BECK MODULES AND ALTERNATIVE ALGEBRAS

NISHANT DHANKHAR, ALI TAHBOUB, HAYNES MILLER, AND VICTOR YIN

Abstract. We set out the general theory of "Beck modules" in a variety of algebras and describe them as modules over suitable "universal enveloping" unital associative algebras. We pay particular attention to the somewhat nonstandard case of "alternative algebras," defined by a restricted associative law, and determine the Poincaré polynomial of the universal enveloping algebra in the homogeneous case.

1. Introduction

The notion of a "module" occupies an important place in the study of general algebraic systems. Most of these diverse notions are united under the theory of "Beck modules." Given an object A in any category $\mathbf{C}$, one may consider the "slice category" $\mathbf{C}/A$ of objects in $\mathbf{C}$ equipped with a map to A. A Beck module for A is then an abelian group object in $\mathbf{C}/A$. If $\mathbf{C}$ is the category of commutative rings, for example, a Beck module for A is simply an A-module, while if $\mathbf{C}$ is the category of associative algebras, a Beck module for A is an A-bimodule. Many other examples occur in the literature: Leibniz algebras [17], λ-rings [14], divided power rings [9], ...

This definition occurs in the thesis [5] of Jonathan Beck written under the direction of Samuel Eilenberg. Eilenberg himself had discussed such objects in [10], at least in the linear context, as the kernel of a "square zero extension." These kernels were understood to constitute "representations" of the algebra, and this structure was made explicit in various cases.

We review below the context of a "variety" $\mathbf{V}$ of algebras over a commutative ring K. In this case, for every $\mathbf{V}$-algebra A the category $\mathbf{Mod}_A$ of Beck A-modules is an abelian category with a single projective generator. As a result, the category $\mathbf{Mod}_A$ is equivalent to the category of right modules over a canonical unital associative K-algebra $U_{\mathbf{V}}(A)$, the "universal enveloping algebra" for A.

This raises the question of identifying the structure of $U_{\mathbf{V}}(A)$ for various varieties $\mathbf{V}$ and $\mathbf{V}$-algebras A. Left and right multiplication determine a K-module map $A \oplus A \to U_{\mathbf{V}}(A)$, and hence a surjection of associative unital K-algebras $\mathrm{Tens}_K(A \oplus A) \to U_{\mathbf{V}}(A)$ (cf. [17]). Each defining equation determines a generator of the kernel of this map, by a process of "noncommutative differentiation" that we describe in detail.

2010 *Mathematics Subject Classification.* 17A30,17.04.

We review some of the standard examples, and then focus on a somewhat less standard one, the variety of "alternative algebras" over K. This example has been considered before, but even over a field basic features of the universal enveloping algebra for an alternative algebra, such as its dimension, have remained obscure. In 1954, Nathan Jacobson [15] wrote "The introduction of the universal associative algebras for the birepresentations [his term for Beck modules] enables one to split the representation problem into two parts: (1) determination of the structure of $U(A)$, (2) representation theory for the associative algebra $U(A)$. In practice, however, it seems to be difficult to treat (1) as a separate problem. Only in some special cases is it feasible to attack this directly." Richard Schafer [20] observed in 1966 that if $\mathbf{V}$ is the variety of alternative K-algebras, with K a field, and $\dim_K A = n$, then $\dim_K U(A) \leq 4^n$. But the precise dimension, even the case of "homogeneous" alternative algebras – those with trivial product – over a field, has eluded analysis.

In that case, the universal enveloping algebra admits a natural grading, by word length, or, as we call it, by weight. Write K^n for the free K-module on n generators, regarded as a homogeneous alternative K-algebra. The principal new result in this paper is the description of an explicit basis for the universal enveloping algebra of K^n when K is a field.

Theorem 1.1. *Let K be any commutative ring. For each n and k, the K-module $U(K^n)_k$ is free, and*

$$\mathrm{rank}_K U(K^n)_k = \begin{cases} 1 & \text{if } k=0 \\ 2n & \text{if } k=1 \\ \frac{3n^2-n}{2} & \text{if } k=2 \\ 2\binom{n}{k} & \text{if } k \geq 3\,. \end{cases}$$

In particular,

$$U(K^n)_k = 0 \quad \text{for} \quad k > n\,.$$

These calculations show that at least in the homogeneous case, the growth rate of the universal enveloping algebra is indeed exponential in the dimension of the algebra, but much slower than the upper bound observed by Schafer.

Our tool is the theory of Gröbner bases for noncommutative graded algebras. We employ hand calculation and Python to determine a Gröbner basis for the ideal of relations defining $U(K^n)$, with K any prime field, for $n \leq 5$. The structure of this basis for these small values of n turns out to imply that a basis with the same structure exists for all n. The set of normal monomials with respect to this basis (which projects to a basis for $U(K^n)$) is then easy to determine. A base-change result then shows that the $U(K^n)$ is a free K-module of the same rank for any commutative base ring K.

After a review of the theory of varieties of algebras in §2, and a reminder of some particular features of alternative algebras in §3, we describe in §4 the theory of Beck modules in this generality, and describe the corresponding universal enveloping algebras. In §5 we discuss the form of non-commutative differentiation that leads from the equations in a variety **V** to the relations in a universal enveloping algebra for a **V**-algebra. In §6 we specialize to the case of alternative algebras. Finally, in an Appendix, we review some of the essential features of the theory of Gröbner bases.

This work is intended as a first step in the study of the Quillen homology and cohomology of algebraic systems such as alternative algebras.

Acknowledgements. This work was carried out under the auspices of a program, supported by MIT's Jameel World Education Laboratory, designed to foster collaborative research projects involving students from MIT and Palestinian universities. We acknowledge with thanks the contributions made by early participants in this program – Mohammad Damaj and Nasief Khlaif of Birzeit University, Hadeel AbuTabeekh of An-Najah National University, and Heidi Lei of MIT – as well as the support of Palestinian faculty – Reema Sbeih and Mohammad Saleh at Birzeit and Khalid Adarbeh and Muath Karaki at NNU. The first author acknowledges support by the MIT UROP office.

2. Varieties of algebras

We will work with algebras defined by a product operation, though much of this work can be carried out in much greater generality. Following the lead of Bourbaki [6, §7.1], we make the following definition.

Definition 2.1. *A* magma *is a set X with a binary operation $X \times X \to X$ (written as juxtaposition).*

We will also restrict our attention to linear examples, and work over a commutative ring K. So a *magmatic K-algebra* (or just *K-algebra*) is a K-module A equipped with a K-bilinear product $A \otimes A \to A$ (written as juxtaposition).

