GMS systems

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This is a workup of the algebraic structure called a "vertex algebroid" by Gorbounov, Malikov, and Schechtman [2] and Bressler [1].

Definition. A Palais algebra over a commutative ring k is a commutative k-algebra A and a Lie algebra T over k together with k-linear maps $A \otimes T \to T$ and $T \otimes A \to A$ which establish T as an A-module and A as a T-module and which satisfy

$$s(bc) = (sa)b + a(sb), \quad [s, bt] = (sb)t + b[s, t],$$

for $a, b, c \in A$ and $s, t \in T$.

A module over a Palais algebra (A, T) is a k-module M together with k linear maps $A \otimes M \to M$ and $T \otimes M \to M$ which establish M as an A-module and as a T-module and which satisfy

$$s(ax) = (sa)x + a(sx)$$

for $s \in T$, $a \in A$, $x \in M$.

The characteristic example is given by taking any commutative k-algebra A and letting $T = \mathrm{Der}_k(A, A)$ with its obvious structures as Lie algebra of operators by derivations on A and as left A-module.

The A-module of Kähler differentials $\Omega_{A/k}$ forms the characteristic example of an (A, T)-module, with $T = \mathrm{Der}_k(A, A)$. The T-module structure $T \otimes \Omega_{A/k} \to \Omega_{A/k}$ is given by the "Lie derivative," characterized by the equation

$$t(a\,\partial b) = (ta)\partial b + a\,\partial(tb).$$

To verify that this map is well defined one can use the fact that the module of Kähler differentials is given by dividing the free A-module generated by the set A (in which an element $a \in A$ of the generating set is written as ∂a) by the relations

$$\partial(ab) = a\,\partial b + b\,\partial a, \quad \partial k = 0.$$

In fact, this example has two additional bits of structure: (1) the universal derivation $\partial : A \to \Omega_{A/k}$; and (2) an A-bilinear pairing

$$T \otimes \Omega_{A/k} \to A,$$

defined as the adjoint of the canonical isomorphism $T \cong \mathrm{Hom}_A(\Omega_{A/k}, A)$, or by the formula

$$\langle t, b\,\partial c\rangle = b(tc).$$

This gives us the following structure.

Definition. A pre-GMS algebra over a commutative ring k consists in a Palais algebra (A, T) and a module Ω for it, together with a map $\partial : A \to \Omega$ of T-modules and an A-bilinear pairing $\langle -, -\rangle : T \otimes \Omega \to A$ satisfying the identities

$$\langle t, \partial b\rangle = tb$$

$$(at)\omega = a(t\omega) + \langle t, \omega\rangle \partial a$$

$$s\langle t, \omega\rangle = \langle [s, t], \omega\rangle + \langle t, s\omega\rangle.$$

We now come to the main definition.

Definition. A GMS algebra over a commutative ring k consists of k-modules A and V together with an element $1 \in A$ and k-linear maps

$$\partial : A \to V, \quad A \otimes A \to A,$$

$$\cdot : A \otimes V \to V, \quad \langle -, -\rangle : V \otimes V \to A, \quad [-, -] : V \otimes V \to V$$

subject to the following axioms.

$$1a = a, \quad a(bc) = (ab)c, \quad ab = ba,$$

so A forms a commutative k-algebra with unit 1; and

$$1 \cdot x = x, \quad \langle x, y\rangle = \langle y, x\rangle$$

$$a \cdot (b \cdot z) = (ab) \cdot z + \langle \partial b, z\rangle \cdot \partial a + \langle \partial a, z\rangle \cdot \partial b$$

$$\partial(ab) = a \cdot \partial b + b \cdot \partial a, \quad [x, y] + [y, x] = \partial\langle x, y\rangle$$

$$\langle \partial a, \partial b\rangle = 0, \quad [\partial a, y] = 0$$

$$[x, b \cdot z] = b \cdot [x, z] + \langle x, \partial b\rangle \cdot z$$

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

$$\langle a \cdot y, z\rangle = a\langle y, z\rangle - \langle y, [z, \partial a]\rangle$$

$$\langle [x, y], z\rangle + \langle [x, z], y\rangle = \langle x, \partial\langle y, z\rangle\rangle.$$

The structure given by GMS includes three further structure maps, which are given in terms of our choice of primitive operations by

$$A \otimes V \to A \quad \text{by} \quad a \otimes y \mapsto -\langle \partial a, y \rangle$$

$$V \otimes A \to A \quad \text{by} \quad x \otimes b \mapsto \langle x, \partial b \rangle$$

$$V \otimes A \to V \quad \text{by} \quad x \otimes b \mapsto b \cdot x + [x, \partial b].$$

We will give the second of these operations a notation:

$$xb = \langle x, \partial b \rangle.$$

Then it's easy to check the equations

$$\partial(xb) = [x, \partial b],$$

$$x(bc) = b \cdot (xc) + c \cdot (xb),$$
$$[x, y]c = x(yc) - y(xc),$$
$$[x, b \cdot z] = b \cdot [x, z] + (xb) \cdot z.$$

