Kervaire invariant and Hopf invariant for the Moore space
Haynes Miller
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Theorem. The following are equivalent for $j \geq 3$.
(1) The element h_j^2 in the Adams spectral sequence for the sphere survives to a stable homotopy class of order 2.
(2) There is a 3-cell complex with a mod 2 Moore space at the bottom, the relative attaching map of the top cell is detected by h_j^2, and $\mathrm{Sq}^{2^{j+1}}$ is nontrivial from bottom to top.
(3) The element $h_{j+1}e_0$ in the Adams spectral sequence for the mod 2 Moore space survives to a stable homotopy class (where e_0 is the bottom homology class in the Moore space).

The fact which ties these together is the Adams differential

$$d_2 h_{j+1} = h_0 h_j^2\,, \quad j \geq 3$$

Proof. (2) clearly implies each of the other claims.

We start by proving that (3) implies (2). That is to say, the relative attaching map of the top cell of the mapping cone of an element of the homotopy of a Moore space represented by $h_{j+1}e_0$ is represented by h_j^2.

Let

$$S^0 = F^0 S^0 \leftarrow F^1 S^0 \leftarrow F^2 S^0 \leftarrow \cdots$$

be the Adams tower corresponding to the minimal resolution

$$0 \leftarrow \mathbb{F}_2 \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

of $\mathbb{F}_2$ over the Steenrod algebra. Thus

$$H^*(F^s/F^{s+1}) = \Sigma^{-s} P_s$$

The composite

$$F^{s-1}/F^s \to \Sigma F^s \to \Sigma F^s/F^{s+1}$$

induces the differential in cohomology, and (by minimality) zero in homotopy. Thus

$$\pi_i(F^s/F^{s+1}) = \mathrm{Ext}_A^{s,s+i}(\mathbb{F}_2, \mathbb{F}_2)$$

An Adams tower for any spectrum X may be obtained by smashing X with this Adams tower for the sphere. Smashing is compatible with cofiber sequences, so if M denotes the mod 2 Moore space, the cofiber sequence

$$S^0 \to M \to S^1$$

induces a 3×3 diagram

$$\begin{array}{ccccc}
F^2S^0 & \longrightarrow & F^1S^0 & \longrightarrow & F^1S^0/F^2S^0 \\
\downarrow & & \downarrow & & \downarrow \\
F^2M & \longrightarrow & F^1M & \longrightarrow & F^1M/F^2M \\
\downarrow & & \downarrow & & \downarrow \\
F^2S^1 & \longrightarrow & F^1S^1 & \longrightarrow & F^1S^1/F^2S^1
\end{array}$$

Let $n = 2^{j+1} - 1$, and let $\alpha \in \pi_n(M)$ be an element detected by $h_{j+1}e_0$. Write $\alpha \in \pi_n(F^1M)$ also for any lift of this element.

By definition, the image of α in $\pi_n(F^1M/F^2M)$ is a cycle representing $h_{j+1}e_0$.

There is a stronger "compatibility" between the smash product and the triangulated structure, described for example in [1], that results in the following assertion: There exist elements $\gamma \in \pi_n(F^1S^0/F^2S^0)$ and $\beta \in \pi_n(F^2S^1)$ which
—map to the same element as α does, in $\pi_n(F^1M/F^2M)$ and $\pi_n(F^1S^1)$, respectively, and
—both map to the same element in $\pi_{n-1}(F^2S^0)$.

Any element of $\pi_n(F^1S^0/F^2S^0)$ is a cycle, by minimality, so γ is a cycle. The assumption that α is represented by $h_{j+1}e_0$ amounts to the assertion that $\gamma = h_{j+1}$.

The boundary $\partial h_{j+1} \in \pi_{n-1}(F^2S^0)$ maps to zero in $\pi_{n-1}(F^2S^0/F^3S^0)$, since h_{j+1} is a cycle (or since the boundary map is zero in homotopy), so it lifts to an element $\zeta \in \pi_{n-1}(F^3S^0)$. The image of ζ in

$$\pi_{n-1}(F^2S^0/F^3S^0) = \mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2) = 0$$

represents $d^2h_{j+1} = h_j^2$.

The boundary map $F^2S^1 \to \Sigma F^2S^0$ is given by multiplication by 2, so it increases filtration; it factors through a map $h_0 : F^2S^1 \to \Sigma F^3S^0$. Thus we can take

$$\zeta = h_0\beta$$

The element $\beta \in \pi_n(F^2S^1)$ lifts the relative attaching map of the $n+1$ cell in the mapping cone of $\alpha \in \pi_n(M)$, so the relative attaching map is represented in the Adams spectral sequence by the image of $\overline{\beta} \in \pi_n(F^2S^1/F^3S^1)$ of β. The map $h_0 : F^2S^1 \to \Sigma F^3S^0$ descends to a map

$$h_0 : F^2S^1/F^3S^1 \to \Sigma F^3S^0/F^4S^0$$

which thus sends $\overline{\beta}$ to $h_0 h_j^2$.

But multiplication by h_0 is a monomorphism, so the relative attaching map is represented by h_j^2.

The proof that (1) implies (2) is similar, and we retain the notations from the above work. We now suppose that h_j^2 survives to an element $\theta_j \in \pi_{n-1}(S^0)$ of order 2. Let $\alpha \in \pi_{n-1}(F^2 S^0)$ be a lift of θ_j to filtration 2. Note that it does not have order 2 there, since $h_0 h_j^2$ is no longer killed by a differential. Its image in $\pi_{n-1}(F^1 S^0)$ does have order 2, though, so in the 3×3 diagram

$$\begin{array}{ccccc}
\Sigma^{-2} F^1 M / F^2 M & \longrightarrow & \Sigma^{-1} F^2 M & \longrightarrow & \Sigma^{-1} F^1 M \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} F^1 S^0 / F^2 S^0 & \longrightarrow & F^2 S^0 & \longrightarrow & F^1 S^0 \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} F^1 S^0 / F^2 S^0 & \longrightarrow & F^2 S^0 & \longrightarrow & F^1 S^0
\end{array}$$

we have an element α in the middle which maps to zero at the lower right corner. Thus there are elements $\gamma \in \pi_n(F^1 M)$ and $\beta \in \pi_n(F^1 S^0 / F^2 S^0)$ which map to the same elements in $\pi_{n-1}(F^1 S^0)$ and $\pi_{n-1}(F^2 S^0)$, respectively, and to the same element as each other in $\pi_n(F^1 M / F^2 M)$.

The element γ is a lift to filtration 1 of the attaching map for the top cell in the mapping cone of the lift of θ_j to a Moore space, and its boundary in $\pi_n(F^1 M / F^2 M)$ is a representative of that attaching map in the Adams E^1 term for the Moore space.

The map $\Sigma^{-1} F^1 S^0 / F^2 S^0 \to F^2 S^0$ induces a differential in the Adams spectral sequence hitting 2α. The Adams differential shows that a choice of β (the only choice in fact) is given by h_{j+1}. The image of this element in $\pi_n(F^1 M / F^2 M)$ is the image of this class in the E_1 term for the Moore space, under the inclusion of the bottom cell.

Thus the attaching map is represented by $h_{j+1} e_0$. $\square$

References

[1] J. P. May, The additivity of traces in triangulated categories, Adv. Math. 163 (2001) 34–73.