The Sullivan fixed-point conjecture and Brown-Gitler spectra

by Haynes Miller*

Conjecture 0.0. If G is a finite group with classifying space BG, and X is a connected finite complex, then the space X^{BG} of pointed maps from BG to X has the weak homotopy type of a point.

In [15], Dennis Sullivan proposed a conjecture overlapping with this, and pointed out its relevance to the study of real algebraic varieties. In this paper, I combine observations due to G. Carlsson and J. R. Harper to verify the conjecture for G an elementary Abelian 2-group and X a sphere. More generally, I prove:

Theorem 0.1. Let V be an elementary Abelian 2-group and X a finite-dimensional connected CW complex. Assume that the mod 2 cohomology of the universal cover $\tilde{X}$ of X is very nice. Then X^{BV} is weakly contractible.

Here I am following [6] in calling an unstable algebra over the Steenrod algebra A very nice if it admits a simple system of generators spanning a vector space closed under the A-action. This rather awkward restriction on the form of $H^*(\tilde{X})$ is no doubt an artifact of the method of proof. However, it does encompass all spheres and more generally all Stiefel varieties.

Remarks 0.2. (1) Of course, the homotopy lifting property shows that the conjecture remains valid for spaces in the smallest class $\mathcal{C}$ containing such X and containing the total space of any fibration whose fiber and base already belong to $\mathcal{C}$.

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(2) One also has the option of proving Theorem 0.1 in the rank 1 case only, for the general case follows easily from the following general remarks. Let A, B, and X be CW complexes. The cofibration sequence $A \vee B \to A \times B \to A \wedge B$ induces a fibration sequence $X^{A \wedge B} \to X^{A \times B} \to X^{A \vee B}$. Standard properties of the compactly generated topology imply that the fiber is $(X^A)^B$ and the base is $X^A \times X^B$; so if X^A and X^B are both weakly contractible, $X^{A \times B}$ is also.

(3) It is amusing to contrast this result against one of its stable analogues, the conjecture of G. B. Segal. Thus, for any $n \geq 0$,

$$[\Sigma^n BV, S^n] = \pi_n((S^n)^{BV}) = 0$$

according to Theorem 0.1, while the stable cohomotopy $[\Sigma^\infty BV, \Sigma^\infty S^0]$ is the completion of the Burnside ring of V [10], [1]. This is, if you like, a dramatic instance of J. F. Adams' dictum, "Cells now, maps later."

These methods yield information about maps to infinite complexes as well. For instance:

Theorem 0.3. Let V be an elementary Abelian 2-group and X a simply connected CW complex whose mod 2 cohomology is of finite type and very nice. Then the map

$$H^* : [BV, X] \to \mathrm{Hom}(H^*(X), H^*(BV))$$

is bijective. Here [-,-] indicates the set of pointed homotopy classes of pointed maps, and Hom indicates the set of maps of unstable A-algebras.

This result applies for instance when X is the classifying space of the Lie group SU_2. In that case it may be rephrased:

Corollary 0.4. If V is an elementary Abelian 2-group, then

$$B : \mathrm{Hom}(V, SU_2) \to [BV, BSU_2]$$

is bijective.

On the other hand, Theorem 0.3 requires only cohomological information about X. For instance, let X be a simply connected CW complex of finite type, whose mod 2 cohomology is a polynomial algebra on a single 4-dimensional class. One would like to conclude that when X and BSU_2 are localized at 2, they become homotopy equivalent. From Theorem 0.3, we find that they are not distinguishable by means of maps from BV:

Corollary 0.5. Let X be a simply connected CW complex whose mod 2 cohomology is a polynomial algebra on a 4-dimensional class. Then $[B\mathbb{Z}/2, X]$ has exactly two elements, one of which is nontrivial in mod 2 cohomology. Moreover, any map from BV to X, for V an elementary Abelian 2-group, factors up to homotopy through $B\mathbb{Z}/2$.

The proof of these theorems uses a form of obstruction theory due to W. S. Massey and F. P. Peterson [14], [3], [13]. The requirements of this theory are the source of the artificial restrictions on the cohomology of the target space. The possibility of using Massey-Peterson towers in this way was shown to me by J. R. Harper and A. Zabrodsky, and the required technology regarding completions (described in Section 5 below) was provided by A. K. Bousfield. This reduces the problem to an algebraic one of showing the triviality of certain Ext groups in the category $\mathcal{U}^*$ of unstable left A-modules. Now Harper had pointed out to me in 1976 that the n-sphere $S(n) = \Sigma^n \mathbb{F}_2$ has an injective envelope in the category $\mathcal{U}^*$; and

M. E. Mahowald immediately recognized that this injective envelope is given by the cohomology module of the Spanier-Whitehead dual of a certain Brown-Gitler spectrum [8], [11]. In 1980-81, G. Carlsson [10] showed that the cohomology of an elementary Abelian group G splits off of a suitable direct limit of these A-modules. Thus, up to $\varprojlim^1$ problems, $\bar{H}^*(BV)$ is injective as well:

Theorem 0.6. The cohomology of an elementary Abelian 2-group is injective in the category of unstable left A-modules of finite type.

This theorem is proved in Section 3. As an appendix to that section I also indicate a new categorical characterization of Brown-Gitler spectra, inspired by Mahowald's observation.

Now an easy exercise in homological algebra, carried out in Section 4, shows that Theorem 0.6 implies the vanishing of the Ext groups required for the obstruction theory of Section 2.