Magmatic K-algebras constitute the objects in a category $\mathbf{Mag}_K$. The forgetful functor to sets has a left adjoint $Mag_K : \mathbf{Set} \to \mathbf{Mag}_K$, so we can define the free magma on a set. The free magma $Mag(S)$ generated by a set S is the set of bracketed strings of elements of S; see [6, §7.1]. The free magmatic K-algebra on a set S is the free K-module generated by $Mag(S)$: $Mag_K(S) = KMag(S)$.

We can adjoin axioms using the following device [2]. An *equation* is an element ω of the free magmatic K-algebra on a finite set S, which we may denote by $S(\omega)$ if there are several equations in play. Given a magmatic K-algebra A, we will say that an equation $\omega \in Mag_K(S)$ is *satisfied by A* if for any set map $S \to A$ the induced map $Mag_K(S) \to A$ sends ω to 0. A set of equations defines a *variety* of K-algebras, namely the subcategory of $\mathbf{Mag}_K$ cut out by (that is, satisfying) these equations. An object of a variety of K-algebras **V** is a "**V**-algebra."

A variety of K-algebras is an "algebraic category" [11]. It is complete and cocomplete. Any subalgebra of a $\mathbf{V}$-algebra is again a $\mathbf{V}$-algebra. The forgetful functor $u : \mathbf{V} \to \mathbf{LMod}_K$ to the category of left K-modules has a left adjoint

$$F : \mathbf{LMod}_K \to \mathbf{V}\,.$$

Examples 2.2. Here are four standard examples, beyond $\mathbf{Mag}_K$ itself.

- $\mathbf{Ass}_K$, the variety of associative algebras, is defined by the equation
$$(xy)z - x(yz) \in Mag_K\{x, y, z\}\,.$$
- Adding the further equation
$$xy - yx \in Mag_K\{x, y\}$$
gives us the variety of commutative K-algebras, $\mathbf{Com}_K$.
- A Lie algebra (in $\mathbf{Lie}_K$) is a K-algebra satisfying the equations
$$xx \in Mag_K\{x\}\,, \quad (xy)z + (yz)x + (zx)y \in Mag_K\{x, y, z\}\,.$$
- An alternative algebra is a magmatic K-algebra satisfying the equations
$$(xx)y - x(xy)\,,\, (xy)y - x(yy) \,\in\, Mag_K\{x, y\}\,.$$
These are the objects in the variety $\mathbf{Alt}_K$.

Note that we do not assume a unit element in any of these examples.

There is a reversal involution $\overline{(-)} : \mathbf{Mag} \to \mathbf{Mag}$. It comes with a natural bijection of underlying sets $X \to \overline{X}$ that we will also denote with an overline, and $\overline{x}\,\overline{y} = \overline{yx}$. It extends to an involution of $\mathbf{Mag}_K$. To any variety $\mathbf{V}$ of K-algebras we can associate an "opposite" variety $\overline{\mathbf{V}}$, with defining equations given by reversing the defining equations of $\mathbf{V}$. By sending a $\mathbf{V}$-algebra to the same K-module with opposite multiplication, you get a natural equivalence of categories

$$\mathbf{V} \to \overline{\mathbf{V}}\,, \quad A \mapsto \overline{A}\,.$$

A variety $\mathbf{V}$ is *symmetric* $\overline{\mathbf{V}} = \mathbf{V}$. All the examples above are symmetric, but, for example, the variety of "left alternative K-algebras," satisfying $(xx)y - x(xy)$ but perhaps not $(xy)y - x(yy)$, is not symmetric; its opposite is the variety of right alternative K-algebras. If $\mathbf{V}$ is a symmetric variety, the isomorphism $\mathbf{V} \to \overline{\mathbf{V}}$ becomes an involution on $\mathbf{V}$, sending an algebra to the same K-module with the opposite multiplication.

We end with a discussion of base-change. Let $\alpha : K \to L$ be homomorphism of commutative rings. An equation in $\mathbf{Mag}_K(S)$ induces an equation in $\mathbf{Mag}_L(S)$. So a variety of K-algebras, say $\mathbf{V}_K$, induces a variety of L-algebras, which we'll denote by $\mathbf{V}_L$. This is a transitive operation. Moreover, the functor $L \otimes_K - : \mathbf{LMod}_K \to \mathbf{LMod}_L$ lifts to a functor $L \otimes_K - : \mathbf{V}_K \to \mathbf{V}_L$.

3. Alternative algebras

The example of alternative algebras is less familiar than the others and we spend a moment introducing it. Schafer's book [20] provides a good reference.

Any associative K-algebra is alternative, and Emil Artin proved that any alternative K-algebra with two generators is associative [7]. The alternative identities imply that the further "flexible" equation

$$(xy)x - x(yx)$$

is satisfied. The algebra of octonions [3] is a well-known example of a nonassociative alternative algebra.

The *associator* in a magmatic K-algebra is the trilinear form

$$(x, y, z) = (xy)z - x(yz)$$

In an alternative algebra the associator is an alternating form: transpositions reverse the sign. This suggests adding a further basic example, one defined by weight 3 equations:

- An almost alternative K-algebra is a magmatic K-algebra for which the associator is an alternating form; that is to say, satisfying the equations

$$(xy)z - x(yz) + (xz)y - x(zy)\,, \quad (xy)z - x(yz) + (yx)z - y(xz)\,.$$

If 2 is invertible in K these axioms are equivalent to the alternative axioms. In some respects this "almost alternative" condition is better behaved than the alternative condition itself; it is operadic, for example. If K is a commutative $\mathbb{F}_2$-algebra, multiplication table

	a	b	c
a	a	b	0
b	0	0	0
c	c	0	b

defines an almost alternative K-algebra that is not alternative.

A monoid X in **Set** defines a unital associative algebra in $\mathbf{LMod}_K$ by forming the free K-module on X. The equations for alternative algebras make sense in **Set**, so one can talk about "alternative sets." An alternative product on X determines a magmatic K-algebra structure on KX, but it is not necessarily alternative. For example the multiplication table

	a	b	c
a	a	a	c
b	a	b	b
c	c	b	c

is commutative and alternative, and hence even flexible, but the K-module that it generates is not alternative. We thank Hadeel AbuTabeekh for this example.

4. Beck modules

Let $\mathbf{V}$ be a variety of K-algebras and A a $\mathbf{V}$-algebra. The "slice category" $\mathbf{V}/A$ has as objects morphisms in $\mathbf{V}$ with target A, and as morphisms maps compatible with the projections to A. This slice category again has good properties; in particular it is complete and cocomplete. We can thus speak of abelian group objects in $\mathbf{V}/A$.

