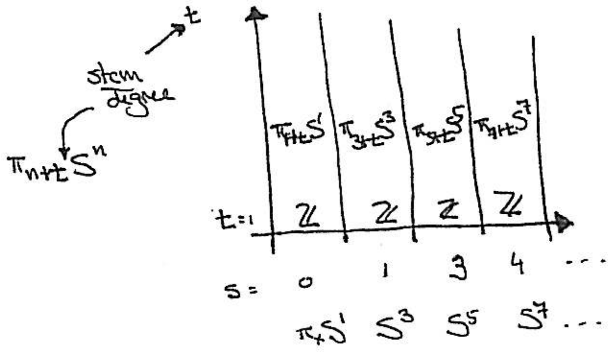
So, here it is

$$\begin{tikzcd}
s=-1 & s=0 & s=1 & s=2 & \\
* \arrow[r] & \pi_*\Omega S^1 \arrow[r, "e"] \arrow[d] & \pi_*\Omega^2 S^2 \arrow[r, "e"] \arrow[d, "\Omega h"] & \pi_*\Omega^3 S^3 \arrow[r] \arrow[d, "\Omega^2 h"] & \cdots \\
 & \pi_*\Omega S^1 & \Omega^2 S^3 \arrow[ul, "p"] & \pi_*\Omega^3 S^5 \arrow[ul, "p"] & E^1 \arrow[l] \arrow[ll, bend left] \arrow[lll, bend left]
\end{tikzcd}$$

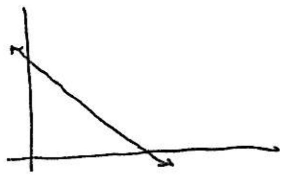
It converges to $\pi_* QS^0$.

$$E^1_{st} = \pi_{s+t}(\Omega^{s+1} S^{2s+1}) = \pi_{2s+1+t}(S^{2s+1})$$

So the columns of the spectral sequence are homotopy groups of odd spheres. So here it is:

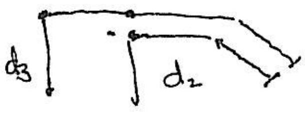


And the diagonal $s+t=k$



reflects $\pi_k \Omega^5 S^5$, i.e. is the total degree k line for $\pi_k QS^0$.

The differentials are of degree : $d_r : E_{st} \longrightarrow E_{s-r, t-r+1}$



so they are like the differentials in the usual homology spectral sequence.

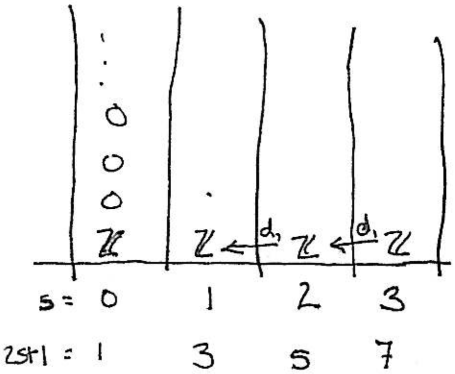
Now an obvious question is this: if you think of a spectral sequence as a way of computing the E^∞ term from the E^1 term, well, why isn't this game hopeless? We have a spectral sequence converging to stable homotopy whose input is <u>unstable</u> homotopy, which could very well be much more difficult to compute. But one really neat feature of this game is that in

$$\begin{tikzcd}
* \arrow[r] & \Omega S^1 \arrow[r] & \Omega^2 S^2 \arrow[r] \arrow[d] & \Omega^3 S^3 \arrow[r, "\equiv"] \arrow[d] & \Omega^3 S^3 \arrow[r, "\equiv"] \arrow[d] & \Omega^3 S^3 \arrow[r] \arrow[d] & \cdots \\
 & & \Omega^2 S^3 & \Omega^3 S^5 & \cancel{\Omega^3 S^5} & \cancel{\Omega^3 S^5} & \\
 & & & & 0 & 0 &
\end{tikzcd}$$

we can just <u>stop</u> the exact couple anywhere, and get an identical SS to the one we just saw except that we'll have zeros in the columns beyond where we stop; otherwise, the picture is the same. And the spectral sequence we get will converge to the homotopy of the sphere in the last column; in the case above, for example, $\pi_* \Omega^3 S^3$. So actually we have a whole family of spectral sequences converging to the input of our original spectral sequence.

Well, that still doesn't sound very good, except that you can play off all of these facts against each other and often you can get pretty far. For example, let's look at d_1. For example

$$\begin{tikzcd} d_1 : \pi_2 \Omega^3 S^5 \arrow[r, "p"] & \pi_1 \Omega^2 S^2 \arrow[d, "h"] \\ & \pi_1 \Omega^2 S^3 \end{tikzcd}$$



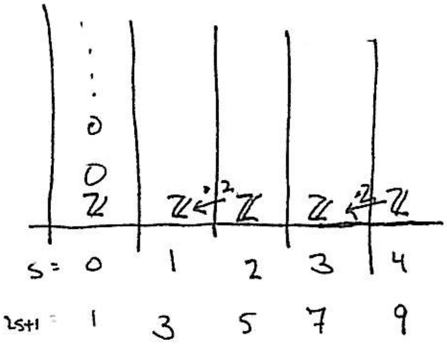
Well, on the bottom row all we need to know is what happens to $z_{2s+1} \in \pi_s \Omega^{s+1} S^{2s+1} = \pi_{2s+1} S^{2s+1}$
$$\shortparallel$$
$$\mathbb{Z}\langle z_{2s+1} \rangle.$$

But we already computed that considering $z_{2s+1} \in \pi_{2s-1} \Omega^2 S^{2s+1}$,

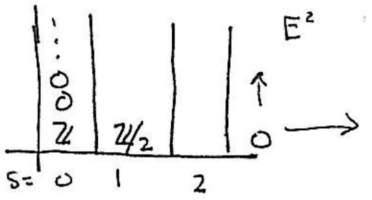
we get
$$\begin{tikzcd} z_{2s+1} \arrow[r, mapsto] & w_s \\ \pi_{2s-1} \Omega^2 S^{2s+1} \arrow[r, "p"] & \pi_{2s-1} S^s . \end{tikzcd}$$
$d_1(z_{2s+1})$ is thus

$$h(w_s) \in \pi_{2s-1} S^{2s-1} = \left\{ \begin{array}{ll} \pm 2 z_{2s-1} , & s = \text{even} \\ 0 , & s = \text{odd} \end{array} \right\}.$$

So in fact we can write

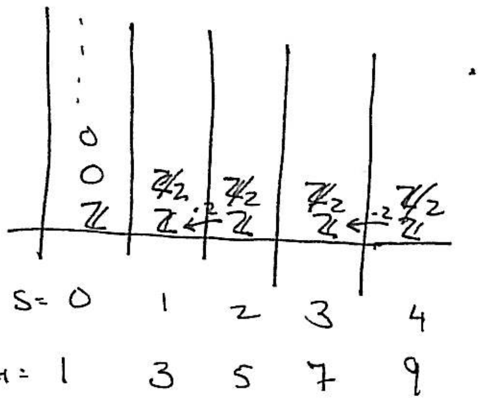


But we can get more out of this information: notice that this is the only differential *into* the bottom row, and the only differential *out* of the bottom row for columns $s \leq 2$. So we can truncate this spectral sequence and nothing more happens:

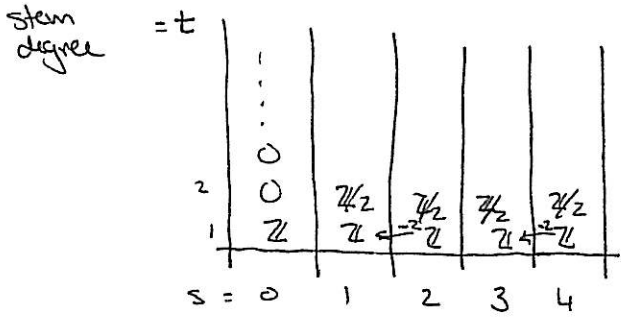


This spectral sequence computes $\pi_*\Omega^3S^3$, so we have found that $\pi_4S^3 \cong \mathbb{Z}/2$. But by the Freudenthal theorem, for stem degree 1 S^3 is already in the stable range. So in fact $\pi_{n+1}S^n = \mathbb{Z}/2$ for $n \geq 3$.

And this in turn lets us fill in a whole row of the original spectral sequence:



Last Friday (4/7) saw the introduction of the EHP spectral sequence. Remember the E^1 term looked like this:



$$2s+1 = 1 \quad 3 \quad 5 \quad 7 \quad 9 \leftarrow E^1_{st} = \pi_{s+t}\Omega^{s+1}S^{2s+1} = \pi_{2s+1+t}S^{2s+1}$$

In introducing it the claim was made that studying this spectral sequence would help to attack the theorem about desuspending w_n. In order to see why this might be true, let's take a look at how to interpret the differentials in the spectral sequence. Note that this discussion will apply in general to any spectral sequence arising from an exact couple.

To start with, recall how the spectral sequence arises. It came from the exact couple we got by applying homotopy to the fibrations

$$\begin{tikzcd}
* \arrow[r] & \Omega S^1 \arrow[r, "e"] \arrow[d] & \Omega^2 S^2 \arrow[r, "e"] \arrow[d, "h"] & \Omega^3 S^3 \arrow[r] \arrow[d, "h"] & \cdots & QS^0 \\
 & \Omega S^1 & \Omega^2 S^3 & \Omega^3 S^5 & &
\end{tikzcd}$$

getting the exact couple

$$\begin{tikzcd}
\pi_*\Omega S^1 \arrow[rr, "e"] \arrow[d] & & \pi_*\Omega^2 S^4 \arrow[rr, "e"] \arrow[d] & & \pi_*\Omega^3 S^3 \arrow[r] \arrow[d, "h"] & \cdots \\
\pi_*\Omega S^1 & & \pi_*\Omega^2 S^3 \arrow[ull, "P"] & & \pi_*\Omega^3 S^5 \arrow[ull, "P"] &
\end{tikzcd}$$

Two lessons are of great importance in sorting everything out.

① We can truncate the sequence of triangles anywhere we want, obtaining a spectral sequence, compatible in some sense with the first, that converges to the homotopy of a finite sphere.

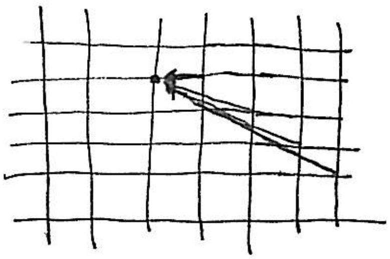
② The second lesson concerns how an element of $\pi_* S^n$ is recorded in the truncated spectral sequence of ① (and therefore in the bigger spectral sequence if you take n large enough). Since everything is recorded in terms of its (James) Hopf invariants, and this is the obstruction to desuspending an element of $\pi_* \Omega^i S^n$, the recipe is

a) Desuspend as far as possible, and then

b) compute the (James) Hopf invariant h_j

this is the filtration at which a given element appears.

Now when you study all this at the E^1 term, there is indeterminacy associated with the choice in desuspension, but this precisely corresponds to the various differentials coming into the target group from elsewhere:



Now in this context, what do the differentials mean? (And I want to advertise this as a great way to understand how various homotopy groups interact)

$$\begin{tikzcd}
\pi_* \Omega^1 S^1 \arrow[r, "e"] \arrow[d] & \pi_* \Omega^2 S^2 \arrow[r, "e"] \arrow[d] & \pi_* \Omega^3 S^3 \arrow[r, "e"] \arrow[d] & \pi_* \Omega^4 S^4 \arrow[r] \arrow[d] & \cdots \\
\pi_* \Omega 1 S^1 & \pi_* \Omega^2 S^3 \arrow[ul] & \pi_* \Omega^3 S^5 \arrow[ul] & \pi_* \Omega^4 S^7 \arrow[ul, "p"] \arrow[l, bend left, "d_1"] \arrow[ll, bend left, "d_2"] \arrow[lll, bend left, "d_3"] &
\end{tikzcd}$$

So the significance of the rth differential is: take the class in $\pi_* \Omega^{s+1} S^{2s+1}$, apply p; then <u>disuspend</u> $r-1$ times and record the result in the agreed way, that is, take the James-Hopf invariant.

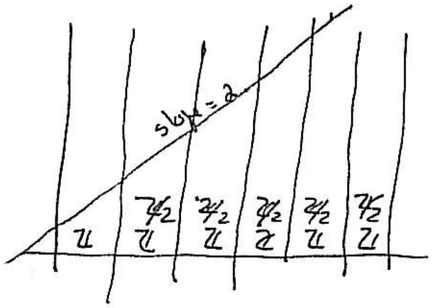
And the collection of d_r's together can be understood as the obstruction to lifting from $\Omega^{s+1} S^{2s+1}$ to $\Omega^{s+1} S^{s+1}$.

Before we go on, note that this helps explain why this SS might be useful in studying how far we can disuspend w_n: $p(\iota_{2n+1}) = \pm w_n$, so studying how far you can disuspend w_n is the same as looking for the first non-zero differential in the EHP SS on the fundamental class in $\pi_s \Omega^{s+1} S^{2s+1}$.

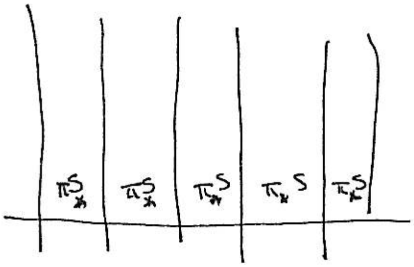
Now usually when you attack a problem like desuspending a class, a standard approach is to convert the problem to a stable one and hope that things become more cohomological. Now suspension gives homomorphisms

$$\pi_* S^1 \xrightarrow{e^2} \pi_{*+2} S^3 \xrightarrow{e^2} \pi_{*+4} S^5$$

which provide a horizontal operation on the EHPSS which is an isomorphism up to $* = 2 \cdot$ (dimension of the sphere). So there is a line of slope 2 on the E' term



beneath which the columns are the same and in fact represent stable homotopy. So this grid maps to a grid whose columns are stable homotopy



and the map is an isomorphism below the line of slope 2.

And the question is: is this the E' term of some spectral sequence which is compatible with the first? Our next goal is to construct this spectral sequence and the map of spectral sequences. Then we can play off the differentials on either side and learn about desuspending the Whitehead square.

4/12 cont'd.

In order to set the spectral sequence up it's a good idea to talk about stable homotopy a bit; for more information, see Adams' blue book.

Stable Homotopy

The notation $[X,Y]$ will mean pointed homotopy classes of pointed maps. There is a map $[X,Y] \xrightarrow{\Sigma} [\Sigma X, \Sigma Y]$, and the idea is to study this game in detail. One nice thing about it is that when you suspend you get a group: $[\Sigma X, Z]$ is a group naturally in Z, and $[\Sigma^2 X, Z]$ is abelian. The group structure comes from

$$g_1+g_2 : \Sigma X \xrightarrow{\text{pinch}} \Sigma X \vee \Sigma X \xrightarrow{g_1 \vee g_2} Z \vee Z \xrightarrow{\text{fold}} Z .$$

"Naturally in Z" means that given $f: Z \to Y$, in

$$\begin{tikzcd} \Sigma X \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \arrow[r, "f"] & Y \end{tikzcd}$$

we have $f_*(g_1+g_2) = f_*g_1 + f_*g_2$ in $[\Sigma X, Y]$.

Warning $[\Sigma X, Z]$ is not a natural in ΣX!

As an example, consider the map $S^{2n-1} \xrightarrow{w_n} S^n$, the Whitehead square. Then $w_n^* : [S^n, S^n] \longrightarrow [S^{2n-1}, S^n]$ is not a homomorphism for n even. To see this, recall that $w_n = [\iota_n, \iota_n] \in \pi_{2n-1} S^n$ is defined by

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "w_n"] \arrow[dr, "W"'] & S^n \\
 & S^n \vee S^n \arrow[u, "\text{fold}"']
\end{tikzcd}$$

where W is the attaching map for $S^{2n-1} \xrightarrow{W} S^n \vee S^n \to C[W] = S^n \times S^n$.

This diagram commutes, by definition of ι_n.

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "w_n"] \arrow[dr, "W"'] & S^n \arrow[r, "2\iota_n"] & S^n \\
 & S^n \vee S^n \arrow[u, "\text{fold}"'] \arrow[r, "2\iota_n \vee 2\iota_n"'] & S^n \vee S^n \arrow[u, "\text{fold}"']
\end{tikzcd}$$

The top row represents $(\iota_n + \iota_n) \circ w_n = w_n^*(2\iota_n)$. The lower composite represents $[2\iota_n, 2\iota_n] = 4[\iota_n, \iota_n]$ $\{$by bilinearity of the Whitehead product$\}$ $= 4w_n = 4w_n^*(\iota_n)$. So w_n^* cannot be a homomorphism unless $w_n = w_n^*(\iota_n)$ has finite order in $\pi_{2n-1}S^n$. But for n even, $h(w_n) = 2\iota_n$, so the image of w_n under the Hopf map is infinite cyclic! So for example $\mathrm{End}(\Sigma^n X)$ is not a ring.

The way to get around this is by continuing to suspend.

In what follows we take X and Y to be finite complexes. Then we define $\{X, Y\} \overset{\mathrm{def}}{=} [\Sigma^n X, \Sigma^n Y]$ for $n \gg 0$, in particular for n sufficiently large $[\Sigma^n X, \Sigma^n Y] \cong [\Sigma^{n+1} X, \Sigma^{n+1} Y]$ by the Freudenthal suspension theorem, and we define $\{X, Y\}$ to be this "stable" group, "stable homotopy classes of maps" between X and Y. Now composition gives a product

$$\begin{array}{ccccc} \{X,Y\} & \times & \{Y,Z\} & \longrightarrow & \{X,Z\} \\ \Sigma^n X \xrightarrow{f} \Sigma^n Y & & \Sigma^n Y \xrightarrow{g} \Sigma^n & \rightsquigarrow & \Sigma^n X \xrightarrow{g \circ f} \Sigma^n Z \end{array}$$

which is bilinear since we can assume all maps involved are suspensions.

So we get a category whose objects are finite complexes and whose morphisms are stable homotopy classes of pointed maps. Some important properties of this category are:

1.- We've just seen that the category is pre-additive.

2.- It has coproducts: if X and Y are finite complexes, $X \vee Y$ is the coproduct in this category:

$$\{X \vee Y, Z\} = \{X, Z\} \times \{Y, Z\}$$

This is true because of the homotopy equivalence $\Sigma^n(X \vee Y) \simeq \Sigma^n X \vee \Sigma^n Y$.

3.- Well, these two facts together mean that <u>now</u> $X \vee Y$ is the <u>product</u> in our category!

Using the maps out of

$$\begin{tikzcd} X \vee Y \arrow[r, "\text{pinch}"] \arrow[d, "\text{pinch}"] & X \\ Y & \end{tikzcd}$$

we have to show that

$$\{W, X \vee Y\} \xrightarrow[\cong]{\text{pinch}_* \times \text{pinch}_*} \{W, X\} \times \{W, Y\}$$ is an isomorphism.

a) Surjectivity. If $f: \Sigma^n W \longrightarrow \Sigma^n X$ and $g: \Sigma^n W \to \Sigma^n Y$ represent an element of $\{W, X\} \times \{W, Y\}$, then

$$\Sigma^n W \longrightarrow \Sigma^n W \vee \Sigma^n W \xrightarrow{f \vee g} \Sigma^n X \vee \Sigma^n Y$$ pushes forward to (f, g).

b) injectivity. We have $[\Sigma^n W, \Sigma^n X] \times [\Sigma^n W, \Sigma^n Y] \cong [\Sigma^n W, \Sigma^n X \times \Sigma^n Y]$. So what we have to do is lift a null-homotopy

$$\begin{tikzcd} \Sigma^n W \arrow[r] \arrow[d, hook] & \Sigma^n X \vee \Sigma^n Y \arrow[d] \\ C\Sigma^n W \arrow[r] \arrow[ur, dashed] & \Sigma^n X \times \Sigma^n Y \end{tikzcd}$$

by taking n large enough. The fiber of $\Sigma^n X \vee \Sigma^n Y \longrightarrow \Sigma^n X \times \Sigma^n Y$ is $(2n-1)$-connected, so taking $n > \dim W$ will do it.

So, surprise, the wedge is the product in this category. Well, that's nice; now let's add "formal desuspensions." Somehow the point is that now

$$\{\Sigma X, \Sigma Y\} \xleftarrow[\Sigma]{\cong} \{X, Y\}$$

we've really obliterated the difference between the morphism sets here; we've made Σ a fully faithful functor. Now the idea is to make it into an isomorphism. To fix it up, we'll simply put in formal desuspensions.

The new category, the S-category has as objects pairs (X,n) where X is a finite complex and n is an integer, the "formal n-fold suspension of X."

And $\text{morph}\{(X,m),(Y,n)\} = [\Sigma^{m+k}X, \Sigma^{n+k}Y]$ for $k>>>0$, namely k big enough that $k>m$, $k>n$, and larger still so that one is in the stable range. i.e., $\text{morph}\{(X,m),(Y,n)\} = \{\Sigma^{m+k}X, \Sigma^{n+k}Y\}$ for $k>>0$, i.e. $k>m$ and $k>n$.

