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THE ELLIPTIC CHARACTER AND THE WITTEN GENUS

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The elliptic cohomology of Landweber, Ravenel, and Stong [7] is constructed so as to serve as the third term in the sequence beginning with rational cohomology and KO–theory. Just as there is a multiplicative natural transformation

$$\mathrm{chc} : KO^*(X) \longrightarrow H^*(X;\mathbb{Q})[\![v^{\pm 1}]\!],$$

given by the Chern character of the complexification, so there is an "elliptic character"

$$\lambda : E\ell^*(X) \longrightarrow KO^*(X;\mathbb{Z}[\tfrac{1}{2}])[\![q]\!] .$$

In this addendum to [7], we construct this transformation and use the corresponding Riemann–Roch formula to obtain Witten's elliptic genus and to interpret the rigidity result conjectured by Witten and proven by Taubes (with later improvements by Bott and Taubes [3]).

If $\phi : MU_* \longrightarrow R$ is a ring–homomorphism, we may consider the functor $X \longmapsto R \otimes_{MU_*} MU_*(X) = R_*(X)$ on spaces. Landweber [6] gives conditions on ϕ guaranteeing that $R_*(X)$ is a homology theory. All three cohomology theories of interest to us here — $H\mathbb{Q}^*$, $KO^*[\frac{1}{2}]$, and $E\ell^*$ — are defined in this way. We begin by observing the naturality of this construction. We use the usual machinery [1] of formal group laws, etc., associated with MU^*, and consider the following situation. We have formal group laws F

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over R and F′ over S, a ring homomorphism $\lambda : R \to S$, and a strict isomorphism $\Theta : F' \longrightarrow \lambda F$. Then this data determines a natural transformation

$$\lambda_* : R^*(X) \longrightarrow S^*(X)$$

such that:

(1) On $X = *$, $\lambda_* = \lambda$.

(2) If $L \downarrow X$ is a complex line bundle, then

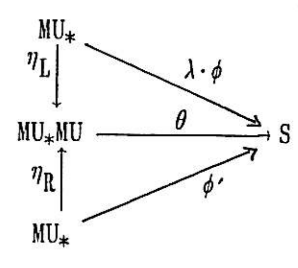
$$\lambda_*(e_R(L)) = \Theta(e_S(L)),$$

where $e_R(L)$ is the image in $R^2(X)$ of the MU–Euler class of L.

The construction of λ_* uses the coaction

$$\psi : MU_*X \longrightarrow MU_*MU \otimes_{MU_*} MU_*X$$

together with the fact that MU_*MU carries the universal strict isomorphism of formal groups. Thus, $MU_*MU = MU_*[b_1, b_2, \ldots]$, $b_0 = 1$, and the formal groups $\eta_L\, G$ and $\eta_R\, G$ over MU_*MU, obtained by pushing the coefficients of the universal group law G over MU_*, are strictly isomorphic by $B(t) = \sum_{i\geq 0} b_i t^{i+1}$. There is a unique ring homomorphism $\theta :$ $MU_*MU \longrightarrow S$ carrying B to Θ and such that



The natural transformation λ_* is the R–linear extension in the diagram

$$\begin{array}{ccc} MU_*(X) & \xrightarrow{\psi} & MU_*MU \otimes_{MU_*} MU_*(X) \\ \phi\otimes 1 \downarrow & & \downarrow \theta\otimes 1 \\ R\otimes_{MU_*} MU_*(X) & \xrightarrow{\lambda_*} & S\otimes_{MU_*} MU_*(X) \end{array}$$

We now specify the complex orientations of our examples. For H^* we take

(3) $$e_H(L) = -c_1(L) = c_1(\bar{L})\ ;$$

its formal group law is $G_a(X,Y) = X + Y$. For $KO^*[\frac{1}{2}]$ we choose the Â–orientation, for which

(4) $$e_{\hat{A}}(\bar{L}) = -e_{\hat{A}}(L)$$

(5) $$ce_{\hat{A}}(L \otimes \bar{L}) = v^{-2}(1 - L)(1 - \bar{L})$$

where $c : KO^* \longrightarrow KU^*$ is complexification, v is the Bott class in KU^{-2}, and 1 denotes the trivial complex line bundle. The Chern character of the complexification,

$$chc : KO^*(X)[\tfrac{1}{2}] \longrightarrow H^*(X; \mathbb{Q})[\![v^{\pm 1}]\!]\ ,$$

arises from the embedding

$$\mathbb{Z}[\tfrac{1}{2}][v^{\pm 2}] \longrightarrow \mathbb{Q}[v^{\pm 1}]$$

together with the isomorphism

$$\exp_{\hat{A}} : G_a \longrightarrow G_{\hat{A}}$$

given by

(6) $$\exp_{\hat{A}}(x) = 2v^{-1}\sinh(vx/2).$$

Of the various versions of elliptic cohomology described in [7] we take for $E\ell^*$ the ring of modular forms for $\Gamma_0(2)$ with Fourier coefficients in $\mathbb{Z}[\frac{1}{2}]$ and poles only at the cusp at 0. Thus

$$E\ell^* = \mathbb{Z}[\tfrac{1}{2}, \delta, \epsilon, (\delta^2 - \epsilon)^{-1}].$$

Its orientation e_E is determined by the Euler formal group law G_E (see [7]). According to Landweber, Ravenel, and Stong, $E\ell_*(X)$ is then a homology theory.