An abelian group structure on a $\mathbf{V}$-algebra over A, $p : B \downarrow A$, begins with a unit: a map from the terminal object of $\mathbf{V}/A$, that is, a section $\eta : A \uparrow B$ of p. This unit defines an "axis inclusion" $i : B \coprod B \to B \times_A B$ in $\mathbf{V}/A$. A magma structure on B is an extension of the "fold map" $\nabla : B \coprod B \to B$ over the product. In these algebraic situations, the map i is an epimorphism, so such an extension is unique if it exists: Being a unital magma object in $\mathbf{V}/A$ is a *property* of a pointed object, not further structure on it. Furthermore, the unique unital magma structure with given unit, when it exists, is an abelian group structure. We call an object of this type an *abelian object.*

Definition 4.1. [5] *Let A be a $\mathbf{V}$-algebra. A* Beck A-module *is an abelian object in the slice category $\mathbf{V}/A$:*

$$\mathbf{Mod}_A = \mathrm{Ab}(\mathbf{V}/A)\,.$$

Proposition 4.2. [1, Theorem 3.16] *and* [4, Chapter 2, Theorem 2.4] $\mathbf{Mod}_A$ *is a complete and cocomplete abelian category.*

In our K-linear situation, write M for the kernel of $p : B \downarrow A$. Suppose $B \downarrow A$ has the structure of a unital magma in $\mathbf{V}/A$. This consists of two pieces of structure: the "unit" is a map from the terminal object in $\mathbf{LMod}_K/A$, that is, a section of $p : B \downarrow A$, and the "addition," a map $\alpha : B \times_A B \to B$ over A. Since

$$B \times_A B = (A \oplus M) \times_A (A \oplus M) = A \oplus M \oplus M$$

the structure map has the form $\alpha : A \oplus M \oplus M \to A \oplus M$. Using linearity and unitality it's easy to see that the "addition" is actually determined by the addition in M:

$$\alpha(a, x, y) = (a, x + y)\,.$$

The K-algebra structure on $A \oplus M$ is described by left and right "actions"

$$A \otimes M \to M\,, \quad M \otimes A \to M$$

both of which we denote by juxtaposition. Together they determine the multiplication on $A \oplus M$ by

$$(a, x)(b, y) = (ab, ay + xb)\,.$$

Absent further axioms, these action maps satisfy no properties. This describes the category of magmatic Beck A-modules. It is equivalent to the category of right modules over $\mathrm{Tens}_K(A \oplus A)$. Let $\lambda : A \to \mathrm{Tens}_K(A \oplus A)$

denote the inclusion of the first factor, and ρ the inclusion of the right factor. Then the action of $\mathrm{Tens}_K(A \oplus A)$ on M is given by

$$x\lambda(a) = ax\,, \quad x\rho(a) = xa\,.$$

If we are working with a general variety of K-algebras $\mathbf{V}$, the axioms of $\mathbf{V}$ will determine further properties of these two actions. For example, with $\mathbf{V} = \mathbf{Ass}$, these left and right "actions" are required to satisfy

$$(xb)c = x(bc)\,, \quad (ay)c = a(yc)\,, \quad (ab)z = a(bz)$$

for all $a, b, c \in A$ and $x, y, z \in M$. In other words, $\mathbf{Mod}_A$ is the usual category of bimodules over A_+ (for which K acts the same way on both sides), where

$$A_+ = K \oplus A$$

with product given by $(p, a)(q, b) = (pq, pb + qa)$ is the unital K-algebra associated to A.

Forming the underlying K-module of a Beck A-module gives a functor

$$u : \mathbf{Mod}_A \to \mathbf{LMod}_K\,.$$

Lemma 4.3. *The functor u has a left adjoint*

$$F_A : \mathbf{LMod}_K \to \mathbf{Mod}_A$$

sending a K-module V to the "free A-module generated by V."

Proof. We appeal to the Freyd adjoint functor theorem, [18, p. 117]. The functor u reflects limits: If $M : D \to \mathbf{Mod}_A$ is a diagram of A-modules, the limit of $uM : D \to \mathbf{LMod}_K$ has a unique A-module structure that serves as the limit in $\mathbf{Mod}_A$. The solution set condition is this: For any $V \in \mathbf{LMod}_K$, we require a set Σ of pairs (M, f), where $M \in \mathbf{Mod}_A$ and $f : V \to uM$, with the property that for any $g : V \to uN$ there is $(M, f) \in \Sigma$ and an A-module map $t : M \to N$ such that $g = (ut) \circ f$. To describe an appropriate set Σ, we employ the following language. A V-*generated* A-module is a pair (M, f) where M is an A-module, $f : V \to uM$, and M is the minimal A-module containing the image of f. We claim that for a given K-module V, there is a set of V-generated A-modules such that any V-generated A-module is isomorphic to a member of this set. This is clear if $\mathbf{V} = \mathbf{Mag}_K$, since then the category of A-modules is equivalent to the category of right modules over the unital associative K-algebra $\mathrm{Tens}_K(A \oplus A)$, and a V-generated right module over this K-algebra is isomorphic to one of the form

$$V \to V \otimes_K \mathrm{Tens}_K(A \oplus A) \to M$$

where the first map sends v to $v \otimes 1$ and the second is a quotient map of right modules. But if $A \in \mathbf{V}$, then the isomorphism classes of V-generated A-modules in $\mathbf{V}$ are among the isomorphism classes of V-generated A-modules in $\mathbf{Mag}_K$, and so also form a set.

We can now take Σ to be a set of representatives of isomorphism classes of V-generated Beck A-modules. $\square$

Spelling out the adjunction, we have a bijection, natural in the pair $V \in \mathbf{LMod}_K$ and $M \in \mathbf{Mod}_A$:

$$\mathrm{Hom}_A(F_A V, M) = \mathrm{Hom}_K(V, uM)\,.$$

In particular,

$$\mathrm{Hom}_A(F_A K, M) = uM\,.$$

Since u is exact, the object $F_A K$ is a projective generator of $\mathbf{Mod}_A$. This lets us apply another theorem of Freyd's, the embedding theorem [12, p. 106], to identify the category $\mathbf{Mod}_A$ with the category of right modules over a certain unital associative K-algebra.