Note that $(X,1) \overset{\simeq}{\rightleftarrows} (\Sigma X, 0)$, i.e. the "formal suspension" is isomorphic to the informal suspension. So we write (X,n) as $\Sigma^n X$ and $\text{morph}\{(X,m),(Y,n)\}$ as $\{\Sigma^m X, \Sigma^n Y\}$. But now we have the object $(X,-1) = \Sigma^{-1}X$. And so we talk about formerly pathological objects like $\{S^0, S^1\} = [S^k, S^{k-1}]$ $k>>0$ $= \mathbb{Z}/2$.

As an example of how things work, remember earlier we studied the group $J(X)$ which had to do with stable fiber-homotopy equivalence. Let $V \downarrow X$ be an $(n-1)$-sphere bundle. A long time ago we saw that $V\downarrow X$'s being fibre-homotopy trivial implied a splitting

$$\begin{tikzcd} T(V) \arrow[r] & S^n \\ S^n \arrow[u] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Also we found out that such a coreduction meant that

$$\begin{tikzcd} S^0 \underset{X}{*} V \arrow[d] \\ X \end{tikzcd}$$

is fiber-homotopy trivial. And so now in our new category we find

<u>Lemma</u> $V\downarrow X$ is stably fiber-homotopy trivial $\Leftrightarrow$ $T(V)$ has a stable retraction.

$$\begin{tikzcd} T(V) \\ S^n \arrow[u] \end{tikzcd}$$

In other words, in the S-category $T(V) \simeq \Sigma^n(X_+)$ implies that $T(V) \simeq S^n V$ (something)! Now earlier we also sketched a proof of Atiyah that $\tilde{J}(X)$ is finite over a finite complex; so for any $V \downarrow X$ there is an n so $\overbrace{V * \cdots * V}^{n} \downarrow X$ is stably fibre-homotopy trivial. So that says that in the S-category, $\{T(V^{*k})\}$ is periodic up to suspension. This is "James periodicity."

As a final remark, note that now we can talk about ~~vec~~ T(virtual bundle): for example, $T(V - \epsilon_n) = \Sigma^{-n} T(V)$.

Well, let's continue with the discussion of stable homotopy, and really enter the stratosphere at this point, at least for a short while. So remember last time we ended with the notion of a category $\mathcal{C}$, the S-category of finite complexes.

In this category there is an important notion of "exact triangles":

$X \to Y \to Z \to \Sigma X$ is an exact triangle if it is homotopy equivalent to $X \xrightarrow{f} Y \to C(f) \to \Sigma X$ on the level of spaces, perhaps after suspending a lot. Two important facts about exact triangles are

① a sequence $X \to Y \to Z \to \Sigma X$ is exact if and only if $\{W, -\}$ is exact for all $W \in \mathcal{C}$.

② $Y \xrightarrow{g} Z \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y$ is an exact Δ if and only if $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$ is an exact Δ.

This sounds like (co)homology... to make that hint precise, consider now a category (containing the category $\mathcal{C}$) $\mathcal{S}$, "the stable category", such that, well, it has a bunch of properties

- $\mathcal{S}$ has exact triangles, and the inclusion of $\mathcal{C}$ in $\mathcal{S}$ takes exact triangles to exact triangles.
- $\mathcal{S}$ is additive: it has products Π and coproducts $\vee$.
- the equivalence Σ on $\mathcal{C}$ extends to one on $\mathcal{S}$.
- $\mathcal{S}$ has smash products with nice properties, e.g. $W \wedge : \mathcal{S} \to \mathcal{S}$ preserves exact Δ's, and $S^0 \wedge$ is a unit.

- "Brown Representability" holds
- "Whitehead Representability" holds.

What do these last two mean? They have to do with cohomology and homology mean, so we'd better know what those are:

A functor $E^0 : \mathcal{S}^{op} \longrightarrow \mathcal{A}b^{\leftarrow \text{abelian gps.}}$ is cohomological if

① E^0 sends exact Δ's to exact sequences.

② $\mathcal{S}$ is going to contain infinite complexes, so we'd better know what E^0 does to limits, i.e. to wedges: E^0 must satisfy the "Milnor axiom"

$$E^0(\bigvee_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha E^0(X_\alpha).$$

(note that in this setting we define $E^q(X) = E^0(\Sigma^{-q}X)$).

And "Brown Representability" says: any cohomology theory is representable: there is a spectrum E (the objects of $\mathcal{S}$ are called "spectra") such that $E^0(X) \underset{\text{naturally}}{\cong} \{X, E\}$. ($\underline{\underline{\text{NB}}}$ in this context $E^0|_{\text{spaces}}$ always refers to $\underline{\text{reduced}}$ $\overset{E-}{\text{cohomology}}$, so usually the twiddle is left out).

Analogously, $E_0 : \mathcal{S} \longrightarrow \mathcal{A}b$ is homological if

① E_0 sends exact Δ's to exact sequences

② Milnor's axiom holds: $\bigoplus_\alpha E_0(X_\alpha) \xrightarrow{\cong} E_0(\bigvee_\alpha X_\alpha)$;

Whitehead Representability says any homology theory is representable: there is a spectrum E so that $E_0(X) \underset{\text{naturally}}{\cong} \underset{\pi_0^s(X\wedge E)}{\underset{\shortparallel}{\{S^0, X\wedge E\}}}$.

If you looks at Adam's blue book you'll see how to construct $\mathcal{S}$, with lots f technical details. For now, we'll just see what an object, i.e., a spectrum is: one construction takes a spectrum to be a sequence

$\ldots, E_{n-1}, E_n, E_{n+1}, \ldots$ f pointed spaces, with pointed

maps $\Sigma E_n \longrightarrow E_{n+1}$.

Two examples:

① Take X a pointed space, define a spectrum $(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ * & n < 0 \end{cases}$

This gives a name to the inclusion functor $\begin{pmatrix}\text{homotopy category f} \\ \text{pted spaces}\end{pmatrix} \longrightarrow \mathcal{S}$

which has an adjoint $\Omega^\infty : \mathcal{S} \longrightarrow \begin{pmatrix}\text{homotopy category} \\ \text{f pointed spaces}\end{pmatrix}$, in particular

there are maps

$$X \xrightarrow{\alpha} \Omega^\infty \Sigma^\infty X$$

$$E \xleftarrow{\beta} \Sigma^\infty \Omega^\infty E .$$

② Fix an abelian group A; define the spectrum HA by

$$(HA)_n = \begin{Bmatrix} K(A,n) & n \geq 0 \\ * & n < 0 \end{Bmatrix};$$

the map $\Sigma K(A,n) \longrightarrow K(A,n+1)$ being the adjoint f the identity $K(A,n) \longrightarrow \Omega K(A,n+1)$.

Now, let's talk about a notion f <u>duality</u> that makes sense in the context f spectra. This is where we lift off a bit; Michael Artin says this the hardest thing in the world to write down.

Definition A duality is a map $X \wedge Y \xrightarrow{\alpha} S^0$ such that for all $W \in \mathcal{S}$,

$$\{W, Y\} \xrightarrow{X\wedge} \{X \wedge W, X \wedge Y\} \xrightarrow{\alpha_*} \{X \wedge W, S^0\}$$

is an isomorphism.

The first thing to note is that dualities exist:

Theorem For all X there is a Y with a duality $X \wedge Y \to S^0$.

Proof $W \mapsto \{X \wedge W, S^0\}$ is a cohomology theory, so it has a representing spectrum Y; hence there is a natural isomorphism $\{X \wedge W, S^0\} \underset{BR}{\cong} \{W, Y\}$. The map α comes from taking for W the spectrum Y itself: $\{X \wedge Y, S^0\} \cong \{Y, Y\}$. Let α be the element of $\{X \wedge Y, S^0\}$ corresponding to the identity. The naturality of the representability gives us that the isomorphism comes in the manner described in the definition of duality: for any $W \in \mathcal{S}$, $\gamma \in \{W, Y\}$:

$$\begin{tikzcd}
\gamma \arrow[r, mapsto] & \mathrm{id} \wedge \gamma \arrow[r, mapsto, "\alpha"] & ? \arrow[r, mapsto] & \gamma \\
\{W, Y\} \arrow[r, "X\wedge"] & \{X \wedge W, X \wedge Y\} \arrow[r, "\alpha_*"] & \{X \wedge W, S^0\} \arrow[r, "BR", "\cong"'] & \{W, Y\} \\
\{Y, Y\} \arrow[r, "X\wedge"] \arrow[u, "\gamma^*"] & \{X \wedge Y, X \wedge Y\} \arrow[r, "\alpha_*"] \arrow[u, "\gamma^*"'] & \{X \wedge Y, S^0\} \arrow[r, "BR", "\cong"'] \arrow[u, "\gamma^*"'] & \{Y, Y\} \arrow[u, "\gamma^*"'] \\
\mathrm{id} \arrow[r, mapsto] \arrow[uuu, mapsto, bend left] & \mathrm{id} \arrow[r, mapsto] & \alpha \arrow[r, mapsto] & \mathrm{id} \arrow[uuu, mapsto, bend right]
\end{tikzcd}$$

The first two squares obviously commute; the third commutes by naturality of Brown Representability.

In fact there's more to be had from this use of Brown Representability: the correspondence $X \rightsquigarrow Y = DX$ can be made contravariantly functorial in the sense that $X \wedge DX \xrightarrow{\alpha_X} S^0$ is a natural transformation: for $f : X \to Y$,

$$\begin{tikzcd}
X \wedge DY \arrow[r, "1\wedge Df"] \arrow[d, "f\wedge 1"'] & X \wedge DX \arrow[d, "\alpha_X"] \\
Y \wedge DY \arrow[r, "\alpha_Y"'] & S^0
\end{tikzcd}$$

commutes.

To make the dualities this way, first use a massive axiom of choice to pick a DX for every X; obtain the α_X's in the manner described above. Now for any map $X \xrightarrow{f} Y$ we get, for any $W \in \mathcal{S}$, a map γ via

$$\begin{tikzcd}
\{W, DY\} \arrow[r, "\gamma"] \arrow[d, equal, "BR"'] & \{W, DX\} \arrow[d, equal, "BR"] \\
\{Y \wedge W, S^0\} \arrow[r, "f\wedge 1^*"] & \{X \wedge W, S^0\}
\end{tikzcd}$$

Take $W = DY$; then define Df by $\begin{matrix} 1 \overset{\gamma}{\longmapsto} Df \\ \{DY, DY\} \xrightarrow{\gamma} \{DY, DX\}. \end{matrix}$

Then we have

$$\begin{tikzcd}
Df \arrow[r, mapsto] & 1\wedge Df \arrow[r, mapsto] & \alpha_X \circ (1\wedge Df) \arrow[r, mapsto, "BR"] & Df \\
\{DY, DX\} \arrow[r, "X\wedge"] & \{X\wedge DY, X\wedge DX\} \arrow[r, "\alpha_{X*}"] & \{X\wedge DY, S^0\} \arrow[r, "\cong"', "BR"] & \{DY, DX\} \\
 & & \alpha_Y \circ (f\wedge 1) & \\
\{DY, DY\} \arrow[r, "Y\wedge"] & \{Y\wedge DY, Y\wedge DY\} \arrow[r, "\alpha_{Y*}"] & \{Y\wedge DY, S^0\} \arrow[r, "\cong", "BR"'] \arrow[uu, "(f\wedge 1)^*"'] & \{DY, DY\} \arrow[uu] \\
id \arrow[r, mapsto] & id \arrow[r, mapsto] & \alpha_Y \arrow[uu, mapsto] \arrow[r, mapsto] & 1 \arrow[uuuu, mapsto, bend right]
\end{tikzcd}$$

The maps in the rows are determined by the construction of the dualities involved; the last square commutes by definition of Df, and the two composites at $\{X \wedge DY, S^0\}$, which are equal, are the two ways of going around the square that we wanted to show commutes.

All right, enough; let's see what having D does for us.

Lemma

⓪ You can choose $DS^0 = S^0$

① $X \to Y \to Z \to \Sigma X$ an exact Δ $\Rightarrow$ $DX \leftarrow DY \leftarrow DZ \leftarrow \Sigma^{-1} DX$ is an exact Δ.

② $D(\Sigma X) = \Sigma^{-1} DX$ (um, well, so this explains the end of ①).

Corollary A finite complex is built out of spheres using exact Δ's, so we get

① D (finite spectrum) is finite

② If K is finite then $D(K \wedge X) \simeq DK \wedge DX$ (but not if K is infinite; in general D isn't nice when there are limits around).

③ X can be taken for DDX.

Remark The defining property of D was

$$\{W, DX\} \cong \{X\wedge W, S^0\}.$$

Now there's nothing particularly special about S^0: If Y is any spectrum, $W \mapsto \{X\wedge W, Y\}$ is a cohomology theory for W, so it's represented by a spectrum $F(X,Y)$. If you think of $X\wedge W$ as a tensor product, $F(X,Y)$ is like a hom-space. So think of $F(X,Y)$ as a "function spectrum," and $DX = F(X,S^0)$. It's not a function space in any way; we got it out of Brown representability.

Notice that there is a map $DX\wedge Y \xrightarrow{f} F(X,Y)$ defined by

$$\begin{tikzcd}
\{X\wedge(DX\wedge Y), Y\} \arrow[d, "\cong", "\left(\substack{\text{definition of}\\ F(X,Y)}\right)"'] & \ni & \left(X\wedge DX\wedge Y \xrightarrow{\alpha_\pi\wedge 1} S^0\wedge Y \xrightarrow{id} Y\right) \arrow[d, mapsto] \\
\{DX\wedge Y, F(X,Y)) & & f
\end{tikzcd}$$

Now we'd like f to be an equivalence; this isn't always true, but for some reasonably nice condition, say X or Y a finite complex, then it is.

4/19 ①

We were talking about duality; the last thing mentioned was the "function spectrum" $F(X,Y)$ satisfying $\{W, F(X,Y)\} \cong \{X\wedge W, Y\}$ $\forall W \in \mathcal{S}$; if X is finite then $F(X,Y) \cong (DX)\wedge Y$. Now call $Y = E$, take $W = S^n$, and assume X is finite; then

$$\begin{tikzcd}
\pi_n(DX\wedge E) \arrow[r, phantom, "\cong"] \arrow[d, equal] & \{X\wedge S^n, E\} \arrow[d, equal] \\
E_n(DX) \arrow[r, phantom, "\cong"] & E^n(X)
\end{tikzcd}$$

I want to apply this sort of duality, and relate it to other sorts of duality you're familiar with. For example, how can we ~~reg~~ recognize a duality?

Suppose X is finite and we have a map $\hat{f} : X\wedge Y \to S^0$; then $\hat{f}$ corresponds to a map $f : Y \longrightarrow DX$; one could ask whether f is an equivalence.

$$\begin{tikzcd}
 & X\wedge DX \arrow[d, "\epsilon_X"] \\
X\wedge Y \arrow[ru, "1\wedge f"] \arrow[r, "\hat{f}"'] & S^0
\end{tikzcd}$$

Look at the diagram below:

$$\begin{tikzcd}
\{S^n, Y\wedge E\} \arrow[r, "f_*"] \arrow[d, "X\wedge"'] \arrow[dd, bend right=60, "\hat{f}/"'] & \{S^n, DX\wedge E\} \arrow[d, "X\wedge"] \arrow[dd, bend left=60, "\cong"] \\
\{X\wedge S^n, X\wedge Y\wedge E\} \arrow[r, "f_*"] \arrow[d, "\hat{f}_*"'] & \{X\wedge S^n, X\wedge DX\wedge E\} \arrow[d, "\epsilon_{X*}"] \\
\{X\wedge S^n, E\} \arrow[r, "="'] & \{X\wedge S^n, E\}
\end{tikzcd}$$

from equivalence $F(X,E) = DX\wedge E$

So if we can conclude that $\hat{f}/ : E_n(Y) \longrightarrow E^n(X)$ is an isomorphism, then $f_* : E_n(Y) \longrightarrow E_n(DX)$ is also, so at least f is an ~~isomorphism~~ equivalence as far as E-homology is concerned.

Note that if X and Y are finite spectra, and $X \xrightarrow{f} Y$ is an isomorphism on $H_*(;\mathbb{Z})$ then f is an equivalence (there's no π_1 problem here). So taking $E = H(;\mathbb{Z})$ will be enough. That's what the standard duality theorems give you.

Alexander Duality

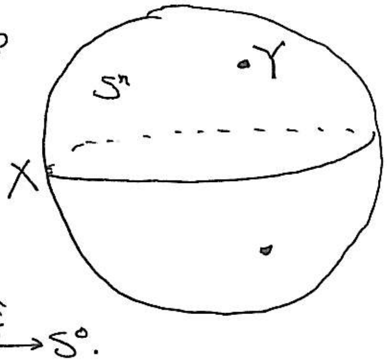
Let X be a finite complex embedded in S^n; let Y be another complex disjoint from X in S^n. Consider the map

$$\begin{aligned} X \times Y &\longrightarrow S^{n-1} \\ (x,y) &\longmapsto \frac{x-y}{\|x-y\|} \end{aligned}$$

{differences in $\mathbb{R}^n$ using stereographic projection from a fixed point not in Y or X}

and apply the Hopf construction to get a map

$$\begin{tikzcd} X * Y \arrow[r] \arrow[d, "\simeq"] & S^n \\ \Sigma(X \wedge Y) & \end{tikzcd}$$



So in the $\mathcal{S}$ category you get a map $X \wedge \Sigma^{1-n} Y \xrightarrow{\hat{f}} S^0$.

Alexander duality says

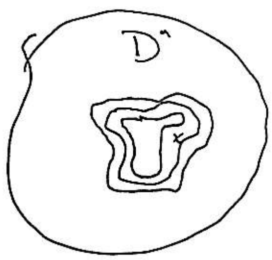
Theorem If $Y \simeq S^n - X$ then $\hat{f}$ is a duality; $DX \simeq \Sigma^{1-n} Y$.

The proof consists of showing that $\hat{f}/ : \bar{H}_i(\Sigma^{1-n} Y) \xrightarrow{\cong} H^i(X)$ is iso for $i \in \mathbb{Z}$.

Remark If you think of X and Y as embedded in $\mathbb{R}^n$ instead of S^n and $Y \simeq \mathbb{R}^n - X$, then $Y \simeq S^n - X_+$, so $D(X_+) \simeq \Sigma^{1-n} Y$.

Milnor-Spanier Duality

For example, take $X \subseteq \mathring{D}^n \cong \mathbb{R}^n$, and suppose $N \subseteq \mathring{D}^n$ is a regular neighborhood so that [illegible] the inclusion $X \overset{\simeq}{\hookrightarrow} N$ is a homotopy equivalence.



$$\text{Then } D(X_+) \simeq \Sigma^{1-n}(D^n - N).$$

The virtue of the regular neighborhood is that you can identify ~~the~~ what Σ(complement) is: all you need to find the suspension is an inclusion into a contractible space.

So in

$$\begin{tikzcd} D^n - N \arrow[r] & D^n \arrow[r] & D^n/D^n-N \arrow[d, phantom, "\cong" rotate=90] \\ & & \Sigma(D^n - N) \end{tikzcd}$$

Hence $\Sigma^{1-n}(D^n - N) \simeq \Sigma^{-n}(D^n/D^n-N) \simeq \Sigma^{-n}(\bar{N}/\partial\bar{N})$.

Well that's most useful when you can say something about N; if X is a d-dimensional manifold without boundary then the normal bundle ν of the inclusion $X \hookrightarrow D$ is n-d-dimensional.

$N \simeq$ the disk bundle of ν, and $\bar{N}/\partial\bar{N} \simeq T(\nu)$, so

$D(X_+) \simeq \Sigma^{-n} T(\nu) \simeq T(\nu - n\epsilon)$. This is called "Milnor-Spanier duality" although it's not called that very often. In homology,

$$E^{-i}(X_+) = E_i(D(X_+)) = E_{i+n}(T(\nu)).$$

Poincaré Duality

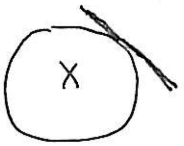
If now ν is oriented for E, then the Thom isomorphism says $E_{i+n}(T(\nu)) \cong E_{i+d}(X_+)$; converting to unreduced homology

$$E^{-i}(X) \cong E_{i+d}(X) \qquad \text{"Poincaré Duality."}$$

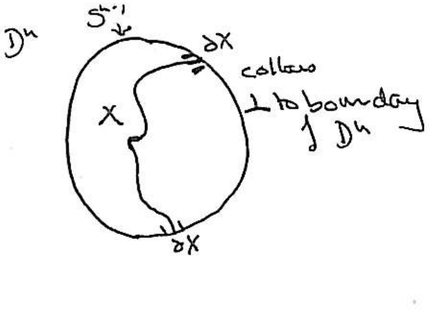
Note that $\nu + \underset{\substack{\uparrow \\ \text{tangent} \\ \text{bundle}}}{\tau} = n\epsilon$, so $D(X_+) = T(\nu - n\epsilon) = T(-\tau)$.

