**Verdier systems**

July 16, 2001

Let

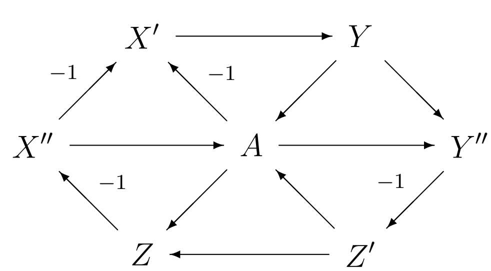
$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram in a triangulated category. Verdier proved that this diagram can be enlarged to

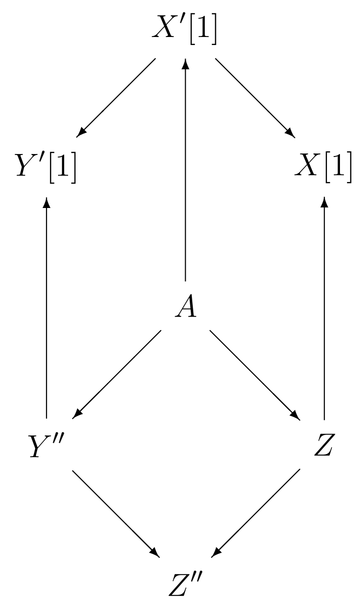
$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & Y'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z' & \longrightarrow & Z & \longrightarrow & Z'' & \longrightarrow & Z'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'[1] & \longrightarrow & X[1] & \longrightarrow & X''[1] & \longrightarrow & X'[2] \end{array}$$

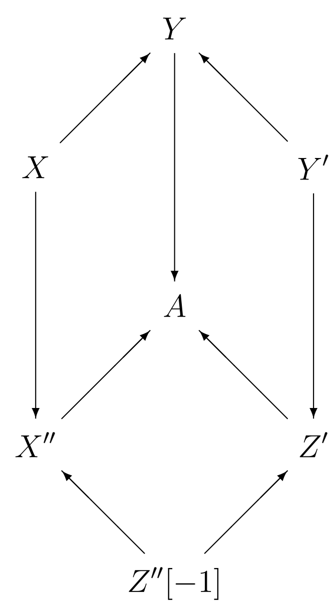
which is commutative except that the lower right square is anti-commutative, and in which each row and each column is an exact triangle, and the bottom row is the suspension of the top row and the right column is the suspension of the left column.

His proof (reproduced in Beilinson, Bernstein, and Deligne, *Faisceaux pervers*, Astérisque 100 (1982)) gives more, namely: there is, in addition, an object $A$ which fits into the diagram



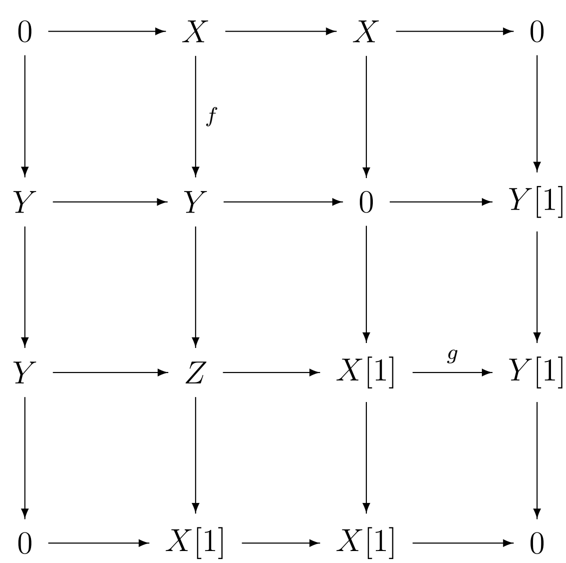
in which the arrows around the outside are from the diagram and each triangle is either exact or commutative, and these arrows fit into the commutative diagrams



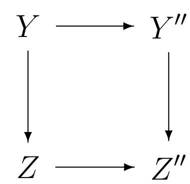

The first hexagon expresses $A$ as the cofiber of any of the three diagonal maps $X' \to Y$, $Y''[-1] \to Z'$, or $Z[-1] \to X''$.

Not every $3 \times 3$ extension satisfies these requirements. For example, let $Y \to Z \to X[1]$ be a cofiber sequence admitting distinct maps $f, g : X \to Y$ completing it to an exact triangle, and build the diagram

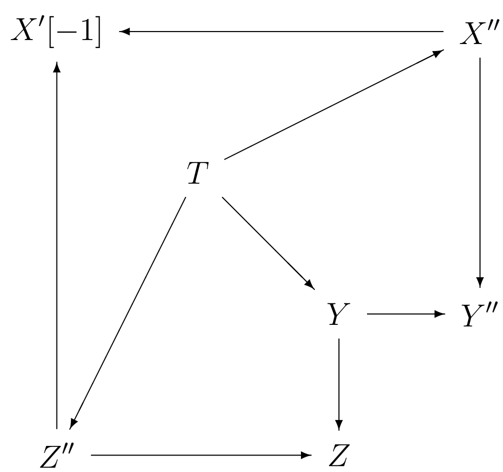


in which morphisms with source and target equal are identities. Then the bottom square in the right hand hexagon fails to commute.

The dual statement is also true, and this verifies a version of an old claim of mine. A commutative square




can be enlarged to the bottom right corner of a $4 \times 4$ diagram, in such a way that the following holds: For any map $T \to Y$ whose composite into $Z''$ is null, there exist maps $T \to X''$ and $T \to Z'$ such that




commutes. The universal example is provided by taking $T \to Y$ to be the fiber of $Y \to Z''$. In the dual, the two maps are the two other maps to $A$, and the required commutativity is given by the lower right square in the second of the pair of diagrams.