Definition 4.4. Let $\mathbf{V}$ be a variety of K-algebras and $A \in \mathbf{V}$. The *universal enveloping algebra* of A, $U_{\mathbf{V}}(A)$, is the unital associative K-algebra

$$U_{\mathbf{V}}(A) = \mathrm{End}_A(F_A K)\,.$$

For any $M \in \mathbf{Mod}_A$, the K-module underlying M thus admits a natural right module structure over $U_{\mathbf{V}}(A)$, given by precomposing with the endomorphism of $F_A(K)$. To summarize:

Proposition 4.5. *This construction provides a natural equivalence of abelian categories*

$$\mathbf{Mod}_A \to \mathbf{RMod}_{U_{\mathbf{V}}(A)}\,.$$

This construction is of course natural in the $\mathbf{V}$-algebra A. Any variety of K-algebras has a terminal object, the trivial K-module 0, and the defining property of the universal enveloping algebra implies that $U_{\mathbf{V}}(0) = K$. So a universal enveloping algebra always has a canonical augmentation

$$\epsilon : U_{\mathbf{V}}(A) \to K\,.$$

Let Ω' and Ω be two sets of equations, cutting out varieties $\mathbf{V}'$ and $\mathbf{V}$ of K-algebras. If $\Omega' \subseteq \Omega$, then any $\mathbf{V}$-algebra is a $\mathbf{V}'$-algebra; write $i : \mathbf{V} \to \mathbf{V}'$ for the inclusion functor. Fix $A \in \mathbf{V}$. There is then a functor

$$i_* : \mathbf{Mod}_A \to \mathbf{Mod}_{iA}$$

that sends M to itself as a K-module, with the same left and right actions by A but now regarded as giving a Beck iA-module structure. This functor is induced by a map of unital associative algebras

$$i^* : U_{\mathbf{V}'}(iA) \to U_{\mathbf{V}}(A)$$

In particular we might take Ω' to be empty, so that $\mathbf{V}' = \mathbf{Mag}_K$ and $U_{\mathbf{V}'}(A) = \mathrm{Tens}_K(A \oplus A)$. For any $\mathbf{V}$ we thus receive a map of unital associative K-algebras

$$\pi : \mathrm{Tens}_K(A \oplus A) \to U_{\mathbf{V}}(A)$$

Denote the composite $\pi \circ \lambda$ by $l : A \to U_{\mathbf{V}}(A)$ and $\pi \circ \rho$ by $r : A \to U_{\mathbf{V}}(A)$. These are K-linear maps, and the sum of their images generates $U_{\mathbf{V}}(A)$ as an associative unital K-algebra; the map π is surjective.

5. Noncommutative partial differentiation

Let **V** be a variety of K-algebras. The universal enveloping algebra of a **V**-algebra A is the quotient of $U_{\mathbf{Mag}_K}(A) = \mathrm{Tens}_K(A \oplus A)$ by an ideal generated by elements determined by the equations defining **V**. These elements are derived by a process of "noncommutative partial differentiation," which we now describe.

Let $Mag(S)$ be the free magma on a set S, and $Ass(X)$ the free associative algebra on the a X. Write

$$\lambda\,,\, \rho : Mag(S) \to Ass(Mag(S) \sqcup Mag(S))$$

for the inclusions of the left and right summands.

Lemma 5.1. *For each $x \in S$ there is a unique map*

$$\frac{\partial}{\partial x} : Mag(S) \to Ass(Mag(S) \sqcup Mag(S))$$

such that

$$\frac{\partial x}{\partial x} = 1\,;$$

for any $y \in S$ with $y \neq x$

$$\frac{\partial y}{\partial x} = 0\,;$$

and for any $\alpha, \beta \in Mag(S)$,

$$\frac{\partial \alpha\beta}{\partial x} = \frac{\partial \alpha}{\partial x}\rho_\beta + \frac{\partial \beta}{\partial x}\lambda_\alpha\,.$$

Proof. This is immediate, since any element of $Mag(S)$ is built up from elements of S by a unique sequence of multiplications. □

For any commutative ring K, this map extends by linearity to

$$\frac{\partial}{\partial x} : KMag(S) \to \mathrm{Tens}_K(KMag(S) \oplus KMag(S))\,.$$

Example 5.2. Define the left- and right-bracketed powers of x by

$$x^{(1} = x\,,\, x^{(n} = x^{(n-1}x\,, \quad x^{1)} = x\,,\, x^{n)} = xx^{n-1)}$$

Then for $n > 1$ we have the following noncommutative analogues of the familiar formula for the derivative of a power:

$$\frac{\partial x^{(n}}{\partial x} = \lambda_{x^{(n-1}} + \lambda_{x^{(n-2}}\rho_x + \cdots + \lambda_x\rho_x^{n-2} + \rho_x^{n-1}$$
$$\frac{\partial x^{n)}}{\partial x} = \rho_{x^{n-1)}} + \rho_{x^{n-2)}}\lambda_x + \cdots + \rho_x\lambda_x^{n-2} + \lambda_x^{n-1}\,.$$

The application of this operation to the determination of the structure of universal enveloping algebras is this: Recall that a variety of K-algebras is cut out by a set Ω of equations. An element of Ω is a pair (S, ω) where S is a finite set and $\omega \in KMag(S)$. If A is a K-algebra, a set map $a : S \to A$ determines a map of K-algebras $KMag(S) \to A$, which we denote by $\omega \mapsto$

$\omega(a)$. Putting this map on both factors gives us $KMag(S) \oplus KMag(S) \to A \oplus A$, and hence to a map of associative unital K-algebras

$$\mathrm{Tens}_K(KMag(S) \oplus KMag(S)) \to \mathrm{Tens}_K(A \oplus A)$$

which we denote again by $\mu \mapsto \mu(a)$. Here is the description of the ideal defining $U_{\mathbf{V}}(A)$.

Proposition 5.3. *Suppose $\mathbf{V}$ is a variety of K-algebras cut out by a set of equations Ω. Let A be a $\mathbf{V}$-algebra. Then the universal enveloping algebra $U_{\mathbf{V}}(A)$ is the quotient of $\mathrm{Tens}_K(A \oplus A)$ by the ideal I generated by the set*

$$\left\{ \frac{\partial \omega}{\partial x}(a) \,:\, (S,\omega) \in \Omega \,,\, x \in S \,,\, a : S \to A \right\} .$$

Proof. Let A be a magmatic K-algebra and M a module for it: so $A \oplus M$ has the structure of a magmatic K-algebra with product given by $(a,m)(b,n) = (ab, an+mb)$. M is then a right module for the magmatic universal enveloping algebra

$$U_{\mathbf{Mag}}(A) = \mathrm{Tens}_K(A \oplus A) \,.$$

Given $\omega \in KMag(S)$, we claim that for any $(a,m) : S \to A \oplus M$:

$$\omega(a,m) = \left(\omega(a), \sum_{x \in S} m_x \frac{\partial \omega}{\partial x}(a) \right) \in A \oplus M \,.$$

Since the definition of the partial derivative is inductive, we proceed by induction. For the base case, we note that it is true if $\omega = x$ for some $x \in S$: then $x(a,m) = (a_x, m_x)$, which agrees with the right hand side since all but one term vanishes in the sum. Then, given $\alpha, \beta \in K\mathbf{Mag}(S)$, we compute

$$\begin{aligned} \alpha\beta(a,m) &= \alpha(a,m)\beta(a,m) \\ &= \left(\alpha(a), \sum m_x \frac{\partial \alpha}{\partial x}(a) \right) \left(\beta(a), \sum m_x \frac{\partial \beta}{\partial x}(a) \right) \\ &= \left(\alpha(a)\beta(a), \sum m_x \frac{\partial \beta}{\partial x} \lambda_{\alpha(a)} + \sum m_x \frac{\partial \alpha}{\partial x} \rho_{\beta(a)} \right) \\ &= \left(\alpha\beta(a), \sum m_x \left(\frac{\partial \beta}{\partial x}(a) \lambda_{\alpha(a)} + \frac{\partial \alpha}{\partial x}(a) \rho_{\beta(a)} \right) \right) \end{aligned}$$

and the factor in the sum is indeed $\dfrac{\partial \alpha\beta}{\partial x}(a)$.