I begin the paper with an argument, developed in conversation with D. S. Kahn, which allows one to replace the target space by its universal cover in Theorem 0.1:

Theorem 0.7. Let X be a connected CW complex of finite category, G a group of finite exponent, and $f : BG \to X$ a continuous map. Then $f_\# : G \to \pi_1(X)$ is trivial.

In this paper I have generally not tried to push to the limits allowed by the techniques, since it seems likely that a variant of this approach will lead to a general result in the elementary Abelian case. In particular, I have not dealt with odd primes, although such an extension is certainly possible.

In addition to the indispensable assistance of John Harper, which is apparent throughout the paper, I am grateful to Pete Bousfield, Ralph Cohen, Gunnar Carlsson, Dan Kahn, Mark Mahowald, Jeff Smith, and Alex Zabrodsky for their help. I also wish to thank the Mathematics Departments of Northwestern University and the University of Cambridge for their hospitality.

§1. The fundamental group.

A pointed subspace $(A,*)$ of $(X,*)$ is contractible in X if the identity map of X can be deformed rel $*$ to a map sending A to $*$. G. W. Whitehead [18], [17], says that X has category less than n if X admits a covering by n subspaces each contractible in X. It is equivalent to require that the n-fold diagonal $\Delta_n : X \to X^n$ compresses rel $*$ into the fat wedge

$$X^n_{n-1} = \{(x_1,\ldots,x_n) \in X^n : x_i = * \text{ for some } i\} .$$

Three easy consequences of this definition are:

(1.1) Category is a pointed homotopy invariant.

(1.2) If $h^*(-)$ is a multiplicative cohomology theory and X has category less than n, then the ideal $\bar{h}^*(X)$ has trivial $n^{\underline{th}}$ power.

(1.3) The category of a connected CW complex is at most its dimension. This is true for a reduced CW complex (one with a single vertex) by the skeletal approximation theorem, and follows in general by (1.1) since a connected CW complex is pointed homotopy equivalent to the reduced CW complex obtained by collapsing out a maximal tree.

The terms of Theorem 0.7 are now defined. To begin its proof, I need:

Proposition 1.4. Let $*$ be a vertex of the CW complex B, let $p : E \to B$ be a covering projection with E connected, and pick a basepoint $* \in p^{-1}(*)$ for E. Then the category of E is at most the category of B.

Proof. Let $h : B \times I \to B^n$ be a compression rel $*$ of Δ_n into B^n_{n-1}. By the homotopy lifting property there is a compression $\bar{h} : E \times I \to E^n$ of $\Delta_n : E \to E^n$ into $p^{-1}(B^n_{n-1})$, and by uniqueness of path lifting, this compression fixes $* \in E$. For each $e \in p^{-1}(*) = F$, pick a path to $*$, being careful to take the trivial path from $*$ to itself. This gives a map $f : F \times I \to E$. Now F is a subcomplex of E, so the inclusion is a cofibration and we may extend f to a contraction $g : E \times I \to E$ of F in E. Let g_n be the composite

$$g_n : E^n \times I \overset{1\times\Delta_n}{\to} E^n \times I^n = (E\times I)^n \overset{g^n}{\to} E^n .$$

Then g_n compresses $p^{-1}(B^n_{n-1})$ to E^n_{n-1} rel $*$, and the map $k : E\times I \to E^n$ defined by $k(e,t) = g_n(\bar{h}(e,t),t)$ is the required compression of Δ_n rel $*$. □

Proof of Theorem 0.7. Suppose to the contrary that $\sigma \in G$ maps nontrivially under $f_{\#}$. Since G has finite exponent, σ has finite order; say $f_{\#}\sigma$ has order m and σ order mn. Let $p : \tilde{X} \to X$ be the covering projection such that $p_{\#}\pi_1(\tilde{X})$ is the subgroup $\langle f_{\#}\sigma \rangle$ of $\pi_1(X)$ generated by $f_{\#}\sigma$. There results a commutative diagram

$$\begin{array}{ccccc} B\langle\sigma\rangle & \to & \tilde{X} & \to & B\langle f_{\#}\sigma\rangle \\ \downarrow & & \downarrow & & \downarrow \\ BG & \overset{f}{\to} & X & \to & B\pi_1(X) \end{array} .$$

By Proposition 1.4, then, the natural map $Bh : B\mathbf{Z}/mn \to B\mathbf{Z}/m$ factors through a space of finite category. Thus, by (1.2), $(Bh)^*x$ has finite height for any $x \in \bar{K}^*(B\mathbf{Z}/m)$, where $K^*(-)$ is complex K-theory. But this is false. Let ξ_m be the representation of $\mathbf{Z}/m$ on $\mathbb{C}$ in which 1 acts by $e^{2\pi i/m}$. Then $h^*\xi_m = \xi_{mn}$. Now $\xi_m - 1 \in R(\mathbf{Z}/m) \cong \mathbf{Z}[\xi_m]/(\xi_m^m - 1)$ has augmentation 0, and $h^*(\xi_m - 1) = \xi_{mn}^n - 1 \in R(\mathbf{Z}/mn)$ has infinite height. Since an elementary argument ([2] §8) shows that the complex representation ring $R(G)$ of a cyclic group G embeds naturally in $K^0(BG)$, this provides a contradiction. □

Theorem 0.7 allows one to suppose, in considering the Sullivan conjecture, that the target complex is simply connected, provided one is willing to allow it to be merely finite-dimensional rather than finite. It indicates, in fact, that the proper level of generality for the conjecture, at least if one sticks to trivial G-spaces, involves groups of finite exponent and spaces of finite category.