This is a little mystical, so perhaps we should look at it more closely in the context of

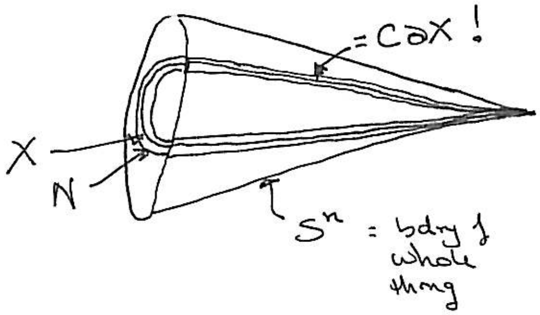
Atiyah Duality (see Atiyah, "Thom Complexes": It's a good thing to look at) where we take for a change $(X, \partial X)$ to be a manifold with boundary. You have to be careful about interpreting $\tau\left(\substack{\text{manifold with} \\ \text{boundary}}\right)$. $\tau(X)$ is best defined to be d-dimensional everywhere, but with an identified $d{-}1$-dim subbundle on the boundary:



So the new picture of embedded X is



Well this isn't quite what ~~we~~ we had before. The nice idea is to take the cone on what we had before.



So now $X/\partial X \simeq X \cup C\partial X$

The new regular neighborhood of the whole model is $(N \cup C(N \cap S^{n-1}))$;

its complement is $S^n - (N \cup C(N \cap S^{n-1}))$. And now by projecting down onto D^n, this is $\cong D^n - N$. So

$$D(X/\partial X) \simeq D(X \cup C\partial X) \simeq \Sigma^{1-n}(S^n - N \cup C(N \cap S^{n-1})) \simeq \Sigma^{1-n}(D^n - N)$$

$$\simeq \Sigma^{-n}(\bar{N}/\partial\bar{N}) \simeq \Sigma^{-n}T(\nu) = T(-\tau).$$

and this is "Atiyah Duality." Now if X is oriented, i.e. there is a Thom isomorphism, we get $E_i(X) \simeq E^{d-i}(X, \partial X)$ and this is "Lefschetz Duality."

Now suppose X is a compact closed manifold and $\xi \downarrow X$ is a smooth vector bundle; then $(\underset{\substack{\uparrow\\ \text{disk bundle}}}{D(\xi)}, \underset{\substack{\uparrow\\ \text{sphere bundle}}}{S(\xi)})$ is a compact manifold-with-boundary. Atiyah duality says

$$D(T(\xi)) = D(D(\xi)/S(\xi)) \simeq T(-\tau(D(\xi))) \simeq T(-\pi^*(\tau(X) \oplus \xi))$$

$$= T(-\tau(X) - \xi)$$ (where π is the projection $\begin{array}{c} D(\xi) \\ \downarrow \pi \\ X \end{array}$).

Here's an example: $X = \mathbb{R}P^{k-1}$ $\quad \xi = nL$ (take $n \geq 0$ for the moment; we'll see it doesn't make any difference soon).

Then

① $T(nL) = \mathbb{R}P^{n+k-1} / \mathbb{R}P^{n-1} \simeq \mathbb{R}P_{n}^{n+k-1}$

② $\tau(\mathbb{R}P^{k-1}) + \epsilon = kL.$

So $$D(\mathbb{R}P_{n}^{n+k-1}) = D(T(nL)) \simeq T(-nL - \tau) \simeq T(-nL + \epsilon - kL)$$

$$= \Sigma T(-(n+k)L) \overset{(\text{def})}{=} \Sigma \, \mathbb{R}P_{-n-k}^{n-1}$$

Summarizing

$$D(\mathbb{R}P_{-b}^{t-1}) \simeq \Sigma \, \mathbb{R}P_{-t}^{b-1} \qquad b, t \in \mathbb{Z}, \; t-1 \geq -b.$$

(taking $b = -n$, $t = n+k$ in the above. The condition $t-1 \geq -b$ requires that $\mathbb{R}P_{-b}^{t-1}$ has a cell).

or $$\mathbb{R}P_{-b}^{t-1} = T(-bL \downarrow \mathbb{R}P^{t+b-1}).$$

By the way if you don't like these somewhat ethereal spaces, you can use James Periodicity: let a be the periodicity of L on $\hat{J}(\mathbb{R}P^{t+b-1})$. Then $T((ja-b)L) \simeq \Sigma^{ja} T(-bL)$; for j large enough, $T((ja-b)L)$ is an actual stunted projective space!

Today we look at the attaching maps for stunted projective spaces. In fact the attaching maps we're going to look at are the "stable relative attaching maps", so perhaps we should begin by saying what that means. Suppose X is a CW complex; then it has a skeleton filtration $Sk_k X$. The $k+1$-skeleton is obtained from the k-skeleton by attaching $k+1$-cells as in

$$\begin{tikzcd}
Sk_k X \arrow[r, phantom, "\subseteq"] & Sk_{k+1} X \\
\vee S^k \arrow[u] \arrow[r, hook] & \vee D^{k+1} \arrow[u]
\end{tikzcd}$$

Although the attaching maps $S^k \longrightarrow Sk_k X$ are to the k-skeleton, it's certainly possible that an attaching map pulls back to a lower skeleton. A "relative attaching map" for a $k+1$-cell is an a factorization of this kind through the *lowest* possible skeleton; that is, the map a below:

$$\begin{tikzcd}
Sk_{k-j-1} X \arrow[r, hook] & Sk_{k-j} X \arrow[r, hook] & Sk_k X \arrow[r] & Sk_{k+1} X \\
 & & \vee S^k \arrow[u] \arrow[r, hook] & \vee D^{k+1} \arrow[u] \\
 & & S^k \arrow[u] \arrow[uul, "a"] \arrow[uull, dashed, "/" marking]
\end{tikzcd}$$

A "stable relative attaching map" is a stable one of these. That is, the cells of the $k+2$-skeleton of ΣX are the suspensions of the cells of the $k+1$-skeleton of X:

$$\begin{tikzcd}
Sk_k X \arrow[r] & Sk_{k+1} X & & Sk_{k+1}\Sigma X \arrow[r] & Sk_{k+2}\Sigma X \\
\vee S^k \arrow[u, "\vee f"] \arrow[r] & \vee D^{k+1} \arrow[u] \arrow[r, squiggly, "\Sigma"] & & \vee S^{k+1} \arrow[u, "\vee\Sigma f"] \arrow[r] \arrow[d, phantom, "\shortparallel"] & \vee D^{k+2} \arrow[u] \\
 & & & \vee\Sigma S^k &
\end{tikzcd}$$

so one could ask how far down an attaching map factors, perhaps after suspending very often

$$\begin{tikzcd}
Sk_{k-j-1+n}\Sigma^n X \arrow[r, hook] & Sk_{k-j+n}\Sigma^n X \arrow[r, leftrightarrow] & Sk_{k+n}\Sigma^n X \arrow[r] & Sk_{k+1+n}\Sigma^n X \\
 & & \vee S^{k+n} \arrow[u] \arrow[r] & \vee D^{k+1+n} \arrow[u] \\
 & & S^{k+n} \arrow[u] \arrow[uul, "a"'] \arrow[uull, dashed, "\times" description] &
\end{tikzcd}$$

The inclusion of the $k-j-1$-skeleton in the $k-j$-skeleton is a cofibration; we have a cofibration sequence

$$Sk_{k-j-1} X \hookrightarrow Sk_{k-j} X \xrightarrow{b} \vee S^{k-j}$$

Stably, this is also a fibration sequence, so being unable stably to factor the map a through $Sk_{k-j-1} X$ means that the composite ba in the diagram below is non-trivial:

$$\begin{tikzcd}
 & \vee S^{k-j} & & \\
Sk_{k-j-1}X \arrow[r, hook] & Sk_{k-j}X \arrow[u, "b"] \arrow[r, hook] & Sk_k X \arrow[r, hook] & Sk_{k+1}X \\
 & & \vee S^k \arrow[u] \arrow[r, hook] & \vee D^{k+1} \arrow[u] \\
 & & S^k \arrow[u] \arrow[uul, "a"] &
\end{tikzcd}$$

So a stable relative attaching map gives a nontrivial element of $\pi^s_k(\vee S^{k-j}) = \bigoplus \pi^s_k(S^{k-j})$. <u>Warning</u>: there's indeterminacy in how you factor the attaching map, so the element you get may not be well-defined.

In any case, we want to understand the stable relative attaching maps for $\mathbb{R}P_j$ for example, in this diagram we ask

$$\begin{tikzcd}
S^{n-2} \arrow[r, "\pi"] \arrow[dr] \arrow[ddr, "\times" marking] & \mathbb{R}P^{n-2} \arrow[r] & \mathbb{R}P^{n-1} = C(\pi) \\
 & \mathbb{R}P^{n-j} \arrow[u, hook] \arrow[r] & S^{n-j} \\
 & \mathbb{R}P^{n-j-1} \arrow[u, hook] &
\end{tikzcd}$$

① What is j, stably?

② What is the compression onto the top cell S^{n-j}?

The answer will come out in terms of the image of J, actually its stable version $j : \pi_{*-1}(O) \longrightarrow \pi_{*-1}(QS^0) = \pi^S_{*-1}$. Remember $\pi_{*-1}(O) = \pi_*(BO) = \widetilde{KO}^*$. Here's a table, leaving out the degrees where $\widetilde{KO}^* = 0$:

e	$\rho(2^e) = *$	$KO^e = \pi_{*-1}(O) \ni$	generator		$\pi^S_{*-1} \ni$	
0	1	$\mathbb{Z}/2$	g_0	$\longmapsto$	$j_0 = -2z$	
1	2	$\mathbb{Z}/2$	g_1		$j_1 = \eta$	Hopf invariant 1 elements.
2	4	$\mathbb{Z}$	g_2		$j_2 = \nu$	
3	8	$\mathbb{Z}$	g_3		$j_3 = \sigma$	
4	9	$\mathbb{Z}/2$	g_4		$j_4 = \eta\sigma$	
5	11	$\mathbb{Z}/2$	g_5		$j_5 = \eta^2\sigma$	

Note that in degree zero, $\pi_0^S(U) \cong \mathbb{Z}/2$ but $\pi_0^S(S) = \mathbb{Z}$, so j can't be a homomorphism; in fact, in other degrees it is.

To describe the attaching maps the discussion will begin where the answer is, which may seem like a funny place at first, so have patience. Recall that

$$\widetilde{KO}(\mathbb{R}P^{p(n)-1}) \cong \mathbb{Z}/2^{\nu(n)}\langle L-1\rangle$$

$$\widetilde{KO}(\mathbb{R}P^{p(n)}) \cong \mathbb{Z}/2^{\nu(n)+1}\langle L-1\rangle$$

Now these two fit into a cofiber sequence

$$\mathbb{R}P^{p(n)-1} \hookrightarrow \mathbb{R}P^{p(n)} \to S^{p(n)}$$

which gives an exact sequence

$$\begin{tikzcd}
\widetilde{KO}(S^{p(n)}) \arrow[r] & \widetilde{KO}(\mathbb{R}P^{p(n)}) \arrow[r, two heads] & \widetilde{KO}(\mathbb{R}P^{p(n)-1}) \\
 & \mathbb{Z}/2^{\nu(n)+1}\langle L-1\rangle \arrow[r, two heads] & \mathbb{Z}/2^{\nu(n)}\langle L-1\rangle \\
\begin{matrix}\mathbb{Z} \text{ or} \\ \mathbb{Z}/2\end{matrix}\langle g_{\nu(n)}\rangle \arrow[r] & \text{ker of } \curvearrowright \cong \mathbb{Z}/2 &
\end{tikzcd}$$

$\uparrow$
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previous
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$$g_{\nu(n)} \longmapsto \text{generator of kernel} = 2^{\nu(n)}(L-1).$$

Notice that in $\mathbb{Z}/2^{\nu(n)+1}\langle L-1\rangle$,

$$2^{\nu(n)}(L-1) = (\text{anything odd}) \cdot 2^{\nu(n)}(L-1).$$

But $n = \text{odd} \cdot 2^{\nu(n)}$, so for the generator of kernel we can take $n(1-L)$.

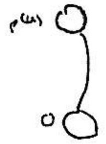
In terms of bundles this means that $n(1-L) \downarrow \mathbb{R}P^{p(n)}$ is classi~~ef~~fied by a map into BO that factors through the map $\mathbb{R}P^{p(n)} \to S^{p(n)}$ by the map $S^{p(n)} \xrightarrow{g_{\nu(n)}} BO$:

$$\begin{tikzcd}
n(1-L) \arrow[r] \arrow[d] & E(g_{\nu(n)}) \arrow[d] & \leftarrow \text{virtual bundle classified by } g_{\nu(n)} \\
\mathbb{R}P^{\rho(n)} \arrow[r] & S^{\rho(n)} \arrow[r, "g_{\nu(n)}"] & BO \\
& \uparrow \text{iso on } H_{\rho(n)}(-;\mathbb{Z}/2). &
\end{tikzcd}$$

That's the starting point. Now study the Thom spaces to get stunted projective spaces. In the $\mathcal{S}$-category, we have

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d] & S^0 \arrow[d] \\
\Sigma^n(\mathbb{R}P^{\rho(n)-n}_{-n}) \arrow[r] & T(g_{\nu(n)})
\end{tikzcd}$$

and the first question is, what is $T(g_{\nu(n)})$? Well, by the Thom isomorphism, $T(g_{\nu(n)})$ has a zero-cell and a $\rho(n)$-cell; the attaching map is an element of $\pi_{\rho(n)-1}(QS^0)$; the fact is



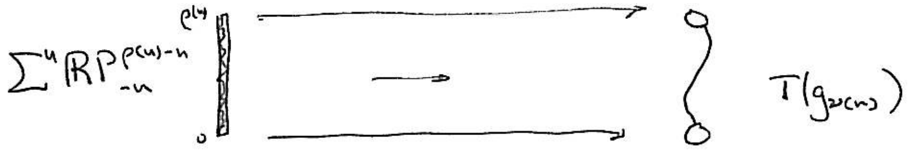
<u>Theorem</u> The attaching map for $T(g_{\nu(n)})$ is $j_{\nu(n)} = j(g_{\nu(n)}) \in \pi_{\rho(n)-1}(QS^0)$.

This has the status almost of a folk theorem; it's due to Toda and Adams. We will prove it next time; for now, what else could it be?

Now we can line up two cofiber sequences vertically:

$$\begin{tikzcd}
S^0 \arrow[r, "="] \arrow[d] & S^0 \arrow[d] \\
\Sigma^n(\mathbb{R}P^{\rho(n)-n}_{-n}) \arrow[r, "①"] \arrow[d] & T(g_{2\nu(n)}) \arrow[d] \\
\Sigma^n(\mathbb{R}P^{\rho(n)-n}_{-n+1}) \arrow[r, "② = c"] & S^{\rho(n)}
\end{tikzcd}$$

① is an isomorphism in dimension $\rho(n)$ by naturality of the Thom isomorphism, so ② is ± the collapse map. In other words, $\Sigma^n \mathbb{R}P^{\rho(n)-n}_{-n}$ has cells between dimensions 0 and $\rho(n)$, and the map to $T(g_{2\nu(n)})$ strips away the cells in between.



Well, that's pretty good, only attaching maps are supposed to go the other way... So let's dualize this picture. Two facts about the Spanier-Whitehead dual we will use are

$$D\,\mathbb{R}P^{t-1}_{-b} = \Sigma\,\mathbb{R}P^{b-1}_{-t}$$

$$S^p \xrightarrow{f} S^0 \quad \rightsquigarrow \quad S^0 \xrightarrow{\overset{\pm\Sigma^{-p}f}{\overset{\shortparallel}{D(f)}}} S^{-p}.$$

The only space whose dual we have to compute is $T(g_{2\nu(n)})$. Since (by the Folk Theorem) $T(g_{2\nu(n)}) = C(j_{2\nu(n)})$ we have the cofiber sequence

$$S^{\rho(n)-1} \xrightarrow{j_{\nu(n)}} S^0 \longrightarrow C(j_{\nu(n)}) = T(g_{\nu(n)}).$$

Dualizing, we get

$$\begin{tikzcd}
C(\Sigma^{-\rho(n)+1} j_{\nu(n)}) \arrow[d, phantom, "\parallel"] & S^{-\rho(n)+1} \arrow[l] & S^0 \arrow[l, "\Sigma^{-\rho(n)+1} j_{\nu(n)}"'] & DT(g_{\nu(n)}) \arrow[l] \\
\Sigma^{-\rho(n)+1} C(j_{\nu(n)}). & & &
\end{tikzcd}$$

So $DT(g_{\nu(n)}) = \Sigma^{-\rho(n)} C(j_{\nu(n)}) = \Sigma^{-\rho(n)} T(g_{\nu(n)})$, so $T(g_{\nu(n)})$ is nearly self-dual. So the dual diagram is

$$\begin{tikzcd}
S^0 & S^0 \arrow[l] \\
\Sigma^{1-n}\mathbb{R}P^{n-1}_{n-\rho(n)-1} \arrow[u] & \Sigma^{-\rho(n)} C(j_{\nu(n)}) \arrow[l] \arrow[u] \\
\Sigma^{1-n}\mathbb{R}P^{n-2}_{n-\rho(n)-1} \arrow[u] & S^{-\rho(n)} \arrow[l] \arrow[u] \\
S^{-1} \arrow[u] & S^{1}. \arrow[l] \arrow[u, "\Sigma^{-\rho(n)} j_{\nu(n)}"']
\end{tikzcd}$$

Suspend this n-1 times and you get,

$$\begin{tikzcd}
 & S^{n-1} & S^{n-1} \arrow[l, "="'] \\
 & \mathbb{R}P^{n-1}_{n-\rho(n)-1} \arrow[u] & \Sigma^{n-\rho(n)-1} C(j_{\nu(n)}) \arrow[l] \arrow[u] \\
 & \mathbb{R}P^{n-2}_{n-\rho(n)-1} \arrow[u] & S^{n-\rho(n)-1} \arrow[l, "②"'] \arrow[u] \\
① & S^{n-2} \arrow[u, "\pi"] & S^{n-2} \arrow[l, "="] \arrow[u, "j_{\nu(n)}"']
\end{tikzcd}$$

(actually, $\Sigma^{n-\rho(n)-1}$? $j_{\nu(n)}$, but, as a map of spectra $j_{\nu(n)}$)

① is the attaching map $S^{n-2} \xrightarrow{\pi} \mathbb{R}P^{n-2}_{n-\rho(n)-1} \longrightarrow \mathbb{R}P^{n-1}_{n-\rho(n)-1}$; this follows because the left vertical line is a cofiber sequence. ② is the dual of ± the collapse map, and is ± the inclusion of the bottom cell. Well so we've done it: we've factored π as $j_{\nu(n)}$ followed by inclusion of the bottom cell. In other words, the attaching map for

$$\begin{tikzcd}
S^{n-2} \arrow[r, "\pi"] \arrow[dr, "j_{\nu(n)}"'] & \mathbb{R}P^{n-2}_{n-\rho(n)-1} \\
 & S^{n-\rho(n)-1} \arrow[u, hook]
\end{tikzcd}$$

the top cell of $\mathbb{R}P^{n-2}_{n-\rho(n)-1}$ pulls all the way back to the $n-\rho(n)-1$-skeleton. It goes no further: that would mean that $\pi \simeq *$, but if π is null homotopic, stably, then we'd get,

$$\begin{tikzcd}
 & & S^0 \arrow[d, "id"] \arrow[dl, dashed, "\exists"'] & \\
\Sigma^{-n+1}\mathbb{RP}^{n-2}_{n-\rho(n)-1} \arrow[r] & \Sigma^{-n+1}\mathbb{RP}^{n-1}_{n-\rho(n)-1} \arrow[r] & S^0 \arrow[r, "\pi"] & \Sigma^{-n+2}\mathbb{RP}^{n-2}_{n-\rho(n)-1}
\end{tikzcd}$$

because the sequence is an exact triangle, a map going back as above. Looking at what that means in terms of the dual, it means there is a stable splitting of the 0-sphere $S^0 \rightleftarrows \Sigma^n \mathbb{RP}^{\rho(n)-n}_{-n} = T(n(1-L)\downarrow\mathbb{RP}^{\rho(n)})$ which would mean that $n(1-L)$ is stably fiber-homotopy trivial, equivalently that nL is stably fiber-homotopy trivial. But by the vector field problem, its not: $n(1-L) = n(1-L) \neq 0$ in $\tilde{J}(\mathbb{RP}^{\rho(n)}) = \mathbb{Z}/2^{\nu(n)+1}\langle L-1\rangle$.