Now suppose that $\mathbf{V}$ is cut out by Ω, that A is a $\mathbf{V}$-algebra, and that M is a Beck A-module, and let $\omega \in \Omega$ and $(a,m) : S(\omega) \to A \oplus M$. Since $A \oplus M$ is a $\mathbf{V}$-algebra, $\omega(a,m) = 0$, so the right hand side of the above equation vanishes. Let $x \in S$, and take $m : S \to M$ be a function that vanishes except at x. We discover that $\partial\omega/\partial x(a)$ vanishes on M. Since M was an arbitrary Beck A-module, this element lies in the ideal I.

Since the noncommutative partial derivatives of the defining equations for $\mathbf{V}$, evaluated on maps to A, simply record the validity of those equations on a Beck A-module, there are no further relations in the ideal I. $\square$

Example 5.4. The equation xy cuts out the variety of K-algebras with trivial multiplication; this is just the category $\mathbf{LMod}_K$ of K-modules. We find

$$\frac{\partial xy}{\partial x} = \rho_y \quad , \qquad \frac{\partial xy}{\partial y} = \lambda_x$$

so for any algebra A in this variety, $U(A) = K$; the category of Beck A-modules is again just the category of K-modules.

Example 5.5. $\mathbf{Ass}_K$ is cut out by $(xy)z - x(yz)$. Compute:

$$\frac{\partial (xy)z}{\partial x} = \frac{\partial xy}{\partial x}\rho_z = \rho_y\rho_z \quad , \qquad \frac{\partial x(yz)}{\partial x} = \rho_{yz}$$
$$\frac{\partial (xy)z}{\partial y} = \frac{\partial xy}{\partial y}\rho_z = \lambda_x\rho_z \quad , \qquad \frac{\partial x(yz)}{\partial y} = \frac{\partial yz}{\partial y}\lambda_x = \rho_z\lambda_x$$
$$\frac{\partial (xy)z}{\partial z} = \lambda_{xy} \quad , \qquad \frac{\partial x(yz)}{\partial z} = \frac{\partial yz}{\partial z}\lambda_x = \lambda_y\lambda_x$$

so the defining relations for $U_{\mathbf{Ass}}(A)$ are

$$r_a r_b = r_{ab}\,, \quad l_a r_b = r_b l_a\,, \quad l_{ab} = l_b l_a$$

for $a, b \in A$. This shows that Beck A-modules in $\mathbf{Ass}_K$ are precisely A_+-bimodules. The quotient of $\mathrm{Tens}_K(A \oplus A)$ by these relations is the "extended K-algebra"

$$U_{\mathbf{Ass}}(A) = A_+^{op} \otimes_K A_+\,.$$

Example 5.6. Commutativity is specified by $xy - yx$. Compute

$$\frac{\partial xy}{\partial x} = \rho_y \quad , \qquad \frac{\partial yx}{\partial x} = \lambda_y$$
$$\frac{\partial xy}{\partial y} = \lambda_x \quad , \qquad \frac{\partial yx}{\partial y} = \rho_x$$

and both equations give us $l_a = r_a$: The corresponding universal enveloping algebra is just $\mathrm{Tens}_K(A)$, independent of the algebra structure on A. A Beck module for a commutative magmatic K-algebra is simply a K-module V together with a K-linear map $V \otimes_K A \to V$.

If we combine these two, we get the category $\mathbf{Com}_K$ of commutative, associative, nonunital K-algebras. Combining $l_a = r_a$ with the the relations in $U_{\mathbf{Ass}}(A)$, we find that a Beck A-module is simply an A_+-module in the usual sense, and

$$U_{\mathbf{Com}}(A) = A_+\,.$$

Example 5.7. The variety $\mathbf{Leib}_K$ of Leibniz algebras [17] over K is cut out by

$$(xy)z - x(yz) - (xz)y\,.$$

Compute:

$$\frac{\partial((xy)z - x(yz) - (xz)y)}{\partial x} = \rho_y\rho_z - \rho_{yz} - \rho_z\rho_y$$
$$\frac{\partial((xy)z - x(yz) - (xz)y)}{\partial y} = \lambda_x\rho_z - \rho_z\lambda_x - \lambda_{xz}$$
$$\frac{\partial((xy)z - x(yz) - (xz)y)}{\partial z} = \lambda_{xy} - \lambda_y\lambda_x - \lambda_x\rho_y\,.$$

This leads to the relations in $U_{\mathbf{Leib}}(A)$:

$$r_{ab} = [r_a, r_b]\,, \quad l_{ab} = [l_a, r_b]\,, \quad (l_a + r_a)l_b = 0$$

(where the bracket denotes the commutator in this associative algebra). If we adjoin the relation xx, to get the variety $\mathbf{Lie}_K$, we find $\lambda_x = -\rho_x$ and so $l_a = -r_a$ in $U_{\mathbf{Lie}}(A)$, and $U_{\mathbf{Lie}}(A)$ is the quotient of $\mathrm{Tens}_K(A)$ by the relations

$$r_{ab} = [r_a, r_b]\,,$$

giving us the usual Lie universal enveloping algebra.

Remark 5.8. This use of the term "universal enveloping algebra" differs from the classical Lie perspective. In that case, one has a "forgetful" functor $\mathbf{Ass}_K \to \mathbf{Lie}_K$ that sends an associative K-algebra to the Lie algebra structure on the K-module A given by $[a, b] = ab - ba$; and U is the left adjoint of this functor. In our generality, there is no "underlying" associative algebra; the universal enveloping algebra has a different defining property. But it turns out to produce the same result in the case of $\mathbf{Lie}_K$. This meaning for the term was explored in the operadic case by Ginzburg and Kapranov in [13], and in the example of Leibniz algebras by Loday and Pirashvili in [17].