§2. Obstruction theory.

I shall begin by recalling briefly the theory of Massey and Peterson [14], [3], with improvements due to Harper [13] and Zabrodsky. In this section, p is an arbitrary prime.

Mod p cohomology in its richest form is a functor from pointed spaces to the category $\mathcal{A}^*$ of augmented unstable algebras over the Steenrod algebra A. Let $\mathcal{A}^*_{ft}$ denote the full subcategory of those of finite type. Formation of the augmentation ideal gives a functor I to the category $\mathcal{U}^*_{ft}$ of unstable left A-modules of finite type, and this functor has a left adjoint U [14]. It is easy to verify that an object of $\mathcal{A}^*_{ft}$ is very nice in the sense of the introduction iff it is of the form U(M) for some $M \in \mathcal{U}^*_{ft}$.

The functor U helps to relate the category $\mathcal{U}^*_{ft}$ to geometry. This category has enough projective objects, and there is a contravariant association $P \mapsto K(P)$ of a mod p generalized Eilenberg-MacLane space to a projective in $\mathcal{U}^*_{ft}$, equipped with natural isomorphisms

$$\pi_t(K(P)) \cong \mathrm{Hom}_A(P,S(t)) \tag{2.1}$$

$$H^*(K(P)) \cong U(P) \ . \tag{2.2}$$

There is a functor $\Omega : \mathcal{U}^*_{ft} \to \mathcal{U}^*_{ft}$ left adjoint to suspension Σ :

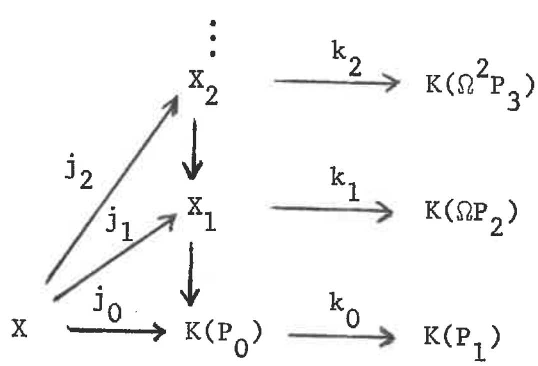
$$\mathrm{Hom}_A(\Omega M,N) \cong \mathrm{Hom}_A(M,\Sigma N) \ . \tag{2.3}$$

Since Σ is exact, Ω carries projectives to projectives, and the isomorphism (2.2) is naturally compatible with Ω:

$$K(\Omega P) \cong \Omega K(P) \ . \tag{2.4}$$

Now let X be a simply connected space such that $H^*(X) \cong U(M)$, and let $M \leftarrow P_\bullet$ be a projective resolution of M in $\mathcal{U}^*_{ft}$. There is a tower of principal fibrations under X :

(2.5)



such that

$$\ker(H^*(X_s) \to H^*(X)) = \ker(H^*(X_s) \to H^*(X_{s+1})) \ ; \tag{2.6}$$

(2.7) k_s is induced by a null-homotopy of $d_s k_{s-1}$. That is, there exists a commutative square

$$\begin{array}{ccc} X_{s-1} & \xrightarrow{h_s} & PK(\Omega^{s-1}P_{s+1}) \\ \Big\downarrow k_{s-1} & & \Big\downarrow \pi \\ K(\Omega^{s-1}P_s) & \xrightarrow{d_s} & K(\Omega^{s-1}P_{s+1}) \end{array} ,$$

in which π is the path-space fibration, such that the induced map $X_s \to K(\Omega^s P_{s+1})$ of homotopy fibers is homotopic to k_s . Here and below I write d_s for any map induced by d_s .

Property (2.7) was suggested by Zabrodsky. It appears to be a fundamental feature of Massey-Peterson towers, and it may be possible to give a treatment of the subject in which it occupies a central position. For the present, however, I give a derivation of it from other known properties at the end of this section.

By applying π_* to (2.5) one obtains a spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}^s(M,S(t)) \underset{s}{\Longrightarrow} \pi_{t-s}(X) .$$

The Ext group here, and below, is computed in the category $\mathcal{U}^*_{ft}$, and $S(t)$ is the reduced cohomology of the t-sphere. This is not quite the use I wish to make of (2.5) here, however. Rather I will use it, together with some completion techniques, to prove:

<u>Theorem 2.8</u>. Let Y be a connected CW complex such that $\bar{H}_*(Y; \mathbf{Z})$ is of finite type and p-torsion, and let X be a simply connected space such that

$H^*(X)$ is of finite type and isomorphic to $U(M)$. Consider the map

$$H^* : [Y,X] \to \operatorname{Hom}_A(M,\bar{H}^*(Y)) .$$

Then H^* is (a) monic if $\operatorname{Ext}^s(M,\bar{H}^*(\Sigma^s Y)) = 0$ for all $s > 0$ and (b) epic if $\operatorname{Ext}^{s+1}(M,\bar{H}^*(\Sigma^s Y)) = 0$ for all $s > 0$.

This theorem and the method of proof presented below are for the most part due to Harper ([13] 2.2.1, for example). I have chosen a different set of convergence conditions. Moreover, an improvement will be noticed in part (a), for Harper proves only that, under the stated assumptions, $f \simeq *$ if $f^* = 0$. That proof is easier, requiring, aside from (2.6), only the elementary fact that $k_s i_s \simeq d_s$, where $i_s : K(\Omega^s P_s) \to X_s$ is the inclusion of the fiber over $*$. This restricted form of Theorem 2.8(a) is in fact all that is needed to prove the Sullivan Conjecture, Theorem 0.1. The full strength of Theorem 2.8 is required, however, to prove Theorem 0.3.