Now it's an easy matter to get back to the general case: consider $\mathbb{RP}^{n-1}_{-N}$, where $N \geq -(n-\rho(n)-1)$. Then we have the diagram

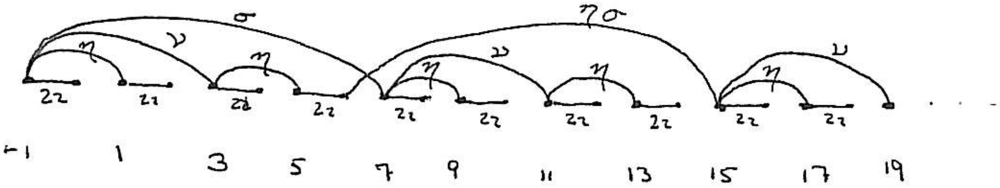
$$\begin{tikzcd}
 & \vdots & \vdots \\
 & \mathbb{RP}^{n-2}_{n-\rho(n)} \arrow[u] \arrow[r, "="] & \mathbb{RP}^{n-2}_{n-\rho(n)} \arrow[u] \\
S^{n-2} \arrow[r, "\pi"] \arrow[dr, dashed, "\bar{\pi}"'] \arrow[drr, bend right=60, "j_{\nu(n)}"'] & \mathbb{RP}^{n-2}_{-N} \arrow[u] \arrow[r, "collapse"] & \mathbb{RP}^{n-2}_{n-\rho(n)-1} \arrow[u] \\
 & \mathbb{RP}^{n-\rho(n)-1}_{-N} \arrow[u] \arrow[r] & S^{n-\rho(n)-1} \arrow[u, "\Sigma^{-1}pinch"'] \\
 & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

Now ~~was~~ taking $[S^{n-2}, -]$ on this diagram is exactly what we need to get a Mayer-Vietoris sequence: two long exact sequences with ~~an isomor~~ maps between them which are isomorphisms in every third position. The two composites $S^{n-2} \overset{collapse\circ\pi}{\underset{\Sigma^{-1}pinch\circ j_{\nu(n)}}{\rightrightarrows}} \mathbb{RP}^{n-2}_{n-\rho(n)-1}$ are equal; this is what we've just shown, and this gives us the map $\bar{\pi}$ above, and this is the best possible. So we get:

Theorem The relative attaching map for the $n-1$-cell of $\mathbb{R}P^{n-1}_{-N}$ can be taken to be $j_{\nu(n)}$; that is, a $\bar{\pi}$ so that

$$\begin{tikzcd}
S^{n-2} \arrow[r, "\pi"] \arrow[dr, "\bar{\pi}"'] \arrow[drr, bend right=40, "j_{\nu(n)}"'] & \mathbb{R}P^{n-2}_{-N} \arrow[r] & \mathbb{R}P^{n-1}_{-N} \\
& \mathbb{R}P^{n-\rho(n)-1}_{-N} \arrow[u, hook] \arrow[r] & S^{n-\rho(n)-1}.
\end{tikzcd}$$

It's time to draw some pictures. First let's see where all these relative attaching maps go. Here's $\mathbb{R}P^{\infty}_{-1}$: ~~horiz~~ lines show where the top cell of $\mathbb{R}P^{n-1}_{-1}$ is attached.



For example, if n is odd then $\rho(n) = 1$, $\nu(n)=0$. So the attaching map for $\mathbb{R}P^{2k}_{-N}$'s top-dimensional cell is $j_0 = 2\iota$. For $n-1 = 1$, 3, or 7, the relative attaching map is to the -1-cell -- so if it weren't there, stably these cells wouldn't be attached; these splittings correspond to the Hopf-invariant 1 elements.

The second picture is a spectral sequence; it's the Atiyah-Hirzebruch spectral sequence for stable homotopy of $\mathbb{R}P^{\infty}_{-1}$. The exact couple comes from

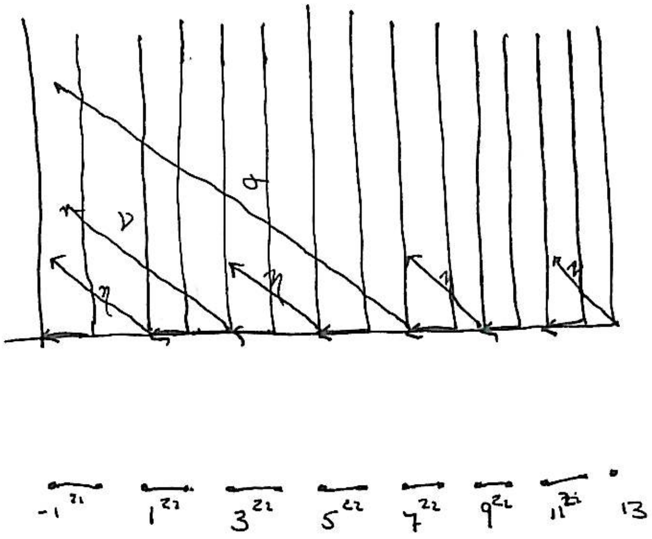
$$\begin{tikzcd}
\cdots \arrow[r, hook] & \mathbb{R}P_{-1}^{k-1} \arrow[r, hook] \arrow[d] & \mathbb{R}P_{-1}^{k} \arrow[r] \arrow[d] & \mathbb{R}P_{-1}^{k+1} \arrow[r] \arrow[d] & \cdots \\
 & \cancel{\Sigma} S^{k-1} & \cancel{\Sigma} S^{k} & \cancel{\Sigma} S^{k+1} &
\end{tikzcd}$$

Apply π_*^S

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*^S \mathbb{R}P_{-1}^{k-1} \arrow[r] \arrow[d] & \pi_*^S \mathbb{R}P_{-1}^{k} \arrow[r] \arrow[d] & \pi_*^S \mathbb{R}P_{-1}^{k+1} \arrow[r] \arrow[d] & \cdots \\
 & \pi_*^S \cancel{\Sigma} S^{k-1} \arrow[ul] & \pi_*^S \cancel{\Sigma} S^{k} \arrow[ul] & \pi_*^S \cancel{\Sigma} S^{k+1} \arrow[ul] &
\end{tikzcd}$$

Warning: it isn't totally clear to what this SS converges! However, the restriction principle applies, so any finite piece will converge to $\pi_*^S \mathbb{R}P_{-1}^{N}$.

As in the EHP sequence, the differentials record how far back you can pull a class. So our stable relative attaching maps tell us about non-zero differentials on this spectral sequence. The columns at E_1 are π_*^S. A class is recorded in the usual way, that



is, you pull it back as far as possible, and then push it out to the top dimensional cell; this is exactly the picture from our theorem.

$$\begin{tikzcd}
 & & S^{n-2} \arrow[lld] \arrow[d] \\
\mathbb{R}P^{n-\rho(n)-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{R}P^{n-2} \\
S^{n-\rho(n)-1} & &
\end{tikzcd}$$

The Folk Theorem

Last time we used a "folk theorem" due to Toda and Adams to the effect that the $\overset{\text{virtual}}{\wedge}$ bundle over $S^{\rho(n)}$ classified by $g_{\nu(n)}$

$$\begin{tikzcd} S^0 \arrow[r] & E(g_{\nu(n)}) \arrow[d] & \\ & S^{\rho(n)} \arrow[r, "g_{\nu(n)}"] & BO \end{tikzcd}$$

has as its Thom space the space $C(j_{\nu(n)})$ from the cofibration sequence

$$S^{\rho(n)-1} \xrightarrow{j_{\nu(n)}} S^0 \longrightarrow C(j_{\nu(n)})$$

where $j_{\nu(n)}$ is the image of $g_{\nu(n)}$ under the stable version of J.

This is a good time to remember how J is defined. The action of $O(n)$ on $\mathbb{R}^n$ restricts to a map $O(n) \times S^{n-1} \xrightarrow{\text{action}} S^{n-1}$. If α represents a class in $\pi_k(O(n))$, we get

$$S^k \times S^{n-1} \xrightarrow{\alpha \times id} O(n) \times S^{n-1} \xrightarrow{\text{action}} S^{n-1}.$$

Applying the Hopf construction yields a map

$$S^{n+k} = S^k * S^{n-1} \xrightarrow{J\alpha} S^n = \Sigma S^{n-1}$$

which represents a class in $\pi_{n+k}(S^n)$.

In proving the theorem it pays to set up the geometry very precisely. For this purpose, define

$$CX = [0,1]\times X \big/_{0,x \sim 0,x'} \hookleftarrow X$$
$$(1,x) \longmapsfrom x$$

$$\Sigma X = C(X)/X .$$

Well study the problem in somewhat astonishing generality; our data will be a map $A\times X \xrightarrow{\varphi} X$ which we'll think of as a group action, although it doesn't have to be one. φ has an obvious extension $\hat{\varphi}: A\times CX \longrightarrow CX$.
$(a,t,x) \longmapsto (t,ax)$

With this data we'll try to form a bundle, the first of two important constructions for today:

① Associated "bundle" construction

This will be a "bundle" $$\begin{tikzcd} E(\varphi) \arrow[d, "p"] \\ \Sigma A \end{tikzcd}$$ formed using φ as the clutching map.

$$\begin{tikzcd} E(\varphi) \arrow[d, "r"'] & = CA\times X \amalg X \big/_{(1,a,x)\sim ax} \\ \Sigma A \end{tikzcd}$$

The fiber is $p^{-1}(0,a) \cong X$. Note that when φ is a group action this construction in fact does determine an isomorphism between the fibers over the clutched coordinates, so this is genuinely a fiber bundle.

Also we can apply this construction to $\hat{\varphi}$, getting $E(\hat{\varphi})$ the "fiberwise cone on $E(\varphi)$"; we'll call it $\hat{E}(\varphi)$.

Then we get

$$\begin{tikzcd}
E(\varphi) \arrow[r, hook] & \hat{E}(\varphi) \arrow[r] & T(\varphi) \\
X \arrow[u] \arrow[r, hook] & CX \arrow[u] \arrow[r] & CX/X = \Sigma X \arrow[u]
\end{tikzcd}$$

In our case, we had a class $g_{\rho(n)} \in \pi_{\rho(n)} BO$. Remember though that we think of $g_{\rho(n)}$ as a class in $\pi_{\rho(n)-1} O$. It can be thought of as a map $S^{\rho(n)-1} \xrightarrow{g_{\rho(n)}} O(N)$. Now $O(N)$ acts on S^{N-1}, so we can use this to clutch a bundle $E(g_{\rho(n)}) \downarrow S^{\rho(n)}$ by

$$\begin{tikzcd}
S^{\rho(n)-1} \times S^{N-1} \arrow[r, "g_{\rho(n)} \times 1"] \arrow[rr, bend right, "\varphi"'] & O(N) \times S^{N-1} \arrow[r, "\text{action}"] & S^{N-1}
\end{tikzcd}$$

whose fiber is S^{N-1}. This is the sphere bundle of the $\mathbb{R}^N$-bundle classified by $S^{\rho(n)} \xrightarrow{g_{\rho(n)}} BO(N)$, so the Thom space we were studying last time arises from the constructions above applied to this composite, i.e. taking $A = S^{\rho(n)-1}$ and $X = S^{N-1}$.

This looks very good: it looks a lot like the definition of J. In order to state things correctly, we need a careful definition of the Hopf construction that matches our definitions of CX and ΣX:

② The Hopf Construction.

on φ is the quotient

$$\begin{tikzcd}
A \times CX \arrow[r, "\hat{\varphi}"] \arrow[d] & CX \arrow[d] \\
\dfrac{A \times CX}{(a,1,x) \sim (a',1,x)} = A * X \arrow[r, "J\varphi"] & \Sigma X
\end{tikzcd}$$

And the theorem is

<u>Theorem</u>

$$\begin{tikzcd} A * X \arrow[r, "J\varphi"] & \Sigma X \arrow[r] \arrow[d] & C(J\varphi) \arrow[dl] \\ & T(\varphi) & \end{tikzcd}$$

∃ this map, and it's a <u>homeomorphism</u> rel. to ΣX.

Note first of all that this is what we want: φ for us is the composite

$$\begin{tikzcd} S^{\rho(n)-1} \times S^{N-1} \arrow[r, "g_{\nu(n)} \times 1"] \arrow[rr, bend right, "\varphi"'] & O(N) \times S^{N-1} \arrow[r, "\text{action}"] & S^{N-1} \end{tikzcd}$$

and we are showing that $T(\varphi)$ ($= T(g_{\nu(n)})$ from last time) $= C(J\varphi)$ ($= C(j_{\nu(n)})$ from last time) rel. $\Sigma S^{N-1} = S^N$. In fact we were interested last time in the relation between $T(g_{\nu(n)})$ and $C(j_{\nu(n)})$ stably, and we can make the fiber S^N as connected as we want. So we get a stable equivalence

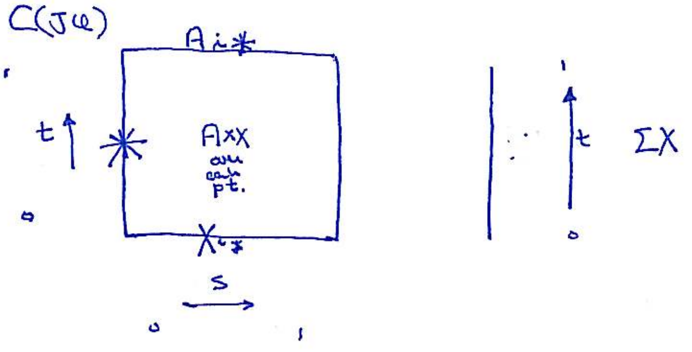
$T(g_{\nu(n)}) \simeq C(j_{\nu(n)})$.

<u>Proof</u>

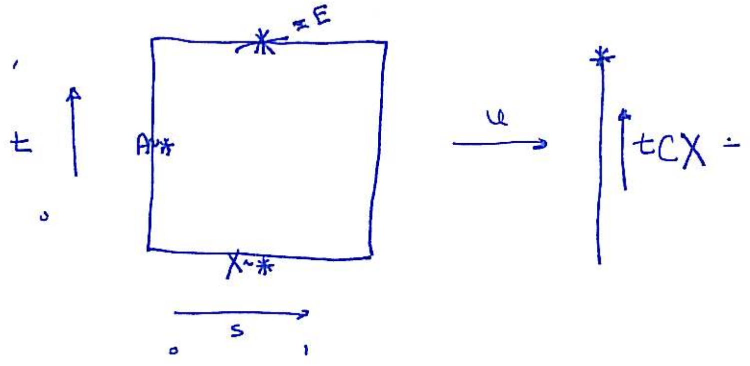
The hard part was parameterizing things right; now it's just a matter of drawing pictures. In these pictures, we draw A as $\circ$ and X as $\circ$.

Then we can keep track of the suspension, join, and cone coordinates.

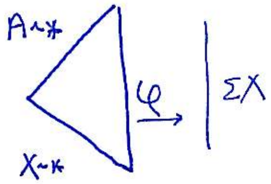
$$C(J\varphi) = \Sigma X \cup_{J\varphi} \overset{s\ a\ t\ x}{C(A * X)}$$



$$T(\varphi) = \hat{E}/E = \frac{\overset{s\ \ a}{CA} \times \overset{t\ \ x}{CX} \amalg CX}{\begin{matrix}(1,a,t,x) \sim (t,ax) \\ (s,a,1,x) \sim (s',a',1,x')\end{matrix}}$$



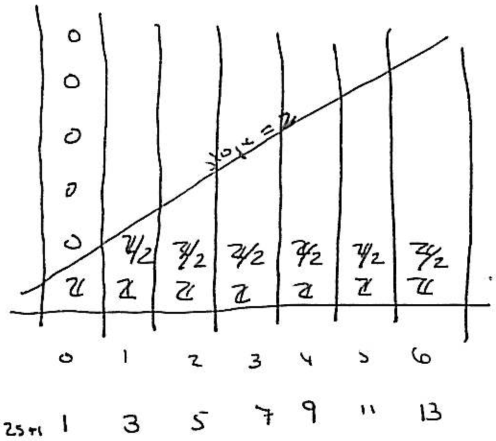
So both pictures really are the same; they both look like the triangle



And homeomorphisms with this triangle for the two spaces above are given by

$$f(s,a,t,x) = \left(\frac{s-st}{1-st}, a, st, x\right), \quad g(s,a,t,x) = \left(s+t-st, a, \frac{t}{s+t-st}, x\right)$$

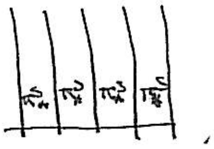
Remember how we got here: we were studying the EHP spectral sequence. This spectral sequence has as its columns the homotopy groups of odd spheres, and converged to $\pi_* QS^0 = \pi_*^S S^0$; moreover we had arranged it so that beneath a line of slope 2, the entries in each column were the stable homotopy of spheres:



This feature suggested the question: is there a spectral sequence whose columns are the stable homotopy of spheres, and a map of spectral sequences from the EHPSS to this SS, such that the map is an isomorphism below the celebrated line of slope 2?

On Friday we constructed a candidate, our Atiyah-Hirzebruch Spectral sequence for the attaching maps on $\mathbb{R}P^\infty$: $\tilde{H}^*(\mathbb{R}P^\infty_+; \pi_*^S) \Rightarrow$ (one hopes) $\pi_*^S(\mathbb{R}P^\infty_+)$.

whose E^1-term is



Sure enough, there is a SS-map between these two spectral sequences which is an isomorphism below the line at E^1. To see this, remember the EHP sequence: $\Omega^{n-1}S^{n-1} \longrightarrow \Omega^n S^n \longrightarrow \Omega^n S^{2n-1}$. Here we looped it n-1-times as this is the form in which it went into the EHPSS. This is a fibration, strictly if n is even or localized at 2 if n is odd. Linking these together and applying π_* gave us the EHPSS. On the other hand, the AHSS from Friday came from taking the cofibration sequences $\mathbb{RP}^{n-2}_+ \to \mathbb{RP}^{n-1}_+ \to S^{n-1}_+$ and applying $\pi_*^S = \pi_* Q$. (Recall $QX = \bigcup_k \Omega^k \Sigma^k X$). The sequence $Q\mathbb{RP}^{n-2}_+ \to Q\mathbb{RP}^{n-1}_+ \to QS^{n-1}_+$ is a fibration, since π_*^S is a homology theory and hence exact on $\mathbb{RP}^{n-2}_+ \to \mathbb{RP}^{n-1}_+ \to S^{n-1}_+$. Next, note that we have $S^{n-1}_+ \xrightarrow{e^n} \Omega^n S^{2n-1}_+ \xrightarrow{``e^{\infty-n}"} Q(S^{n-1}_+)$, and $e^{\infty-n}$ is an isomorphism on π_* for $* < n$.

Theorem There are maps S_n ("Snaith Maps" or (oh, dear) "Hopf-James" maps)

$$\begin{tikzcd}
\Omega^{n-1}S^{n-1} \arrow[r] \arrow[d, "S_{n-1}"] & \Omega^n S^n \arrow[r] \arrow[d, "S_n"] & \Omega^n S^{2n-1} \arrow[d, "``e^{\infty-n}"\ "] \\
Q(\mathbb{RP}^{n-2}_+) \arrow[r] & Q(\mathbb{RP}^{n-1}_+) \arrow[r] & Q(S^{n-1}_+)
\end{tikzcd}$$

So this theorem does it. This is a wonderful theorem; we'll try to prove it. It was probably first proved by Nick Kuhn, although the maps were constructed by Snaith.

Before we go on though, we should note two corollaries which answer an old question we've been trying to answer for a long time now.

<u>Corollary</u> In the portion of these sequences

$$\begin{tikzcd}
\pi_{n-1}\Omega^{n}S^{2n-1} \arrow[r, "P"] \arrow[d, "e^{\infty-n}"'] & \pi_{n-2}\Omega^{n-1}S^{n-1} \arrow[d, "S_{n-1}"] \\
\pi^{S}_{n-1}(S^{n-1}) \arrow[r] & \pi^{S}_{n-2}(\mathbb{R}P^{n-2}_{+})
\end{tikzcd}$$

we have

$$\begin{tikzcd}
2 \arrow[r, mapsto, "P"] \arrow[d, mapsto] & W_{n-1} \arrow[d, mapsto, "S_{n-1}"] \\
2 \arrow[r, mapsto] & \pi
\end{tikzcd}$$

$= $ stable homotopy class of attaching map for the top cell on $\mathbb{R}P^{n-1}$

<u>Prf.</u> The top row we've known for a while; the left leg is obvious. The bottom row is almost as obvious; we have the Barratt-Puppe sequence

$$S^{n-2} \xrightarrow{\pi} \mathbb{R}P^{n-2} \longrightarrow \overset{C(\pi)}{\overset{\shortparallel}{\mathbb{R}P^{n-1}}} \longrightarrow S^{n-1} \xrightarrow{\Sigma\pi} \mathbb{R}P^{n-2}$$

in which the middle two maps are the cofibration on which the snaith maps above were defined.

Well, we found out on Friday what happens to π in the AHSS: the differentials on the various π hit elements in the image of the J-homomorphism.

<u>Thm.</u> So W_{n-1} desuspends to an element in $\pi_{2n-p(n)-2}(S^{n-p(n)})$ and no further.

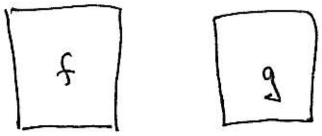
Today will examine where the Snaith maps come from, but you're going to have to believe some things. Some References for this are

J.P. May, Geometry of Iterated loop Spaces
F.R.Cohen, Unstable Decomposition of $\Omega^2\Sigma^2 X$ (Math Zeit. 182 (1983))
N. Kuhn, The Geometry of James-Hopf Maps (Pacific J. (1982))

Remember, we're trying to construct maps

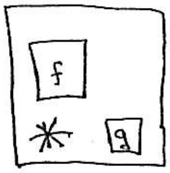
$$\Omega^n S^n \xrightarrow{S_n} Q\mathbb{R}P^{n-1}.$$

And since constructing maps out of loop spaces is hard, we'd like a tractable model for $\Omega^n S^n$. Fortunately, there are some hints as to how to proceed. First, we do have the map $\Omega^2 X \times \Omega^2 X \longrightarrow \Omega^2 X$ giving the H-space structure; if we represent $f \in \Omega^2 X$ and $g \in \Omega^2 X$ by boxes



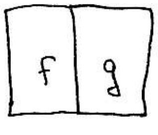
where the edges go to the basepoint,

then their product could be represented by



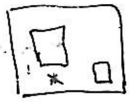
,

for example. Of course, the usual representative is



, but you have to fiddle with this anyway, for example to show the product is associative or commutative up to homotopy.