The q-expansion $f \longmapsto \tilde{f}$, $\tilde{f}(q) = f(\tau)$, $q = e^{2\pi i\tau}$, determines a ring homomorphism

$$\lambda : E\ell^* \longrightarrow KO^*[\tfrac{1}{2}][\![q]\!]$$

by sending a form $f \in E\ell^{-4k}$ of weight 2k to $v^{2k}\tilde{f}$. Explicitly [10]

$$\delta \longmapsto v^2\left[-\frac{1}{8} - 3\sum_{n\geq 1}\left[\sum_{2\nmid d\,|\,n} d\right]q^n\right]$$

$$\epsilon \longmapsto v^4\left[\sum_{n\geq 1}\left[\sum_{2\nmid d\,|\,n} \left(\frac{n}{d}\right)^3\right]q^n\right]$$

Then we have the key

Proposition. There is an isomorphism $G_{\hat{A}} \longrightarrow \lambda G_E$ over $KO^*[\frac{1}{2}][\![q]\!]$ given by

$$\Theta(y) = y\prod_{n\geq 1}\left[1 - \frac{q^n v^2 y^2}{(1-q^n)^2}\right]^{(-1)^n}$$

This was proven by the Chudnovskys [4] and later by Zagier [10], but it represents an explicit form of the well–known fact that the Tate curve has multiplicative reduction, and as such is due to Jacobi and Tate and was known to Morava [8] in 1973. Zagier's formulas (5) and (6) are equivalent to the proposition since they show that

$$\begin{array}{ccc} & G_a & \\ {}^{\exp_{\hat{A}}}\swarrow & & \searrow^{\exp_E} \\ G_{\hat{A}} & \xrightarrow{\ \theta\ } & \lambda G_E \end{array}$$

commutes.

We now recall (from Dyer [5], for example) the general "Riemann–Roch formula" associated to a multiplicative transformation of cohomology theories. Let $\lambda : h^* \longrightarrow k^*$ be such a map and let $\xi \downarrow X$ be a vector bundle. Assume given orientations – i.e., Thom classes – $u_h \in h^d(X^\xi)$ and $u_k \in k^d(X^\xi)$. Then $\lambda u_h \in k^d(X^\xi)$ is another Thom class, so by the Thom isomorphism theorem there is a unique class $\rho(\xi) \in k^0(X)$ such that if $\epsilon : k^0(X) \longrightarrow k^0$ is induced by the inclusion of a point then

$$\epsilon(\rho(\xi)) = 1$$

and

$$\rho \cup \lambda u_h = u_k.$$

Now suppose $p : E \longrightarrow B$ is a smooth fiber bundle with compact d–dimensional fiber and bundle of tangents along the fiber τ. Assume given orientations u_h and u_k for τ. These determine Gysin homomorphisms $p_!^h : h^*(E) \longrightarrow j^{*-d}(B)$ and $p_!^k : k^*(E) \longrightarrow k^{*-d}(B)$. The Riemann–Roch formula then asserts that

$$\lambda p_!^h(\alpha) = p_!^k(p(\tau) \cup \lambda(\alpha)).$$

In particular, if h^* and k^* are equipped with complex orientations, then complex vector bundles are naturally oriented, and $u(\xi \oplus \xi') = u(\xi) \cup u(\xi')$. The class ρ is thus "exponential," and is determined by its value on the universal line bundle $L \downarrow BU(1)$. Since the homotopy–equivalence $BU(1) \longrightarrow MU(1)$ pulls u back to the Euler class e, we may as well define $\rho(L)$ by the equation

$$\rho(L) \cup \lambda e_h(L) = e_k(L)\ . \tag{7}$$

Our theories are all localized away from 2 and the complex orientations are odd: $e(\bar{L}) = -e(L)$. This permits a (unique) factorization of $MU \to h$ through $MU \to MSO$: so oriented real vector bundles are canonically oriented.