Theorems 0.1 and 0.3 follow immediately from Theorems 2.8 and 0.7 and the following completely algebraic result, which will be proved in Section 4.

Theorem 2.9. Let V be an elementary Abelian 2-group and M an unstable left A-module of finite type. Then

$$\operatorname{Ext}^s(M, \bar{H}^*(\Sigma^n BV)) = 0$$

(a) for any $s > n \geq 0$ and for $s = n > 0$, and

(b) for any $s,n \geq 0$ if M is finite.

Before starting the proof of Theorem 2.8, it is convenient to record a couple of consequences of Zabrodsky's observation (2.7). They both involve principal actions, for which I need some notation. Given a map $k : X \to B$, I shall write $\alpha_k : \Omega B \times E_k \to E_k$, or just α, for the action of ΩB on the homotopy fiber E_k of k. Also, given $f : Y \to E_k$ and $h : Y \to \Omega B$, I shall write $h * f$ for the composite

$$Y \xrightarrow{\Delta} Y \times Y \xrightarrow{h \times f} \Omega B \times E_k \xrightarrow{\alpha} E_k$$

The following lemma is a restatement of "primitivity of the principal action" [13] 1.2.6.

<u>Lemma 2.10</u>. The k-invariants are linear over the algebraic differential. That is, the following diagram is homotopy commutative.

$$\begin{array}{ccc} K(\Omega^s P_s) \times X_s & \xrightarrow{\alpha} & X_s \\ \downarrow d_s \times k_s & & \downarrow k_s \\ K(\Omega^s P_{s+1}) \times K(\Omega^s P_{s+1}) & \xrightarrow{\mu} & K(\Omega^s P_{s+1}) \end{array} .$$

<u>Proof</u>. Use naturality of the principal actions resulting from (2.7), and the fact that $\mu \simeq \alpha_\pi$. □

<u>Corollary 2.11</u>. Let $f : Y \to X$ and $h : Y \to K(\Omega^s P_s)$. Then

$$k_s(h * f) \simeq d_s h * k_s f \ . \quad \square$$

<u>Lemma 2.12</u>. The following diagram is homotopy-commutative.

$$\begin{array}{ccc} K(\Omega^s P_{s-1}) \times X_s & \xrightarrow{d_s \times 1} & K(\Omega^s P_s) \times X_s \\ & \searrow^{pr_2} & \downarrow \alpha \\ & & X_s \end{array}$$

The proof of this lemma involves a technical result about compatibility of various principal actions which, in order not to further delay presentation of the proof of Theorem 2.8, I have placed at the end of this section.

Proof of Theorem 2.8. I shall prove part (a), and leave the proof of part (b), which is similar and somewhat easier, to you. So let Y be a connected CW complex such that $H_*(Y)$ is of finite type, let X and M be as in the statement of the theorem, and suppose that $f, g : Y \to X$ induce the same map in cohomology. Then the composites $f_0, g_0 : Y \to X \to K(P_0)$ are homotopic. I will now show that $f_s, g_s : Y \to X \to X_s$ are homotopic provided f_{s-1} and g_{s-1} are. By principality of $X_s \to X_{s-1}$, there is a map $h : Y \to K(\Omega^s P_s)$ such that $g_s \simeq h * f_s$. Thus by (2.11), $k_s g_s \simeq d_s h * k_s f_s$. Now f_s and g_s both lift to X_{s+1}, so $k_s f_s$ and $k_s g_s$ are both null-homotopic, and it follows that $d_s h \simeq *$. Thus $h^*|\Omega^s P_s \in \mathrm{Hom}_A(\Omega^s P_s, \bar{H}^*(Y)) = \mathrm{Hom}_A(P_s, \bar{H}^*(\Sigma^s Y))$ is a cocycle. By assumption, it is therefore also a coboundary; that is, h factors through $d_{s-1} : K(\Omega^s P_{s-1}) \to K(\Omega^s P_s)$. Lemma 2.12 then implies that $h * f_s$ is homotopic to f_s, as claimed.

Now the issue of whether the homotopies $f_s \simeq g_s$ together yield a homotopy $f \simeq g$ is a question of convergence, and will be dealt with in Section 5. This finishes my treatment of Theorem 2.8. □

It is now time to prove (2.7). The proof is based on:

Lemma 2.13. The composite

$$\delta : K(\Omega^s P_s) \times X \xrightarrow{1 \times j_s} K(\Omega^s P_s) \times X_s \xrightarrow{\alpha} X_s$$

induces a monomorphism in cohomology.

Proof. This follows from a comparison of the "fundamental sequences" [13] associated to the fibrations in the homotopy commutative diagram

$$\begin{array}{ccccc}
K(\Omega^s P_s)\times X & \to & K(\Omega^s P_s)\times X_s & \overset{\alpha}{\to} & X_s \\
\downarrow & & \downarrow & & \downarrow \\
PK(\Omega^{s-1}P_s)\times X & \to & PK(\Omega^{s-1}P_s)\times X_{s-1} & \overset{pr_2}{\to} & X_{s-1} \\
\downarrow & & \downarrow & & \downarrow \\
K(\Omega^{s-1}P_s) & \overset{in_1}{\to} & K(\Omega^{s-1}P_s)\times K(\Omega^{s-1}P_s) & \overset{\mu}{\to} & K(\Omega^{s-1}P_s) ,
\end{array}$$

by analogy with the proof of [13] 1.2.6. □

Lemma 2.14. The k-invariant k_s may be characterized as the unique map k such that (a) the diagram

$$\begin{array}{ccc}
K(\Omega^s P_s)\times X_s & \xrightarrow{\alpha} & X_s \\
\downarrow{\scriptstyle d_s\times k} & & \downarrow{\scriptstyle k} \\
K(\Omega^s P_{s+1})\times K(\Omega^s P_{s+1}) & \xrightarrow{\mu} & K(\Omega^s P_{s+1})
\end{array}$$

is homotopy commutative and (b) $kj_s : X \to K(\Omega^s P_{s+1})$ is null-homotopic.

