Elliptic Moduli in Algebraic Topology

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This is a report on joint work in progress with M. J. Hopkins.

The theme. Recently topology has been interacting in new ways with algebra via the following pull-back diagram.

$$\begin{array}{ccc} \text{topology} & \longleftarrow & \text{new stuff} \\ \downarrow & & \downarrow \\ \text{sets} & \longleftarrow & \text{algebra} \end{array}$$

The "new stuff" here forms a natural enrichment of the algebra under it. The algebra today is moduli of elliptic curves.

Here's the pattern. If I is some shape of diagram—some small category—and D is a functor from it to groups, I can form $\lim D$, the group of of compatible families of elements of the $D(i)$. For example a pull-back is a limit, as is the fixed point set of a group action.

Now suppose that $D_* = \pi_*(X)$, for a diagram of *spaces*. A characteristic feature of homotopy theory is that one can form a modification of the notion of limit which is both more homotopy-invariant and more interesting: the *homotopy limit* $\operatorname{holim} X$. For example,

$$\operatorname{holim}\left\{ \begin{array}{ccc} & & X \\ & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \right\} = \left\{ \begin{array}{cc} & x \\ & \\ y & (\omega : g(y) \sim f(x)) \end{array} \right\}$$

If $X = * = Y$ then the holim is empty if they land in different path components, and homotopy equivalent to the space of pointed loops on Z if the are in the same component. The homotopy limit of a group action is the *homotopy fixed point set* $\operatorname{map}_G(EG, X)$. The homotopy limit is *not* the limit in the homotopy category, but rather a more sophisticated construction which in general depends not just on the image of the diagram in the homotopy category but rather on the diagram of spaces itself. On the other hand, unlike the actual limit, a map of diagrams which is a homotopy equivalence on each object induces a homotopy equivalence of the homotopy inverse limits.

There is a map

$$\pi_* \operatorname{holim} X \to \lim \pi_* X$$

which is in general neither injective nor surjective. It's the edge homomorphism of a spectral sequence.

Model example. In this work we'll replace spaces by *spectra*. These objects should be better-known outside Topology since they they make life so much easier. They behave like spaces which can have homotopy groups in negative dimensions. In compensation the homotopy in each dimension is an abelian group. They represent cohomology theories. For example, topological K-theory is represented by a spectrum K. $\pi_n K = \bar{K}(S^n)$, which is zero for n odd and $\mathbb{Z}$ for n even. In fact K-theory is a "ring-spectrum," and $\pi_* K = \mathbb{Z}[u^{\pm 1}]$. It is a "periodic ring spectrum," in the sense that it has no odd homotopy and has a unit in dimension 2 (the Bott class).

Periodic ring spectra represent computable cohomology theories. For example if E is a periodic ring spectrum then

$$E^0(\mathbb{C}\mathrm{P}^\infty) \cong E^0[[x]]$$

so there is a "first Chern class" or Euler class for complex line bundles (Novikov and Quillen). It satisfies a product law

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2)).$$

The power series F is a *formal group law*. For example we can take $e(L) = 1 - L$ in the case of K-theory, and then the formal group law is the *multiplicative formal group* $G_m(x, y) = x + y - xy$. I should point out that the formal group *law* depends upon the parameter x, while the formal group is canonically determined by E.

Complex conjugation acts on this spectrum; this is a diagram of spectra. $Tu = -u$ so the fixed subring is $\mathbb{Z}[u^{\pm 2}]$. On the other hand

$$\operatorname*{holim}_{\mathbb{Z}/2} K = KO.$$

There is a natural map $\pi_* KO \to (\pi_* K)^{\mathbb{Z}/2}$, which is neither onto (u^{4k+2} is not in the image) nor one-to-one (there is 2-torsion in $\pi_{8k+1,2} KO$). There is a spectral sequence

$$H^*(\mathbb{Z}/2; \pi_* K) \Longrightarrow \pi_* KO.$$

Why spectra? Perhaps I should say a word about why you should care about spectra. I'll motivate from index theory. A *genus* is an additive and multiplicative bordism invariant of manifolds with some geometric structure; for example a complex structure, or, better, a complex structure on the normal bundle: a *U-manifold*. Novikov and Milnor showed that a genus on U-manifolds with values in a $\mathbb{Q}$-algebra is determined by its values on $\mathbb{C}\mathrm{P}^n$. The *Todd genus* for example is such that

$$\mathrm{Td}(\mathbb{C}\mathrm{P}^n) = 1$$

for every n. According to Thom, the ring of bordism classes of U-manifolds is isomorphic to the homotopy ring of a spectrum, the unitary Thom spectrum MU. The Todd genus thus defines a homomorphism of graded rings $\pi_*(MU) \to \mathbb{Z}[u^{\pm 1}]$.