Example 5.9. Chataur and Livernet [8] consider "level algebras," defined by the equations

$$xy - yx\,,\ (wx)(yz) - (wy)(xz)\,.$$

The second equation leads to

$$\rho_x\rho_{yz} = \rho_y\rho_{xz}\,,\ \lambda_w\rho_{yz} = \rho_z\lambda_{wy}\,,\ \rho_z\lambda_{wx} = \lambda_w\rho_{xz}\,,\ \lambda_y\lambda_{wx} = \lambda_x\lambda_{wy}\,.$$

Commutativity leads to $\lambda_x = \rho_x$, and hence to the attractive relations in $U_{\mathbf{Lev}}(A)$:

$$U_{\mathbf{Lev}}(A) = \mathrm{Tens}_K(A)/(r_a r_{bc} = r_b r_{ca} = r_c r_{ab}\,,\ a, b, c \in A)\,.$$

Since they are multilinear, the relations can be restricted to hold for a, b, c belonging to a basis for A. The universal enveloping algebra of a trivial level algebra is the tensor algebra on the underlying module.

Example 5.10. Even the variety of K-algebras cut out by $(xx)x$ has interesting universal enveloping algebras: The relations are

$$r_a^2 + l_a r_a + l_{aa} = 0\,.$$

Replacing a by $a+b$ gives

$$r_a r_b + r_b r_a + l_a r_b + l_b r_a + l_{ab} + l_{ba} = 0\,,$$

and since these relations are multilinear it suffices to require them for a, b in a set of K-module generators for A. In the homogeneous case of K^n, with basis $e_1, \ldots, e_n$, let $l_i = l_{e_i}$ and $r_i = r_{e_i}$. The relations are neater if we let $q_i = l_i + r_i$, for then they are

$$q_i r_i\,,\; q_i r_j + q_j r_i\,, \quad i = 1, \ldots, n\,.$$

Example 5.11. The alternative equations differentiate to

$$\begin{gathered}\rho_{xy} - \rho_x \rho_y = \lambda_x \rho_y - \rho_y \lambda_x\,, \quad \lambda_{xx} = \lambda_x \lambda_x \\ \rho_y \rho_y = \rho_{yy}\,, \quad \lambda_{xy} - \lambda_y \lambda_x = \rho_y \lambda_x - \lambda_x \rho_y\,.\end{gathered}$$

so $U_{\mathbf{Alt}}(A)$ is $\mathrm{Tens}_K(A \oplus A)$ modulo the ideal generated by

$$\begin{gathered}r_{bb} = r_b r_b\,, \quad l_{aa} = l_a l_a \\ l_{ab} - l_b l_a = r_b l_a - l_a r_b = r_a r_b - r_{ab}\,.\end{gathered}$$

Remark 5.12. We have three closely related varieties of K-algebras, related by forgetful right adjoints

$$\mathbf{Com}_K \xrightarrow{u} \mathbf{Ass}_K \xrightarrow{u} \mathbf{Alt}_K\,.$$

The left adjoints are given by forming the maximal associative quotient of an alternative algebra, and the maximal commutative quotient of an associative algebra. These functors induce right adjoints

$$\mathbf{Com}_K/A \to \mathbf{Ass}_K/uA$$

for $A \in \mathbf{Com}_K$ and

$$\mathbf{Ass}_K/A \to \mathbf{Alt}_K/uA$$

for $A \in \mathbf{Ass}_K$. As right adjoints, they preserve products, and hence induce functors

$$\mathrm{Ab}(\mathbf{Com}_K/A) \to \mathrm{Ab}(\mathbf{Ass}_K/uA)\,, \quad \mathrm{Ab}(\mathbf{Ass}_K/A) \to \mathrm{Ab}(\mathbf{Alt}_K/uA)\,.$$

These functors can be described by means of ring homomorphisms between the corresponding universal enveloping algebras: There is a K-algebra surjection natural in $A \in \mathbf{Ass}_K$

$$U_{\mathbf{Alt}}(uA) \to U_{\mathbf{Ass}}(A)\,, \quad l_a \mapsto a \otimes 1\,, \quad r_b \mapsto 1 \otimes b\,,$$

and a K-algebra surjection natural in $A \in \mathbf{Com}_K$

$$U_{\mathbf{Ass}}(uA) \to U_{\mathbf{Com}}(A)\,, \quad a \otimes b \mapsto ab\,.$$

Remark 5.13. The reversal endomorphism of **Mag** induces natural isomorphisms

$$U_{\overline{\mathbf{V}}}(\overline{A}) = U_{\mathbf{V}}(A)^{op}$$

that swaps the K-module maps r, l from the K-module A.

Formation of the universal enveloping algebra enjoys a strong base-change property. A homomorphism of commutative rings $f : K \to L$ induces a multiplicative map $f : KMag(S) \to LMag(S)$. By applying this map to the equations defining a variety $\mathbf{V}_K$ of K-algebras, we obtain a base-changed variety $\mathbf{V}_L$ of L-algebras, and a functor $L \otimes_K - : \mathbf{V}_K \to \mathbf{V}_L$.

Proposition 5.14. *Fix this notation, and let $A \in \mathbf{V}_K$. There is a natural map $U_{\mathbf{V}_K}(A) \to U_{\mathbf{V}_L}(L \otimes_K A)$ of unital K-algebras that extends to an isomorphism*

$$L \otimes_K U_{\mathbf{V}_K}(A) \to U_{\mathbf{V}_L}(L \otimes_K A)$$

of unital L-algebras.

Proof. To begin with, $f : K \to L$ induces a multiplicative map

$$f_* : \mathrm{Tens}_K(KMag(S) \oplus KMag(S)) \to \mathrm{Tens}_L(LMag(S) \oplus LMag(S)) .$$

Base-change is compatible with partial differentiation: Given $\omega \in K\mathbf{Mag}(S)$ and $x \in S$,

$$f_* \frac{\partial \omega}{\partial x} = \frac{\partial f \omega}{\partial x} .$$

So for any $\mathbf{V}_K$-algebra A, f_* carries the equations defining $U_{\mathbf{V}_K}(A)$ to the equations defining $U_{\mathbf{V}_L}(L \otimes_K A)$. We obtain the desired homomorphism on universal enveloping algebras. Moreover, the relations in $\mathrm{Tens}_K(A \oplus A)$ defining $U_{\mathbf{V}_K}(A)$ are carried precisely to the relations in $\mathrm{Tens}_L((L \otimes_K A) \oplus (L \otimes_K A))$ defining $U_{\mathbf{V}_L}(L \otimes A)$, so the result follows. □

6. Universal enveloping algebras for $\mathbf{Alt}_K$

Let K^n denote the free K-module of rank n regarded as an alternative K-algebra with trivial product. Write l_i and r_i, $1 \leq i \leq n$, for the images in $U(K^n)$ of the standard basis elements under $l, r : K^n \to U(K^n)$. This associative K-algebra is graded by weight. Clearly $U(K^0) = K$ and $U(K^1)$ has basis $\{1, l_1, r_1, l_1 r_1\}$; in fact $l_1^2 = r_1^2 = l_1 r_1 - r_1 l_1 = 0$.