Proof. The k-invariant k_s satisfies (a) by virtue of primitivity of the principal action, [13], and (b) since j_s lifts to j_{s+1} . On the other hand, (a) and (b) together allow one to compute that $\delta^* k_s = d_{s+1} pr_1 : K(\Omega^s P_s)\times X \to K(\Omega^s P_{s+1})$; but δ^* is monic by Lemma 2.13. □

Proof of (2.7). It follows easily from the compatibility of the splitting of the fundamental sequence with the k-invariant k_{s-1} that $d_s k_{s-1} \simeq *$. Pick a null-homotopy h, and look at the commutative diagram

$$\begin{array}{ccccc}
 & & X_s & \xrightarrow{k} & K(\Omega^s P_{s+1}) \\
 & \overset{j_s}{\nearrow} & \downarrow & & \downarrow \\
X & \xrightarrow{j_{s-1}} & X_{s-1} & \xrightarrow{h} & PK(\Omega^{s-1} P_{s+1}) \\
 & & \downarrow k_{s-1} & & \downarrow \pi \\
 & & K(\Omega^{s-1} P_s) & \xrightarrow{d_s} & K(\Omega^{s-1} P_{s+1}) \ .
\end{array}$$

Any such k satisfies (a) of Lemma 2.14, as noted in the proof of Lemma 2.10. To complete the proof, it therefore suffices to alter h to another null-homotopy h_s such that the map k' induced on homotopy fibers satisfies $k'j_s \simeq *$. Since j_{s-1} is epic in cohomology, kj_s factors as ℓj_{s-1} for some $\ell : X_{s-1} \to K(\Omega^s P_{s+1})$. If χ reverses paths and $*$ juxtaposes them, then $h_s = \chi\ell * h$ has the desired property. □

Proof of Lemma 2.12. This is based on the following technical result about principal actions.

Proposition 2.15. Let h be a null-homotopy of a composite gf, and construct homotopy fibers to produce a commutative diagram

$$\begin{array}{ccccc}
 & & F & \xrightarrow{k} & \Omega Z \\
 & & \downarrow & & \downarrow \\
H & \to & X & \xrightarrow{h} & PZ \\
\downarrow \ell & & \downarrow f & & \downarrow \pi \\
G & \to & Y & \xrightarrow{g} & Z
\end{array}$$

Then the homotopy fibers of k and of ℓ are identical, and if we call this common space E, then the following diagram is homotopy commutative.

$$\begin{array}{ccc} \Omega^2 Z\times\Omega G\times E & \xrightarrow{\Omega\alpha_g\times 1} & \Omega G\times E \\ \downarrow{\scriptstyle 1\times\alpha_\ell} & & \downarrow{\scriptstyle \alpha_\ell} \\ \Omega^2 Z\times E & \xrightarrow{\alpha_k} & E \end{array}$$

To prove this proposition, draw pictures of the elements of the spaces involved; you will see that the homotopy required is similar to the one showing that a double loop space is homotopy commutative. It is convenient to remember that when ΩZ is regarded as the homotopy fiber of π, it maps to PZ by sending a loop to the reverse of its second half.

By including $*$ into ΩG, we find that α_k factors as $\alpha_\ell(\Omega i\times 1)$ where $i : \Omega Z \to G$ is the natural map. Since $i\circ\Omega g \simeq *$, this implies:

Corollary 2.16. Let h be a null-homotopy of a composite gf, and construct homotopy fibers to produce a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{k} & \Omega Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & PZ \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{g} & Z \end{array}$$

Then

$$\begin{array}{ccc} \Omega^2 Y \times E & \xrightarrow{\Omega^2 g \times 1} & \Omega^2 Z \times E \\ & \searrow^{pr_2} & \downarrow \alpha_k \\ & & E \end{array}$$

is homotopy-commutative. □

Lemma 2.12 follows from an application of this Corollary to (2.7). □

§3. Projective objects.

In this section, I shall show how to apply work of G. Carlsson [10] to prove Theorem 0.6; the object of Section 4 will be to show how this in turn implies 2.9. In these two sections, it will be convenient to work in the homology setting; so let $\mathcal{U}$ denote the category of right unstable A-modules, and $\mathcal{U}_{ft}$ its full category of modules of finite type. Thus the Steenrod algebra A is considered to be nonpositively graded, and a right A-module M is unstable provided that $xSq^i = 0$ for all $i > \frac{|x|}{2}$ and all $x \in M$. In these two sections I also reserve the symbol $P(r)$ to denote the reduced mod 2 homology of an elementary Abelian 2-group of rank r, viewed as an object in $\mathcal{U}_{ft}$.

By duality, Theorem 0.6 is equivalent to:

Theorem 3.1. $P(r)$ is projective in $\mathcal{U}_{ft}$.