It is also useful to note the equation

$$[a \cdot \partial b, z] = \langle \partial b, z \rangle \cdot \partial a - \langle \partial a, z \rangle \cdot \partial b.$$

Let Ω denote the sub k module of V generated by the elements $a \cdot \partial b$ as a and b run over A. Then it is easy and fun to check the following statements. The operation $\cdot$ defines an A-module structure on Ω, the operation $[-,-]$ defines a Lie algebra structure on the k-module quotient $T = V/\Omega$, and the operation $x, b \mapsto xb$ defines a T-module structure on A, in such a way that (A, T) forms a Palais algebra. Moreover, the operation $[-,-]$ defines a T-module structure on Ω in such a way that Ω becomes a module for this Palais algebra. For the last one checks that the bracket of two elements of Ω is trivial.

The Palais algebra structure underlies a natural GMS algebra structure, in which $\partial : A \to \Omega$ is the corestriction of $\partial A \to V$ and the pairing $\langle -, - \rangle : T \otimes \Omega \to A$ descends from the pairing on V. For the last, one checks that $\Omega \subset V$ is self-orthogonal with respect to the pairing.

This work defines the functor in the

Proposition. A GMS algebra has an underlying pre-GMS algebra.

Let (A, V) be GMS algebra, and assume that the map $V \to T$ has a k-linear section. Use this section to express

$$V = \Omega \oplus T$$

and teh structure maps accordingly. We will use this decomposition to describe what information must be added to the pre-GMS system in order to specify the GMS system. We will simply identify T with its image in V.

For $a \in A$, $a\cdot$ induces actions on Ω and on T, but may have a component sending T into Ω: so we need to give a map

$$\alpha : A \otimes T \to \Omega.$$

In terms of it,

$$a \cdot \begin{pmatrix} \omega \\ t \end{pmatrix} = \begin{pmatrix} a\omega + \alpha(a, t) \\ at \end{pmatrix}.$$

The pairing $\langle -, - \rangle$ is symmetric, Ω is self-orthogonal, and we are given the induced pairing $T \otimes \Omega \to A$, so what remains to specify is the restriction of the pairing to T in V: a map

$$\gamma : T \otimes T \to A.$$

In terms of it,

$$\left\langle \begin{pmatrix} \zeta \\ s \end{pmatrix}, \begin{pmatrix} \omega \\ t \end{pmatrix} \right\rangle = \langle s, \omega \rangle + \langle t, \zeta \rangle + \gamma(s, t).$$

Finally, the bracket $[-, -]$ takes the form

$$\left[\begin{pmatrix} \zeta \\ s \end{pmatrix}, \begin{pmatrix} \omega \\ t \end{pmatrix} \right] = \begin{pmatrix} \partial\langle t, \zeta \rangle - t\zeta + s\omega + \beta(s, t) \\ [s, t] \end{pmatrix}$$

where

$$\beta : T \otimes T \to \Omega.$$

These three maps satisfy various relations:

$$\alpha(1, t) = 0, \qquad \gamma(s, t) = \gamma(t, s),$$

$$\alpha(a,bt)-\alpha(ab,t)+a\alpha(b,t)=(tb)\,\partial a+(ta)\,\partial b$$
$$\partial\gamma(s,t)=\beta(s,t)+\beta(t,s)$$
$$\alpha(sb,t)-s\alpha(b,t)+\alpha(b,[s,t])=\beta(s,bt)-b\beta(s,t).$$
$$s\beta(t,u)-t\beta(s,u)+u\beta(s,t)-\beta([s,t],u)+\beta(s,[t,u])-\beta(t,[s,u])=\partial\langle u,\beta(s,t)\rangle.$$
$$a\gamma(s,t)-\gamma(as,t)=\langle t,\alpha(a,s)\rangle+[s,t]a$$
$$s\gamma(t,u)=\langle u,\beta(s,t)\rangle+\gamma([s,t],u)+\langle t,\beta(s,u)\rangle+\gamma([s,u],t).$$

One can change the splitting by means of a k-linear map $f:T\to\Omega$, replacing $t\in V$ with $t+f(t)$. The effect on the three maps is given by

$$(f\cdot\alpha)(a,t)=\alpha(a,t)+af(t)$$
$$(f\cdot\beta)(s,t)=\beta(s,t)+\partial\langle t,f(s)\rangle-tf(s)+sf(t)$$
$$(f\cdot\gamma)(s,t)=\gamma(s,t)+\langle s,f(t)\rangle+\langle t,f(s)\rangle.$$

References

[1] Paul Bressler, Vertex Algebroids I, https://arxiv.org/abs/math/0202185

[2] Vassily Gorbounov, Fyodor Malikov, and Vadim Schechtman, Gerbes of chiral differential operators. II. Vertex algebroids. Invent. Math. 155 (2004) 605680.