Theorem 6.1. *Let K be any commutative ring. For each n and k, the K-module $U(K^n)_k$ is free, and*

$$\mathrm{rank}_K U(K^n)_k = \begin{cases} 1 & \text{if } k = 0 \\ 2n & \text{if } k = 1 \\ \frac{3n^2 - n}{2} & \text{if } k = 2 \\ 2\binom{n}{k} & \text{if } k \geq 3 . \end{cases}$$

In particular, $U(K^n)_k = 0$ for $k > n$ as long as $n > 1$. The growth of $\dim_K U(K^n)$ with n is exponential;

$$\dim_K U(K^n) = 2 \cdot 2^n + \frac{n^2 + n - 2}{2} .$$

Proof. In fact we can make a more precise statement, specifying for each k a set of monomials in $S = \{l_1, r_1, \dots, l_n, r_n\}$ forming a basis for $U(K^n)_k$. The first step is to determine a Gröbner basis for the ideal

$$I = \ker(\mathrm{Tens}_K(K^n \oplus K^n) \to U(K^n)) .$$

Order S as shown; order monomials first by weight and within a given weight left-lexicographically. The "leading monomial" in a polynomial will be the least term. The expressions for the generators of I, when applied to the elements of S, are

$$l_i l_i \text{ and } r_i r_i \text{ for all } i$$
$$l_i r_i - r_i l_i \text{ for all } i$$
$$l_j l_i + r_i r_j \text{ and } r_j r_i + l_i l_j \text{ for } j < i$$
$$l_j r_i + r_j r_i - r_i l_j \text{ and } r_j l_i - l_i r_j - r_i r_j \text{ for } j < i$$

The first identities do not imply that $l_a^2 = 0 = r_a^2$ for all a, however, and to get a complete set of relations involving just the basis elements we have to adjoin $l_j l_i + l_i l_j$ and $r_j r_i + r_i r_j$ for $j < i$. Comparing leading terms with the existing relations, we can drop $l_j l_i + l_i l_j$ and $r_j r_i + l_i l_j$ for $j < i$ at the expense of adjoining $l_i l_j - r_i r_j$ for $j < i$. We can also use $r_j r_i + r_i r_j$ to replace $l_j r_i + r_j r_i - r_i l_j$ with $l_j r_i - r_i r_j - r_i l_j$, so that all the relations except one have the effect of replacing an increasing sequence of subscripts by a decreasing one. We then have a "reduced" set of relations, in the sense that there are no repeated leading terms:

- weight 2, length 1: $l_i l_i$ and $r_i r_i$ for all i
- weight 2, length 2: $l_i r_i - r_i l_i$ for all i
- weight 2, length 2: $l_j l_i + r_i r_j$, $r_j r_i + r_i r_j$, and $l_i l_j - r_i r_j$ for $j < i$
- weight 2, length 3: $l_j r_i - r_i l_j - r_i r_j$ and $r_j l_i - l_i r_j - r_i r_j$ for $j < i$

Write R_0 for this basis for I. Notice that there are monomials in the ideal that are not divisible by any leading entry in this list: for example

$$r_i r_j l_j = l_i(l_j l_j) - (l_i l_j - r_i r_j) l_j .$$

So this is not a Gröbner basis. To get one, we need to inspect overlaps and adjoin any overlap differences that can't be reduced to zero using the leading monomials of elements of R_0, and then repeat that process if necessary. This is not hard to do by hand, and it leads to the following additional basis elements.

- weight 3, length 1: $r_i r_j l_j$, $r_i l_i r_j$, $l_i r_j l_j$ for $j < i$
- weight 3, length 2: $l_i r_j l_k - r_i l_j r_k$, $l_i r_j r_k - r_i r_j l_k$ for $k < j < i$
- weight 3, length 3: $r_i l_j r_k + r_i r_j l_k + r_i r_j r_k$ for $k < j < i$

Let R_1 be the set R_0 with these new weight 3 relations adjoined. One must now inspect overlaps of the relations with the old ones and with each other. The result of a hand calculation is that all the overlap differences reduce to 0 using R_1; we have obtained a Gröbner basis for I. Both these calculations were checked using a Python script for $n \leq 5$. This suffices to verify the general case, since at most 5 indices are involved in any overlap of elements of R_1, and the $n = 5$ case is general enough to cover all possibilities.

Suppose now that K is a field. As described in §7, the set of "normal monomials," that is, those not divisible by the leading entry of any element of the Gröbner basis, projects to a vector space basis for the quotient K-algebra $U(K^n) = \mathrm{Tens}_K(K^n \oplus K^n)/I$.

Since all the relations are of weight at least 2, we find that $\{1\}$ is a basis for $U(K^n)_0$ and $\{l_1, r_1, \ldots, l_n, r_n\}$ is a basis for $U(K^n)_1$.

The first relation forbids repeated letters. The second implies that if a subscript is repeated it must be in the order rl. The third and fourth relations force the indices in a normal monomial to be decreasing, and the $l_i l_j$ relations imply that there can be no repeated l's. The other relations are of weight 3, so we find that $U(K^n)_2$ has basis

$$\{r_i l_i\} \sqcup \{l_i r_j,\, r_i l_j,\, r_i r_j \,:\, i < j\}$$

Thus

$$\dim_K U(K^n)_2 = n + 3\binom{n}{2} = \frac{3n^2 - n}{2}\,.$$

The first and third weight 3 length 1 relations imply that $r_j l_j$ can only occur at the beginning of a normal monomial, but since ll can never occur the second relation in that list rules that out as well. So all normal monomials have strictly increasing subscripts.

Suppose an l occurs in a monomial of weight at least 3 but not at the end. If it is the next-to-last entry, it must be in the pattern rlr. If it is earlier, it must be in rlr or lrx for x either r or l. All these are excluded by the final three relations.

So we discover that the normal monomials of weight $k \geq 3$ are exactly those with strictly increasing subscripts and which consist entirely of r's except possibly for the last entry. There are $2\binom{n}{k}$ of these.