To see what is involved in this theorem, I make some general remarks about projectives in $\mathcal{U}$ and $\mathcal{U}_{ft}$. Let $\mathcal{M}$ be the category of nonnegatively graded $\mathbf{F}_2$-vector spaces. Since every object of $\mathcal{M}$ is projective, projectives in $\mathcal{U}$ may be produced by constructing a left adjoint G to the forgetful functor $\mathcal{U} \to \mathcal{M}$. For $V \in \mathcal{M}$, $G(V)$ is the largest unstable quotient of the right A-module $V \otimes A$. Evidently one should have, for instance, $G(S(n)) = G(n)$ where $G(n)$ is defined as

(3.2) $$G(n) = \Sigma^n A/A\{Sq^i : 2i > n\} ;$$

and to see that this is indeed the case one must only carry out the easy check that $G(n)$ is unstable. Generally, then,

(3.3) $$G(V) = \bigoplus_{n \geq 0} V_n \otimes G(n) .$$

For $M \in \mathcal{U}$, $G(M)$ is a projective surjecting to M : $\mathcal{U}$ has enough projectives, though $\mathcal{U}_{ft}$ does not, since, for instance, as we shall see, $\bigoplus_{i \geq 0} G(2^i)$ is not of finite type.

Consider now the projective $G(n)$. Write x_n for its generator over A. By adjointness,

(3.4) $$\mathrm{Hom}_A(G(n),M) = M_n .$$

Thus for instance there is a unique A-linear map $G(n) \to G(n+k)$ carrying x_n to $x_{n+k}Sq^k$. Using these maps, define

(3.5) $$G_n = \varinjlim\{G(n) \to G(2n) \to G(4n) \to \ldots\} .$$

By (3.4)

$$\mathrm{Hom}_A(G_n,M) = \varprojlim \{M_n \xleftarrow{Sq^n} M_{2n} \xleftarrow{Sq^{2n}} M_{4n} \leftarrow \ldots\} .$$

It follows that while G_n is not projective in $\mathcal{U}$, $\mathrm{Hom}_A(G_n,-)$ has but one nontrivial right-derived functor:

Lemma 3.6. For any $M \in \mathcal{U}$,

$$\mathrm{Ext}^s(G_n, M) = \lim_{\leftarrow} M_{2^i n} \qquad s = 0$$
$$= \lim_{\leftarrow}{}^1 M_{2^i n} \qquad s = 1$$
$$= 0 \qquad s > 1 . \quad \square$$

If for example M is bounded above and $n > 0$, then obviously all these Ext groups are trivial. Alternatively, suppose M is of finite type. Then $M_{2^i n}$ is finite for each i, so $\lim_{\leftarrow}{}^1 M_{2^i n} = 0$. It is not hard to show that G_n is itself of finite type; so we have

Lemma 3.7. G_n is projective in $\mathcal{U}_{ft}$. $\square$

Notice that there is a unique nontrivial map $f : G_1 \to P(1)$ in $\mathcal{U}_{ft}$, where $P(1) = \bar{H}^*(B\mathbf{Z}/2)$. I come now to the following amazing observation of G. Carlsson [10]:

Proposition 3.8. This map f is a split epimorphism in $\mathcal{U}_{ft}$.

Since summands of projectives are projective, Theorem 3.1 follows in the rank one case. I pause to recall Carlsson's argument. Let $\Delta : G(2^i) \to G(2^i) \otimes G(2^i)$ be the A-linear map sending x_{2^i} to $(x_{2^i} \otimes x_{2^i})Sq^{2^i}$. Dually, $G(2^i)^*$ is an "algebra" over the Steenrod algebra, albeit without unit and nonassociative. An observation due to Mahowald is tantamount to the assertion that $G(2^i)^*$ is generated by a single element, in dimension 1. These products are compatible over i, and make G_1^* into an "algebra" with a single generator x. It follows from this that G_1 is of finite type, as claimed above. Moreover, a dimension count shows that G_1^* is the free "algebra" on a generator x of dimension 1. It therefore maps A-linearly to the free associative "algebra" on x, which is $P(1)^*$. The dual of this projection splits f.

I must deal also with the case of higher rank (though not to prove Theorem 0.1, as remarked in (0.2)). Carlsson shows that $G_1^{\otimes r}$ is a suitable direct limit of $G(n)$'s ; so the same arguments apply to show that $P(1)^{\otimes r}$ is projective in $\mathcal{U}_{ft}$. Theorem 3.1 then follows in general, since $P(r)$ is a direct sum of such A-modules. □

Appendix: Dual Brown-Gitler spectra as projective covers of spheres.

The unstable A-module $G(n)$ is the projective cover of $S(n) = \bar{H}_*(S^n)$ in $\mathcal{U}$. Recall, e.g. from [16] pp.89 ff., that a morphism $p : P \to X$ in a category is a projective cover provided that (a) P is projective, (b) p is epi, and (c) any $f : Y \to P$ such that pf is epi is itself epi. In general, given X, (a) - (c) characterize p (if it exists) up to a noncanonical isomorphism. In the present context, the isomorphism is actually canonical, since $G(n)$ is monogenic.

Now $G(n)$ is the homology of the n-dual $T(n)$ of the Brown-Gitler spectrum $B[\frac{n}{2}]$, [8]. One is thus compelled to ask whether this spectrum admits a characterization as a projective cover.

