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Correction to "The Sullivan conjecture on maps from classifying spaces"

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I am very grateful to J. Lannes and Lionel Schwartz for calling to my attention the following two errors in the paper [3]. Neither error changes the plan of the proof, but each necessitates minor changes in several parts of the paper.

I.

On page 49, I assert that given an unstable coalgebra C over the mod p Steenrod algebra A, $C \in \mathbf{CA}$, the module of primitives PC is the suspension of an unstable A-module : $\Sigma^{-1}PC \in \mathbf{U}$. This is true for $p = 2$, but certainly false for $p > 2$: Consider, for instance, the A-coalgebra $\overline{H}_* BZ_p$ which plays such a large role in this paper.

This error makes nonsense of the spectral sequence (2.5),

$$\mathrm{Ext}^s_{\mathbf{U}}(M, \Sigma^{-1}R^tP(C)) \Rightarrow \mathrm{Ext}^{s+t}_{\mathbf{CA}}(\Sigma M, C),$$

which is central to the proof of the Sullivan conjecture. However, we may proceed as follows. Consider, instead of the category $\mathbf{U}$, the category $\mathbf{V}$ of A-modules satisfying the modified unstable condition

$$xP^t = 0 \text{ for } |x| \leq 2pt,$$

and no extra condition on $x\beta$. Since $P^0 = 1$, $M_i = 0$ for $i \leq 0$ if $M \in \mathbf{V}$. The primitives PC in $C \in \mathbf{CA}$ are in $\mathbf{V}$. Thus $R^t_G P : \mathbf{CA} \to \mathbf{V}$ for each t. Since $P : \mathbf{CA} \to \mathbf{V}$ carries injectives to injectives, the proof given in Section 2 leads to the following corrected form of (2.5).

Theorem 2.5. (i) *There is a convergent cohomological spectral sequence*

$$\mathrm{Ext}^s_{\mathbf{V}}(\Sigma M, R^t_G P(C)) \Rightarrow \mathrm{Ext}^{s+t}_{\mathbf{CA}}(\Sigma M, C),$$

natural in $M \in \mathbf{U}$, $C \in \mathbf{CA}$.

(ii) *There is for each t an isomorphism of graded vector spaces*

$$R^t_G P(C) \cong R^t_{S'} P(C),$$

natural in $C \in \mathbf{CA}$.

Theorem 2.6 is true and useful as it stands. Theorem 2.7, while true, correctly proved, and useful in certain contexts [2], is no longer exactly what is needed here. Rather, one has:

THEOREM 2.7′. *If $N \in \mathbf{V}$ is bounded, then*

$$\operatorname{Ext}^s_{\mathbf{V}}\left(\bar{H}_*(\Sigma^n BZ_p), N\right) = 0$$

for all $s \geq 0$ and $n > 0$.

As before, the proof breaks up into two steps. First we prove the restricted form

THEOREM 6.1′. *If N is a bounded object of $\mathbf{V}$, then for each $s \geq 0$*

$$\operatorname{Ext}^s_{\mathbf{V}}\left(\bar{H}_*(\Sigma BZ_p), N\right) = 0.$$

To prove this, recall that in Section 6 it was shown that the A-module $\bar{H}_* BZ_p$ is a summand of

$$\bigoplus_{n=1}^{p-1} \lim_{k \to \infty} G(2np^k).$$

Theorem 6.1′ will follow as before if $\Sigma G(2n)$ is a projective in $\mathbf{V}$.

Now $\mathbf{V}$ is a full subcategory of $\mathbf{U}$, and an epimorphism in $\mathbf{V}$ is an epimorphism in $\mathbf{U}$. Thus if $P \in \mathbf{V}$ is projective in $\mathbf{U}$, then it is projective in $\mathbf{V}$. But $\Sigma G(2n) = G(2n + 1)$ is projective in $\mathbf{U}$. I owe this simple argument to J. Lannes.

One next must modify the work of Section 8. The functor $\Sigma : \mathbf{V} \to \mathbf{V}$ again has a right adjoint, which we denote by Ω', and one needs:

LEMMA 8.2′. *In* $\mathbf{V}$:
(i) *The right derived functors $R^i\Omega'$ are trivial for $i > 1$.*
(ii) *If N is bounded then so are $\Omega' N$ and $R^1\Omega' N$.*
(iii) *If N is of finite type then so are $\Omega' N$ and $R^1\Omega' N$.*

Part (iii) follows of course from the fact that A is of finite type. For the others, we set up the analogue in $\mathbf{V}$ of [1], p. 103. Thus, define a graded vector space $D'M$, for $M \in \mathbf{V}$, by

$$(D'M)_{2pi+1} = M_{2i+1} \qquad \bar{x} \leftrightarrow x,$$

$$(D'M)_{2pi+2} = M_{2i+1} \qquad \bar{\bar{x}} \leftrightarrow x,$$

$$(D'M)_j = 0 \qquad \text{otherwise.}$$

Declare

$$\bar{\bar{x}}\beta = \bar{x},$$
$$\bar{x}P^{pi} = \overline{xP^i},$$
$$\bar{\bar{x}}P^{pi} = \overline{\overline{xP^i}}.$$

Map M to $D'M$ by

$$\begin{aligned} \lambda' x &= \overline{xP^i} && |x| = 2pi+1, \\ &= \overline{\overline{x\beta P^i}} && |x| = 2pi+2, \\ &= 0 && \text{otherwise.} \end{aligned}$$

LEMMA. (i) $D'M \in \mathbf{V}$.
(ii) λ' *is A-linear.*
(iii) $\ker \lambda'$ *and* $\operatorname{coker} \lambda'$ *are suspensions in* $\mathbf{V}$.

