Beck modules over a Poisson algebra

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Let K be a commutative ring and A a Lie algebra over K. A Lie algebra over A with a cross-section is of the form $A \oplus M$, with projection killing M and cross-section sending a to $(a, 0)$. The Lie algebra structure on $A \oplus M$ is determined by a K-linear map $\cdot : A \otimes_K M \to M$, as follows.

$$[(a,0),(b,0)] = ([a,b],0)$$
$$[(a,0),(0,y)] = (0,a\cdot y)$$
$$[(0,x),(b,0)] = -(0,b\cdot x)$$
$$[(0,x),(0,y)] = (0,[x,y])\,.$$

We have built in anti-symmetry, as long as we assume anti-symmetry of $[x, y]$.

The Jacobi identity says that the cyclic sum of

$$a\cdot(b\cdot z) - a\cdot(c\cdot y) + a\cdot[y,z] - [b,c]\cdot x + [x,b\cdot z] - [x,c\cdot y] + [x,[y,z]]$$

is zero. With $a = b = c = 0$ we get the Jacobi identity for $[x, y]$. With $y = z = 0$ we get

$$[b,c]\cdot x = b\cdot(c\cdot x) - c\cdot(b\cdot x)\,.$$

These identies imply the general case.

Now if $A \oplus M$ also has the structure of an abelian object over A (with unit given by the cross-section), then the addition map must be

$$A\oplus M\oplus M = (A\oplus M)\times_A(A\oplus M) \to A\oplus M$$

by $(a, x, y) \mapsto (a, x + y)$, and so this map must be a Lie algebra homomorphism. This is equivalent to requiring $[x, y] = 0$, and assuming this makes $A \oplus M$ into an abelian object.

We have proven:

Lemma. Let A be a Lie algebra over K. The category of abelian objects over A is equivalent to the category of K-modules M equipped with a K-linear map $\cdot : A \otimes_K M \to M$ such that

$$[a,b]\cdot z = a\cdot(b\cdot z) - b\cdot(a\cdot z)\,.$$

That is to say, it is equivalent to the category of modules over the universal enveloping algebra of A,

$$U(A) = \text{Tens}_K(A)/([a,b] - a \otimes b + b \otimes a)\,.$$

Let $i : A \to U(A)$ be the natural map. Let M be a $U(A)$-module. A *Lie derivation* with values in M is a K-linear map $\sigma : A \to M$ such that

$$\sigma([a,b]) = i(a)\sigma(b) - i(b)\sigma(a)\,.$$

This is the same as a section of the abelian object $A \oplus M \downarrow A$.

The $U(A)$-module of "Lie-Kähler differentials" supports the universal Lie derivation out of A. It is gven by

$$\Omega^{\text{Lie}}_{A/K} = \frac{U(A) \otimes_K A}{\{1 \otimes [a,b] - i(a) \otimes b + i(b) \otimes a\}}$$

with universal derivation given by $\sigma(a) = 1 \otimes a$.

Following Cartan and Eilenberg, as long as A is free over K there is a resolution of K by $U(A)$ modules given by the Chevalley-Eilenberg complex $U(A) \otimes_K \Lambda^*(A)$, with $d : U(A) \otimes \Lambda^2(A) \to U(A) \otimes A$ extending

$$x \wedge y \mapsto ix \otimes y - iy \otimes x - 1 \otimes [x,y]$$

Thus the indicated quotient is simply

$$I_{A/K} = \ker\left(\epsilon : U(A) \to K\right).$$

In these terms, the universal derivation $A \to I_{A/K}$ is given by $a \mapsto i(a)$.

Let A be a Poisson algebra over K. A Beck module over A will be a Beck module over A regarded as a commutative K-algebra and as a Lie algebra over K, separately, so consists of an A-module M with a bilinear map $\cdot : A \otimes_K M \to M$ as above, subject to the Poisson identity, which amounts to

$$a \cdot bz + a \cdot cy - bc \cdot x =$$
$$b(a \cdot z) - b(c \cdot x) + [a,c]y + c(a \cdot y) - c(b \cdot x) + [a,b]z$$

With $y = z = 0$ this gives

$$bc \cdot x = b(c \cdot x) + c(b \cdot x)$$

With $x = y = 0$ it gives

$$[a,b]z = a \cdot bz - b(a \cdot z)\,.$$

These identities imply the general case.

We have proven:

Lemma. Let A be a Poisson algebra over K. The category of abelian objects over A is equivalent to the category of A-modules M equipped with a K-linear map $\cdot : A \otimes_K M \to M$ such that

$$\begin{aligned} [a,b] \cdot z &= a \cdot (b \cdot z) - b \cdot (a \cdot z) \\ ab \cdot z &= a(b \cdot z) - b(a \cdot z) \\ [a,b]z &= a \cdot bz - b(a \cdot z)\,. \end{aligned}$$

A section of $A \oplus M \to A$ is precisely a derivation with respect to each of the structures: a K-linear map $\sigma : A \to M$ such that

$$\sigma(ab) = a\sigma b + b\sigma a \quad , \quad \sigma[a,b] = a \cdot \sigma b - b \cdot \sigma a\,.$$

Such a map σ is a "Poisson derivation."