Let $\mathcal{S}_H$ be the full subcategory of the stable category consisting of spectra which (a) are connective, (b) have trivial homology with $\mathbf{Z}[\frac{1}{2}]$ coefficients, and (c) admit a mod 2 homology monomorphism to the suspension spectrum of some space. Condition (b) serves to eliminate from consideration all primes (even 0) other than 2. Given a connective spectrum X there is a map $\tilde{X} \to X$ terminal among maps from $H\mathbf{Z}[\frac{1}{2}]$-acyclic spectra, namely, the fiber of the localization map $X \to X[\frac{1}{2}]$. The map $\tilde{X} \to X$ is a mod 2 homology isomorphism, and is called the cocompletion of X at 2; cf. [5].

The proof of Theorem B of [9] implies that $T(n)$ lies in $\mathcal{S}_H$, provided that we replace $T(0) = S^0$ and $T(1) = S^1$ by their cocompletions.

The natural notion of "epi" in $\mathcal{S}_H$ is not the categorical one, which is, of course, what I had in mind above, but rather this: a map $f : X \to Y$ in $\mathcal{S}_H$ is an H-epi provided that $H_*(f)$ is epi. Then H-projectives and H-projective covers may be defined accordingly, and one has:

Theorem 3.9. $T(n)$ is an H-projective cover of $\tilde{S}^n$ in the category $\mathcal{S}_H$.

Proof. Condition (b) of the definition is trivial, and (c) follows from the cyclicity of $H_*(T(n))$. It remains to check that $T(n)$ is projective in $\mathcal{S}_H$; and by duality this is contained in the assertion that if $f : X \to Y$ is an H-epi in $\mathcal{S}_H$ then $\pi_r(B(k)\wedge X) \to \pi_r(B(k)\wedge Y)$ is epi for $r \le 2k+1$. I leave the case $k = 0$ to you; remember that $B(0)$ is the 2-adically completed 0-sphere. Recall (from [11]) that in [8] Brown and Gitler construct, for given $k > 0$, a diagram of cofibration sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & * \\ & & \nwarrow & & \nwarrow & & \nwarrow & & \\ & & H_2 & & H_1 & & H_0 & & \end{array} \tag{3.10}$$

and compatible maps $B(k) \to E_s$, in which (a) each H_s is a mod 2 generalized Eilenberg MacLane spectrum, (b) $B(k) \to E_s$ becomes highly connected as s becomes large, and (c) for any space K, $\pi_r(H_s\wedge K) \to \pi_r(E_s\wedge K)$ is monic for $r \le 2k$. If $X \to \Sigma^\infty K$ is a mod 2 homology monomorphism, then clearly $\pi_r(H_s\wedge X) \to \pi_r(E_s\wedge X)$ is monic for

$r \leq 2k$ as well. Thus $\pi_r(E_s \wedge X) \to \pi_r(E_{s-1} \wedge X)$ is epic for $r \leq 2k+1$. Now let $f : X \to Y$ be an H-epi in $\mathcal{S}_H$. It then follows that $\pi_r(E_s \wedge X) \to \pi_r(E_s \wedge Y)$ is epi for $r \leq 2k+1$, by induction on s using the fact that in the diagram

$$\begin{array}{ccccccc} \pi_r(H_s \wedge X) & \to & \pi_r(E_s \wedge X) & \to & \pi_r(E_{s-1} \wedge X) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_r(H_s \wedge Y) & \to & \pi_r(E_s \wedge Y) & \to & \pi_r(E_{s-1} \wedge Y) & & \end{array}$$

surjectivity of the end vertical arrows implies surjectivity of the middle vertical arrow. The result now follows from property (c) of (3.10). □

The usual proof that projective covers are unique shows here that any two projective covers of X are mod 2 homology-isomorphic over X. But for $H\mathbb{Z}[\frac{1}{2}]$-acyclic spaces, this gives an integral homology equivalence, and hence, using connectivity, a homotopy equivalence, by the Whitehead theorem. Thus Theorem 3.9 provides a characterization of $T(n)$, and hence of its dual, $B[\frac{n}{2}]$.

The category $\mathcal{S}_H$ is of course very artificial. A more natural category $\mathcal{S}$ is afforded by replacing (c) in the definition by: (c') admit a split monomorphism to the suspension spectrum of some space. If $T(n) \in \mathcal{S}$, then the analogue of Theorem 3.9 holds in $\mathcal{S}$. Thus I propose the

Conjecture 3.10. The n-dual of $B[\frac{n}{2}]$ is a summand of the suspension spectrum of some space.

§4. Homological algebra.

I shall next show how to obtain Theorem 2.9 from Theorem 3.1. This requires consideration of derived functors of Ω^n, the n-fold iterate of $\Omega : \mathcal{U} \to \mathcal{U}$. In this homology context, Ω may be defined as follows. For $M \in \mathcal{U}$, define $DM \in \mathcal{U}$ by $(DM)_{2n} = M_n$ (via $\bar{\ }$), $(DM)_{2n+1} = 0$, and $\bar{x}Sq^{2i} = \overline{xSq^i}$. Then there is an exact sequence in $\mathcal{U}$

$$0 \to \Sigma\Omega M \to M \xrightarrow{\lambda} DM$$

where $\lambda x = 0$ if $|x|$ is odd and $\lambda x = \overline{xSq^i}$ if $|x| = 2i$. Clearly Coker λ is a suspension; write it as $\Sigma\Omega_1 M$. Then on general principles Ω_1 is the first right derived functor of Ω, and there are no higher derived functors: $\Omega_s = 0$ for $s > 1$.