We leave the verification of this lemma to the reader. Clearly $\Sigma\Omega' M = \ker \lambda'$, so that we have an exact sequence in $\mathbf{V}$

$$0 \to \Sigma\Omega' M \to M \xrightarrow{\lambda'} D'M \to \Sigma\Omega'_1 M \to 0$$

defining Ω'_1. One checks that if M is injective in $\mathbf{V}$ then λ' is epi; this amounts to the assertion that $\beta^\varepsilon P^i : QH^{2i+1}K_n \to QH^{2pi+1+\varepsilon}K_n$ is mono for $\varepsilon = 0, 1$. Since D' is exact, standard methods show that $R^1\Omega' = \Omega'_1$ and $R^s\Omega' = 0$ for $s > 1$. Also, D' clearly preserves boundedness, so that $(8.2)'$ holds.

The rest of Section 8 proceeds just as before, with $\mathbf{V}$ used instead of $\mathbf{U}$. Theorem 8.8 needs an obvious modification. Theorem 8.9 remains true as stated, since $\Omega' N$ and $\Omega'_1 N$ display even stronger limits on the dimension of their top nonzero degrees than do Ω and Ω_1.

2.

My second error is "Fact 3.9", which is obvious nonsense. This was used in my proof of (3.12). Of the various ways to fix this up, we choose to modify (3.3) using the following definition.

Definition. The *almost-simplicial category* $\tilde{\Delta}$ is the category of ordered sets $[n] = \{0, 1, \ldots n\}$, $n \geq 0$, together with order-preserving maps which send 0 to 0. An almost simplicial object in a category $\mathbf{C}$, $X \in \tilde{s}\mathbf{C}$, is a contravariant functor from $\tilde{\Delta}$ to $\mathbf{C}$.

Thus an almost simplicial object is almost a simplicial object, except that it lacks d_0's. For any almost simplicial abelian group V, define

$$N_n(V) = \{x \in V_n : d_i x = 0 \text{ for all } i > 0\}.$$

From the simplicial identities, one sees that given $k > 0$:

(1) If $d_i x = 0$ for all $i > k$, then $d_i(x - s_{k-1} d_k x) = 0$ for all $i \geq k$.

(2) If $x = s_i y$ for some $i \geq 0$, then

$$x - s_{k-1} d_k x = \begin{cases} s_i(y - s_{k-2} d_{k-1} y) & \text{if } k > i + 1 \\ 0 & \text{if } k = i + 1. \end{cases}$$

These are the observations leading to the usual normalization theorem for simplicial abelian groups. Here they show that any almost simplicial abelian group V splits canonically as

$$V = \bigoplus_{m \geq 0} V(m)$$

where $V(m)$ is the almost simplicial subgroup with

$$V(m)_n = \bigoplus_{\phi : [n] \twoheadrightarrow [m]} \phi^* N_m(V).$$

One thus sees that $N : \tilde{s}Z \to nZ$ is an equivalence of categories.

We now strengthen (3.3) as follows.

Definition 3.3′. A map $i : A \to X$ in $s\mathbf{A}_k$ is *almost free* provided that there is an almost simplicial subgroup V of X such that for each $n \geq 0$, the natural map $A_n \otimes SV_n \to X_n$ is an isomorphism.

Then Theorem 3.4 is true as stated. We recall parts (i) and (ii) for convenience; the meaning of (iii) is unchanged.

THEOREM 3.4. (i) *Any almost free morphism is a cofibration.*

(ii) *Any map $A \to B$ admits a factorization $A \to X \to B$ in which $A \to X$ is almost free and $X \to B$ is an acyclic fibration.*

The proof of (ii) is as given; one must check that $A \to \operatorname{diag} \bar{S}^A_* X$ is almost free in this new sense.

The proof of (i) runs as in the paper, without (3.9), until we discuss the subspaces W_m and $W_{m,n}$. We then merely let $W_m = N_m(V)$ and $W_{m,n} = V(m)_n$ (as above). It is then easy to check that $F_{m-1}X \to F_m X$ is almost free with generating vector space $W_{m,*}$. The rest of the proof goes as before.

We must also check that $\overline{W}$ preserves almost free morphisms (inadvertantly called "locally free" in the proof of (5.5) (b)). Actually, to make this work, we must replace W and $\overline{W}$ by the "opposite" constructions. Let

$$(W'X)_n = X_n \otimes X_{n-1} \otimes \cdots \otimes X_0,$$

$$d_0(x_n \otimes \cdots \otimes x_0) = d_0 x_n \otimes \cdots \otimes d_0 x_2 \otimes (d_0 x_1)(\eta \varepsilon x_0),$$

$$d_i(x_n \otimes \cdots \otimes x_0) = d_i x_n \otimes \cdots \otimes d_i x_{i+1} \otimes (d_i x_i) x_{i-1} \otimes x_{i-2} \otimes \cdots \otimes x_0 \quad \text{for } i > 0,$$

$$s_i(x_n \otimes \cdots \otimes x_0) = s_i x_n \otimes \cdots \otimes s_i x_i \otimes 1 \otimes x_{i-1} \otimes \cdots \otimes x_0,$$

and let $(\overline{W}'X)_n = k \otimes_{X_n} (W'X)_n$. Then $W', \overline{W}'$ enjoy the same properties as $W, \overline{W}$; but in addition, if $A \to X$ is almost free with generating subspaces V_*, then $\overline{W}'A \to \overline{W}'X$ is almost free with generating subspaces U_*,

$$U_n = V_{n-1} \oplus \cdots \oplus V_0.$$

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[3] H. R. MILLER, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. **120** (1984), 39–87.

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