Hirzebruch's book is devoted to showing that for complex manifolds this coincides with the *arithmetic genus*, i.e. (after Hirzebruch) the alternating sum of the dimensions of the cohomology groups (which are finite dimensional) of the Dolbeault complex

$$0 \longrightarrow C^\infty(M) \xrightarrow{\bar\partial} C^\infty(\bar T^* M) \xrightarrow{\bar\partial} C^\infty(\Lambda^2 \bar T^* M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} C^\infty(\Lambda^n \bar T^* M) \longrightarrow 0.$$

Now suppose that $E \downarrow X$ is a *family* of complex manifolds. Any genus gives us a *number* for each point in X—locally the same number, of course. But the Dolbeault complex gives us more: the cohomology groups form *vector bundles* over X, and their alternating sum is an element of $K(X)$. By pairing the Dolbeault complex with a vector bundle over E you get a map $K(E) \to K(X)$.

On the other hand, the fact that complex vector bundles have K-theory Euler classes leads to a map of ring spectra $MU \to K$. This map induces the Todd genus in homotopy, and also constructs from the bundle $E \to B$ a *Gysin map* $K(E) \to K(B)$. The index theorem for families identifies these two covariant maps.

The homotopy type of the spectrum reflects the geometry (vector bundles here), and the orientation $MU \to K$ reflects the analysis (index theory).

Elliptic curves. The next analogue is much more interesting. Here the diagram is indexed by a certain category of elliptic curves, which I review in pedestrian form.

Look at a smooth cubic plane projective curve E over a field k, and suppose that $o \in E(k)$. The k-valued points form a group by requiring that the sum of the three points at which E meets a line is o. This curve can be normalized so that $o = [0, 1, 0]$ is the unique point at infinity and that the line at infinity is tangent to E. By scaling x and y appropriately this curve is given by a Weierstrass equation

$$E : \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \qquad a_i \in R.$$

I've written R here because this equation makes sense over any ring R. Smoothness is equivalent to a certain polynomial, the discriminant Δ, being a unit in R. There are still some coordinate changes that preserve this form. (I'll omit scaling, which contributes a grading to everything.)

$$\begin{aligned} x &= x' + r \\ y &= y' + sx' + t \end{aligned}$$

The set of Weierstrass equations is a functor of R which is representable by the ring

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}].$$

The group of coordinate changes is represented by a Hopf algebra with underlying ring

$$S = \mathbb{Z}[r, s, t]$$

which co-acts on A: $\psi : A \to A \otimes S$.

We can form a category of Weierstrass curves E/R, with maps $E/R \to E'/R'$ given by a ring homorphism $f : E \to E'$ and a coordinate change $fE \to E'$. Actually, it is better to form the assocated stack $\mathcal{E}ll$ in the flat topology on affine schemes, and we do so.

Now $E/R \mapsto R$ is a functor on this category, and we may form the limit. This will give natural invariants of elliptic curves in the ground-ring: i.e., polynomials in a_i, Δ, which are left fixed by coordinate changes. This ring of "integral modular forms" can also be thought of as the ring

$$H^0(S; A)$$

of primitive elements for the coaction of S by coordinate changes, and was computed by Tate and Deligne:

$$H^0 = \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = 12^3\Delta).$$

Topological modular forms. To begin with one wishes to associate a spectrum to an elliptic curve. I do not know how to do this in general, but if E/R satisfies a simple flatness condition then it can be done. The flatness condition is that the map

$$A \xrightarrow{\psi} A \otimes S \xrightarrow{\text{“}E\text{”} \otimes 1} R \otimes S \tag{1}$$

should be flat. For example the universal case $R = A$ works, as does the "Legendre curve"

$$y^2 = x(x-1)(x-\lambda) \qquad \text{over} \qquad \mathbb{Z}[1/2, \lambda^{\pm 1}, (1-\lambda)^{-1}].$$

Theorem I. There is a lift in

$$\begin{array}{ccc} & & \begin{pmatrix}\text{Periodic} \\ \text{Ring Spectra}\end{pmatrix} \\ & \nearrow & \downarrow \\ \mathcal{E} : \mathcal{E}ll_{\text{flat}} & \underset{\substack{\text{completion} \\ \text{along o}}}{\longrightarrow} & (\text{Formal Groups}) \end{array}$$

This is based on the work of Quillen, relating complex cobordism to the theory of formal groups. Peter Landweber used this to give a general prescription for constructing a spectrum from a formal group. The theorem of Pierre Conner and Ed Floyd relating K-theory to complex bordism was the motivating example. The first example of such a construction starting with an elliptic curve was due long ago to Jack Morava, and more recently to Landweber, Doug Ravenel, and Bob Stong. The possibility of a more general construction was perceived by Jens Franke. The final touches rely on recent work of Mark Hovey and Neil Strickland. (As a technical point, one must restrict still further to insure that the completion at o is a formal group in the usual sense, rather than just locally so, but we won't belabor the issue.)