So we'll study spaces of rectangles. For example,



is a point on $C_2(2)$; in general, the space $C_k(n)$ will

$\uparrow$ in I^2
2 rectangles

be

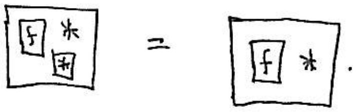
$$C_k(n) = \{\text{space of } k \text{ disjoint parallel-}n\text{-rectangles in } I^n\},$$

so what you're really describing is the space of embeddings $\coprod_k I^n \hookrightarrow I^n$. This is called the space of "little cubes." The point is, this space parameterizes the multiplication on $\Omega^n X$; namely for each k we have a map $C_k(n) \times (\Omega^n X)^k \longrightarrow \Omega^n X$; these piece together to give

$$\coprod_{k\geq 1} C_k(n) \times (\Omega^n X)^k \longrightarrow \Omega^n X.$$

To model the multiplicative structure you have to make some identifications:

1) The constant loop obliterates a cube; e.g.



2) The symmetric group Σ_k acts diagonally on $C_k(n) \times (\Omega^n X)^k$, and the map is equivariant w.r.t. this action.

Now if $X = \Sigma^n A$ then we have the map $A \xrightarrow{\alpha} \Omega^n \Sigma^n A$, and get

$$\begin{tikzcd}
\coprod_{k\geq 1} C_k(n) \times_{\Sigma_k} (\Omega^n X)^k/\sim \arrow[r] & \Omega^n \Sigma^n A \\
\coprod_{k\geq 1} C_k(n) \times_{\Sigma_k} A^k/\sim \arrow[u] &
\end{tikzcd}$$

meaning the identification 1) above, which we won't mark down from now on.

<u>Important Theorem</u> (May) If A is path-connected, this composite is a weak equivalence.

This is the basic theorem in the theory of iterated loop spaces. Note that it bears some resemblance to James' theorem.

Now you can replace each cube with its center; obviously the <u>size</u> of the cube doesn't affect homotopy properties. So we get an Σ_k-equivariant equivalence

$$C_k(n) \xrightarrow[\Sigma_k]{\simeq} F_k(\mathbb{R}^n)$$

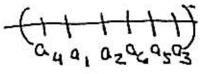
Where $F_k(W) \overset{\text{def}}{=}$ space of ordered k, distinct, points in W. So we have

$$\coprod_{k\geq 1} F_k(\mathbb{R}^n)\times_{\Sigma_k} A^k/\sim \;\xleftarrow{\simeq}\; \coprod_{k\geq 1} C_k(n)\times_{\Sigma_k} A^k/\sim \;\longrightarrow\; \Omega^n\Sigma^n A.$$

What's going on? We're choosing a bunch of points, k of them, and not ordering them, but labelling them with a point of A; sort of a charge; If the charge $= 0$, the basepoint, we're ignoring it. So call this space $C(\mathbb{R}^n; A)$ (this is Fred Cohen's notation).

Well, this is a pretty simple model, and we ought to be able to understand it. For one thing, let's relate it to the James construction: so let $n=1$.

$$\begin{array}{c} F_k(\mathbb{R}^1) \\ \| \; {\scriptstyle \Sigma_k\text{-equivariantly}} \\ \{t_1<\cdots<t_k\}\times\Sigma_k \end{array}$$

In this case the god-given ordering on $\mathbb{R}^1$ tells us the <u>unique</u> permutation to bring a collection of points  into standard order.

So

$$\begin{tikzcd}
C(\mathbb{R},A) = \coprod_k \{t_1 < \cdots < t_k\} \times A^k/\sim \arrow[d, "\simeq"] \arrow[dr, "\simeq"'] & \\
 & \coprod A^k/\sim
\end{tikzcd}$$

$\coprod A^k/\sim$ = James Construction $J(A)$, the "Free monoid" on A.

So $C(\mathbb{R}^n, A)$ really is a generalization of the James construction.

In order to study $C(\mathbb{R}^n; A)$, look at the obvious filtration

$$F(W,A)_k = \coprod_{j \le k} F_j(W) \times_{\Sigma_j} A/\sim .$$

The associated quotient is

$$D_k(W,A) = F_k/F_{k-1} = F_k(W) \times_{\Sigma_k} A^k \Big/ \left\{ \begin{matrix} \text{whenever any entry is} \\ \text{labeled with basepoint} \\ \text{of } A \text{, you get basepoint of} \\ D_k(W,A) \end{matrix} \right\} F_k(W) \times_{\Sigma_k} \left\{ \begin{matrix} \text{Fat wedge} \\ \text{of } A \end{matrix} \right\}$$

$$= F_k(W) \times_{\Sigma_k} A^{(k)} \overset{\leftarrow \text{smash } k}{\Big/} F_k(W) \times_{\Sigma_k} * .$$

Now consider the case $A = S^q$: $(S^q)^{(k)} \underset{\Sigma_k}{=} (\mathbb{R}^k)^q_+$, the 1-pt. compactification.

So if ξ_k is the vector bundle

$$\begin{tikzcd}
\xi_k = F_k(W) \times_{\Sigma_k} \mathbb{R}^k \arrow[d] \\
B_k(W) = F_k(W)/\Sigma_k ,
\end{tikzcd}$$

then $F_k/F_{k-1} = D_k(W, S^q) = T(q\xi_k \downarrow B_k(W))$.

So you're filtering $C(W, A)$ by something whose successive quotients are Thom spaces. Take the example $W = \mathbb{R}^n$, $k=2$. Then

$$\begin{array}{ccccl} F_2(\mathbb{R}^n) & \xrightarrow[\Sigma_2]{\simeq} & \mathbb{R}^n \times (\overset{\mathbb{R}_{>0} \times S^{n-1}}{\overset{\parallel}{\mathbb{R}^n - \{0\}}}) & \leftarrow & \Sigma_2 \text{ acts by fixing the left,} \\ (x,y) & \longmapsto & \left(\frac{x+y}{2}, \frac{x-y}{2}\right) & & \text{change of sign on right; hence antipodally on } S^{n-1}. \end{array}$$

and $B_2(\mathbb{R}^n) \cong \mathbb{R}^{n+1} \times \mathbb{R}P^{n-1}$. Now $\xi_2 = 1 + L$ over $\mathbb{R}^{n+1} \times \mathbb{R}P^{n-1}$, so

$$T(\xi_2) = D_2(\mathbb{R}^n, S^q) = T(q(1+L) \downarrow \mathbb{R}P^{n-1}) = \Sigma^q \mathbb{R}P_q^{n+q-1}.$$

Well that's a good sign; we got a stunted projective space.

Here's another example: we claim that the inclusion $F_k(\mathbb{R}^{n-1}) \hookrightarrow F_k(\mathbb{R}^n)$ is null-homotopic. A basepoint in $F_k(\mathbb{R}^n)$ consists of a choice of k distinct points, and any k points in $\mathbb{R}^{n-1}$ can be smoothly moved to the fixed set of points in $\mathbb{R}^n$.

That means that $F_k(\mathbb{R}^\infty) = \bigcup_n F_k(\mathbb{R}^n)$ is a contractible space with a free Σ_k-action, so it's an E_{Σ_k}, and $F_k(\mathbb{R}^\infty) \downarrow B_k(\mathbb{R}^\infty)$ is a universal Σ_k-bundle, and $Q(A) = \bigcup_n \Omega^n \Sigma^n A = \bigcup_n \coprod_{k \geq 1} F_k(\mathbb{R}^n) \times_{\Sigma_k} A^k/\sim$

$$= \coprod_{k \geq 1} E_{\Sigma_k} \times_{\Sigma_k} A^k/\sim.$$

Let's see now how to use this to produce maps. We're going to do this in blinding generality; namely, the map s_k will be a map

$$s_k : C(W, A) \longrightarrow C(B_k(W), D_k(W, A))$$

So a point of $C(W,A)$ is a finite subset $S \subset W$ and an assignment $f : S \to A$. We have to take this to a finite subset $s_k S$ of points of $B_k(W)$. $B_k(W)$'s points are k-tuples in W, so we take for $s_k S$ the set $\{T \subseteq S \mid \#T = k\}$. In addition to $s_k S$ we need an assignment $s_k f$ of points in $s_k S$ to charges in $D_k(W, A)$. But a charge in $D_k(W,A)$ is an assignment of charges in A to a k-elt. subset of W. So for $T \in s_k S$, we define $s_k f(T) = f|_T$!

Now take the case $W = \mathbb{R}^n$ and A path-connected. Then

$$\begin{tikzcd}
C(W, A) \underset{weak}{\simeq} \Omega^n \Sigma^n A \arrow[r, "s_k"] & C(B_k(\mathbb{R}^n), D_k(\mathbb{R}^n, A)) \arrow[d, "\begin{smallmatrix}\text{(by embedding}\\ B_k(\mathbb{R}^n) \text{ in } \mathbb{R}^N)\end{smallmatrix}"] \\
& C(\mathbb{R}^N, D_k(\mathbb{R}^n, A)) \arrow[d, "\simeq \text{ weak}"] \\
& \Omega^N \Sigma^N D_k(\mathbb{R}^n; A).
\end{tikzcd}$$

Now when $A = S^1$, (you'd like to take S^0 but it's not connected) you get

$$S_2 : \Omega^n S^{n+1} \longrightarrow \Omega^N \Sigma^N (\Sigma \mathbb{R}P^n)$$

so, looping once,

$$\Omega^{n+1} S^{n+1} \xrightarrow{S} \Omega^{N+1} \Sigma^{N+1} \mathbb{R}P^n \longrightarrow Q\mathbb{R}P^n.$$

There's lots of work still to be done: you have to check the compatibility of the maps and so forth. But the model's so simple it's not hard to believe that it works. For more information, see N. Kuhn's paper.

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5/5

One of the advantages of a topics course is that you can change direction in midstream. So now let's study the Adams conjecture for a while, starting with a tool by Becker and Gottlieb which enables us to get a slicker proof than the original ones of Quillen and Friedlander. It's a construction that's of interest anyway: "transfer" The basic ~~idea~~ construction is so simple that it's hard to concentrate on long enough to appreciate how much information it contains.

① Pontrjagin - Thom Construction

If X is a locally compact Hausdorff space and $\mathcal{U} \subseteq X$ is an open subset, then you get a map $X_+ \longrightarrow X_+/(X_+ - \mathcal{U}) \cong \mathcal{U}_+$ from the one-point compactification of X to that of $\mathcal{U}$, just by collapsing out the complement of $\mathcal{U}$ in X^+. This is called the "Pontrjagin - Thom collapse," and it gives a contravariance you may not have noticed before.

The construction is natural with respect to proper maps: if $X \xrightarrow{f} X'$ is proper, $\mathcal{U}' \subseteq X$, and $\mathcal{U} = f'(\mathcal{U})$, then you get

$$\begin{tikzcd}
X_+ \arrow[r, "f"] \arrow[d, "\text{Pontrjagin-Thom}"'] & X'_+ \arrow[d, "\text{Pontrjagin-Thom.}"] \\
\mathcal{U}_+ \arrow[r, "f"] & \mathcal{U}'_+
\end{tikzcd}$$

② Gysin Map

Next we apply the Pontrjagin-Thom construction to the case of a smooth fibration of compact manifolds

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

We'd like to convert p to an open embedding. By the Whitney embedding theorem you can embed E in $\mathbb{R}^n$ for n sufficiently large, and an embedding $E \hookrightarrow \mathbb{R}^n$ induces an inclusion

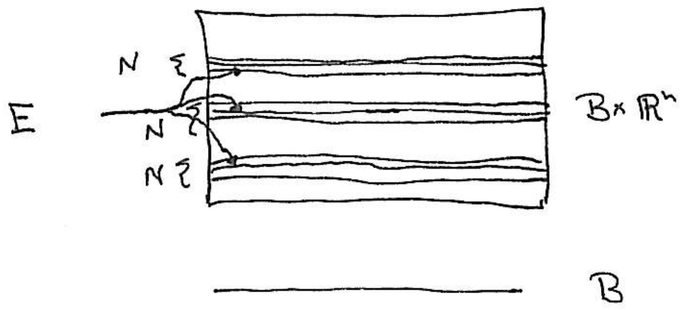
$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "p"'] & B \times \mathbb{R}^n \arrow[dl] \\ B & \end{tikzcd}$$

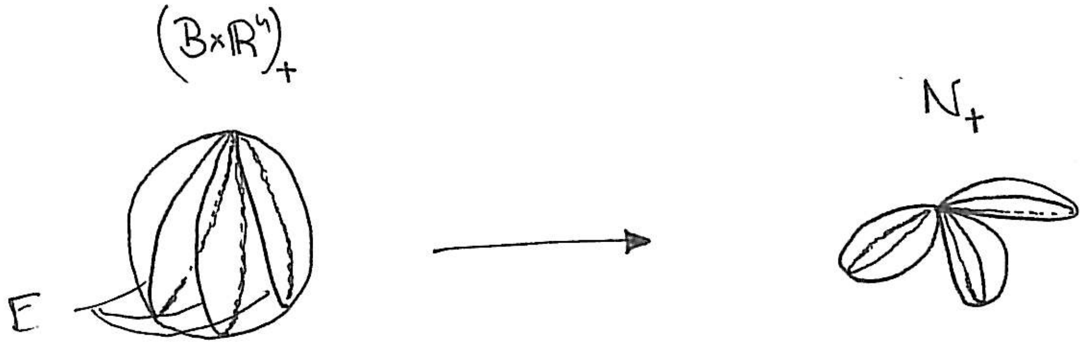
Well, i still isn't an open inclusion, so now consider $\nu(i)$ the normal bundle of the inclusion and get a tubular neighborhood N of E in $B \times \mathbb{R}^n$. Now the Pontrjagin-Thom collapse gives a map

$$\begin{tikzcd} (B \times \mathbb{R}^n)_+ \arrow[r] \arrow[d, equal] & N_+ \arrow[r, phantom, "="] \arrow[dr, equal] & T(\nu(i) \downarrow E) \arrow[d, equal] \\ T(n\varepsilon \downarrow B) \arrow[d, equal] & & \bar{N}/\partial(N) \\ \Sigma^n B_+ & & \end{tikzcd}$$

Lots of Thom spaces are going to appear for a while, so perhaps we should give in and follow Atiyah's convention of writing the bundle as an exponent: $T(\nu(i) \downarrow E) = E^{\nu(i)}$. So we have a map $B^{n\epsilon} \longrightarrow E^{\nu(i)}$

In some sense what you're doing is inverting p, constructing a sort of multivalued inverse. It's instructive to think about the case that $E \xrightarrow{p} B$ is a finite cover.





The next question is: what is $\nu(i)$? Well, $\nu(i) + \tau_E = i^*(\tau_{B\times\mathbb{R}^n})$
$= p^*\tau_B + n\epsilon_E$.

Now $\tau_E = p^*\tau_B + \tau(p)$, the "vertical vectors" or tangent vectors along the fiber. So $\nu(i) + \tau(p) = n\epsilon_E$; this is sort of a tangent bundle/normal bundle "<u>relative</u> to B." So $\nu(i) = n\epsilon - \tau(p)$, and so we get a stable map

$$B_+ \xrightarrow{p!} E^{-\tau(p)}$$

from here on we adopt the French notation "⇸" for stable maps.

the "Gysin Map" denoted $p!$, "p shriek".

③ Becker-Gottlieb Transfer

The inclusion $\nu(i) \hookrightarrow n\epsilon_E$ induces a map of Thom spaces $E^{\nu(i)} \longrightarrow E^{n\epsilon} = \Sigma^n E_+$ which, together with the Gysin map

$$\Sigma^n B_+ = B^{n\epsilon} \xrightarrow{p!} E^{\nu(i)} \longrightarrow E^{n\epsilon} = \Sigma^n E_+$$

is the Becker-Gottlieb transfer

$$t(p) : \quad B_+ \nrightarrow E_+$$

which has the virtue that it doesn't shift dimensions.

④ Euler Characteristic

The Euler characteristic in this context will be defined as $\chi(p) \in \pi^0_S(B)$ an element in the stable cohomotopy of B. Namely, take the transfer

$$B_+ \xrightarrow{t(p)} E_+$$

Now there's a ridiculous map $E_+ \longrightarrow S^0$ which takes the basepoint to the basepoint and pinches everything else, i.e. E, to the other point. On the level of complexes, this isn't much of a map, but the claim is that stably there's a lot going on. So the ~~Eu~~ Euler characteristic is defined as

$$\chi(p) : B_+ \xrightarrow{t(p)} E_+ \xrightarrow{\text{pinch}} S^0 \quad \in \pi_s^0(B) .$$

(i.e. unreduced stable cohomotopy)

Naturality of $\chi(p)$ follows from the naturality of $p_!$ with respect to pull-backs: if $E = f^*E'$ in

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, bend left, "i"] \arrow[dd, "p"'] & & E' \arrow[r, hook, "i'"] \arrow[dd, "p'"] & B' \times \mathbb{R}^n \arrow[ddl] \\
& B \times \mathbb{R}^n \arrow[urr] \arrow[dl] & & \\
B \arrow[rr, "f"] & & B' &
\end{tikzcd}$$

we get $f^*(\nu(i')) = \nu(i)$, and ~~if~~ if N' is a tubular neighborhood for E' in $B' \times \mathbb{R}^n$, we can use $f'(N')$ for one of E in $B \times \mathbb{R}^n$ (we may have to choose N' well, but because B and B' and the fiber are compact, there is no real difficulty)

Then we get the following naturality:

$$\begin{tikzcd}
B^{n\varepsilon} \arrow[r, "f"] \arrow[d, "p!"'] \arrow[dd, bend right, "t(p)"'] & B'^{n\varepsilon} \arrow[d, "p'!"] \arrow[dd, bend left, "t(p')"] \\
E^{\nu(i)} \arrow[r, "f"] \arrow[d] & E^{\nu(i')} \arrow[d] \\
E^{n\varepsilon} \arrow[r, "f"] & E'^{n\varepsilon}
\end{tikzcd}$$

The naturality of $\chi(p)$ follows by pinching out.

We'd like to understand several things; for one, what does the Euler characteristic $\chi(p)$ have to do with the Euler characteristic (in the usual sense)? Here's a start.

<u>Lemma</u> This diagram commutes.

$$\begin{tikzcd}
B_+ \arrow[r, "t(p)"] \arrow[d, "\Delta"'] & E_+ \arrow[r, "p_+"] & B_+ \\
(B\times B)_+ = B_+\wedge B_+ \arrow[rr, "\chi(p)\wedge 1"] & & S^0\wedge B_+ \arrow[u, "\cong"']
\end{tikzcd}$$

Before we prove the lemma, note this corollary. Any cohomology theory has an action of stable cohomotopy: if $\alpha \in \{X, h\}$ is a class in $h^*(X)$ and $\beta \in \pi^S_*(X) = \{X, S\}$ then we get

$\beta \cup \alpha : \Sigma X \longrightarrow \Sigma X \vee \Sigma X \xrightarrow{\alpha \vee \beta} \Sigma h \vee \Sigma S \longrightarrow h$.

Corollary If $x \in h^*(B)$ then

$$t(p)^* p^* x = \chi(p) \cup x.$$

Proof of the Lemma. Consider the diagram

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] \arrow[dd, "p"'] & & E \times B \arrow[dr] \arrow[dd] & \\
& B \times \mathbb{R}^n \arrow[rr, "p \times 1"] \arrow[dl] & & B \times \mathbb{R}^n \times B \arrow[dl] \\
B \arrow[rr, "\Delta"] & & B \times B &
\end{tikzcd}$$

Things on the left are the pull-backs of things on the right.

Then we get

$$\begin{tikzcd}
B^{n\varepsilon} \arrow[r, "\Delta"] \arrow[d] \arrow[dd, bend right=60, "t(p)"'] & B^{n\varepsilon} \wedge B^0 \arrow[d] \arrow[dd, bend left=60] \\
E^{\nu(1)} \arrow[r, "\Delta"] \arrow[d] & E^{\nu(1)} \wedge B^0 \arrow[d] \\
E^{n\varepsilon} \arrow[r, "\Delta"] \arrow[dr, "\Sigma^n p"'] & E^{n\varepsilon} \wedge B^0 \arrow[d, "\left(\substack{\text{pinch } E \text{ to} \\ \text{pt.}}\right) \wedge 1 \;=\; \chi(p) \wedge 1"] \\
& S^n \wedge B^0
\end{tikzcd}$$

(note $B^0 = B_+$!)

$t(p \times 1) = t(p) \wedge 1$. This is because $B \times \mathbb{R}^n \times B$ is suitable as an embedding space for $E \times B$ if $B \times \mathbb{R}^n$ is one for E !

Now focus on $\chi(p)$ as a cohomotopy class. We have

Lemma In $\begin{matrix} F \to E \\ \downarrow p \\ B \end{matrix}$ the Hurewicz map $\pi^0_S(B) \longrightarrow H^0(B)$ sends $\chi(p)$ to $\chi(F)$, the usual Euler characteristic of F.