The multiplicative transformation

$$chc : KO^*(X)[\tfrac{1}{2}] \longrightarrow H^*(X; \mathbb{Q})[\![v^{\pm 1}]\!],$$

together with the orientations chosen above (see (6)), leads to

$$chc\ p_!^{\hat{A}}(\alpha) = p_!^{H}(\hat{A}(\tau) \cup chc(\alpha))\ . \tag{10}$$

The multiplicative transformation

$$\lambda : E\ell^*(X) \longrightarrow KO^*(X)[\tfrac{1}{2}][\![q]\!]\ ,$$

together with the orientations chosen above (see (8)), leads to

$$\lambda\ p_!^{E\ell}(\alpha) = p_!^{\hat{A}}(\rho(\tau) \cup \lambda(\alpha))$$

where ρ is the exponential characteristic class in $KO^*(-)[\frac{1}{2}][\![q]\!]$ such that, with $y = e_{\hat{A}}(L)$,

$$\rho(L) = \frac{y}{\Theta(y)} = \prod_{n \geq 1} \left[1 - \frac{q^n v^2 y^2}{(1-q^n)^2}\right]^{(-1)^{n-1}}$$

If we now complexify, the n^{th} term in the product becomes (by (4) and (5)) the $(-1)^{n-1}$ power of

$$(11) \qquad 1+\frac{q^n(1-L)(1-\bar{L})}{(1-q^n)^2}=\frac{(1-q^nL)(1-q^n\bar{L})}{(1-q^n)^2}$$

K–theory is better equipped than cohomology to succinctly express multiplicative sequences. Thus, (11) determines the multiplicative class $\Lambda_{-q^n}(\xi - \dim \xi) \otimes \mathbb{C}$, and its reciprocal, $S_{q^n}(\xi - \dim \xi) \otimes \mathbb{C}$. Hence, with $\tilde{\xi} = \xi - \dim \xi$,

$$c\rho(\xi) = \bigotimes_{\substack{n>0\\ \text{even}}} S_{q^n}(\tilde{\xi}) \otimes \bigotimes_{\substack{n>0\\ \text{odd}}} \Lambda_{-q^n}(\tilde{\xi}) \otimes \mathbb{C} = R_q(\xi).$$

If we now apply the traditional Riemann–Roch formula (10), we arrive (with $\alpha = 1$) at

$$\mathrm{chc}\ \lambda_* p_!^E(1) = p_!^H(\hat{A}(\tau) \cup \mathrm{ch}\ R_q(\tau))$$

which is Witten's expression [9].

Let the circle group T act smoothly on a compact manifold M, and form the Borel construction $p : ET \times_T M \longrightarrow BT$. We obtain

$$p_!^E : E\ell^*(ET\times_T M) \longrightarrow E\ell^*(BT),$$

and the image classes can be considered equivariant elliptic characteristic numbers for M. The following rigidity result was conjectured by Witten [9] and proved by Taubes and by a subsequent collaboration of Bott and Taubes [3].

Theorem: Suppose M is a spin manifold. Then $p_!^E(1)$ is constant, in the sense that it is pulled back from $E\ell^*(*)$ under $BT \longrightarrow *$.

One may wonder whether elliptic cohomology might be useful in giving a proof of this result closer in spirit (and simplicity) to the proof by Atiyah and Hirzebruch [2] of the analogous result for K–theory. One can indeed set up a "fixed point formula" in elliptic cohomology, and try to play the two sides off against each other. The extra ingredient available in the K–theory setting is a diagram of the form

$$\begin{array}{ccc} K_T(M) & \longrightarrow & K(ET \times_T M) \\ \Big\downarrow p_! & & \Big\downarrow p_! \\ K_T(*) & \longrightarrow & K(BT)\,. \end{array}$$

$K_T(*)$, a Laurent series ring, provides a "rational structure" in the power series ring $K(BT)$, and restricts the location of poles enough to complete the proof. What is missing, then, is a T–equivariant elliptic cohomology, equipped with suitable Thom classes. Note that in the K–theory context consideration of elliptic operators may be confined entirely to the construction of such Thom classes in K_G; and in fact may be eliminated there as well when $G = T$. Note also that one really only needs a rational equivariant theory. One may hope that a suitable geometric construction of elliptic cohomology might lead to an equivariant theory. It has been hinted, by Witten and by Hopkins, that $E\ell_T^*(*)$ might be a ring of Jacobi forms.

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BIBLIOGRAPHY

1. J. F. Adams, Stable Homotopy and Generalised Homology, Univ. of Chicago, 1974.
2. M. F. Atiyah and F. Hirzebruch, Spin–manifolds and group actions, in Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), Springer, 1970, pp. 18–26.
3. R. Bott and C. Taubes, On the rigidity theorems of Witten, to appear.
4. D. V. Chudnovsky and G. V. Chudnovsky, Elliptic modular functions and elliptic genera, Topology, to appear.
5. E. Dyer, Cohomology theories, Benjamin, 1969.
6. P. S. Landweber, Homological properties of comodules over MU_*MU and BP_*BP, Amer. J. Math. 98(1976) 591–610.
7. P. S. Landweber, D. C. Ravenel, and R. E. Stong, Periodic cohomology theories defined by elliptic curves, to appear.
8. J. Morava, Forms of K–theory, manuscript, 1973.
9. E. Witten, The index of the Dirac operator in loop space, Elliptic Curves and Modular Forms in Algebraic Topology/Proceedings, Princeton 1986, P. S. Landweber, ed., Springer Lecture Notes in Math. 1326, 1988, pp. 161–181.
10. D. Zagier, Note on the Landweber–Stong elliptic genus, Ibid., pp. 216–224.