Now we'd like to form a homotopy limit of this diagram, but a diagram up to homotopy is not sufficiently rigid to make this construction (even though the homotopy limit is homotopy invariant!). We have to lift further to some category of spectra and real maps rather than homotopy classes of maps between them. For this it turns out to be useful to use the ring-structure. There is a category of "A_∞-ring spectra," which is a topological version of the theory of associative rings. It forms a topological model category. It is due in different forms to a large group of people. The result I am reporting on is the sort of application one can make of the technical work on spectra and could not be done without it.

Three obstruction theories. First, there is an obstruction theory for the existence of an A_∞ structure. To make the obstructions vanish I must restrict the elliptic curve further, requiring that the map (1) should be not just flat but "étale." Etale means that in a homotopical sense there are no relative differentials; so that R can't be too big relative to A. In fact the universal Weierstrass curve itself is not étale, but the Legendre curve and enough other examples are.

Theorem II: Objects. For E/R étale, $\mathcal{E}^{E/R}$ admits an essentially unique A_∞ structure.

Here we rely on an obstruction theory developed by Alan Robinson and more recently and in different form by Charles Rezk and by Hopkins and Paul Goerss.

Theorem III: Morphisms. There is a lift in

$$\begin{array}{ccc} & & \mathrm{Ho}(\mathrm{Periodic}A_\infty) \\ & \nearrow & \downarrow \\ \mathcal{E}:\mathcal{E}ll_{\text{ét}} & \longrightarrow & (\text{Periodic Ring Spectra}) \end{array}$$

and the lift is fully faithful.

The final job is to lift the diagram from $\mathrm{Ho}A_\infty$ to a diagram in A_∞ itself. For this a key observation is that any $f : E/R \to E'/R'$ induces a homotopy equivalence

$$A_\infty(\mathcal{E}^{E/R}, \mathcal{E}^{E'/R'})_f \xleftarrow{f^*} A_\infty(\mathcal{E}^{E/R}, \mathcal{E}^{E/R})_1 :$$

the diagram is *centric*. This allows us to apply an obstruction theory developed by Bill Dwyer and Dan Kan for use in the theory of p-compact groups. It leads to

Theorem IV: Associativity. There is an essentially unique lift

$$\begin{array}{ccc} & & A_\infty \\ & \nearrow & \downarrow \\ \mathcal{E}:\mathcal{E}ll_{\text{ét}} & \longrightarrow & \mathrm{Ho}A_\infty . \end{array}$$

This rigid diagram *is* elliptic cohomology. It is the sort of structure which would emerge naturally from a construction involving geometric cocycles (analogous to vector bundles).

Now, finally, I can take

$$TMF = \operatorname*{holim}_{\mathcal{E}ll_{\text{et}}} \mathcal{E}$$

in the category A_∞. The result, like KO, is an A_∞ ring spectrum. There is a spectral sequence

$$H^*(S; A) \Longrightarrow \pi_* TMF$$

whose edge homomorphism

$$\pi_* TMF \to H^0$$

is neither one-to-one nor onto. There is nontrivial higher cohomology (all killed by 24). There are nontrivial differentials (on Δ, for example). So Δ is not a *topological* modular form, though 24Δ and Δ^{24} are. Δ^{24} is a unit, giving TMF a periodicity of degree 24^2. The homotopy of TMF (which is hardest to understand at the prime 2) has been studied in detail by Hopkins and Mark Mahowald. In fact, the completion at 2 of this spectral sequence has been known to Mahowald for twenty-five years.

The Witten genus and further questions. The action by complex conjugation on K leads to a variant of the Todd genus $MU \to K$ with values in KO, no longer on U-manifolds but rather on Spin manifolds: the $\hat{A}$ genus, or, more subtly, the "Atiyah invariant" $\alpha : MSpin \longrightarrow KO$. This genus also has an index-theory interpretation, by means of the Dirac operator.

It seems that there should be an analogue for TMF. Witten produced a genus which takes values in modular forms on "String manifolds," that is, manifolds whose structure group reduces to the next connective cover of $Spin(n)$. This amounts to $p_1 = 0$. The connective covering group is a good topological group but is no longer finite dimensional. (It's a bundle over $Spin(n)$ with $\mathbb{C}\mathrm{P}^\infty$ as fiber.) An analogue of the Clifford algebra approach would be nice.

Matthew Ando, Hopkins, and Neil Strickland have shown that there is a *canonical* ring-spectrum map

$$MStr \to \mathcal{E}^{E/R}$$

for any flat object E/R (or indeed for any "elliptic spectrum"). The proof uses the "theorem of the cube." This is a beautiful result, but doesn't quite do what we want. We would like to lift this to a map from the constant diagram $MStr$ to the diagram $\mathcal{E}$ in A_∞. If this can be done, then we get an orientation

$$\omega : MStr \longrightarrow TMF$$

enriching the Witten genus. This would put interesting restrictions on the image of the Witten genus and also define a whole batch of new torsion invariants for string manifolds.

The motivating question remains: what is the corresponding theory of geometric cocycles? What is the analogue of index theory?