To describe the Beck module of Poisson differentials it is useful to discuss the universal enveloping algebra of a Poisson algebra. A Beck module structure over A on a K-module M is determined by two K-linear maps $A \to \mathrm{End}_K(M)$, or by a K-algebra map

$$\mathrm{Tens}_K(A \oplus A) \to \mathrm{End}_K(M)$$

where the first copy of A acts by $(a,x) \mapsto ax$ and the second by $(a,x) \mapsto a \cdot x$. Write $\alpha : A \to \mathrm{Tens}_K(A \oplus A)$ for the inclusion of the first copy and $\lambda : A \to \mathrm{Tens}_K(A \oplus A)$ for the inclusion of the second. The relations on the two "action" maps show that this map factors through the quotient by the ideal I generated by the elements

$$\alpha(1) - 1 \quad , \quad \alpha(ab) - \alpha(a)\alpha(b)$$
$$\lambda([a,b]) - \lambda(a)\lambda(b) + \lambda(b)\lambda(a)$$
$$\lambda(ab) - \alpha(a)\lambda(b) + \alpha(b)\lambda(a) \quad , \quad \alpha([a,b]) - \lambda(a)\alpha(b) + \alpha(b)\lambda(a)$$

as a, b range over A.

Write

$$U_{\text{Pois}}(A) = \text{Tens}_K(A \oplus A)/I\,,$$

so that a Beck module over A is the same thing as a left $U_{\text{Pois}}(A)$-module.

There is a universal Poisson derivation $\sigma : A \to \Omega^{\text{Pois}}_{A/K}$ where

$$\Omega^{\text{Pois}}_{A/K} = \frac{U_{\text{Pois}}(A) \otimes_K A}{\{1 \otimes ab - \alpha(a) \otimes b - \alpha(b) \otimes a, 1 \otimes [a,b] - \lambda(a) \otimes b + \lambda(b) \otimes a\}}$$

and $\sigma : a \mapsto 1 \otimes a$. It would be interesting to know more about this Poisson module.

If $K = \mathbb{F}_2$ one may wish to require that the Lie algebra A comes equipped with a *restriction*, that is, a function $\xi : A \to A$ such that

$$[\xi(a), b] = [a, [a, b]] \quad \text{and} \quad \xi(a + b) = \xi(a) + [a, b] + \xi(b)\,.$$

Note that by taking $a = b$ we find $\xi(0) = 0$. A *restricted Lie algebra* is a Lie algebra together with a restriction. One should think of $\xi(a)$ as half of $[a, a]$.

A Beck module over the restricted Lie algebra A is a Beck module over A as a Lie algebra, together with a restriction, which is given on (a, x) by $(\xi a, \zeta(a, x))$. Compatiblity with the section forces $\zeta(a, 0) = 0$. The addition formula is equivalent to

$$\zeta(a + b, x + y) = \zeta(a, x) + a \cdot y + b \cdot x + \zeta(b, y)\,.$$

Taking $a = 0$ and $y = 0$ gives

$$\zeta(b, x) = \zeta(0, x) + b \cdot x\,.$$

Write

$$\varphi(x) = \zeta(0, x)$$

so that $\zeta(b, x) = \varphi(x) + b \cdot x$. The addition formula now implies that φ is linear, and this in turn implies the general case.

The bracket formula for the restriction is equivalent to

$$\xi a \cdot y + b \cdot \varphi(x) + b \cdot (a \cdot x) = a \cdot (a \cdot y) + a \cdot (b \cdot x) + [a, b] \cdot x$$

or

$$\xi a \cdot y = a \cdot (a \cdot y) \quad , \quad b \cdot \varphi(x) = 0\,.$$

The addition morphism now automatically commutes with the restriction map.

We have proven:

Lemma. A Beck module over a restricted Lie algebra A is an $\mathbb{F}_2$ vector space M together with linear maps

$$\cdot : A \otimes M \to M \quad , \quad \varphi : M \to M$$

such that

$$[a,b] \cdot z = a \cdot (b \cdot z) - b \cdot (a \cdot z)$$
$$a \cdot \varphi(y) = 0 \quad , \quad \xi a \cdot y = a \cdot (a \cdot y)\,.$$

A *restriction* on a Poisson algebra over $\mathbb{F}_2$ is a restriction ξ on the underlying Lie algebra which satisfies also the Poisson condition

$$\xi(xy) = x^2\xi(y) + x[x,y]y + \xi(x)y^2\,.$$

A Beck module over a restricted Poisson algebra A will be a Beck module over A as a Poisson algebra, which is also a Beck module over A as a restricted Lie algebra, such that the Poisson condition on the restriction is satisfied. This is an A-module M together with linear maps

$$\cdot : A \otimes M \to M \quad , \quad \varphi : M \to M$$

such that the above conditions hold; the Poisson condition is then

$$ab \cdot ay + ab \cdot bx + \varphi(ay) + \varphi(bx) =$$
$$a^2\varphi(y) + a^2(b \cdot y) + b^2\varphi(x) + b^2(a \cdot x) + a[a,b]y + ab(a \cdot y) + ab(b \cdot x) + [a,b]bx\,.$$

With $a = 0$ this gives

$$\varphi(bx) = b^2\varphi(x)$$

With $y = 0$ and $a = 1$ it gives

$$b \cdot bx = b(b \cdot x)\,.$$

These (along with the Poisson module identities) imply the general case.

We have proven:

Lemma. A Beck module over the restricted Poisson algebra A consists in a Poisson module $(M, \cdot)$ over A such that

$$a \cdot ay = a(a \cdot y) \quad , \quad \xi a \cdot y = a \cdot (a \cdot y)$$

together with an $\mathbb{F}_2$-linear endomorphism $\varphi : M \to M$ such that

$$a \cdot \varphi(y) = 0 \quad , \quad \varphi(ay) = a^2 \varphi(y)\,.$$