This concludes the proof in case K is a field. For the general case, note first that since the tensor algebra is a finitely generated K-module in each weight, we know that $U(K^n)$ is too. Suppose now that $K = \mathbb{Z}$. By base change 5.14, $U(\mathbb{Z}^n) \otimes \mathbb{Q} = U(\mathbb{Q}^n)$, and for any prime p, $U(\mathbb{Z}^n) \otimes \mathbb{F}_p = U(\mathbb{F}_p^n)$. Consequently, if $U(\mathbb{Z}^n)_k$ had a nonzero element of order p, the ranks of $U(\mathbb{F}_p^n)_k$ and $U(\mathbb{Q}^n)_k$ would differ. But our calculation showed that these dimensions were independent of the field.

Finally, we can appeal to base-change again to pass to an arbitrary commutative base ring K. □

7. Appendix: Gröbner bases

We elaborate briefly on the "Buchberger algorithm" for expanding a basis of an ideal in a tensor algebra to a Gröbner basis.

Let S be a set equipped with a well-founded partial order: a partial order in which every strictly decreasing sequence is finite. The free monoid B generated by S inherits a partial order – first by weight, and left-lexicographically within a given weight – that is again well-founded.

Let K be a field. The free K-module generated by B is the tensor algebra on S, $T = KB$.

Any nonzero element in T has a "leading monomial," the least monomial occurring with nonzero coefficient in its expression as a linear combination of elements of B. Write

$$LM : T^* \to B$$

for this function, where we write $I^* = I - \{0\}$ for any ideal I in T. The partial order on B pulls back to a well-founded weak ordering on T^*.

For $u, v \in B$, say that u divides v, $u|v$, if there are monomials s, t such that $v = sut$. Divisiblity is transitive.

Definition 7.1. *Let I be an ideal in T. A subset G of I^* is a* Gröbner basis *for I if for all $r \in I^*$ there is $g \in G$ such that $LM(g)|LM(r)$.*

A Gröbner basis G for I yields an efficient algorithm for deciding whether $z \in T^*$ lies in I^*. We may assume z is monic; that is, the coefficient of $LM(z)$ is 1. If $LM(z)$ is not divisible by $LM(g)$ for any $g \in G$, then $z \notin I^*$. If instead

$$LM(z) = sLM(g)t$$

for some $g \in G$ and $s, t \in B$, then form

$$z' = z - sgt\,.$$

This is in I if and only if z is. If $z' = 0$, we have established that $z \in I$. If not, at least we can say that the leading monomials cancel, so z' is strictly less than z in the well order on T^*. Divide z' by the coefficient of its leading monomial, and repeat this process, which terminates because the ordering is well-founded. The original element z is in I if and only if the element you wind up with is 0, in which case you have written z as an explicit linear combination of terms divisible by elements of G. This shows that G generates I as an ideal.

Given a subset R of I that generates I as an ideal, we attempt to enlarge it to a Gröbner basis for I. We may assume that each element of R is monic. Say that $r, s \in R$ *overlap* if there are $a, b, c \in B$ such that $LM(r) = ab$ and $LM(s) = bc$. For each overlapping pair r, s, form the "overlap difference"

$$rc - as\,.$$

This difference again lies in I, but exhibits a new leading monomial, one that was hidden in the original generating set R.

Now use the reduction process with respect to R to simplify each of the overlap differences.

Then adjoin all the nonzero reduced overlap differences, made monic, to the set R to get a new generating set R'. One may want to precede this step by doing some linear algebra to find a simpler basis for the space spanned by these polynomials.

Proposition 7.2 ([16], Prop. 5.2,p. 95)**.** *If R is finite, this process terminates, and the result is a Gröbner basis for I.*

A Gröbner basis is minimal (no subset is a Gröbner basis for the same ideal) if and only if it is reduced (no divisibility relations among its leading monomials) [16, p. 67]. One may always refine a Gröbner basis to a reduced one.

A monomial u is *normal* mod I if it is not divisible by any element of $LM(I^*)$. If G is a Gröbner basis for I, it suffices to check non-divisibility by the leading monomials of elements of G: Suppose that $r \in I^*$ is such that $LM(r)|u$, and let $g \in G$ be such that $LM(g)|LM(r)$: then $LM(g)|u$.

Proposition 7.3 ([16], Prop. 3.3, p. 70)**.** *The set of monomials that are normal mod I projects to a vector space basis for T/I.*

References

[1] J. Adámek and J. Rosický, *Locally presentable and accessible categories* London Mathematical Society Lecture Note Series 189, 1994.
[2] J. Adámek, J. Rosický, and M.E Vitale, *Algebraic Theories*, Cambridge Tracts in Mathematics 184, 2011.
[3] J. C. Baez, The octonions, Bulletin of the American Mathematical Society 39 (2002) 145–205.
[4] M. Barr, *Acyclic models*, Centre de Recherches Mathématiques Monograph Series 17, 2002.
[5] J. M. Beck, Triples, algebras, and cohomology, Reprints in Theory and Applications of Categories 2 (2003) 1–59.
[6] N. Bourbaki, *Elements of Mathematics: Algebra I, Chapters 1–3* Hermann, 1971.
[7] R. H. Bruck and E. Kleinfeld, The structure of alternative division algebras, Proceedings of the American Mathematical Society 2 (1951) 878–890.
[8] D. Chataur and M. Livernet, Adem-Cartan algebras, Communications in Algebra 33 (2005) 4337–4360.
[9] Ioannis Dokas, Triple cohomology and divided powers algebras in prime characteristic, Journal of the Australian Mathematical Society 87 (2009) 161–173.
[10] S. Eilenberg, Extensions of general algebras, Annales de la Société Polonaise de Mathématique 21 (1948) 125–134.
[11] M. Frankland, Behavior of Quillen (co)homology with respect to adjunctions, Homology, Homotopy, and Applications 17:1 (2015) 67–109.
[12] P. Freyd, *Abelian Categories*, Harper and Row, 1964.
[13] V. Ginzburg and M. Kapranov, Koszul duality for operads, Duke Mathematical Journal 76:1 (1994) 203–272.
[14] L. Hesselholt, The big de Rham-Witt complex, Acta Mathematica 214 (2015) 135–207.
[15] N. Jacobson, Structure of alternative and Jordan bimodules, Osaka Mathematical Journal 6:1 (1954) 1–72.
[16] H. Li, *Gröbner Bases in Ring Theory*, World Scientific, 2012.
[17] J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology, Mathematische Annalen 296 (1993) 139–158.
[18] Saunders Mac Lane, *Categories for the Working Mathematician*

[19] R. D. Schafer, Representations of alternative algebras, Transactions of the American Mathematical Society 72 (1952) 1–17.
[20] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, 1966.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139
E-mail address: dhankhar@mit.edu

Department of Mathematics, Birzeit University, Ramallah, Palestine
E-mail address: aliraeedtahbub@hotmail.com

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139
E-mail address: hrm@math.mit.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139
E-mail address: victor05@mit.edu