Remarks

1) Of course it's easier to prove the lemma if you take the right definition of $\chi(F)$.

2) Notice that $\chi(p)$ is stable cohomotopy has a lot more information; when you project to cohomology you forget about p.

Proof Suppose B is connected; pick a point in B, $*$. Then

$$H^0(B) \xrightarrow[\cong]{j^*} H^0(pt), \text{ and}$$

$$\begin{tikzcd} F \arrow[r] \arrow[d, "p_F"'] & E \arrow[d, "p"] \\ * \arrow[r, "j"'] & B \end{tikzcd}$$

is a pull-back, so $j^*\chi(p) = \chi(p_F)$.

so we can assume $B = *$. The only space left is F.

Given an embedding $F \overset{i}{\hookrightarrow} S^n$, the Euler characteristic $\chi(p)$ is defined by

$$S^n \xrightarrow{p!} F^{\nu(i)} \longrightarrow F^{\nu(i) \oplus \tau(F) = n\epsilon} \xrightarrow[\text{pinch}]{} S^n$$

$\tau(p) = \tau(F)$!

and we must show this composite has degree $\chi(F)$.

Let $N \subset \mathbb{R}^n$ be a tubular neighborhood for the embedded image of F; then $S(\nu) \cong \partial N$, and ∂N is a codimension-1 submanifold with an outward-pointing normal direction. The map $F^\nu \longrightarrow S^n$ is the Thom-space level of a map

$$\begin{tikzcd}
\nu \arrow[r] \arrow[d] & \nu \oplus \tau \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[d] \\
F \arrow[r] & F \arrow[r, "\text{pinch}"] & *
\end{tikzcd}$$

which on the level of sphere-bundles is a map $\partial N \xrightarrow{\gamma} S^{n-1}$, the Gauss map! The degree of γ is a standard definition of the Euler characteristic $\chi(F)$; see Milnor, Topology from a Differentiable Viewpoint for further information.

But that's it:

$$\begin{tikzcd}
 & S(\nu) \arrow[r, "\gamma"] \arrow[d] & S^{n-1} \arrow[d] \\
 & D(\nu) \arrow[r, "\gamma"] \arrow[d] & D^n \arrow[d] \\
S^n \arrow[r, "\text{Pontrjagin-}", "\text{Thom collapse, degree 1 by construction}"'] \arrow[rr, bend right, "\text{degree } \chi(F)"'] & F^\nu \arrow[r] & S^n
\end{tikzcd}$$

It's worth thinking about this on the level of cohomology. The bundle maps

$$\begin{tikzcd}
\nu \arrow[r, "\Delta"] \arrow[d, "\zeta"'] & \{0\} \times \nu \arrow[d, "\zeta \times 1"] \\
n \arrow[r, "\Delta"] & \tau \times \nu
\end{tikzcd}$$

induce on the level of Thom spaces

$$\begin{tikzcd}
F^{\nu} \arrow[r] \arrow[d, "\zeta"'] & F^{0} \wedge F^{\nu} \arrow[d, "\zeta \wedge 1"] \\
F^{n\varepsilon} \arrow[r] & F^{\tau} \wedge F^{\nu}
\end{tikzcd}$$

If $u_{\xi} \in \bar{H}^n(X^{\xi})$ is the Thom class of ξ, define the "Euler class" e_{ξ} as the image under pull-back by ζ:

$$\begin{tikzcd}[row sep=small]
\bar{H}^n(X^{\xi}) \arrow[r, "\zeta^*"] & \bar{H}^n(X^0) \cong \bar{H}^n(X) . \\
u_{\xi} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & e_{\xi}
\end{tikzcd}$$

Then in the above square, the Thom classes go:

$$\begin{tikzcd}
e_{\tau} \cup U_{\nu} & e_{\tau} \wedge U_{\nu} \arrow[l] \\
U_{n\varepsilon} \arrow[u, mapsto, "\zeta"] & U_{\tau} \wedge U_{\nu} \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

Let $\sigma \in \bar{H}^n(S^n)$ be a generator; think of it as the Thom class of $\mathbb{R}^n \downarrow *$. The diagram

$$\begin{tikzcd}
\nu \arrow[r] \arrow[d] & \nu \oplus \tau \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[d] \\
F \arrow[r] & F \arrow[r, "\text{pinch}"] & *
\end{tikzcd}$$

gives maps of Thom spaces under which σ pulls back as:

$$S^n \xrightarrow{c} F^{\nu} \longrightarrow F^{\nu\epsilon} \longrightarrow S^n$$

$$\chi(F)\cdot\sigma \longleftarrow\!\shortmid\; e_\tau \cup U_\nu \longleftarrow\!\shortmid\; U_{\nu\epsilon} \longleftarrow\!\shortmid\; \sigma .$$

Well, now we can use classical theorems about the Euler characteristic to study our new Euler characteristic; for example, we can use Hopf's theorem to compute it in a special case.

Let M^n be a compact Riemannian manifold and let V be a non-degenerate vector field (i.e., $V(M)$ intersects the zero-section $S(M)$ of TM transversally). Then around each zero x of V there is a small sphere $S_\varepsilon(x)$ so that $V\big|_{S_\varepsilon(x)} \neq 0$.

Define $i_x = \text{degree of } \frac{V}{\|V\|}\Big|_{S_\varepsilon(x)} : S_\varepsilon^{n-1} \to S^{n-1} = \pm 1.$

Theorem (Hopf) $\chi(M) = \sum\limits_{\substack{x \in M \\ V(x)=0}} i_x$

(For proof, see Milnor)

For example, suppose G is a compact Lie Group; let $T < G$ be a maximal torus; let $N(T)$ be its normalizer. Then the identity component of $N(T)$ is T itself, and $N(T)/T$ is a finite discrete group, the "Weyl group." The important fact is:

Claim $\chi(G/T) = |W|$
$\chi(G/N(T)) = 1$ if G is compact and connected.

To prove this, we'll come up with a vector field and use Hopf's theorem. There is an action of G on G/T which restricts to an action of T. In $T_eT = L(T)$ the Lie algebra of T there is a vector X such that $\exp X = g$, and $\exp tX$ is a path in T from e to g; let γ_t be the induced flow on G/T; then $\frac{d}{dt}\gamma_t\Big|_{t=0}$ is a vector field on G/T. Now suppose $g \in T$ is such that $\{g^n \mid n \in \mathbb{Z}\}$ is dense in T (i.e. g is irrational on each component of the torus; T is called "topologically cyclic"); let v be the vector field on G/T corresponding to this element. A zero of v is a fixed point of the action of g and therefore (by continuity) a fixed point for the action of T. Such a point on G/T is a coset hT so that

$$thT = hT \;\; \forall t \in T \iff hth^{-1}T = T \;\; \forall t \iff h \in N(T).$$

So zeroes correspond to elements of $N(T)/T = W$. I claim all the zeroes are non-degenerate ~~the~~ and have the same index, so $\chi(G/T) = \#|W|$. Now the action of ~~G~~ g descends to $G/N(T)$ with only one fixed point, so $\chi(G/N(T)) = 1$.

The Transfer in K-theory

Suppose $\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}$ is a finite covering. Then the transfer is a map $B^{ne} \xrightarrow{t(p)} E^{ne}$ which induces a map in KO-theory

$$KO(E) \longrightarrow KO(B).$$

There's an obvious thing to do here, but there's no obvious connection with the map $t(p)$: if $\xi \downarrow E$ is a vector bundle over E, we can form a vector bundle over B by taking as fibre over b the sum of the vector spaces over points in E which cover b:

$$b \in B \rightsquigarrow \bigoplus_{p(x)=b} \xi_x = (p_* \xi)_b.$$

This is more than you might hope for; it says there's an underlying map $\mathrm{Vect}(E) \to \mathrm{Vect}(B)$.

But in fact, the two constructions are the same:

Nontrivial Lemma

$$\begin{tikzcd} KO(E) \arrow[r, "t(p)^*"] & KO(B) \\ Vect(E) \arrow[u, phantom, sloped, "\subset"] \arrow[r, "p_*"] & Vect(B) \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

.

Corollary Let $E \xrightarrow{p} B$ is a finite covering and $\xi \in KO(B)$; then if ξ is stably fiber-homotopy trivial, so is $t(p)^* \xi$.

Remark

Note that this isn't a trivial fact: transfer is a cohomological construction, but stable fiber-homotopy triviality isn't. But it follows from the lemma: a map $S(\xi) \longrightarrow S^{n-1}$ which is degree 1 on each fiber gives a map $S(p_* \xi) \to S^{n \cdot \text{degree } p}$ which is degree one on each fiber.

Remark

The corollary can also be proved by noting the factorization

$$\begin{tikzcd} \widetilde{KO}(X) \arrow[rr] \arrow[dr, two heads] & & \widetilde{Sph}(X) \\ & \widetilde{J}(X) \arrow[ur, tail] & \end{tikzcd}$$

and that $\widehat{Sph} = \widehat{Sph}_0$ is the 0th level of a cohomology theory, using the ∞-loop-space theory of Boardman and Vogt.

So the transfer maps induce maps on this theory, and you can produce the result by a naturality argument.

The second fact about transfer in K-theory is the relation to Adams operations. In fact, ~~the~~ the result really concerns the interaction of Adams operations with arbitrary stable maps.

<u>Lemma</u> Suppose $X_+ \xrightarrow{f} Y_+$; then $f^* \psi^k x - \psi^k f^* x$ has order dividing a power of k (and independent of x). So the ψ^k <u>aren't</u> stable operations.

<u>Proof</u> To study the problem, you have to distinguish between the stable map f and an actual representative. So denote the induced homomorphism by $f^{\circledast} : KO(Y_+) \to KO(X_+)$.

Suppose n is taken large enough that there is an actual map $f : X_+ \wedge S^{8n} \to Y_+ \wedge S^{8n}$. Then

$$\begin{tikzcd}
\underline{x} \arrow[d, mapsto] & \widetilde{KO}(X_+) \arrow[d, "\cong"] & \widetilde{KO}(Y_+) \arrow[l, "f^{\circledast}"'] \arrow[d, "\otimes \gamma^n"] \\
x \otimes \gamma^n & \widetilde{KO}(X_+ \wedge S^{8n}) & \widetilde{KO}(Y_+ \wedge S^{8n}) \arrow[l, "f^*"]
\end{tikzcd}$$

(γ generator of $\widetilde{KO}(S^8) \cong \mathbb{Z}$)

Chasing this, you find

$$\begin{array}{ccc} \psi^k\left[f^{\circledast}(x)\otimes\gamma^n\right] & = & \psi^k\left[f^*(x\otimes\gamma^n)\right] \\ \parallel & & \parallel \\ k^{4n}\,\psi^k f^{\circledast}(x)\otimes\gamma^n & & f^*\left[\psi^k(x\otimes\gamma^n)\right] = f^*\left[(\psi^k x)\otimes k^{4n}\gamma^n\right] \end{array}$$

$$= k^{4n} f^*\left[(\psi^k x)\otimes\gamma^n\right]$$

$$= k^{4n}\left[(f^{\circledast}\psi^k x)\otimes\gamma^n\right]$$

so $k^{4n}\left[\psi^k f^{\circledast} - f^{\circledast}\psi^k\right] = 0$; n depended only on f.

Well, nothing in topology ever proceeds in a straightforward way: usually you end up having to talk about some "relative" version. This time, we'll need a sort of "relative" transfer.

So, as before, let $\begin{matrix} F \to E \\ \ \ \downarrow p \\ \ \ B \end{matrix}$ be a smooth fiber bundle with F and B compact; let $\xi \downarrow B$ be a vector bundle, and consider

$$\begin{tikzcd} p^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

As before, consider an embedding $E \hookrightarrow \mathbb{R}^n$ and get

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, "p"'] & B \times \mathbb{R}^n \arrow[r, hook, "\xi"] \arrow[d, "pr"] & n\epsilon \oplus \xi \\ & B & \end{tikzcd}$$

Now you embed one step farther: $B \times \mathbb{R}^n \overset{\xi}{\hookrightarrow} n\epsilon \oplus \xi$. And we can do the collapse on the inclusion given by a tubular neighborhood for $p^*\xi$ in $n\epsilon \oplus \xi$:

$$t(p) : \Sigma^n B^\xi \underset{\substack{\text{"relative} \\ \text{Gysin map"}}}{\xrightarrow{p\,!}} E^{\nu(i) \oplus p^*\xi} \hookrightarrow E^{n\epsilon \oplus p^*\xi} = \Sigma^n E^{p^*\xi}.$$

So the relative transfer is the stable map $t(p) : B^\xi \to E^{p^*\xi}$.

<u>Lemma</u> Suppose $\chi(F) = \pm 1$. Then if $J(p^*\xi) = 0$ in $J(E)$, then $J(\xi) = 0$ in $J(B)$, so p is a monomorphism in J-theory.

Proof $J(p^*\xi)=0$ means that

$$\begin{tikzcd} S^d \arrow[r] \arrow[rr, bend right, "\text{degree } 1"'] & E^{p^*\xi} \arrow[r, "\exists", "\shortmid" marking] & S^d \end{tikzcd} \qquad (d=\dim\xi)$$

Then we have

$$\begin{tikzcd} S^d \arrow[r] & E^{p^*\xi} \arrow[r, "\shortmid" marking] & S^d \\ S^d \arrow[r] \arrow[urr, bend right=60] & B^\xi \arrow[u, "t(p)"] & \end{tikzcd}$$

and we'd like to show the degree f this is ±one.

Well, choose a basepoint in B; then you have

$$\begin{tikzcd} & d\epsilon \arrow[ld] \arrow[rr] \arrow[dd] & & p^*\xi \arrow[ld] \arrow[dd] \\ F \arrow[rr] \arrow[dd] & & E \arrow[dd] & \\ & R^d \arrow[ld] \arrow[rr] & & \xi \arrow[ld] \\ * \arrow[rr, hook] & & B & \end{tikzcd}$$

and from this you get

$$\begin{tikzcd} & & S^d \\ F^{d\epsilon} \arrow[r] & E^{p^*\xi} \arrow[ur, "\shortmid" marking] & \\ S^d \arrow[r] \arrow[u, "t", "\shortmid" marking] & B^\xi \arrow[u, "t"', "\shortmid" marking] & \end{tikzcd}$$

But $S^d \xrightarrow{+} F^{d\epsilon}$ isn't just the inclusion f the bottom cell; in fact we found that it was $\chi(F)\cdot$ inclusion f bottom cell! So if $\chi(F)=\pm 1$,

$S^d \longrightarrow B^\xi \xrightarrow{t(p)} E^{p^*\xi} \xrightarrow{+} S^d$ has degree ±one.

Now I'm ready to tell you what the Adams Conjecture says, although not what it means.

Adams Conjecture

Let B be a finite complex, and let $\xi \in KO(B)$, and $k \geq 1$ an integer. Then some power of k kills $J(\psi^k \xi - \xi)$ in $J(B)$.

Admittedly, this is an obscure statement, but take it from me that it's important and worth proving. We'll prove it bit by bit.

① ξ is a linear combination of line bundles.

If k is odd then $\psi^k \xi = \xi$ so $\psi^k \xi - \xi = 0$.

If k is even, and ξ is a line bundle (by additivity it's sufficient to consider this case), then $\psi^k \xi - \xi = \underline{1} - \xi \in \widetilde{KO}(B)$. ξ is classified by a map $B \xrightarrow{f} \mathbb{RP}^N$ for large enough N:

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & L \arrow[d] \\
B \arrow[r, "f"] & RP^N
\end{tikzcd}$$

so $\underline{1} - \xi = f^*(\underline{1} - L)$, and $\widetilde{KO}(\mathbb{RP}^N)$ is a finite 2-group generated by $(\underline{1} - L)$.

② The next case is that ξ is a 2-dimensional bundle.

Let P be the associated principal bundle so that $\xi = P \times_{O(2)} \mathbb{R}^2$. We described the Adams operations ψ^k in terms of representations, so now we have to think a little about the representation theory of $O(2)$: what is $O(2)$? Well, there are two parts:

$$\begin{tikzcd}
SO(2) \arrow[r, hook] \arrow[d, equal] & O(2) \arrow[r, shift left] & \mathbb{Z}/2 \arrow[l, shift left, "r"] \\
S^1 & &
\end{tikzcd}$$

↖ splitting comes from choosing any reflection, say r is reflection through x-axis.

So what are the representations of $O(2)$? Let $V = \mathbb{R}^2$ be the basic representation; then we can take exterior powers:

$\lambda^0(V) = 1$ trivial 1-dim. representation

$\lambda^1(V) = V$

$\lambda^2(V) = \det$ 1-dimensional "determinant" representation

More interestingly, we have representations μ_k: S^1 acts on $\mathbb{C}^{\otimes_{\mathbb{C}} k}$ by, for $z \in S^1$, $z(w_1 \otimes \cdots \otimes w_k) = zw_1 \otimes \cdots \otimes zw_k = z^k(w_1 \otimes \cdots \otimes w_k)$, and we can extend to $O(2)$ by letting r act by complex conjugation. Note that $\mu_1 = V$.

Next we want to express the ψ^k's in terms of λ^j's and μ_ℓ's, so that we have some chance of being able to compute. In order to do that, we should write down the

Character Table for O(2).

Conjugacy classes in O(2):

Let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Conjugacy classes are

$\{R_\theta, -R_\theta\}, 0 < \theta < \pi$

$\{1\}$, $\{-1\}$, $\{R_\theta r : \text{reflections}\}$

$tr\,\psi^k(g) = tr(g^k)$ so

	R_θ	r
λ^0	1	1
$\lambda^1 = V$	$2\cos\theta$	0
λ^2	1	-1
μ_k	$2\cos k\theta$	0
ψ^k	$2\cos k\theta$	$\begin{Bmatrix} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{Bmatrix}$

μ_k is the natural candidate for ψ^k, but it's wrong: the formula for ψ^k is

$$\psi^k = \mu_k \quad \text{if } k \text{ is odd}$$
$$= \mu_k + (\lambda_0 - \lambda_2) \quad \text{if } k \text{ is even.}$$

Now we can use the μ_k's to check the Adams conjecture: $\psi^k V - V$ differs from $\mu_k V - V$ (if k is even) by a linear combination of line bundles with virtual dimension 0; these are killed by a power of 2 by the above argument for the line-bundle case; if k is odd, $\psi^k V - V = \mu_k V - V$. The claim is that for some e, $k^e(\mu_k V - V) = 0$ in J.

The trick -- which is the key point in understanding the Adams Conjecture -- is the map

$$V \xrightarrow{f_k} \mu_k$$

$$f(z) = z^k$$

which is equivariant w.r.t. the $O(2)$ action on either side, but certainly not linear; it won't induce a map of vector bundles, but it will induce one of sphere bundles, which is degree k on each fiber. The result then follows from

Adams' Mod k Dold Lemma (Adams, $J(X)$ I, Topology ~1965)

Let B be a finite complex and let $\xi \downarrow B$, $\xi' \downarrow B$ be vector bundles; suppose $f: S(\xi) \to S(\xi')$ is degree k on each fiber. Then $k^e[J(\xi) - J(\xi')] = 0$ for some e.

Well, this is as far as Adams got; he decided it was unreasonable of others to expect him to do more... Now you need a trick to be able to finish in a finite amount of time.

To do the general case we'll make a few reductions, namely to oriented $2n$-dimensional bundles. A vector bundle $\xi \downarrow B$ has a Stiefel-Whitney class $w_1(\xi) \in H^1(B; \mathbb{Z}/2)$ which is the obstruction to the orientability of ξ. Now elements of $H^1(B; \mathbb{Z}/2)$ are in 1-1 correspondence via w_1 with line bundles

Now perform the L and $\tilde{L}$ constructions fiberwise on the bundle $\xi \downarrow B$, and you get a fiber bundle

$$\begin{tikzcd}
 & L(\mathbb{R}^{2n}) \arrow[r] & L(\xi) \arrow[d, "p"] & p^*\xi \arrow[l] \\
\text{has } \chi = 1 \arrow[ur] & & B & \xi \arrow[l]
\end{tikzcd}$$

Moreover this fits into

$$\begin{tikzcd}
 & \tilde{L}(\xi) \arrow[d, "q"'] & \zeta \arrow[l] & \text{2-plane bundle} \arrow[l] \\
L(\mathbb{R}^{2n}) \arrow[r] & L(\xi) \arrow[d, "p"] & p^*\xi \arrow[l, "\cancel{\xi}"'] & \\
 & B & \xi \arrow[l] &
\end{tikzcd}$$

The claim is that this wonderful formula is geometrically clear:

$$\boxed{\begin{array}{ccc} q_* \zeta & = & p^*\xi \\ \| & & \\ t(q)^* \zeta & & \end{array}}$$

over B, so in particular there is a line bundle $\lambda \downarrow B$ so that $w_1(\xi \oplus \lambda) = 0$, i.e. $\xi \oplus \lambda$ is orientable. The additivity of the Adams conjecture ~~means~~ implies ~~then that we can check~~ (plus the fact that we've already proved it for line bundles) implies that we can assume ξ is an oriented, $2n$-dimensional vector bundle over B; moreover B is compact so we can give ξ a metric.