Now Ω^n has derived functors too, which we write Ω^n_s. In the study of them and their relationship with Ext, two spectral sequences play a role. They derive from the compositions

$$\Omega^m \circ \Omega^n = \Omega^{m+n}$$

$$\mathrm{Hom}_A(M,-) \circ \Omega^n = \mathrm{Hom}_A(\Sigma^n M,-)$$

and the fact that Ω^n, being right adjoint to the exact functor Σ^n, carries injectives to injectives. They have the form

(4.1) $$\Omega^m_s \Omega^n_t N \underset{s}{\Longrightarrow} \Omega^{m+n}_{s+t} N$$

(4.2) $$\mathrm{Ext}^s(M, \Omega^n_t N) \underset{s}{\Longrightarrow} \mathrm{Ext}^{s+t}(\Sigma^n M, N) .$$

The "Singer" spectral sequence (4.1), together with the above construction of Ω and Ω_1, gives by induction

(4.3) $$\Omega^n_t = 0 \quad \text{for} \quad t > n .$$

(4.4) $$\Omega^n_n = \Omega_1 \Omega^{n-1}_{n-1} .$$

(4.5) If N is of finite type, so is $\Omega^n_t N$.

(4.6) If N is finite, so is $\Omega^n_t N$.

Now take for M the A-module $P(r) = \bar{H}_* BV$, with V an elementary Abelian 2-group of rank r. Since $P(r)$ is projective in $\mathcal{U}_{ft}$, the "EHP" spectral sequence (4.2) collapses to a natural isomorphism

(4.7) $$\mathrm{Ext}^s(\Sigma^n P(r),N) \cong \mathrm{Hom}_A(P(r), \Omega^n_s N) .$$

It is thus trivial for $s > n$, by (4.3), and this gives us Theorem 2.8(a) except when $s = n > 0$. For the remaining case, note that $\Omega_1 M$, being concentrated in odd dimensions, is always a suspension. Thus, by (4.4), $\Omega^n_n M$ is a suspension, so the last case falls to the trivial

Lemma 4.8. $\mathrm{Hom}_A(P(r), \Sigma N) = 0$ for any $N \in \mathcal{U}$.

Proof. Suppose $f(x) \neq 0$, $|x| = i$. Since $H^* BV$ is polynomial, there exists y such that $y Sq^i = x$. But Sq^i acts trivially on $(\Sigma N)_{2i} = N_{2i-1}$, by unstability.

Similarly, Theorem 2.8(b) follows from (4.6), (4.7), and the following equally trivial

Lemma 4.9. $\mathrm{Hom}_A(P(r),N) = 0$ for any $N \in \mathcal{U}$ which is bounded above. □

§5. Convergence.

The final task is to prove a convergence theorem. While Massey and Peterson [14] did important work on this issue, it seems better to appeal to the now standard work of Bousfield and Kan [7]; so move to the simplicial framework by passing to singular simplicial sets. To relate a Massey-Peterson tower for X to the p-adic completion $(\mathbb{Z}/p)_\infty X$ of [7], we have:

Lemma 5.1. Let X be a simply-connected space such that $H^*(X)$ is of finite type and very nice. Let (2.5) be a Massey-Peterson tower for X. Then $\{X_i\}$ and $\{(\mathbb{Z}/p)_i X\}$ are weakly equivalent prosystems.

Proof. By [7] III §5.5, p.84, each X_i is $\mathbb{Z}/p$-nilpotent. By (2.6), the first image prosystem $\{\mathrm{Im}(H_*(X_i) \to H_*(X_{i-1}))\}$ is the constant system $\{H_*(X)\}$. Thus $\{X_i\}$ is a $\mathbb{Z}/p$-tower for X, so the result follows from [7] III §6.4, p.88. □

According to [7] VIII §3, homotopy classes of maps agree in the categories of CW complexes and of simplicial sets; so the following theorem is sufficient for our purpose.

Theorem 5.2. Suppose that X is connected and nilpotent and that Y is connected with $\bar{H}_*(Y; \mathbb{Z}[\frac{1}{p}]) = 0$. Then the map $X \to (\mathbb{Z}/p)_\infty X$ induces an equivalence of pointed mapping spaces

$$X^Y \to ((\mathbb{Z}/p)_\infty X)^Y .$$

The statement of the theorem in this generality and the proof given here are both due to A. K. Bousfield, and I am grateful to him for allowing me to reproduce them.

<u>Proof</u>. Recall from [12] that there is, up to homotopy, a fiber square

$$\begin{array}{ccc} X & \to & \prod_{\ell} X_{\mathbf{Z}/\ell} \\ \downarrow & & \downarrow \\ X_{\mathbf{Q}} & \to & (\prod_{\ell} X_{\mathbf{Z}/\ell})_{\mathbf{Q}} \end{array}$$

where X_G denotes the Bousfield $H_*(-;G)$-localization of X [4]. (As noted in [12], the alleged counterexample to this in [7] VI §8.5, p.195, is false.) Thus there is, up to homotopy, an analogous fiber square of pointed function spaces with source space Y. Now Proposition 12.2 of [4] easily implies that C^B is contractible whenever B is h_*-acyclic and C is h_*-local. Taking $h_*(-) = H_*(-; \mathbf{Z}[\frac{1}{p}])$, it follows that $(Z_G)^Y \simeq *$ for any space Z, where $G = \mathbf{Q}$ or $G = \mathbf{Z}/\ell$ with ℓ prime to p. Thus the fiber square implies that the map

$$X^Y \to (X_{\mathbf{Z}/p})^Y$$

is an equivalence, and the proposition follows since $X_{\mathbf{Z}/p} \simeq (\mathbf{Z}/p)_\infty X$ by §4 of [4]. $\square$

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