Now suppose V is a $2n$-dimensional inner-product space over $\mathbb{R}$; let $F(V)$ = the space of sequences of n mutually orthogonal 2-planes in V. Σ_n acts freely on $F(V)$; let $L(V) = F(V)/\Sigma_n$. Claim: we've constructed this space before; it's $SO(2n)/NT$. So $\chi(L(V)) = 1$.

Now let Σ_{n-1} act on $F(V)$ by fixing the last element; then there is an n-sheeted cover $\tilde{L}(V) = F(V)/\Sigma_{n-1} \xrightarrow{q} L(V)$. Moreover, there is a 2-plane bundle $\zeta \downarrow \tilde{L}(V)$ defined by

$$\begin{tikzcd} \zeta \arrow[d] & \zeta_{(\{v_1,\dots,v_{n-1}\}, v_n)} = v_n \\ \tilde{L}(V) & \ni \quad (\{v_1,\dots,v_{n-1}\}, v_n) \end{tikzcd}$$

Now from the lemma at the beginning of this lecture, it suffices to check the Adams conjecture on $p^*\xi$. But by the formula,

$$\begin{aligned} \psi^k p^*\xi - p^*\xi &= \psi^k t(q)^*\zeta - t(q)^*\zeta \\ &= \psi^k t(q)^*\zeta - t(q)^*\psi^k\zeta + t(q)^*\psi^k\zeta - t(q)^*\zeta \\ &= \underbrace{\left(\psi^k t(q)^* - t(q)^*\psi^k\right)}_{\text{killed by } k^e}\zeta + t(q)^*\left(\underbrace{\psi^k\zeta - \zeta}_{\substack{\text{2-plane bundle,} \\ \text{killed by } k^f \text{ for} \\ \text{some } f \text{ by proof of} \\ \text{Adams conjecture for} \\ \text{2-plane bundles.}}}\right) \end{aligned}$$

5/12
5/15
5/17

All right, today well start studying the meaning of the Adams conjecture; in particular I'm going to tell you about the space J.

We've talked about the KO spectrum, representing real K-theory:

$$KO^*(X) = \{X, KO\}.$$

It is a sequence of spaces $\{KO_n\}$ with maps $\Sigma KO_n \xrightarrow{\alpha} KO_{n+1}$, with

$$KO_{8n} = \mathbb{Z} \times BO$$

$$KO_{8n-i} = \Omega^i(\mathbb{Z} \times BO)$$ with the map

$$\Sigma^8 KO_{8n} \to KO_{8n+8}$$ given by

$$\begin{tikzcd}
S^8 \wedge (\mathbb{Z}\times BO) \arrow[r] \arrow[d] & \mathbb{Z}\times BO \\
(\mathbb{Z}\times BO)\wedge(\mathbb{Z}\times BO) \arrow[ur, "\text{multiplication}"'] &
\end{tikzcd}$$;

by Bott Periodicity $\mathbb{Z} \times BO \xrightarrow{\simeq} \Omega^8(\mathbb{Z} \times BO)$.

Now $KO(X)$ is a ring with unit; $\widetilde{KO}^0(S^0) \cong \mathbb{Z} \ni 1$. So there is a map of spectra

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d, equal] & KO \\
\Sigma^\infty S^0 &
\end{tikzcd}$$

representing $1 \in \widetilde{KO}(S^0)$. One question is, how big is the image of the induced map $\pi_*^S \longrightarrow KO_*$?

Another way to ask this question is this. Every spectrum has a space associated with it; in this case we get a map

$$\begin{tikzcd} \Omega^\infty \Sigma^\infty S^0 \arrow[r] \arrow[d, phantom, "\parallel"] & \mathbb{Z}\times BO \\ QS^0 & \end{tikzcd}$$

whose effect in unstable homotopy is the stable homotopy of the map above. Viewed in this light, the answer depends on self-maps of $\mathbb{Z}\times BO$. We have the ψ^k's, so we should use them. The operation $\psi^k : KO \to KO$ corresponds to a map

$$\psi^k : \mathbb{Z}\times BO \longrightarrow \mathbb{Z}\times BO.$$

We know the effect of ψ^k on π_*:

$*$	$\pi_*(\mathbb{Z}\times BO)$	ψ^k	
0	$\mathbb{Z}$		
1	$\mathbb{Z}/2\langle\eta\rangle$	$\eta \mapsto k\eta$	i.e. $\begin{Bmatrix} 0 & k \text{ even} \\ \eta & k \text{ odd} \end{Bmatrix}$
2	$\mathbb{Z}/2\langle\eta^2\rangle$	$\eta \mapsto k^2\eta$	
3	0		
4	$\mathbb{Z}\langle g_4\rangle$	$g_4 \mapsto k^{2}g_4$	
5	0		
6	0		
7	0		
8	$\mathbb{Z}\langle g_8\rangle$	$g_8 \mapsto k^4 g_8$	

Everything else on the table follows from periodicity.

These operations suffer from a bad defect: they're not stable. But we've already seen how to study this problem when we were studying transfer: namely,

$$\begin{tikzcd}
S^8 \wedge (\mathbb{Z}\times BO) \arrow[r, "k^4 \wedge \psi^k"] \arrow[d, "g_8 \wedge 1"'] & S^8 \wedge (\mathbb{Z}\times BO) \arrow[d] \\
(\mathbb{Z}\times BO)\wedge(\mathbb{Z}\times BO) \arrow[r, "\psi^k \wedge \psi^k"] \arrow[d, "\mu"'] & (\mathbb{Z}\times BO)\wedge(\mathbb{Z}\times BO) \arrow[d] \\
\mathbb{Z}\times BO \arrow[r, "\psi^k"] & \mathbb{Z}\times BO .
\end{tikzcd}$$

By adjunction, you get

$$\begin{tikzcd}
KO_0 \arrow[r, "k^4 \psi^k"] \arrow[d, "\simeq"'] & KO_0 \arrow[d, "\simeq"] \\
\Omega^8 KO_8 \arrow[r, "\psi^k"] & \Omega^8 KO_8
\end{tikzcd}$$

The diagram measures how far ψ^k is from being stable. The fix we shall use here is to localize so that k becomes a unit.

The point is, KO^* is a nice ring, so localization is easy to do: $R \subseteq \mathbb{Q}$ is flat over $\mathbb{Z}$, so we can take $KO^*(X;R)$ to be defined as $KO^*(X) \otimes_{\mathbb{Z}} R$; e.g. $R = \mathbb{Z}[\frac{1}{k}]$. Then the diagram becomes

$$\begin{tikzcd}
KO_0 \arrow[r, "\psi^k"] \arrow[d] & KO_0 \arrow[d] \\
\Omega^8 KO_8 \arrow[r, "k^{-4} \Omega^8 \psi^k"] & \Omega^8 KO_8 [\frac{1}{k}] \ ;
\end{tikzcd}$$

i.e. you can define a new operation $\Psi^k : KO \longrightarrow KO[\frac{1}{k}]$ so that

$$\Psi^k_{8n} = k^{-4n}\psi^k : \mathbb{Z}\times BO \longrightarrow \mathbb{Z}\times BO[\tfrac{1}{k}].$$

Now $\mathbb{Z}[\tfrac{1}{k}] \hookrightarrow \mathbb{Z}_{(2)}$, assuming $2\nmid k$, so you get

$$\begin{tikzcd} KO^* \arrow[r, "\Psi^k"] \arrow[d] & KO^*[\tfrac{1}{k}] \arrow[d] \\ KO^*_{(2)} \arrow[r, "\Psi^k"] & KO^*_{(2)} \end{tikzcd}$$

, k odd. Take the case $k=3$ for the rest of this lecture.

But we're still not done massaging the Adams operations; we need the notion of a "connective cover": if X is a space its n^{th} connective cover is the space

$$\begin{tikzcd} X\langle n,\dots,\infty\rangle \arrow[d, "\leftarrow \text{iso in } \pi_i,\ i\geq n"] \\ X \end{tikzcd}$$

$\pi_i X\langle n,\dots,\infty\rangle = 0,\ i<n.$

For example, you have

$$\begin{tikzcd} \vdots \arrow[d] \\ BSpin \arrow[d] \\ BSO \arrow[d] \\ BO \arrow[d] \\ BO\times\mathbb{Z} \end{tikzcd}$$

as a sequence of connective covers.

You can do this for spectra, in which case it's interesting to take away negative homotopy groups: $E\langle 0,\infty\rangle_n = E_n\langle n,\infty\rangle$ has no negative homotopy groups. For example, KO has negative homotopy groups; its 0-connective cover localized at 2 is

$$KO\langle 0, \ldots, \infty\rangle_{(2)} = bo$$

"connective real K-theory." So

$\pi_*(bo)$

$*$	0	1	2	3	4	5	6	7	(8)
$\pi_*(bo)$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_{(2)}$	0	0	0	

but now we restrict to $* \geq 0$.

The construction of Ψ^k gives an operation on connective real K-theory

$$\Psi^k : bo \longrightarrow bo$$

and we know its effect on homotopy groups ($k=3$ below):

$$\begin{tikzcd}[column sep=tiny]
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \cdots \\
\mathbb{Z}_{(2)} \arrow[d, "="'] & \mathbb{Z}/2 \arrow[d, "="'] & \mathbb{Z}/2 \arrow[d, "="] & 0 & \mathbb{Z}_{(2)} \arrow[d, "\cdot 9"] & 0 & 0 & 0 & \mathbb{Z}_{(2)} \arrow[d, ".81"] & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z}_{(2)} \arrow[d, "\cdot 729 = 9^3"] & \cdot & & & \mathbb{Z}_{(2)} \arrow[d, "\cdot 9^4"] \\
\mathbb{Z}_{(2)} & \mathbb{Z}/2 & \mathbb{Z}/2 & \cdot & \mathbb{Z}_{(2)} & & & & \mathbb{Z}_{(2)} & \mathbb{Z}/2 & \mathbb{Z}/2 & & \mathbb{Z}_{(2)} & & & & \mathbb{Z}_{(2)}
\end{tikzcd}$$

Notice that Ψ^3 induces the identity in dimensions < 4. So consider the map $\Psi^3 - 1 : bo \longrightarrow bo$; now the map acts as

$$0 \quad 0 \quad 0 \qquad \cdot 9 - 1 \qquad \cdot 81 - 1 \quad 0 \quad 0 \qquad \cdot 9^3 - 1$$

in homotopy.

You could just take the fiber of this map, but because Ψ^3-1 induces zero in homotopy in dimensions one and two you would get copies of $\mathbb{Z}/2$ from both ends, and these are in fact unnecessary: the map Ψ^3-1 lifts to the 4-connective cover*

$$\begin{tikzcd} & bo\langle 4,\ldots,\infty\rangle \arrow[d] \\ bo \arrow[ur, dashed, "\exists\,\varphi"] \arrow[r, "\Psi^3-1"'] & bo \end{tikzcd}$$

and that brings us to the space J -- or at least the spectrum j: $j =$ fiber of $\varphi : bo \longrightarrow bo\langle 4,\ldots,\infty\rangle$. The space J comes from the 0-space:

← this should be J, only it's not; instead is 0-component is.

$$\begin{tikzcd} \text{fiber} \arrow[r] & \mathbb{Z}\times J \arrow[r] & \mathbb{Z}\times BO \arrow[r, "\varphi"] & BSpin \\ \text{fiber} \arrow[r] & J \arrow[r] & BO \arrow[r] \arrow[u, hook] & BSpin \end{tikzcd}$$

OK, we can figure out the homotopy of J because we know about all the things that go into it. Recall that $\nu(9^k-1) = \nu(k)+3$, so $(9^k-1) = (\text{odd})\, 2^{\nu(k)+3} = (\text{odd})\, 8k$. ← may be fraction, but only odd things in denominator

Hence ψ^3-1 acts either as 0 or as (mult. by $8k$)$\cdot$(unit in $\mathbb{Z}_{(2)}$).

So we get:

---

* Why? I need to check the obstruction theory here.

$$\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\pi_* J & \mathbb{Z}_{(2)} \arrow[d] & \mathbb{Z}/2 \arrow[d] & \mathbb{Z}/2 \arrow[d] & \mathbb{Z}/8 \arrow[d] & 0 \arrow[d] & 0 & 0 & \mathbb{Z}/16 & \mathbb{Z}/2 \arrow[d, "0"] \\
\pi_* bo & \mathbb{Z}_{(2)} \arrow[d, "0"] & \mathbb{Z}/2 \arrow[d, "0"] & \mathbb{Z}/2 \arrow[d, "0"] & 0 & \mathbb{Z}_{(2)} \arrow[d, "\cdot(9-1)"] & 0 & 0 & 0 & \mathbb{Z}_{(2)} \arrow[d, "\cdot(81-1)"] \\
\pi_* bo\langle 4,\dots,\infty\rangle & 0 & 0 & 0 & 0 & \mathbb{Z}_{(2)} & 0 & 0 & 0 & \mathbb{Z}_{(2)}
\end{tikzcd}$$

You know this fits into $0 \to \mathbb{Z}/2 \to ? \to \mathbb{Z}/2 \to 0$. In fact its $\rightarrow$

$$\begin{tikzcd}
 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\pi_* J & (\mathbb{Z}/2)^2 \arrow[d] & \mathbb{Z}/2 & \mathbb{Z}/8 & 0 & 0 & 0 & \mathbb{Z}/32 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 \\
\pi_* bo & \mathbb{Z}/2 \arrow[d, "0"] & \mathbb{Z}/2 \arrow[d, "0"] & 0 & \mathbb{Z}_{(2)} \arrow[d, "\cdot 9^3-1"] & 0 & 0 & 0 & \mathbb{Z} \arrow[d, "\cdot 9^4-1"] & \text{etc.} \\
\pi_* bo\langle 4,\dots,\infty\rangle & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z}_{(2)} & 0 & 0 & 0 & \mathbb{Z} &
\end{tikzcd}$$

What's the point? The point is, this has to do with homotopy groups of spheres. You have

$$\begin{tikzcd}
 & S^0 \arrow[dl] \arrow[d] & \\
j \arrow[r] & bo \arrow[r, "\varphi"] & bo\langle 4,\dots,\infty\rangle
\end{tikzcd}$$

and the question is, how big is the image $\pi_* S^0 \to \pi_* j$? The Adams conjectures answers this.

How? The Adams conjecture says that for some e, $3^e J(\Psi^3-1)=0$. There is an inclusion

$$O(n) \hookrightarrow G_n = \left(\begin{matrix}\text{all degree } \pm 1\\ \text{self-maps of } S^{n-1}\end{matrix}\right)$$

which is compatible with the inclusions $O(n) \hookrightarrow O(n+1)$ and $G_n \hookrightarrow G_{n+1}$. So you get a map of the limits

← and this is the map inducing the J-homomorphism $\pi_* O \to \pi_*^s$!

$$O \xrightarrow{J} G \simeq Q_{\pm} S^0$$

which is an H-map. So you get a map of classifying spaces

$$BO \xrightarrow{BJ} BG.$$

Then the Adams conjecture implies that the composite

$$\begin{tikzcd}
BSpin \arrow[r] & BO \arrow[r, "BJ"] & BG_{(2)} \\
& BO \arrow[ul, "\varphi"] \arrow[u, "\Psi^3-1"'] \arrow[ur] &
\end{tikzcd}$$

← BG localized at 2

$\simeq *$ is null-homotopic!

This gives you a map α below:

$$\begin{tikzcd}
J \arrow[r] \arrow[d, "\alpha"'] & BO \arrow[r, "\varphi"] \arrow[d] & BSpin \arrow[d] \\
G \arrow[r] & EG \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & BG_{(2)} \\
& * &
\end{tikzcd}$$

Now in fact α factors through the $+1$ component of G, $SG = Q_1S^0$

$$\begin{tikzcd}
 & J \arrow[r] \arrow[dl, "\alpha"'] \arrow[d, "\alpha"] & BO \arrow[r] \arrow[d] & BSpin \arrow[d] \\
SG \arrow[r] & G \arrow[r] & EG \arrow[r] & BG_{(2)}
\end{tikzcd}$$

Well, this looks good, in fact it looks very good: the map $S^0 \to j$ is, after you apply Ω^∞ to it, a map

$$QS^0 \xrightarrow{u} \Omega^\infty j = \mathbb{Z} \times J.$$

And in fact

$$\begin{tikzcd}
 & Q_1S^0 \arrow[d, equal] & \\
J \arrow[r, "\alpha"] \arrow[drr, "\simeq !!"'] & SG \arrow[r, hook] & QS^0 \arrow[d, "u"] \\
 & & J
\end{tikzcd}$$

So if C = fiber of u, then there is a homotopy equivalence

$$Q_1S^0 = SG \simeq J \times C.$$

Thus all of the homotopy of J listed above appears as a direct summand in $\pi_* SG$, and in particular is in the image of $S^0 \to j$.

$\downarrow$

and hence in $\pi_* QS^0 = \pi_*^S$

One game you can play now is to try to get out some information on unstable homotopy. First well need some names: pulling back the sequence $J \to BO \to BSpin$ once, you get

$$\begin{tikzcd} Spin \arrow[r, "J"] & J \arrow[r] & BO \end{tikzcd}$$

← this map is essentially the J-homomorphism, in the sense that it restricts from

$$\begin{tikzcd} Spin \arrow[r] \arrow[d] & J \arrow[d, "\alpha"] \\ O \arrow[r, "J"] & QS^0 \end{tikzcd}$$.

Its time for another table

	$\pi_* Spin$	$\xrightarrow{J}$ $\pi_* J$	$\longrightarrow$ $\pi_* BO$	Names New	Names Traditional
1	0	$\mathbb{Z}/2 \langle \alpha_1 \rangle$	$\mathbb{Z}/2$	α_1	η
2	0	$\mathbb{Z}/2 \langle \alpha_2 \rangle$	$\mathbb{Z}/2$	α_2	η^2.
3	$\mathbb{Z}$	$\mathbb{Z}/8 \langle j_2 \rangle$	0	α_3, $\alpha_3 = 4 j_2$	$j^2 = \nu$
4	0	0	$\mathbb{Z}$		
5	0	0	0		
6	0	0	0		
7	$\mathbb{Z}$	$\mathbb{Z}/16 \langle j_3 \rangle$	0	$\alpha_4 = 8 j_3$	$j_3 = \sigma$
8	$\mathbb{Z}/2$	$\mathbb{Z}/2 \langle j_4 \rangle$	$\mathbb{Z}$		$j_4 = \eta\sigma$
9	$\mathbb{Z}/2$	$\langle j_5 \rangle \mathbb{Z}/2 \oplus \mathbb{Z}/2 \langle \alpha_5 \rangle$	$\mathbb{Z}/2$		μ_9, $j_5 = \eta^2\sigma$
10	0	$\mathbb{Z}/2 \langle \alpha_6 \rangle$	$\mathbb{Z}/2$	α_6	$\eta\mu_9$
11	$\mathbb{Z}$	$\mathbb{Z}/8 \langle j_6 \rangle$	0	$\alpha_7 = 4 j_6 = \eta^2\mu_9$	j_6, $\eta^2\mu_9$
12	0	0	$\mathbb{Z}$		
13	0		0		
14	0		0		
15	$\mathbb{Z}$	$\mathbb{Z}/32 \langle j_7 \rangle$	0	$\alpha_8 = 16 j_7$	j_7

The stuff coming from Spin are essentially classes in the image of the J-homomorphism; we had names for these already (j_i; see 4/21 p.3). Now there's more, elements which come from $\pi_* BO$, sort of "honorary members of the image of J". Note that $\eta = j_1$ has been demoted to an "honorary member" because we used Spin instead of O. The elements α_i all have order 2;

in fact, we've given names to all the elements of order 2 that didn't have names before. All of these really live in π_*^S! So what we can do now is study how they behave in the EHPSS: how far they desuspend, and what their Hopf invariants are. Or at least Mark Mahowald can. We're in Mahowald territory in a big way now; for more information see

Mahowald, Annals of Math (1982) "Image of J in the EHP sequence"
Algebraic Topology Seattle LNM 1286
FR Cohen -- Kervaire Invariant
Barratt, Jones, Mahowald.
Selick, ~~Seattle~~ Pacific J. 1983
Feder, Gitler, Lam Pac. J. 1977.

Recall that there are "Snaith maps" s_n that match up fibrations:

$$\begin{tikzcd}
\Omega^n S^n \arrow[r] \arrow[d, "s_n"'] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d, "s_{n+1}"'] & \Omega^{n+1} S^{2n+1} \arrow[d, "e^{\infty-(n+1)}"] \\
Q\mathbb{R}P^{n-1} \arrow[r] & Q\mathbb{R}P^n \arrow[r] & QS^n
\end{tikzcd}$$

You apply π_* to get a map of exact couples (and so of spectral sequences) (and from here on we write P for $\mathbb{R}P$)

$$\begin{tikzcd}
\pi_* \Omega^n S^n \arrow[r] \arrow[d] & \pi_* \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \pi_* \Omega^{n+1} S^{2n+1} \arrow[d] \\
\pi_*^S P^{n-1} \arrow[r] & \pi_*^S P^n \arrow[r] & \pi_*^S (S^n)
\end{tikzcd}$$

π_*^S is formidable, so now we could try to use something else to detect things; an obvious choice given the above is the spectrum j: $j_*(X) = \pi_*(j \wedge X)$, so $j_*(S^0) = \pi_*(j) = \pi_*(J \times \mathbb{Z})$, which we know, so we might hope to be able to compute other things as well. So by smashing with j, we get a map

$$\begin{tikzcd}
\pi_*^S P^{n-1} \arrow[r] \arrow[d] & \pi_*^S P^n \arrow[r] \arrow[d] & \pi_*^S S^n \arrow[d] & \\
j_*(P^{n-1}) \arrow[r] & j_*(P^n) \arrow[r] & j_*(S^n) & \text{and we know this is a summand of } \pi_*^S\text{ !} \arrow[l]
\end{tikzcd}$$

of exact couples. The bottom exact couple gives the Atiyah-Hirzebruch spectral sequence $H_*(\mathbb{RP}^\infty; j_*) \Rightarrow j_*(\mathbb{RP}^\infty)$.

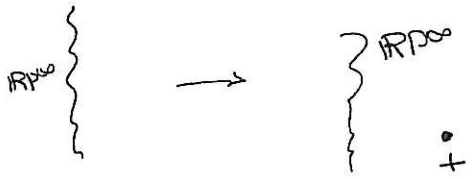
Unfortunately, we can't do all the proofs. Mahowald essentially does this work in the Annals paper, but he doesn't quite say it this way.

Anyway, one thing to do is to try to compute $j_*(\mathbb{RP}^\infty)$ by more direct means. You might expect that it's huge, since j_* has lots of stuff, and $\mathbb{RP}^\infty$ has a cell in each dimension. But in fact its very small; this reflects all the cancellation going on on the EHPSS -- there's lots of it.

To compute $j_*(\mathbb{RP}^\infty)$, note the fibration

$$\begin{tikzcd}
S^\infty \arrow[d] \\
\mathbb{RP}^\infty
\end{tikzcd}$$

induces a transfer map $\mathbb{RP}^\infty_+ \relbar\joinrel\mapstochar\joinrel\rightarrow S^\infty_+ \simeq S^0$. There is an obvious map $\mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty_+$ but be careful: on the level of spaces this <u>isn't</u> a pointed map.



But on the level of spectra, you get a <u>pointed</u> homotopy equivalence

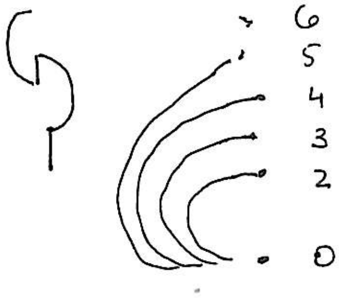
$$\Sigma^\infty(X_+) \underset{\underset{\text{ptd.}}{\uparrow}}{\simeq} \Sigma^\infty X \vee \Sigma^\infty S^0$$

whereas on the level of spaces you certainly don't get a pointed homotopy equivalence $X_+ \underset{X}{\simeq} X \vee S^0$. So any way, you get

and this is an interesting map which doesn't exist unstably. Constructing it involves the same game as the shifting of components we played when constructing the J-homomorphism.

$$\begin{tikzcd}
\mathbb{RP}^\infty \arrow[r, "\shortmid" marking] \arrow[dr, "\shortmid" marking, "\lambda"'] & \mathbb{RP}^\infty_+ \arrow[d, "\shortmid" marking] & & \\
& S^0 \arrow[dr] & & \\
& & R = \text{cofiber of } \lambda. \arrow[dr] & \\
& & & \Sigma\mathbb{RP}^\infty
\end{tikzcd}$$

The cohomology of R has a class x_0 in dimension zero coming from S^0 and then a class x_k in dimension $k \geq 2$ coming from $H^*(\Sigma\mathbb{RP}^\infty)$, and $Sq^k x_0 = x_k \; \forall k$!



The reason for studying R is that $bo_* R$ is very simple; in fact, $bo \wedge R$ is a bouquet of Eilenberg-MacLane spectra:

$$bo \wedge R \simeq \bigvee_{k \geq 0} \Sigma^{4k} H_{(2)} \leftarrow \text{meaning } K(\mathbb{Z}_{(2)})\text{-spectrum.}$$

and $$bo\langle 4, \infty\rangle \wedge R \simeq \bigvee_{k \geq 1} \Sigma^{4k} H_{(2)} \vee \bigvee_{k \geq 1} \Sigma^{4k+2} H_2 \leftarrow \text{meaning } K(\mathbb{Z}/2)\text{-spectrum.}$$

And so we computed $j \wedge R$ from the sequence

$$j \wedge R \longrightarrow bo \wedge R \xrightarrow{\psi^3 - 1} bo\langle 4, \dots, \infty\rangle \wedge R .$$

It's a rational calculation; there's just not that much to do.

The result is striking: $j \wedge R$ is almost not there at all.

$$j\wedge R \longrightarrow bo\wedge R \longrightarrow bo\langle 4,\dots,\infty\rangle\wedge R$$

0	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_{(2)}$		
1				
2				
3		$\mathbb{Z}_{(2)}$	$\xrightarrow{=}$	$\mathbb{Z}_{(2)}$
4				
5	$\mathbb{Z}/2\langle\sigma_1\rangle$			$\mathbb{Z}/2$
6				
7	$\mathbb{Z}/2\langle\tau_1\rangle$	$\mathbb{Z}_{(2)}$	$\xrightarrow{\cdot 2}$	$\mathbb{Z}_{(2)}$
8				
9	$\mathbb{Z}/2\langle\sigma_2\rangle$			$\mathbb{Z}/2$
10				
11		$\mathbb{Z}_{(2)}$	$\xrightarrow{=}$	$\mathbb{Z}_{(2)}$
12				
13	$\mathbb{Z}/2\langle\sigma_3\rangle$			$\mathbb{Z}/2$
14				
15	$\mathbb{Z}/4\langle\tau_2\rangle$	$\mathbb{Z}_{(2)}$	$\xrightarrow{\cdot 4}$	$\mathbb{Z}_{(2)}$
16				
17	$\mathbb{Z}/2\langle\sigma_4\rangle$			

That's it!

$$j_{8k-1}(R) = \mathbb{Z}/2^{\nu(k)+1}\langle\tau_k\rangle \quad , \quad k\geqslant 1$$

$$j_{4k+1}(R) = \mathbb{Z}/2\langle\sigma_k\rangle \quad , \quad k\geqslant 1$$

Now were really interested in $\mathbb{RP}^\infty$.

$$j\wedge \mathbb{RP}^\infty \xrightarrow{\lambda} j\wedge S^0 \longrightarrow j\wedge R$$

0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_{(2)}$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2\langle\alpha_1\rangle$	
2	$\mathbb{Z}/2$	$\mathbb{Z}/2\langle\alpha_2\rangle$	
3	$\mathbb{Z}/8$	$\mathbb{Z}/8\langle j_2\rangle$	
4	$\oplus\mathbb{Z}/2\langle\sigma_1\rangle$		$\mathbb{Z}/2\langle\sigma_1\rangle$
5	0		
6	$\mathbb{Z}/2\langle\tau_1\rangle$		
7	$\mathbb{Z}/16$	$\mathbb{Z}/16\langle j_3\rangle$	$\mathbb{Z}/2\langle\tau_1\rangle$
8	$\mathbb{Z}/2\oplus(\mathbb{Z}/2\langle\sigma_2\rangle)$	$\mathbb{Z}/2\ \langle j_4\rangle$	$\mathbb{Z}/2\langle\sigma_2\rangle$
9	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	
10	$\mathbb{Z}/2$	$\mathbb{Z}/2$	
11	$\mathbb{Z}/8$	$\mathbb{Z}/8\ \langle j_6\rangle$	
12	$\mathbb{Z}/2\langle\sigma_3\rangle$	0	
13	0	0	$\mathbb{Z}/2\langle\sigma_3\rangle$
14	$\mathbb{Z}/2\langle\tau_2\rangle$	0	
15	$\mathbb{Z}/32$	$\mathbb{Z}/32\ \langle j_7\rangle$	$\mathbb{Z}/4\langle\tau_2\rangle$
16	$\mathbb{Z}/2\oplus\mathbb{Z}/2\langle\sigma_4\rangle$	$\mathbb{Z}/2\ \langle j_8\rangle$	
17		$\langle\alpha_9\rangle\ \mathbb{Z}/2\oplus\mathbb{Z}/2\langle j_9\rangle$	$\mathbb{Z}/2\langle\sigma_4\rangle$

Various things could happen, but in fact nothing does:

$$j_*\mathbb{RP}^\infty \cong j_* \oplus \langle \overset{\substack{8k-2\\ \downarrow}}{I_k}, \overset{\substack{4k\\ \downarrow}}{\sigma_k} \rangle$$

$\rightsquigarrow$ i.e. these occur one dimension lower than they do in $\pi_* j \wedge R$.

So now we've computed the E^2-term and the "abutment" of the Atiyah-Hirzebruch spectral sequence

$$H^*(\mathbb{RP}^\infty; j_*) \Rightarrow j_*\mathbb{RP}^\infty.$$

A picture is attached of what the filtration looks like; we've really reached the outer limits of human comprehension here. Note that each of the dots is a $\mathbb{Z}/2$, so a non-zero differential connecting two dots is death to both of them. So you get to E^∞ from E^2 pretty quickly, and you can see a good deal of E^∞ in this picture.

Remember what we had; there were three exact couples

$$\begin{tikzcd}
\pi_* \Omega^n S^n \arrow[r] \arrow[d, "S_{n-1}"'] & \pi_* \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d, "S_n"'] & \pi_* \Omega^{n+1} S^{2n+1} \arrow[d, "e^{\infty-(n+1)}"] \\
\pi_* Q\mathbb{R}P^{n-1} \arrow[r] \arrow[d] & \pi_* Q\mathbb{R}P^n \arrow[r] \arrow[d] & \pi_* Q S^n \arrow[d] \\
j_* \mathbb{R}P^{n-1} \arrow[r] & j_* \mathbb{R}P^n \arrow[r] & j_* S^n
\end{tikzcd}$$

and the last SS can be written out completely; it's the chart.

Today we'll talk about two problems: ① given a class $v \in j_* \mathbb{R}P^\infty$, where can we find a representative, which we will denote $H(v)$, in the SS, i.e. in j_*? ② How do classes in $j_* \mathbb{R}P^\infty$ pull back to $\pi_*^S \mathbb{R}P^\infty$, or even better, to the EHP sequence?

$$\begin{tikzcd}
S^k \arrow[r, "v"] \arrow[dr] \arrow[drr, bend right] & j_* \mathbb{R}P^\infty & \\
 & j_* \mathbb{R}P^s \arrow[u] \arrow[r] & j_* S^s
\end{tikzcd}$$

$H(\alpha)$ is collection of such representatives.

Recall $j_*(\mathbb{R}P^\infty) = j_* \oplus \langle \iota_k, \sigma_k \rangle$, and j_* contained elements α_k of order 2 and elements j_k. Recall also that α_k were often not generators; for example $\alpha_3 = 4 j_2$. So we introduce the notation that

$\alpha_{k/i}$ is an element such that $2^{i-1}\alpha_{k/i} = \alpha_k$

For example, (see table on p. 10)

$$\alpha_4 = 8\sigma, \text{ so } \quad \alpha_{4/2} = 4\sigma, \ \alpha_{4/3} = 2\sigma, \ \alpha_{4/4} = \sigma$$
$$\alpha_{4/1} = \alpha_4$$

With this notation we can get all the classes in the image of J except j_{4k} and $j_{4k+1} \in \pi_{8k} J, \pi_{8k+1} J$. Then the representatives of $j^*(\mathbb{R}P^\infty)$ in this SS are found as follows:

$$\boxed{\begin{aligned} &H(\alpha_{k/i}) = \alpha_{k-i} \\ &H(j_{4k+i}) = j_{4k+i-1}, \quad i = 0, 1 \\ &H(\sigma_k) = \nu = j_2 \in \pi_3(J) \\ &H(2^i \tau_k) = j_{i+1}, \quad i \leq \nu(k) \end{aligned}}$$

Some patterns you will observe when you compare this with the picture:

① Elements in the image of J are born as early as possible, so they are concentrated on the left in the table.

② σ_k's are born as late as possible so they occur to the right on their total degree lines

③ τ_k's are born as late as possible, subject to the requirement that $2^i\tau_k$'s are born ~~B~~ as late as possible too.

The next question we posed for ourselves was: how does this picture pull back to the EHP sequence and $\pi_*^S = \pi_* QS^0$?

Mahowald's answer is:

Theorem of Mahowald

$\mathrm{Im}\,(\pi_*^S = \pi_* QS^0 \longrightarrow \pi_*^S \mathbb{R}P^\infty \longrightarrow j_* \mathbb{R}P^\infty)$ contains

→ $\hat{J}_*$ [and we knew this from our splitting $SG \simeq J \times C$]

→ σ_{2^k} $k \geq 1$

→ and may contain $2^k \tau_k$, $k \geq 0$

↗ coming from $\Theta_{k+2} \in \pi^S_{8 \cdot 2^k - 2}$

"Kervaire Invariant Classes"; may exist.

$\Theta_2, \Theta_3, \Theta_4$, and Θ_5 do exist.

→ and nothing else!

What does this tell us about our picture?

For elements of the image of J, the EHPSS looks the same as our chart. Elements in the image $\pi_* J \to \pi_*^S$ are born on the expected spheres (they can't be born earlier than they are here, and Mahowald constructed such elements in the EHPSS in the right dimensions in his paper) and their Hopf invariants have the right image in $j_*(S^n)$ under the map of SS's.

In fact, a great deal of the information from various parts of the course can be discerned in the picture. For example, Hopf Invariant 1 is present: if you project

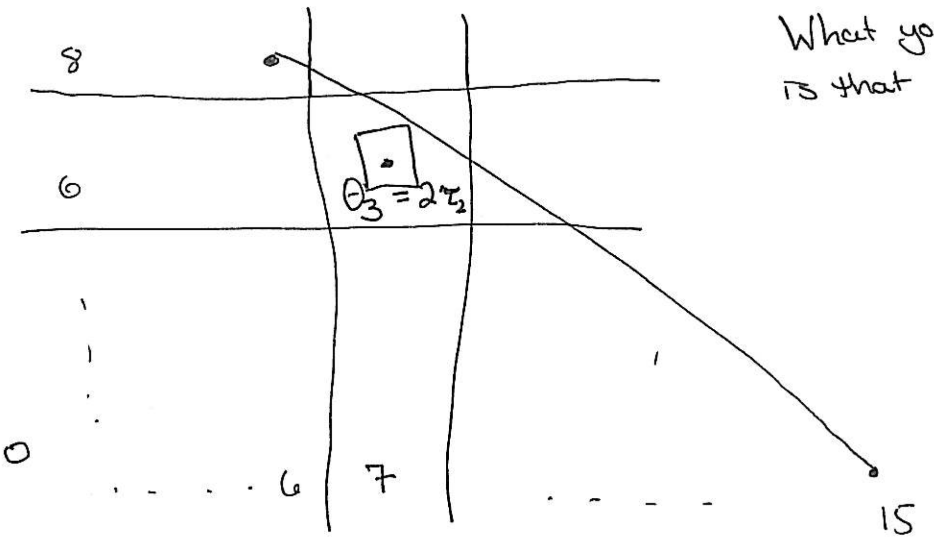
$$\pi_*^S(S^0) \longrightarrow \pi_*^S\mathbb{RP}^\infty \longrightarrow j_*\mathbb{RP}^\infty \longrightarrow \underset{\substack{\nearrow \\ n \text{ odd}}}{j_*\mathbb{RP}^\infty_n}$$

then $\pi_n^S(S^0) \longrightarrow j_*(\mathbb{RP}^\infty_n) = \mathbb{Z}/2$ is the Hopf invariant.

So you have an element of Hopf invariant 1 if you have a survivor on the bottom row.

The issue of the desuspension of w_n (and so of vector fields on spheres) became a picture of differentials off classes on the bottom row of the $\overset{AH}{SS}$ for $\pi_*^S\mathbb{RP}^\infty$; the desusp. of w_n is represented by the <u>ends</u> of the differential.

The desuspension of w_n's brings us to the Kervaire Invariant classes. Notice on the picture at the (7,7) position the differential coming in above from the bottom row:



What you would hope is that $\Theta_3 = \frac{1}{2}\left(\text{desusp. of } W_{15}\right)$.

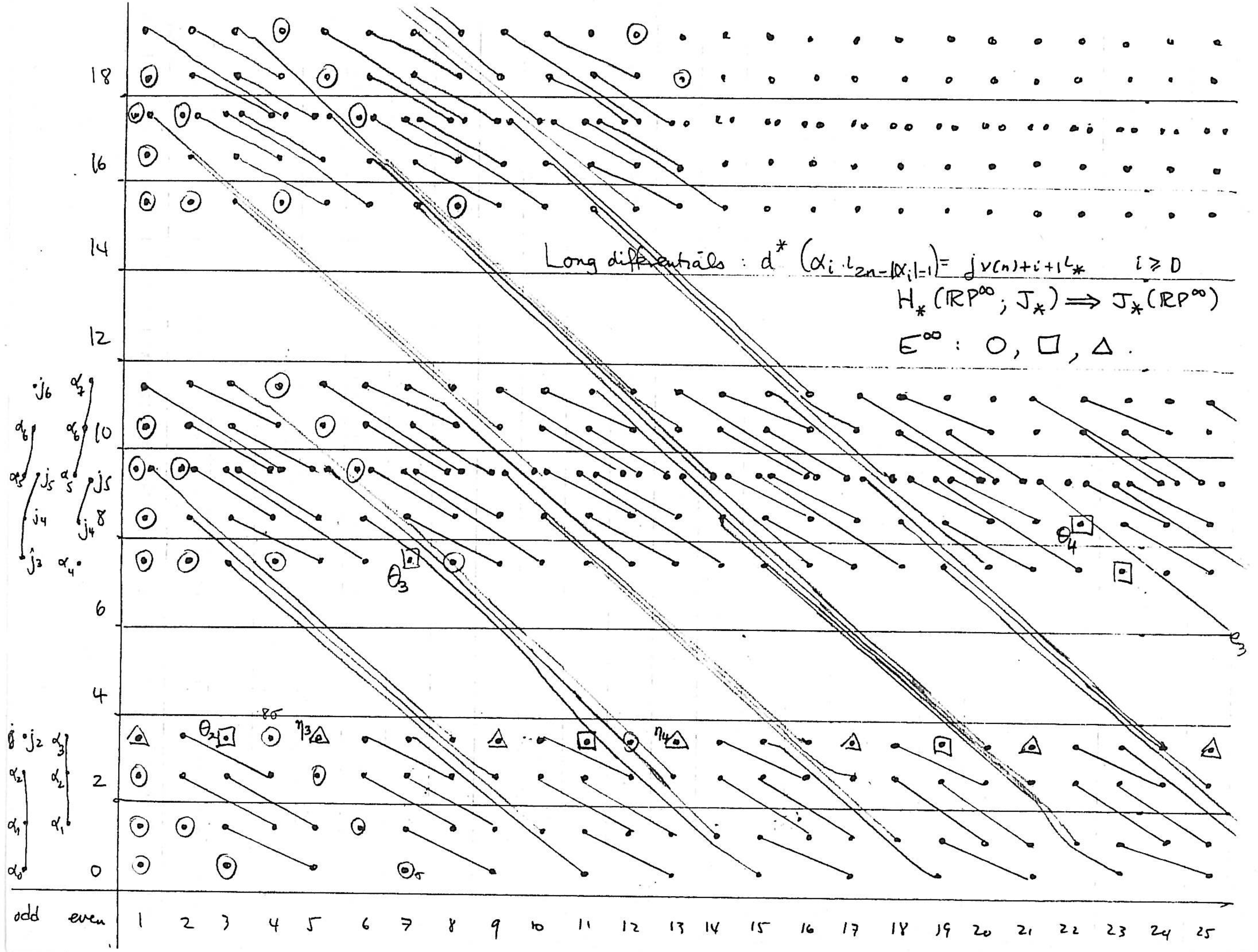
So that's part of the

Wish List for Kervaire Invariant Classes

$\theta_k \in \pi^S_{2^{k+1}-2}$

- born on $S^{2^{k+1}-1-|j_k|}$
- Hopf invariant $= j_k$ (or at least having same image in j_* as j_k)
- order 2 (since in $j_*\mathbb{R}P^\infty$, $2^k \tau_{2^k}$ has order 2)
- θ_k halves a maximal desuspension of $W_{2^{k+1}-1}$
- no further division by 2 is possible, because in $j_*\mathbb{R}P^\infty$ further divisors exist, but these are not in the image of π_*^S
- detected on the Adams 2-line.

Other things have been done: you could investigate further how these classes that are in π_*^S behave in the EHP sequence; for example you could try to compute $p(j_k)$. Mahowald's theorem tells good information about when these are non-zero. Feder, Gitler, and Lam have shown other cases for which $p(j_k)=0$. But the Kervaire Invariant classes represent a real missing case here, about which relatively little is known.



Long differentials : $d^*(\alpha_i \iota_{2n-|\alpha_i|-1}) = j_{\nu(n)+i+1}\iota_*$ $i \geq 0$

$H_*(\mathbb{RP}^\infty; J_*) \Longrightarrow J_*(\mathbb{RP}^\infty)$

E^∞ : ○, □, △.